MODULE 2

Unit 1 Graph of Trigonometric Ratios
Unit 2 Trigonometric Identities and Trigonometric Equations

UNIT 1 GRAPHS OF TRIGONOMETRIC FUNCTION AND THEIR RECIPROCALS

CONTENTS

1.0 INTRODUCTION

Graphs from elementary mathematics help to establish the relation between an independent variable and another variable a dependent variable. Hope you know what independent variables means?

In this unit, graphs of trigonometric ratios are graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ shall be treated. Also the graphs of their reciprocals. Here, the relation between the values of a variable angle and the corresponding trigonometric function can be seen by means of graph. These graphs are applied in physics - radio waves, sound waves, light waves, alternating current, simple harmonic motions etc.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- draw the graphs of trigonometric functions and their reciprocals accurately
- read values of any given angle form the graphs correctly
- determine the periodicity (period) and amplitude of given trigonometric ratios.

3.0 MAIN CONTENT

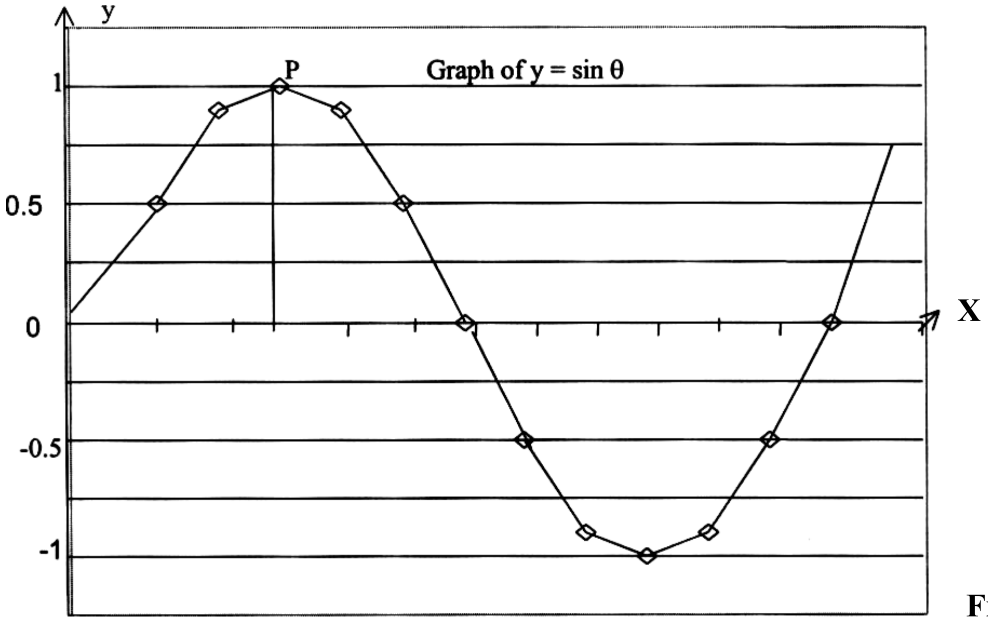


Fig. 4.1: Graph of y = sinθ

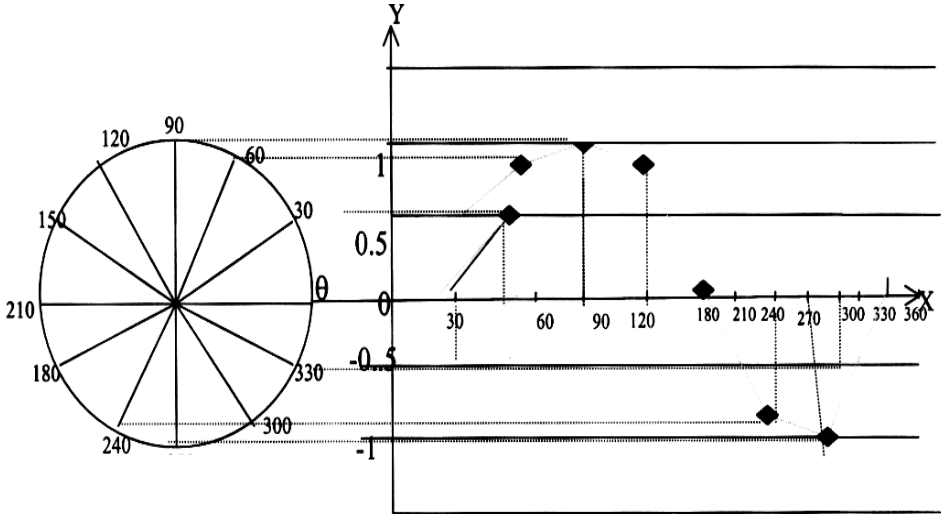


Fig. 4.2: Graph of y = sin θ (by projection)

3.1 Graphs of Trigonometric Functions

In drawing the graphs of trigonometric ratios, abscissa (x-axis) is taken as the independent variable and the ordinate (y-axis) as the dependent variable. This is so because the values of arc dependent on the values of x. The following explains this

3.1.1 Graph of y = sinθ

The graph of y = sin **0** will show the relationship between 0 and sin 0. This graph can be drawn in two possible ways.

Method 1: from table values

Steps:

(a) assign different values to 0° at intervals of 30° to 360 ° i.e. 0 = 0°, 30°, 60°, 90° ...,360°

(b) find the corresponding values of sin 0 , this is used to from the table of values

(c) choose a suitable scale then plot the values

(d) join the plotted points with either the free hand or a broom stick to get a smooth curve. This curve is then the graph of y = sin θ .

Thus following the steps, the table of values approximated to 2 decimal places is.

Table l: Table of Value for y = sin θ for θ≤ 0° ≤ 360

	0^o	30^o	60^o	90^o	120^o 00	150^o 00	180^o 00	210^o 00	240^0	270^o	300^o	330^o	360^o
Sin	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

Scale:

Let 1cm represent 1 unit on the axis i. e. the horizontal or x-axis. Let 4cm represent 1 unit on the sin axis i.e. the vertical or y-axis.
The points arte then plotted on a graph sheet and is joined by a broom stick
(see over leaf graph of y = sin

This means that the length from the height point on the graph to the x-axis is 1.

Method 2: Projection

Steps:

(a) Construct a unit circle and mark out correctly the angles of = 0°, 30°, 60°, ... to 360° (see fig 4.2).

(b) Draw the x and y axis as in other graphs

(c) Draw a horizontal line through the center of the circle to meet the x- axis.

(d) On the x-axis at 30^o interval, mark out the angles 0^o, 30°, 60°, .. to 360°.

(e) Draw dotted horizontal lines from the angles of sectors marked on the unit circle to meet the vertical lines from their corresponding values at the x-axis at a point.

(f) Join these points, then the graph of y = sin is obtained see Figure 4.2.

Properties of the graph of y = sinθ

1. The graph of y = sin or the sine curve is a continuous function i.e. it has no gaps between the values => no break
2. The value of sin increases from at 0° to that 90° and then decreases to -1 at 270 and back to 0° at 360°.

3. The sine curve repeats itself at intervals of 360° [or comes to coincidence with itself upon a translation along the axis of abscissa(x- axis)by some amount]. It is called a period (or cycle) of the