MODULE 1

Unit 1	Trigonometric Ratios I
Unit 2	Trigonometric Ratios II
Unit 3	Inverse of Trigonometric Ratios

UNIT 1 TRIGONOMETRIC RATIOS I

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1.0 INTRODUCTION

Before starting any discussion in trigonometric ratios, you should be able to:

(i) Identify the sides of a right-angled triangle in relation to a marked angle in the triangle. If this is not the case, do not worry. You can quickly go through this now:

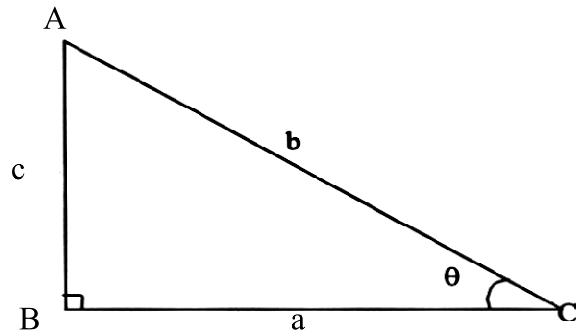


Fig. 1.1: Right-Angled Triangle ABC

Figure 1.1 shows a right-angled triangle ABC, right angled at B, with angle at C marked and the sides marked a, b, c,

AC = b is called the hypothenus

AB = c i.e. the side facing the marked angle at C is called the opposite side of the angle at C adjacent side to the angle at C. opposite side of the angle at C adjacent side to the angle at C.

(ii) Again, you should recall that the ratios of two numbers "x and y" can either be expressed as x/y or $^{y}/_{x}$. If you have forgotten this, please, refresh your memories for this is important in the unit you are about to study.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

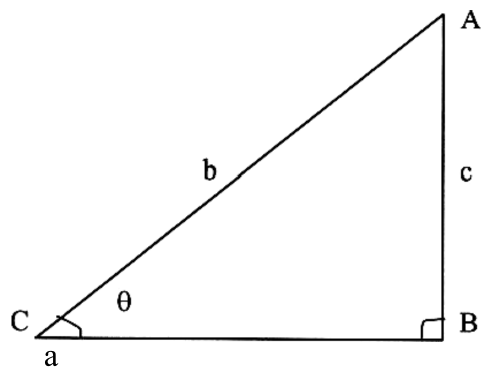
- define trigonometric ratios of a given angle
- state the relationship between the trigonometric ratios
- locate the quadrant of the trigonometric ratios of given angles
- find the basic angles of given angles.

3.0 MAIN CONTENT

3.1.1 Trigonometric Ratios

Having refreshed your minds on the sides of a right-angled triangle and the concept of ratios you are now ready to study the trigonometric ratios (sine, cosine and tangent).

This has to do with the ratio of the sides of a right-angled triangle. Here is an example.



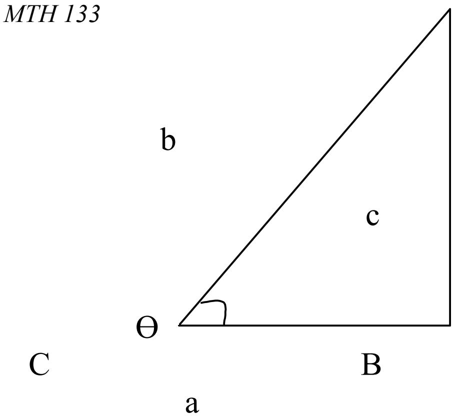
In ΔABC, with A < B = 90° and < C = and the sides of ΔABC, marked a, b, c, respectively, then $\frac{AB}{AC} = \frac{c}{a}$ where AC = b =

$\frac{\text{opposite side to the angle at C}}{\text{Hypotenuse}}$

= sine θ or simply sin θ

In ΔABC see Figure 1.12 below

A



= $\frac{\text{adjacent side to the angle C}}{\text{Hypotenuse}}$ is called Cosine θ or simply Cos θ

Alos, in and in Figure 1.13

$$\frac{AB}{BC} = \frac{c}{a}$$

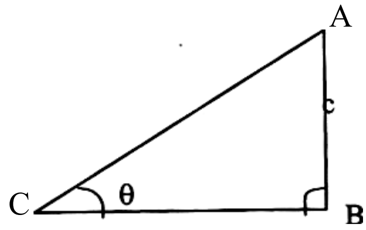


Fig. 1.13

= $\frac{\text{Opposite side to the angle C}}{\text{Adjacent side to the angle C}}$

is called tangent θ or tan θ from the above ratios, you can see that

$$\frac{\sin\theta}{\cos\theta} = \frac{\text{opposite side}}{\text{hypothenus}} \div \frac{\text{adjacent side}}{\text{hypothenus}}$$

Using the notation of the sides of ΔABC

$$\frac{\sin\theta}{\cos\theta} = \left(\frac{c}{b}\right) \div \left(\frac{a}{b}\right)$$

$$\frac{c}{a} = \frac{\text{oppositeside}}{\text{adjacent side}}$$

tan = θ

In the above, at an acute angle and with the knowledge that the sum of the interior angles of a triangle is 180^o. What do you think will happen to the trigonometric ratios? This takes us to the relationships between trigonometric ratios.

3.1.2 Relationship between Trigonometric Ratios

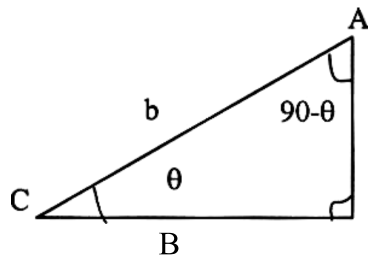


Fig. 1.14

In ΔABC in Figure 1.14 with the usual notations < B = 90^0 and < C = θ, therefore < A = 90 - θ. Once more, finding the trigonometric ratios in relation to the angle at A.

$\text{Sin}\,(90^o - \theta) = \frac{BC}{AC} = \frac{a}{b} = \frac{\text{Opposite side to angle A}}{\text{hypothenus}}$

= COs θ

$\text{Sin}\,(90^o - \theta) = \frac{AB}{AC} = \frac{c}{b} = \frac{\text{opposite side to angle A}}{\text{hypothenus}}$

= Sin θ

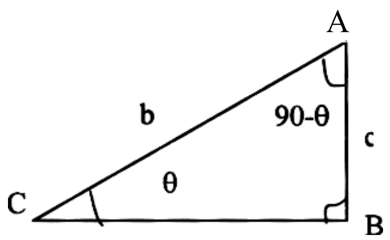
You might be wondered what happens to tan (90^0-0), this will be discussed later.

In summary, given ΔABC as shown

Sin θ = Cos (90^0 - θ)

Cos θ = sin (90° -θ) and

$\frac{\text{Sin}\,\theta}{\text{Cos}\theta} = \tan\theta$



The conclusion from the summary of these trigonometric ratios is that the sine of an acute angle equals the cosine of its complement and vice versa. Thus sin 30^0 = cos 60^0, cos 50° = sin 40^0 etc.(these angles are called complementary angles because their sum is 90^o 1 .e. 30° + 60° = 90°, 50° + 40 = 90^oetc)

Now go through the examples above carefully and try this exercise.

(1) Find the value of θ in the following

(i) $\cos \theta° = \sin \theta$ (ii) $\sin 35° = \cos \theta$

(iii) $\sin 12° = \text{Cos } \theta$

(iv) $\cos 73° = \text{Sin } \theta$. In case you are finding it difficult, the following are the solutions.

Solutions:

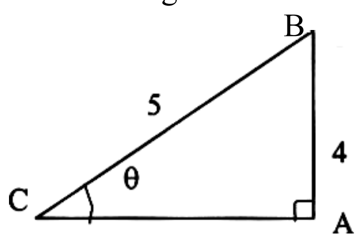
(i) $\text{Cos } 50° = \sin (90° - \theta)$
$= \sin (90° - 50°) = \sin 40°$ (because $50° + 40° = 90°$)

(ii) $\text{Sin } 35° = \cos (90° - 35°)$
$= \text{Cos } 55°$ (since $35° + 55° = 90°$)

(iii) $\text{Sin } 12° = \text{Cos } (90° - 12°)$
$= \text{Cos } 78°$

(iv) $\text{Cos } 73° = \text{Sin } (90° - 73°)$

(2) Find the trigonometric ratios in their following triangle



Solution:

Since there is the measurement of a side missing i.e. AC, and the triangle is right - angled Δ, Using Pythagoras theorem to find the missing side

$BC^2 = AB^2 + AC^2$ (Pythagoras theorem) Substituting for the sides

$5^2 = 4^2 + AC^2$

$25 = 16 + AC^2$

$25 - 16 = AC^2$

$9 = AC^2$

∴ $AC = \sqrt{9} = 3$, then since.

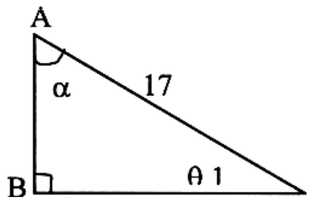
$\sin \theta = \frac{AB}{BC} = \frac{4}{5} = 0.8$

$\cos\theta = \frac{AC}{BC} = \frac{3}{5} = 0.6$

$\tan \theta = \frac{AB}{AC} = \frac{4}{3} = 1.33°$

(3) In the following, angle is acute and angle α is acute. Find the following trigonometric ratios.

8



15 (a) sin α (b) cos α (c) tan α (d) cos θ
(e) sin θ
(f) tan θ

Solutions:

(a) $\sin \alpha = \frac{BC}{AC} = \frac{15}{17}$

(b) $\cos \alpha = \frac{AB}{AC} = \frac{8}{17}$

(c) $\tan \alpha = \frac{BC}{AB} = \frac{15}{8}$

(d) $\cos \theta = \frac{BC}{AC} = \frac{15}{17}$

(e) $\sin \theta = \frac{AB}{AC} = \frac{8}{17}$

(f) tang $= \frac{AB}{BC} = \frac{8}{15}$

You can notice from example (3) that since the sum α and θ is 90° (i.e. α + θ 90°) that:
Sin α= Cos and Cos α= Sin. This again shows that αand are complementary angles.

Having known what trigonometric ratios are, you will now proceed to finding trigonometric ratios of any angle.

3.3 Trigonometric Ratios of Any Angle

It is possible to determine to some extent the trigonometric ratios of all angles using the acute angles in relation to the right-angled triangle. But since all problems concerning triangles are not only meant for right angle triangles, ~it is then good to extend the concept of the trigonometric ratios to angles of any size (i.e. between 0° and any angle).

To achieve the above, you take a unit circle i.e. a circle of radius I unit, drawn

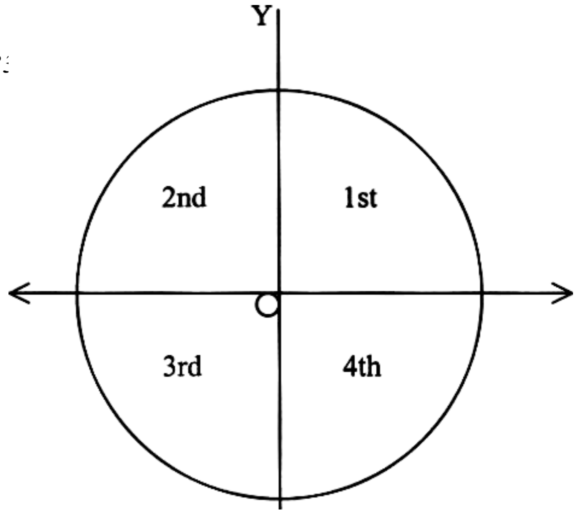


Fig. 2.0

In the Cartesian plane (x and y plane) the circle is divided into four equal parts each of which is called a quadrant (1^{st}, 2^{nd}, 3^{rd}. 4th respectively). Angles are either measured positively in an anti-clockwise direction (Figure 2.1)

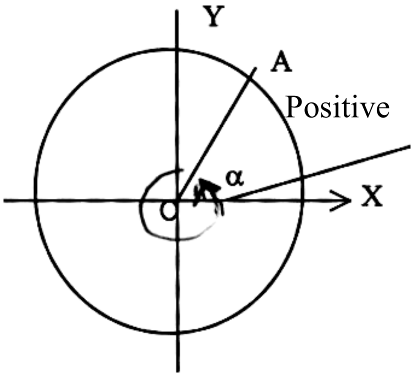


Fig. 2.1

Or negatively in a clockwise direction.

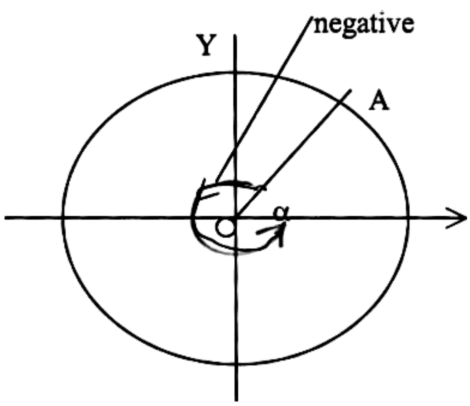
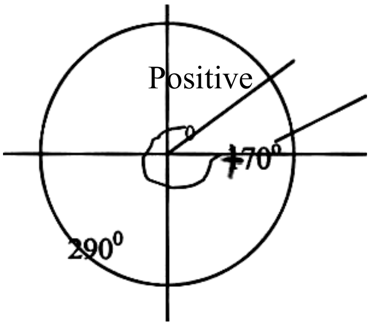
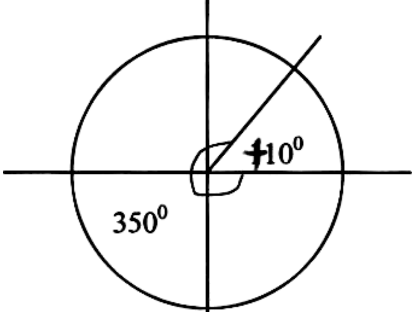


Fig. 2.2

Example. In the diagrams below



+70° = - 290°



+ 10° = -350

Note: Since this concerns angles at a point their sum is 360°. But angles of sizes greater than 360° will always lie in any of the four quadrants. This is determined by first trying to find out how many revolutions (one completed revolution = 360°) there are contained in that angle.

For example, (b) 390° contains 1(360°) plus 30° i.e. 390° = 360° + 30°, 30° is called the basic angle of 390° and since 30° is in the first quadrant, 390° is also in the first quadrant. (a) 600° = 360° + 240°, since 240° is in the third quadrant, 600° is also in the third quadrant.

To find the basic angle of any given angle subtract 360° (1 complete revolution) from the given angle until the remainder is an angle less than 360° , then locate the quadrant in which the remainder falls that becomes the quadrant of the angle.
Now have fun with this exercise,

Exercise:
Find the basic angles of the following and hence indicate the quadrants in which they fell.

(1)	670°	(2)	740°	(3)	1998°
(4)	2002°	(5)	2106°	(6)	544°

Are you happy? Now, move to the next step.

To determine the signs whether positive or negative of the angles and their trigonometric ratios in the four quadrants;

First, choose any point P(x, y) on the circle and O is the center of the circle.

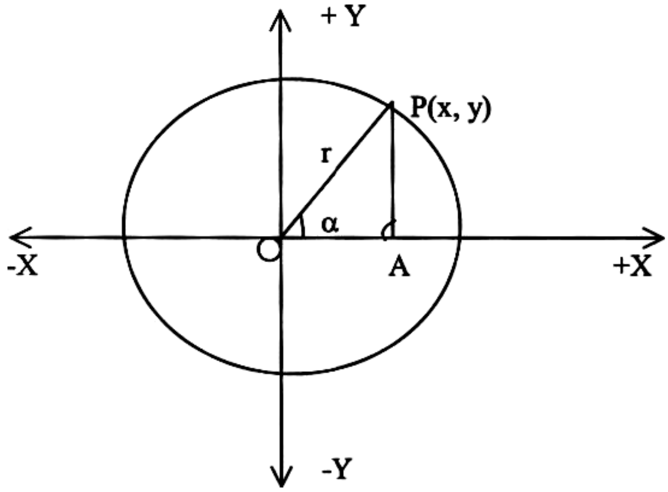


Fig. 2.3

P = r, is the radius and OP makes an angle of αwith the positive x - axis.

Since P is any point, OP is rotated about 0 in the anti-clockwise direction, Hence, in the 1st quadrant ($0° < \theta < 90°$), using your knowledge of trigonometric ratios.

Sin α= $\frac{PA}{P} = \frac{+y}{+r}$ = y/r is positive

cos α = $\frac{A}{P} = \frac{+x}{+r}$ = x/r is also positive

tan α = $\frac{P}{A} = \frac{+y}{+x}$ = y/x is also positive

Therefore in first quadrant (acute angles) all the trigonometric ratios are positive. 2nd quadrant ($90 < \alpha° < 180°$) (obtuse angles)

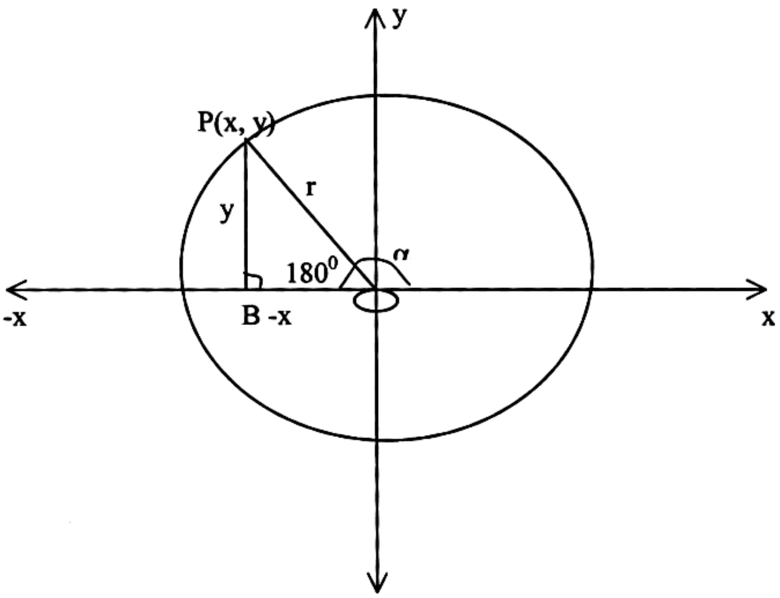


Fig. 2.4

In ΔPBO , < at O is 180 - α, here BO is - x (it lies on the negative x axis) but y and r are positive. The trigonometric ratios are:

$\text{Sin } (180 - \alpha) = \frac{PB}{PO} = \frac{+y}{+r} = y/r$ is positive

$\text{Cos } (180 - \alpha) = \frac{BO}{PO} = \frac{-x}{+r} = -x/r$ is negative

$\text{Tan } (180 - \alpha) = \frac{PB}{BO} = \frac{+y}{-r} = -y/x$ is negative

So, only the sine of the obtuse angle is positive, the other trigonometric ratios are negative. Guess what happens in the 3rd quadrant (reflex angles).

3rd quadrant 180 < α° < 270° (reflex angles)

Note P = r (i.e.) the radius is always positive. Reference is made to 180°, so the angle is (180 + α)° or α- 180°

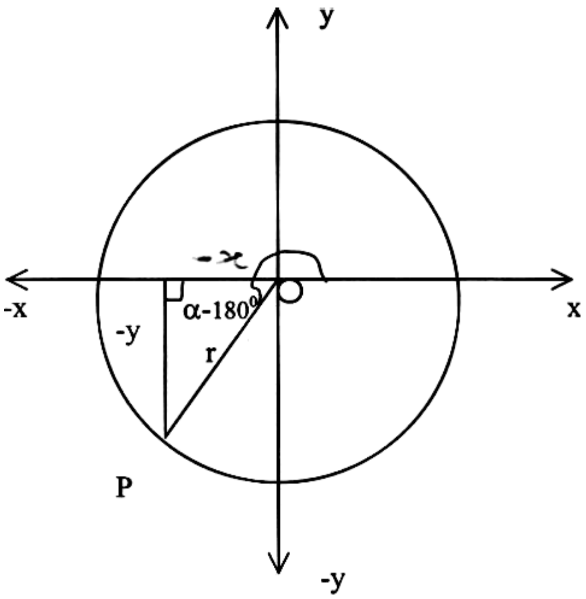


Fig. 2.5

Sin(α-180°) –y/+r = - y/r which is negative

Cos(α-180°) = -x/r = - x/r is negative

tan(α-180°) - - y/-x is positive

so if the angle α lies between 180° and 270° the sine, cosine o9f that angle are negative while the tangent is positive.

4th quadrant 270° <α< 360° (Double Reflex angles) y

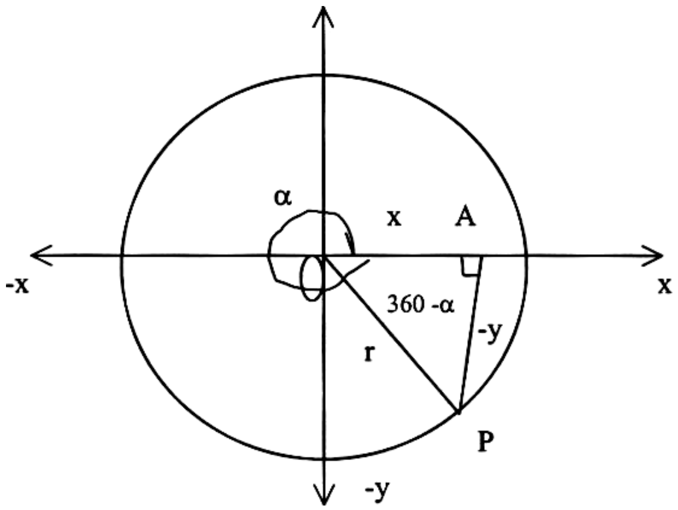


Fig. 2.6

Here PA is negative but OA and OP are positive. Sin (360 – α) = - y/+r = -y/r is negative

Cos (360 – α) +x/+r = x/r is positive

Tan (360 - α) = -y/+x = -- y/x is negative.

Here again sine and tangent of any angle that lies between 270° and 360° are negative the cosine of that angle is positive.

Looking at the figures above, it is seen that the sign of a cosine is similar to the sign of the x - axis(and coordinate) while the sign of a sine is similar to the sign of y - coordinate (i.e. y - axis). The signs can then be written in the four quadrants as shown below see fig: 2.7

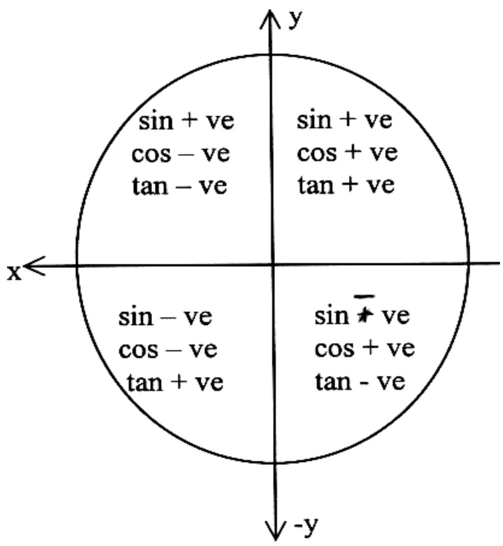


Fig. 2.7

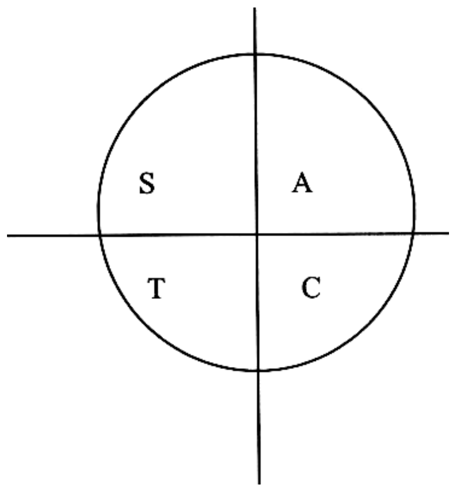


Fig. 2.8

Figure 2.8 is a summary of the signs in their respective quadrants, thus going in the anticlockwise direction, the acronym is;

(1) CAST(from the 4th to 1st to 2nd and then 3rd) (ii) ACTS (from 1st → 4^{th} → 3rd then 2nd)

Clockwise

(iii) All Science Teachers Cooperate (ASTC) (from the 1^{st} 2nd → 3rd → 4th). The letters in figure 2,8 (marked quadrants) show the trigonometric ratios that are positive.

(iv) SACT (2nd → 1st → 4th → 3rd) (v) TASC (3rd → 2nd → 1st → 4th)

Example:
Indicate the quadrants of the following angles and state whether their trigonometric ratios of each is positive or negative.
(1) 155° (11) 525° (iii) 62° (iv) 310° (v) 233°

Solution:
(1) 155° lies between 90° and 180° and therefore is in the 2nd quadrant. The sine of 155° is positive while the cosine and tangent, of 155° are negative. Thus: sin 155° is + ve but cos 155° and tan 155° are negative using the tabular form

No	Angles	Quadrant	Positive trig, Ratios	Negative trig. ratios
1	155^o	2nd	Sin	cos and tan
2	525" =360" +165° the basic angle is 165°	2nd	Sin	cos and tan

3	62^o	Ist	sin, cos and tan	none
4	310^o	4th	Cos	sin, and tan
5	233^o	3rd	Tan	sin and cos

Alternatively, the solution can be thus

1. 155° is in the 2nd quadrant, here only the sin and cosec are positive. s in(1550) = +sin (180 - 155°) =sin 25°
   cos 155° = - cos (180 - 155°) = - cos 25°
   tan 155° = - tan (180 - 155) = - tan 25°

2. 525°; the basic angle of 525° is gotten by
   525° = 360° + 160° (one complete revolution plus 165°)
   .'. 525 = 165 the basic angle lies in the 2nd quadrant and so 525 is in the 2nd quadrant where only the sin is positive
   sin 525° = sin 165° = sin (180 - 165) = sin 15°
   cos 525° = - cos (180-165) - cos 15°
   tan 525° = - tan(180 - 165) = - tan 15°

3. 62°, this is in the first quadrant, where all the trig. Ratios are positive, therefore
   sin 62° = + sin 62°; cos 62° = + cos 62° ; tan 62° = + tan 62°;

4. 310° is in the 4th quadrant where only the cosine is positive, thence. sin 310^0 = - sin (360 - 310) = - sin 50°
   cos 310° = + cos (360 - 310) = + cos 50°
   tan 310° = - tan (360 -310) = - tan 50°

5. 233° is in the 3rd quadrant, only tan is positive, so:
   sin 233° = - sin (233 - 180°) = - sin 53° cos 233° = -cos (233 - 180°) = - cos 53° tan 233° = + tan (233 - 180°) = + tan 53°

Exercise 2.1
Show in which of the quadrant each of the following angles occur and state whether the trigonometric ratio of the angle is positive or negative.

(1)	100^o	(2)	110^o	(3)	123^o	(4)	42^o	(5)	20^o
(6)	231^o	(7)	268^o	(8)	312^o	(9)	591^o	(10)	1999^o

Solutions:

(1)	2^{nd}, only sin + ve	(2)	2^{nd}, only sin + ve
(3)	2ns, only sin + ve	(4)	1^{st} all + ve
(5)	1^{st} all positive	(6)	3^{rd}, only tan + ve

(7)	3^{rd}, only tan +ve	(8)	4^{th}, only cos + ve
(9)	3^{rd}, only tan +ve	(10)	3^{rd}, only tan +ve

4.0 CONCLUSION

In this unit, you have learnt the definition of the trigonometric ratios sine, cosine and tangent and how to find the trigonometric ratios of any given angle. You should have also learnt that the value of any angle depended on its basic angle and its sign depends on the quadrant in which it is found. Thou now understand that the most commonly used trigonometric ratios are the sine , cosine and tangent; and the basic angle lies between O° and 360° i.e. O° ← θ ← 360°

5.0 SUMMARY

In this unit, you have seen that the trigonometric ratios with respect to a right- angled triangle is:

Sin = $\frac{Opposite}{Hypothenus}$ i.e. SOH

Cos = $\frac{Adjacent}{Hypothenus}$ i.e. CAH

Tan = $\frac{Opposite}{adjacent}$ i.e. TOA

Hence, the acronym SOH CAH TOA which is a combination of the above meaning can be used to remember the trigonometric ratios Again, you saw the relationships between the trigonometric ratios 0 the sine of cosine of an acute angle equals The cosine or sine of its complementary angle. That i.e.

(1) $\sin\theta = \cos(90 - \theta)$
(2) $\cos\theta = \sin(90 - \theta)$
(3) $\sin(90 + \theta) = \cos\theta$
(4) $\cos(90 + \theta) = -\sin\theta$

for obtuse angle

(2) $\sin(180 - \theta) = \sin\theta$
$\cos(180 - \theta) = -\cos\theta$
$\sin(180 + \theta) = \sin\theta$
$\cos(180 + \theta) = -\cos\theta$

(3) $\sin(\theta - 180) = -\sin\theta$
$\cos(\theta - 180) = -\cos\theta$

(5) $\sin(360 - \theta) = -\sin\theta$
(6) $\cos(360 - \theta) = \cos\theta$

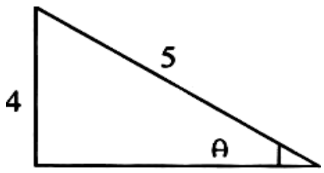
6.0 TUTOR-MARKED ASSIGNMENT

Find the values of y in the following equations

1. sin y = cos 48°
2. cos y = sin 280 33^{1}
3. sin (90 - y) = cos 72° 31
4. cos(90 - y) = sin 56° 47^{1}
5. find the value of sin 0 and cos 0 if tan 0 = 43

TUTOR MARKED ASSIGNMENT: MARKING SCHEME

(1) Y = 42
(2) Y =61 27
(3) 72 31 (4) 56 47
(5) (a)sin θ = 4/5



(b) cos θ = 3/5
2 points each.

7.0 REFERENCES/FURTHER READING

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This list is not exhaustive, you can use any mathematics textbook no matter the level it is written for, to enable you have a good understanding of the unit. There are a lot of mathematics texts in the market and libraries, feel free to use any.

UNIT 2 TRIGONOMETRIC RATIOS II

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1.0 INTRODUCTION

In the previous unit, you learnt about the basic trigonometric ratios - sine, cosine and tangent. You also learnt the relationship between the sine and cosine of any angle, nothing was mentioned about the relationship of the tangent except that it is the sine of an angle over its cosine. Also in our discussion, from our definition of ratios only one aspect is treated i.e. Xy or x : y what happens when it is y : x **or** y/x. An attempt to answer this question will take us to the unit on the reciprocals of trigonometric ratios - secant, cosecant and cotangent.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the reciprocals of trigonometric ratios in relation to the right- angled triangle
- establish the relationship between the six trigonometric ratios
- use trigonometric tables to find values of given angles.

3.0 MAIN CONTENT

3.1 Trigonometric Ratios II

From the previous units, using DABC, right-angled and B and with the usual notations fig. 2.1 (a) the knowledge of the ratio of two numbers "x and y" expressed as x/y was used to find the sine, cosine and tangent of θ . In this unit, the expressed as y/x will be used thus in Figure 2.1 (a)

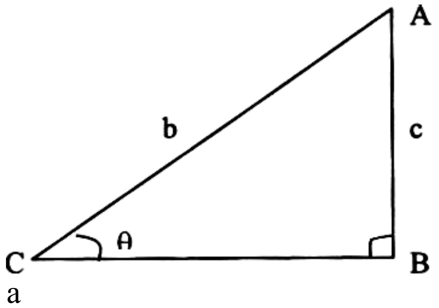


Fig. 2.1

$$\text{Sin } \theta = \frac{AB}{AC} = \frac{c}{b}$$

$$\text{Cos } \theta = \frac{BC}{AC} = \frac{a}{b}$$

$$\text{Tan } \theta = \frac{AB}{BC} = \frac{c}{a}$$

Now if this relationship is viewed in this order.

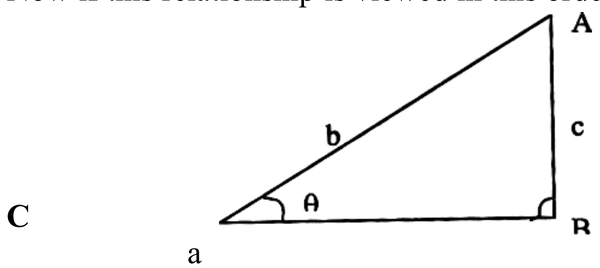


Fig. 2.1 (b)

$$\frac{AC}{AB} = \frac{hypothenus}{opposite} = \frac{b}{c}$$ it is called cosecants or cosecs

$$\frac{BC}{AB} = \frac{adjacent}{opposite} = \frac{a}{c}$$ = is called cotangent opposite or Cot

Now study the above ratios carefully, what can you say of their relationship? This leads us to the following sub-heading

3.2 Relationships between the Trigonometric Ratios

As you can see $\sin\theta$ and cosec θ for example are related in the sense that

$$\text{Sin}\,\theta = \frac{opposite}{hypothenus} = \frac{c}{b}$$ from Figure 2.1(a)

and cosec $\theta = \frac{hypothenus}{opposite} = \frac{b}{c}$ from Figure 2.1(b)

which means that

cosec θ =

$$\frac{\frac{1}{opposite}}{hypothenus} = \frac{\frac{1}{c}}{b}$$

$$= \frac{hypothenus}{opposite} = \frac{b}{c}$$

This then means that cosec θ is the reciprocal of sin θ and sin θ is the reciprocal of cosec θ.

Exercise l:

(i) Find the other reciprocals. Now try this; the above example serves as a guide.

(ii) Verify that cos θ / sin θ = cot θ for any triangle. Is this surprising?

That is the beauty of the trigonometric ratios.

Note from the sum of angles of a triangle giving 180°, the following relations can be proved.

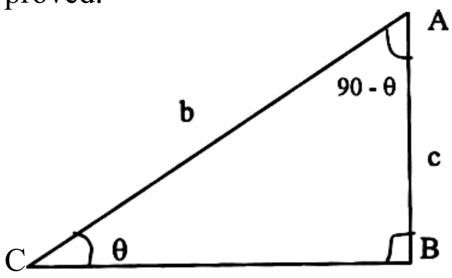


Fig. 2.22

You should recall that in Unit 1,

Sin $(90^{0} - \theta) = \cos\theta$ and

Cos $(90° - \theta) = \sin\theta$ now let us, see the tangent. Tan $(90° - \theta)$ = BC/AC in Figure 2.2

i.e. = a/c = cot θ

Also sec $(90^{0} - \theta)$ = cosec θ. This brings us to the conclusion that the tangent of an acute angle is equal to the cotangent of its complement. I.e. cot 30° = tan 60° and tan 10° = cot 80°; also sec 10° = cosec 80°.

Now go through these examples:

1. Find the value of θ in the following

(a) sec = cosec 30° (b) sin 50 = ?
(c) cot 20° = tan (d) sec 40° = cosec

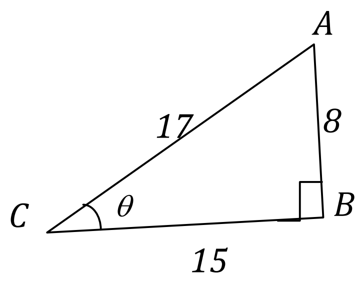
Solution:

(a) cosec 30° = sec (90 - θ)
cosec 30° = sec (90 - 30°) = sec 60° (b) 1/sin 50° = cosec 50°

(c) cot 20° = tan (90° - θ)
=tan (90° - 20°) = tan 70°

(d) sec 40° = cosec (90 - θ)
= cosec(90 - 40°) = cosec 50°

2. In the diagram, on the right find the following: (a) sec θ
(b) cosec θ
(c) cot θ



(a) $\sec\theta = \frac{hypothenus}{adjacent} = \frac{1}{\cos\theta} = \frac{17}{15}$

(b) $\cos ec\theta = \frac{hypothenus}{opposite} = \frac{1}{\sin\theta} = \frac{17}{8}$

(c) $\cot\theta = \frac{adjacent}{opposite} = \frac{1}{\tan\theta} = \frac{15}{8}$

Now move to the next step, the relationship between trigonometric ratios of other angles.

It has been established that:

The secant of any angle is the reciprocal of the cosine of the angle i.e.

(1) sec θ = 1/cos θ
(2) cosec θ = 1/sin θ and

(3) cotan θ = $1/\tan\theta$

It then means that whatever applies to the trigonometric ratios their reciprocals, so the following are true in the first quadrant i.e.; O $° \leq \theta \leq 90°$ (acute) all the reciprocals trigonometric ratios are positive.
Sec θ > cosec θ and cotan θ

In the second quadrant $90^0 \leq \theta < 180^0$ (obtuse) since only the sine is positive only its reciprocal the cosecant will also be positive in the third and fourth quadrants respectively only the tangent and cotangent for $180^0 \leq \theta < 270^0$ are positive and cosine and secant in $270^0 \leq \theta < 360^0$ are positive respectively.

So the following relationships are established

1. $\sec\theta$ = cosec (90° - θ)
   cosec θ = sec (90° - θ)
   tan θ = cot (90° - θ)
   cot θ = tan (90° - θ)

2. sec (180 - θ) is negative, cosec (180 - θ lies between 90° and 180°
) is positive
   cot (180 - θ) is negative.

3. sec (θ - 180) is negative, cosec (θ - 180)
   is negative cot (θ - 180) is positive lies between 180° and 270°

4. sec (360° - θ) is positive, cosec (360 - θ)
   is negative cotan (360 - θ) is negative lies between 270° and 360 °

Having seen the relationships between the trigonometric ratios and their reciprocals, let us move on to find angles using the trigonometric tables.

3.2.1 Use of Trigonometric Tables

In the trigonometric tables, sine, cosine and tangent of angles can be used to find the values of their reciprocals. In the Four Figure Tables, only the tables for sine, cosine and tangent are available so whatever obtains in their case also applies to their reciprocals.

The exact values of the trigonometric ratios obtained using the unit circle may not be accurate due to measurement errors. So to obtain the exact values of the trigonometric ratios, you use the four figure tables or calculators.

The tables to be used here are extract of the natural sine and cosine of selected angles between 10^0 and 89^0 at the interval of 6^1 or 0.1^0 . The full trigonometric tables will be supplied at the end (are tables for log sine, log cos and log tan).

	0'	6 '	12 '	18,	24'	30'	36'	42'	48'	54'
X°	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9
20"	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	3567
30	0.5000	0.5015	5030	5045	5060	5075	5090	5105	5120	5135
40"	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547
50"	0.7660	7672	7683	7694	9705	7716	7727	7738	7749	7760
60"	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738
70"	0.9397	9403	9409	9415	9421	9426	9432	9432	9444	9449
80"	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874
89°	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000

Note that the difference column always at the extreme right - hand corner of the table is omitted

Extracts from natural cosine for cos x° (WAEC, four figure table)

	0	6	12	18	24	30	36	42	48	54
X"	0".0	0".1	0".2	-6'-3-0'4		0".5	0".6	0".7	0".8	0".9
10"	0.9848	9845	9842	9839	9836	9833	9829	9826	9823	9820
20"	0..9397	9391	9385	9379	9373	9367	9361	9354	9348	9342
30"	0.8660	8652	8643	8634	8635	8616	8507	8599	8590	8581
40"	0.7660	76649	7639	7627	7615	7604	7593	7581	7570	7559
50"	0.6428	6414	6401	6399	6374	6361	6347	6334	6320	6307
60"	0.5000	4985	4970	4955	4939	4924	4909	4894	4879	4863
70"	0.3420	3404	3387	3371	3355	3338	3322	3305	3289	3272
80"	0.1736	1719	1702	1685	1668	1650	1633	1616	1599	1583
89"	0.0175	0157	0140	01222	0105	0087	0070	0052	0035	0017

Again the difference column is omitted.

Example:

Find the value of the following angles:

(i) sin 20 . 6^0 (ii) cos 30^0 12^1 (iii) sin 70^0 48^1 (iv) cos 40. 7^0

Solutions:

(I) From the sine table to find sin 20.6^0 look at the left hand column marked x get to the number 20 and move across to 0. 6 on the top now their intense gives 0.3518 .'. sin 20.6^0 =0.3518

(II) For cos (30^0 12^1), go to the natural cosine table look for 30^0 along the first column (x^0) , either and move across unit 1 you fet to 12^1. The value at this intersection is 0.8643.
cos (30^0 12^1) = 0.8643.

(III) Sin (70^0 48^1) = 0.9444 (iv)

(IV) Cos (40 . 7^0) = 0.7581

A times, there might have problems involving minutes or degrees other than the one given in the table. You have to use the difference table when such is the case.

For example:

Find (1) sin (20^0. 15^1) (II) cos (50^0 17^1)

Solutions:

(i) From the sine table (WAEC)
sin (20 12) = 0.3453
plus the difference for 3 = 8 (from the difference column at the extreme right of the sine table)
sin (20 15^1) = 0.3461

Alternatively you can look for sin (20° 18^1) and then subtract the difference of 3^1 thus

Sin (20° 18^1) = 0.3469

Is the difference for 3^1 = -8

Sin (20^0 15^1) 0.3461

You see that either way the value of sin (20° 15^1) is 0. 3461

Note that the values of then (20° 15^1) is the same in the two methods above but in most cases, the values are not there are slight differences at times.

(ii) From the cosine table

cos (50° 17^1) is nearer cos 50 18 cos (50° 18) = 0. 6388
plus the diff. Forte= + 2 0.6390 cos (500 171) = 0.6390

OR

cos (50° 12^1) = 0. 6401
minus the diff. For 5^1 = -12
cos (50° 17^1) = 0. 6389

Observe that the difference was added to the first method is cos 50° 18^1 and subtracted from the second method i.e. cos 50° 12^1 . This is because the angle increases, the value reduces in cosine. You can have a critical look at the tables for cosine. It is good to note that the values of sine increases form O to
1 while the values of cosine decreases from 1 to O.

The same methods as used in finding the tangent of angles from their tangents tables.

For angles greater than 90°, the same tables are used in finding their trigonometric ratios but firstly, you de4termine the quadrant and sign of the angle and treat accordingly.

Examples:

Find (1) sin 120° (2) sin (- 30)°
(3) cos (- 10°) (4) cos 260°

Solutions:

(1) Sin 120° is in the second quadrant and sine is positive = sin (180 - 1200 = sin 60° and since sin 60 is positive, from the sine table. Sin 120° = + sin 60° = 0.8660

(2) Sin (-30°) is in the fourth quadrant, where the sine is negative.

Sin (- 30°) = - sine (360 - 30) = - sin 330° = - sin 30° from the sine table

Sin 30° = 0.5000
- sin 300 _ - 0.5000

(3) Cos (- 10) lies in the fourth quadrant, where cosine is positive. cos (-10) = cos (360 - 10)= cos 350° = cos 10°
cos (-10) the cosine table is 0.9848 cos (- 10) = 0.9848

(4) Cos (260°) is in the 3rd quadrant where cosine is negative = cos (260 - 180°)

= cos 80°. From the cosine table cos 80 = 0.1736 and since cosine is negative in the 3rd quadrant cos 260^0 = - cos 80^0 = - 0.1736.

3.3.2 Use of Logarithms of Trigonometric Functions

Atimes you might be faced with problems which require multiplication and direction in solving triangles. Here the use of tables of trigonometric functions becomes time consuming and energy sapping. It is best at this stage to use the tables of the logarithms of trigonometric functions directly.

Examples:

Find (1) log cos 20° 6^1.

Solution:

The use of the tables of cosine will allow you to (1) find cos 20° 6^1 from the table .

(ii) find this value from the common logarithm table i.e. cos . (20° 6^1.) = 0.9391 (from Natural cosine)

Then log 0.9391 =$\bar{1}$.9727 (from common logarithm)

But using the logarithm table of cosine go straight and find log 20° 6^1. log cos (20° 6^1) = $\bar{1}$. 9727

Here you can see that applying the log cos table is easier and faster.

(ii) Log sin (24° 13^1)

Log sin (24° 13^1) = $\bar{1}$. 6127,

Plus difference for 1^1 = + 0.0002 (cot from the difference table at the right hand extreme column)

log sin 24° 13^1 = $\bar{1}$. 6029

(iii) log tan 40° 17^1

from the log tangent table;

log tan 40° 17^1 = $\bar{1}$. 9269 plus the diff. For 5^1 = + 8

log tan 40° 17^1 = $\bar{1}$. 9277

Alternatively, you can look for the logarithm:

log tan 40° 18^1 = $\bar{1}$. 9 284 minus the deff. for 1^1 = - 2 log tan 40° 18^1 = $\bar{1}$. 9282

The two results in this case are not the same. The second result is preferable because the smaller the difference the more accurate the value of the angle being sort for.

If the angles are in radius convert to degrees. Exercise 2.2
Using the four figure table or calculator:

1. Find the value of each of the following

a.	sin	$32° 17°$	ans. = 0.5341
b.	sin	$126^0.30^0$	ans. = 0.3382
c.	sin	$340° 14^1$	ans. = - 0.3382
d.	cos	$35^o. 7^o$	ans = 0.8121
e.	cos	$137^o 16^1$	ans = -0.7346
f.	cos	(-40^o)	ans = 0.7660
g.	sin	(-40^o)	ans = -0.6428
h.	tan	120^o	ans = -1.7321
i.	tan	265^o	ans = 11.4301
j.	tan	$12^o 46^1$	ans = 0.2266

2. Find the quadrant of the following angles and determine whether the trigonometric ratios (reciprocals) are positive or negative.

(a)	100^o	(b)	110^o	(c)	123^o		
(d)	42^o	(e)	20^o	(f)	231^o		
(g)	268^o	(h)	312^o	(i)	591^o	(j)	1999^o.

Solutions:

a. 2nd, only sine and cosine positive
b. 2nd, only sin and cosec + ve
c. 2nd, only sin and cosec + ve
d. 1st all trig ratios positive
e. 1st, all trig. ratios positive.
f. 3rd, only tan and cot positive
g. 4th, cos and sec positive.
h. 3rd only tan and cot positive
i. 3rd, only tan and cotangent positive.

4.0 CONCLUSION

In Unit 1 and 2, you have learnt the definition of the trigonometric ratios and their reciprocals, and how to find the trigonometric ratios of any given angle and the use of trigonometric tables in finding angles. You should have also learnt that the value of any angle depends on the basic angle and its sign depends on the quadrant in which it is found. However, you need be aware that the most commonly used trigonometric ratios are the sine cosine and tangent and the basic angle θ lies between O° and 360 i.e. $0 \leq \theta < 360$.

5.0 SUMMARY

In these two units, you have seen that the trigonometric ratios and their reciprocals with respect to a right angled triangle is

sin = opposite hypothenus
cos = adjacent hypothenus
tan = opposite adjacent

The acronym SOH CAH TOA meaning

S = sine,	O = opposite over,	H = hypotenuse
C = cosine,	A = adjacent over,	H = hypotenuse
T = tangent,	o = opposite over,	A = adjacent

Can be used to remember the trigonometric ratios their reciprocals are obtained from these.

You have also learnt that:

(i) the sine or cosine or tangent of an acute angle equals the cosine or sine or cotangent of its complementary angle.

Sin θ = cos (90 - θ) sin (90 + θ) = cos θ
Cos θ = sin (90 -θ0 cos (90 + θ) = -sin θ
Tan θ = cot (90 - θ) tan (90 + θ) = - cot θ
This means that you can use the sine table find the cosine of all angles from 90 to 0 at the same interval of 61 or 0° .1°.

(ii) the tables of trigonometric functions can also be used in finding the ratios of given angles by bearing in mind the following where0 is acute or obtuse.

(iii) sin (180 - θ) = - sin θ;
sin (θ - 180) = - sin θ
cos (180 - θ) = -cos θ
cos (θ - 180) = -sin θ
tan (180 - θ) = -tan θ
tan (θ - 180) = tan θ

(iv) sin (180 + θ)= - sin θ
cos (180 + θ)= - cos θ
tan (180 + θ)= tan θ

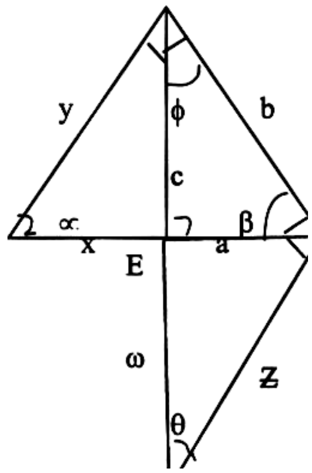
(v) sin (360 - θ) = - sin θ
cos (360 - θ) = cos θ
tan (360 - θ) = -tan θ

In using the table sometimes angles may be expressed in radians, first convert the angles in radians to degrees before finding the trigonometric ratios of the given angles or convert from degrees to radians before finding the trigonometric ratios, if it is in radians.

6.0 TUTOR-MARKED ASSIGNMENT

In the diagram below, find the trigonometric ratios indicated.

A



D

(a) sin θ, cos θ, sec θ, cot θ
(b) cos α, tan α, cosec α, sin α
(c) tan ϕ, cos ϕ, sec ϕ cosec ϕ sin ϕ
(d) sin β cos β tan β, cot β sec β sec β cosec β

2. Express the following interms of the trigonometric ratios of α

(a)	i.	Cos (90 – α)	ii.	Sin (90 + α)
(b)	i.	Cosec (90 – α)	ii.	Sec (90 + α)
(c)	i.	Cos (90 – α)	ii.	Sec (180 – α)
(d)	i.	Sin (360 – α)	ii.	Tan (360 – α)

3. Find the basic angles of the following and their respective quadrants.

(a)	1290^{o}	(b)	-340^{o}	(c)	-220^{o}
(d)	19^{o}	(e)	125^{o}	(f)	214^{o}

4. use trigonometric tables to find the value of the following:
   (a) sin 117° (b) cos 11.1 ° (c) tan
   (d) sin 204 . 7°(e) cos 121°

5. Use the logarithm table for trig. Functions to find the value of the following.
   (a) log cos 34° 17^{1} (b) log sin 23° 25^{1} (c) log tan 11° 6^{1}

7.0 REFERENCES/FURTHER READING

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This list is not exhaustive, you can use any mathematics textbook no matter the level it is written for, to enable you have a good understanding of the unit. There are a lot of mathematics text in the market and libraries, feel free to use any.

UNIT 3 INVERSE TRIGONOMETRIC FUNCTIONS OR CIRCULAR FUNCTIONS

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1.0 INTRODUCTION

Very often you see relations like $y = \sin \theta$ R, it is is possible to find the value of y, if θ is known. On the other hand, the need might arise to find the value of θ when y is known. What do you think can be done in this case?

In the example above i.e. $y = \sin \theta$, sine is a function of an angle and also the angle is a function of sine.

In this unit, you shall learn the inverse trigonometric functions, sometimes called circular functions, the basic relation of the principal value and trigonometric ratios of special angles 0°, 30°, 45°, 60°, 90° , 180°, 270° and 360°

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- define inverse trigonometric functions
- find accurately the inverse trigonometric functions of given values.
- determine without tables or calculators the trigonometric ratios of 0°, 30°, 45°, 60°, 90°, 180°, 270 ° and 360°.
- solve problems involving inverse trigonometric functions and trigonometric ratios of special angles correctly.

3.0 MAIN CONTENT

3.1 Inverse Trigonometric Functions (Circular Functions)

3.1.1 Definition and Notation

The trigonometric ratios of angle are usually expressed as y = sin θ (where y and θ represents any value and angle respectively).
Or y = cos θ

Or y = tan θ.

The above are example when the values of θ is known. When the value of θ is unknown and y is known the above relations can be expressed as:

θ = (Sin^{-1} y) written as arc sin y read as ark sin y
or θ = (Cos^{-1} y), written as arc cosy read as ark cos y
or 0 = (Tan^{-1} y) written as arc tan y read as arc tany

Note that capital letters are used for the first letters of the trig. ratios. These relations arcsin, arccos and arctan are called the inverse trigonometric functions or circular functions. Thus, in the above examples, 0 is called the inverse sine or inverse cosine or inverse tangent of y.

Example

(a) if sin θ = 0.4576, then θ = $\sin^{-1}$(o.4576), meaning that θ is the angle whose sine is 0.4576 or the sine of θ is 0.4576

(b) if cos θ = 0.8594, at then θ = $\cos^{-1}$(0.8954), which implies that θ is the angle whose cosine is 0.8594 or cosine of θ = 0.8954.

(c) if tan θ = 2. 1203, then $\tan^{-1}$ (2. 1203) shows that is the angle whose tangent is 2.1203 or the tangent of is 2.1203.

Procedures for Finding Inverse Trigonometric Functions

Having been conversant with the use of the trigonometric tables, the task here becomes easy.

In finding the inverse trigonometric ratio of any angle, first look for the given value on the body of the stated trigonometric table and, read off the angle and minute under which it appeared. If the exact value is not found, the method of interpolation(i.e. finding the value closest to it and finding the difference between this closest value and the original value, then, look for the difference under the minutes in the difference column) can be adopted.

Example 1

Find the value of the following angles

(a) $\sin^{-1}$ (0.1780) (b) $\cos^{-1}$ (0.2588) (c) $\tan^{-1}$ (1.1777)
this question can also be stated thus: Find y if;
(a) siny = 0.1780 (b) cosy= 0.2588 (C) tan y = 1.1777

Solutions:

1(a) From the sine table (Natural sine table) through the body the value 0.1780 (or a value close to it) is located. The value is 0.1771 found under 10° 12'. The difference between 0.1780 and 0.17711 is 9. This is found under 3 in the difference column. So in tabular form

.. 0.1771	= 10°	121
plus difference for -9	= +	31
0.1780	= 100	15'

the angle whose sine is 0.1780 is $10° \ 15^1$

Alternatively 0.1788 can be located under 10 18 and difference between 0.1788 and 0.1780 is 8 but 8 cannot be found in the difference column so choose the number nearest to 8 i.e. 9 found under 3^1

0.1788 = $10° \ 18^1$
plus difference for 9 = $+ \ 3^1$
0.1779 = $10° \ 15^1$

.'. the angle whose sine is approximately equal to 0.1780 (0.1779) i.e $10° \ 15^1$
= 10° 151 b) Cos θ = 0.2588
from the cosine table, the value 0.2588 is found under $75° \ 0^1$
the angle whose cosine is 0.2588 is 75° .. = 75°
c) tan θ =1. 1777,

from the natural tangent table the value closest to 1.1777 is 1.1750 found under 49° 36^1 . The difference between the two values is 27 which is found under 4^1

.. 0.1750	= 49°	36^1
plus the difference for 27	= +	4^1
0.1777	= 49°	40^1

In the above examples all angles are acute angles. The inverse trigonometric functions can be extended to any angle.

3.1.2 Inverse Trigonometric Functions of Any Angle

The inverse trigonometric functions here are extended to include values of given angles between 0° and 360° and beyond

Example 2:
Find the value of 0 between 0° and 360° in the following:
(a) $\sin\theta = 0.8964$ (b) $\cos\theta = -0.6792$ (c) $\tan\theta = 0.2886$

Solutions:
(a) $\sin\theta = 0.8964 \Rightarrow \theta = \sin^{-1}(0.8964)$
$\therefore \theta = 63° 41^1$; Since $\sin\theta$ is positive then the angle must either be in the 1st or 2nd quadrant thus:

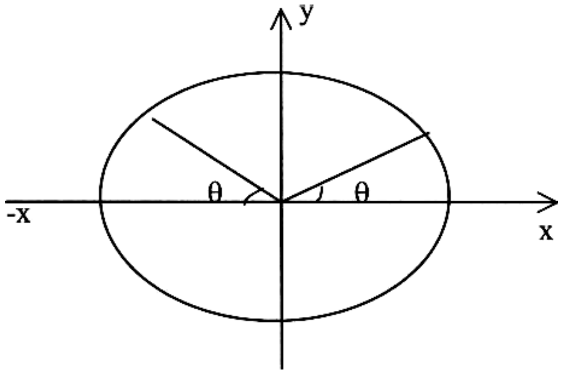


Fig. 3.10
In the first quadrant $\theta = 63^o 41^1$ and in the 2^{nd} quadrant
$\theta = 180 - 63^o 41^1 = 116^o 19^1$.

(b) $\cos\theta = 0.6792$

From the cosine tables $\theta = \cos^{-1} 0.6792 = 47^o 10^1$ but cosine θ is negative, therefore θ lies either in the 2^{nd} or 3^{rd} quadrant.

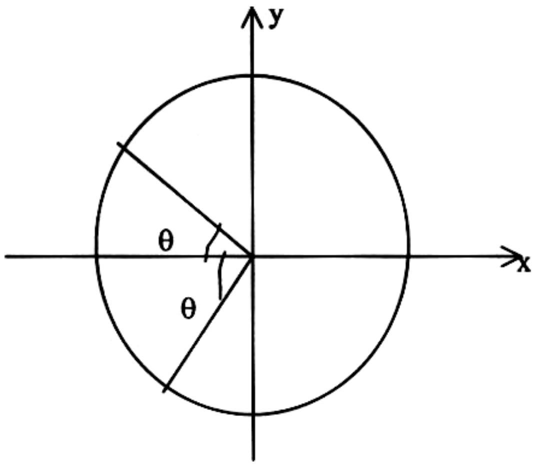


Fig. 3.11

In the 2nd `quadrant

$\theta = 180 - 47^0\ 10 = 132\ °\ 50\sim$

In the 3rd quadrant

$\theta = 180 + 47°\ 10^1 = 227^0\ 10^1$

(c) tan θ = - 0.2886, here $\theta = 16^0\ 6`$ but since tan θ is negative, θ lies either in the 2nd or 4th quadrants.

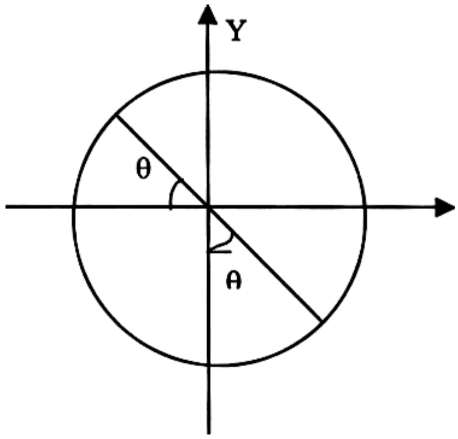


Fig. 3.12

In the 2nd quadrant

$\theta = 180 - 16°\ 61 = 163^0\ 54^1$

in the 4th quadrant $\theta = 360 - 16°\ 6^1$

$\theta = 343°\ 54^1$

Note from the previous units there are several values of θ with the same value but in different quadrants. For example sin 30° = sin 150° = sin 390° = sin 750° etc, hence the inverse trigonometric functions are many valued expressions. This mean that one value of θ is related to an infinite number of values of the function. Hence to obtain all possible angles θ of a given trigonometric ratio either add or subtract (360° K) where K is any integer-positive, negative or zero.

Example 3

Find all possible angles of 0 in example 2(a), (b) and (c) above.

(a) Given sin θ =0.8964 and $\theta = 63°\ 41^1$ and $\theta = 116°\ 19^1$

These two VALUES OF θ ARE THE BASIC ANGLES

∴ ALL POSSIBLE ANGLES OF θ ARE

$63^0\ 41^1 \pm (3\ 60k°)$, where k = …-1,0,+1,+2,+3…..

and $116°\ 19^1 \pm (360k°)$, where k = k = -1,0,1,2,3……

(b) Given cos θ = - 0.6792 and θ was found to be 132° 50' and 227° 10' these are the basic angles, So all possible angles of θ, therefore are 132° 50' ± (360k)° for k = . . ., - - 1,0,1,2,...

(c) Given tan θ = - 0.2886, θ equals 163° 54' and 343° 54'

Here all possible angles of 0 are 163^0 54' ± (360k°) and 243° 54' ± (360k°) for k = ..., -1,0,1,2, ...

Hence to find all possible angles of given angle:

(i) find the basic angles of the given value

(ii) add or subtract (360k°) where k is either a positive negative or zero integer.

3.1.3 Principal Values of Inverse Trigonometric Functions

In this section, attention should be found on the value which lies in a specified range for example:

(i) for $\sin^{-1}(y)$, the range of values are - 1 /2 (-90°) to +1 /2 (-90°). This value is called the principal value of the inverse of sine denoted by $\sin^{-1}y$ (smalls). For example if $\sin^{-1}$ 1/2 = 45° or 1 /4 radians then the principal value of the inverse of sin 1/2 is $\sin^{-1}$ 1/ 2 = 45° or 1/4 (since it is within the range).

(ii) If $y = \cos\theta$, then $\theta = \cos^{-1} y$, is the inverse cosine of y. and the principal value of the inverse of cosine is the value of θ in the range 0° to (180°). This is the same for arc cot θ , and arc sec θ

Example, if $\cos^{-1}$ 1/ 2 = 45°, then arc sec the principal value

$\cos^{-1}$(- 1/ 2) = - 1 /4(-45°) the principal value is $\cos^{-1}$1/ 2) = -1 /4(135°)

(iii) The principal value of the inverse of tangent is the value of θ in the range -π/2 (-90°) to +π/2 (- 90°). This is the same for arc cosec θ.

Example of principal values

The principal value of;

(a) $\tan^{-1}(-1)$ = - /4 = (45°). (b) $\cot^{-1}$ 3= /6

(c) $\sec^{-1}(-2)$ = 2/3 (120°)

The relationship between the values of an inverse function and its principal value is given by the formulae below (Vygodsky 1972, p. 366).

(i) Arc sin x = k + $(-1)^k$ arc sin x

(ii) Arc cos x = 2k ± arc cos x (iii) Arc tan x = k + arc tan x (iv) Arc cot x = k + arc cot x,

where k is any integer positive, negative or zero.

Hence Arc sin, Arc cos, Arc tan denotes arbitary values of inverse trigonometric functions and arcsin, arc cos, arctan denotes principal values of given angles.

Example:

(a) Arc sin½ = k + $(-1)^k$ arc sin ½

= k + $(-1)^k$ x /6 or k(180°) + (-1) k 30°

fork= 0, Arcsin ½ = 0 x + (-1)° /6 = /6(30)°

k=1, = Arcsin 1/2 = l x + (-I)' /6 = - /6 = 5 /6 (150°)

k=2, = Arcsin 1/2 = 2 x + $(-I)^2$ /6 = 2 + /6 = 13 /66 (390°)

k=3, = Arcsin 1/2 = 3 x + $(-I)^3$ /6 = 3+ /6 = 17 /6 (510°)

k=-1, = Arcsin 1/2 = -1 x + (-I)-' /6 = - - /6 = -7 /6 (-210°)

Note the angles in radians can be converted to degrees (see angles in brackets)

Exercise 3.1

(1) Write down the values of

(a) $\sin^{-1}$ (- 1/2) (b) $\cos^{-1}$ (-1) (c) $\tan^{-1}$ (-1)

Ans: (a)-211110°, 330° (b) 180° (c) 135°, 315°

(2) Use tables to evaluate:

(a) $\tan^{-1}$ 2 (b) $\cos^{-1}$ (1/4) (c) $\sin^{-1}$ (3/5)

Ans: (a) 63° 26' (b) 88° 26' (c) 36° 26'

(3) Find the value of the following angles:

(a) $\sin^{-1}$(0.7509) (b) $\cos^{-1}$ (0.9219) (c) $\tan^{-1}$ (2.574)

Ans: (a) 48° 40' (b) 212° 48' (c) 68° 46'

4. a) sin θ = - 0.5120 (b) tan θ = o.9556 (c) cos θ = - 06088

Ans: (a) 210° 48', 329° 12' (b) 43° 42', 223° 42' (c) 127° 30', 230° 53'

5. Find all possible angles in question (4)

Ans: (a) 210° 48'± (360k°) and 329° 12' ± (360k°) (b) 430 72' ± (3 60k)° and 223' 42'+- (3 60k°) (c) 127° 30' ± (360k°) and 230° 53'+- (360k°)

6. Find the value of Arc cot 3

Ans: Arc cot 3 = k +arc cot 3 where k is any integer = k+/6

for k = 0, Arc cot 3 = /6 = 30° (angles in radians)

k = 1, Arc cot 3 = + /6= 6 =210°

k = -l, Arc cot 3 = + /6 = -5 /6 = -150° etc.

3.2 Trigonometric Ratios of Common Angles

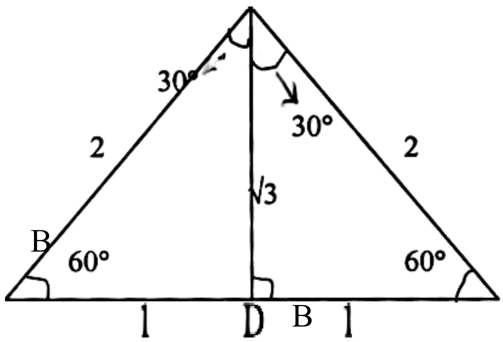
The angles 0°, 30°, 45°, 60°, 90° are called common angles because they are frequently used in mathematics and mechanics in physics.

Although the trigonometric ratios of common angles 0°, 30°, 45°, 60°, 90°, (and multiples of 90° up to 360°) can be found from the trigonometric tables, they can be easily determined and are widely used in trigonometric problems.

3.2.1 The Angle of 30° and 60°

Consider an equilateral triangle ABC of sides 2cm. An altitude AD (Figure 3.2)

A



C

B

Fig: 3.2

An altitude AD (see Figure 3.2)
Bisects < BAC so that < BAD = < CAD = 30°
< ABC = < ACB = 60°
by Pythagoras theorem AD = 3 units.

Hence, the value of the trigonometric ratios of 60° and 30° are

Sin 60°= 3/2 and sin 30° = 1/2

Cos 60° = 1/2 and cos 30° = 3/2

Tan 60° = 3 and tan 30° = I 3

Cot 60° = 1/ 3 and cot 30° 3

Sec 60° = 2 and see 30^{o} =-2/ 3

Cosec 60° = 2/ 3 and cosec 30° = 2

3.2.2 The Angle 45°

Consider a right-angled isosceles triangle ABC with AB= BC = 1 unit,
<B=90° and<A=<C=45°
AC = 2 units (Pythagoras theorem)

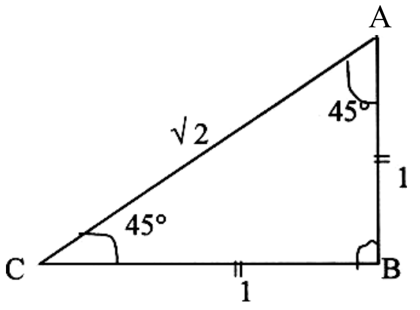


Fig. 3.21

Hence the trigonometric ratios of 45° are sin 45° = 1/ 2
Cos 45° = 1/ 2
Tan 45° = 1
Cot 45' = 1
Sec 45°= 2
Cosec 45° = 2

3.2.3 Angles 0° and 90°

It is difficult in practical problems to find angles 0° and 90° in a right - angled triangle as acute angles but with extended trigonometric functions, these angles are considered. (Figure 3.22 below)

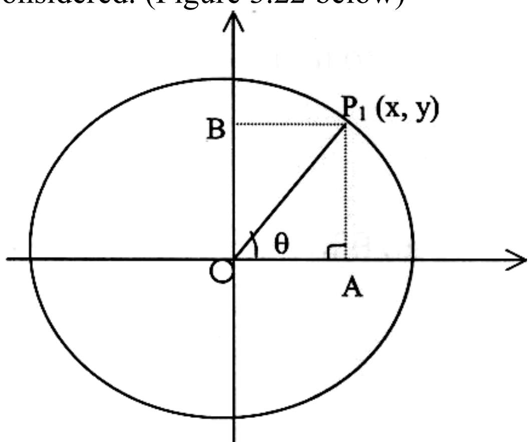


Fig: 3. 2.2.

Using a unit circle let P1 (x, y) be any point on the circle. If P, is rotated about 0 in the anti-clockwise direction through an acute angle, then A is the projection of P, on the X - axis and B is the projection of P, on the Y- axis

In ΔP1 A

Sin θ = $\frac{P1A}{P10}$ but P10 = 1 unit (unit radius)

Sin θ = P1A = y coordinate of P1

= projection of OP, on the Y-axis

cos θ = $\frac{OA}{OP1}$ = $\frac{OA}{1}$= OA but OA = BP1

Therefore, cos θ = BP1= x coordinate of P1

= projection of 0 P1 on the X-axis

Thus if P is any point on a circle with center 0 and unit radius and OP makes an angle with the X-axis, then the sine and cosine of any angle may be defined thus:

Sin θ = y coordinate or the projection of OP on the y-axis and cos θ , x coordinate or the projection of OP on the x-axis.

Thus for angles 0° and 90°

Sin 90° = y coordinate = 1 cos 90° = x coordinate

= 0 (90° has no projection on the x axis)

tan 90° = $\frac{\sin 90^0}{\cos 90^0}$ = $\frac{1}{0}$ = α (infinity)

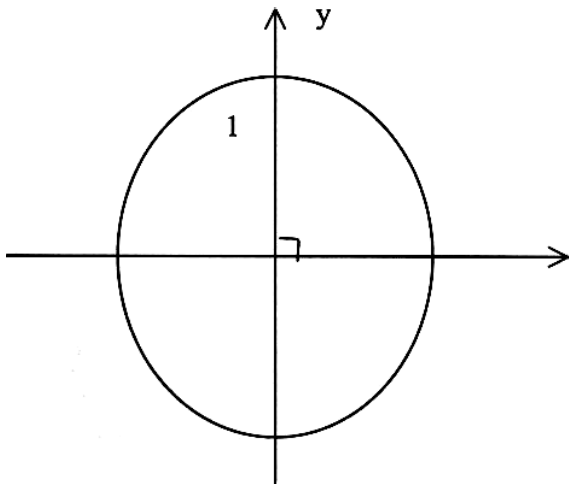


Fig: 3.2.3

Similarly for 0°

Sin 0° = y coordinate

= 0(0° has no projection on y-aixs)

cos 0° = x coordinate

= 1 (0° lies on the x-axis tan 0° = $\frac{\sin 0^0}{\cos 0^\circ}$ = $\frac{0}{1}$ = 0

Alternatively;

In a right-angled triangle ABC, with < A = 90° and < C = which is very small

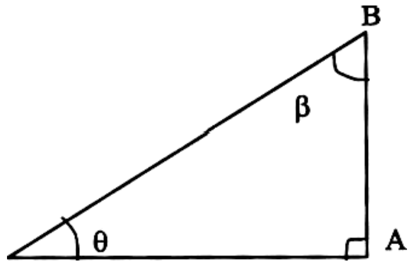


Fig. 3.2.4

The ratios of are: Sin θ = AB BC

Cos θ = AC BC

tan θ = AB AC

also

sin β = AC BC

tan β = AC AB

When gets smaller and smaller, R becomes larger and larger, these are expressed thus as

tends to O i.e. ---- 0

β tends to 90° i.e. β---- 90°

B-->A and BC-->AC as AB--> 0

$\text{Sin } \theta° = \frac{AB}{BC} \rightarrow \frac{0}{AC} = \frac{0}{AC} = 0$ $\text{Cos } \theta° = \frac{AC}{BC} \rightarrow \frac{AC}{AC} = 1$

$\text{Tan } \theta° = \frac{AB}{AC} \rightarrow \frac{0}{AC} = 0$

$\text{Sin } 90° = \frac{AB}{AC} \rightarrow \frac{AC}{AC} = 1$ $\text{Cos } 90° = \frac{AB}{BC} \quad \frac{0}{AC} = 0$ $\text{Tan } 90° = \frac{AC}{AB} \rightarrow \frac{AC}{0}$

Or since 0° and 90° are complementary angles then

Sin = cos (90 - 0) = cos 90° =0

Cos 0° = sin (90 -0) = sin 90° = 1

$\text{Tan } 90^0 = \frac{\sin 90}{\cos 90} = \frac{1}{0} = \alpha$

Here is the summary of the common trigonometric ratios. The trigonometric ratios of these special angles and that of multiples of 90° are presented in the Table 3.1

below.

Table 3.1: Trigonometric Ratios of Special Angles

Angle A°	Sin A	Cos A	Tan A	Cot A	Sec A	Cosec A
0°	0	1	0	α	1	α
30°	½	3/2	1/ 3	3	2/ 3	2
45°	1/ 3	1/ 2	1	1	2	2
60°	3/2	½	3	1/ 3	2	2/ 3
90°	1	0	α	0	α	1
180°	0	-1	0	α	-1	α
270°	-1	0	α	0	α	-1
360°	0	1	0	α	1	α

Example:
Without using tables/calculator find the value of the following:

(1) (i) COs $90^o + 1$ (ii) $\frac{\sin 60°}{\cos 60}$ (iii) $\frac{2}{\sin 30} - \frac{3}{\tan^2 60} + 1$

(2) if θ = 300 evaluate $\frac{\sin^2\theta + \tan^2\theta \ \text{x} \cos\theta}{1-\tan\theta \ \text{x} \cos^2\theta}$

Solutions:

1. (i) cos 90° + 1 from above table, cos 90° = 0
   .. Cos90° + 1 = 0 + 1 = 1
   .. cos 90° + 1 = 1

(ii) $\frac{\text{Ssin}60^0}{\cos 60°}$ = tan 60° and tan 60° = 3

OR

Sin 60° = 3/2 and cos 60 = 1/2

$\frac{\sin 60^o}{\cos 60^o} = \frac{3}{2} = \frac{3}{2} \text{ x } \frac{2}{1} = 3$

$\frac{\sin 60^o}{\cos 60^o}$ = 3

(iii) $\frac{2}{\sin 30} = \frac{3}{\tan^2 60} + 1$ Substituting the values,

$\frac{2}{½} - \frac{3}{(3)2} + 1 = 2 \text{ x } 2 - \frac{-3}{3} + 1$

= 4 – 1 + 1 = 4

2. $\sin^2\theta + \tan^2\theta$ x cosθ , substituting for = 30°

1-tan$^2\theta$ x cosθ

sine 30°
cos30°;
x cos 30°

$$= \frac{\sin^2 30^o + \frac{\sin^2 30^o}{\cos^2 30^o} \times \cos 30^o}{1 - \frac{\sin^2 30^o}{\cos 30^o} \times \cos^2 30^o}$$

and sin 30° = 1/2 , cos 30° = 2 and tan 30° = 1 -73 it then becomes;

Alternatively substituted for tan 30 = 1/,13

$$\frac{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 x \left(\frac{\sqrt{3}}{2}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2 x \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\frac{1}{4}+\left(\frac{\sqrt{3}}{3} x \frac{\sqrt{3}}{2}\right)}{1-\left(\frac{1}{2} x \frac{\sqrt{3}}{2}\right)} = \frac{\frac{1}{4}+\frac{\sqrt{3}}{6}}{1 - ¼}$$

3+2 3
12
3/4

3+2 3 x 4
12 3

3+2 3
9

Exercise 3.2

Simplify the following without using tables or calculators

1. (a) $\frac{\sin^3 330° x \tan^2 240°}{\text{Cos}^4\ 30°}$

Ans; -2/3

(b) $\frac{3 - \sin^2 60°}{2 + \text{cost } 60°}$ + tan^2 60^o -

Ans; 4

2. If θ = 60°, calculate, without table or calculator

(a) sin θ + cos θ Ans; 2(√3 + 1

$1+\cos^2\theta$ 5

(b) $\dfrac{25\cos 3\theta - 2\sin\theta}{\tan\theta\cos\theta}$ Ans: $\dfrac{25 - 2\sqrt{3}}{4\sqrt{3}}$

3. (a) $(\sin 135° + \cos 315°)^2$ Ans; 2

(b) $\dfrac{\tan 240^0}{1+\tan^2 30°}$ $\dfrac{\tan 315°}{1+\tan^2 60°}$ Ans; 2.

4. If sin A = 3/5 and sin B = 5/13. where A and B are acute, find without using tables, the values of

(a) sin A cos B + cos A sin B Ans; 56/65
(b) cos A cos B + sin A sin B Ans; 33/65
(c) $\dfrac{\tan A - \tan B}{1 + (\tan A)(\text{taan}B)}$ Ans; 16/33

5. If A is in the fourth quadrant and cos A = 5/13 find the value of $\dfrac{13\sin A+5\sec A}{5\tan A + 6\text{cosec}A}$ without using tables Ans -2/37

4.0 CONCLUSION

In this unit, you have learnt the inverse trigonometric functions or circular functions, their definitions or meanings and notations, you have also learnt how to find the inverse trigonometric functions from trigonometric tables, the principal value of inverse trigonometric angles, the relation between inverse trigonometric functions and their principal values and also the trigonometric ratios of common angles - how they are derived and how to find their ratios without using tables.

5.0 SUMMARY

In this unit, you have learnt that the inverse of a trigonometric ratios is the angle whose trigonometric ratios, is given. And these values can be found in the body of the trigonometric ratio table from where the angles are read off.

You have also learnt that to find all possible angles of a given problem first find the basic angles then add or subtract (360k°) to it i.e.

(i) All possible angles = basic angle ±(360k°) where k is any integer, positive, negative or zero.
(ii) The relation between the value of an inverse trigonometric function and its principal value are:

Arcsin x = k(180°) + $(-1)^k$ arc sin x

Arccos x = 360k° ± arcos x Arctan x = 180k + Arctan x Arccot x = 180k + acrcot x.

where Arcsin, or Arccos etc represent the values of inverse trigonometric functions and arcsin, arcos etc. represent their principal values.

(iii) The principal values of the following

(a) arcsin is the value between -90° and + 90°

(b) arccos is the value between 0° and 1800. This also applies to arccot and arcsec.

(c) Arctan is the value between -90^0 and $+ 90^0$

(iv) The trigonometric ratios of 0^0, 30^0, 45^0, 60°, 90°,150°, 270° and 360° are presented in the following table.

Angle A°	**In degrees & ratios**	**Sin 0**	**Cos 0**	**Tan 0**	**Cot θ**	**Sec 0**	**Cosec 0**
<u>Degrees</u>							
0^0	0	0	1	0	α	1	α
30^0	π/6	1/2	√3/2	1/√3	√3	2/√3	2
45^0	π/4	1/√3	I/√2	1	1	√2	√2
60^0	π/3	3/2	1/2	√3	1/√3	2	2/√3
90^0	π/2	1	0	α	0	α	1
180^0	π	0	-1	0	α	-1	α
270^0	3π/2	-1	0	α	0	α	-1
360^0	2π	0	1	0	α	1	α

6.0 TUTOR-MARKED ASSIGNMENT

1. write an angle in the first quadrant whose tangent is
   (a) 0.8816 (b) 1.9496 (c) 2.0265
2. Find the values of 0 lying between 0° and 360° when
   (a) sin ½
   (b) cos sin 285° (c) tan = -1
3. find all the angles between 0° and 720° whose tangent is 1/ 3
4. simplify without tables or calculator the following:
   (a) $\frac{\sin 150° - 5\cos 300° + 7\tan 225°}{\tan 135° + 3\sin 210°}$
   (b) sin 60° cos30' + sin 30° cos 60°
5. if tan = 7/24 and θ is reflex, find without tables or calculator the value of,

(a) sec θ (b) sin θ

7.0 REFERENCES/FURTHER READING

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This list is not exhaustive, you can use any mathematics textbook no matter the level it is written for, to enable you have a good understanding of the unit. There are a lot of mathematics texts in the market and libraries, feel free to use any.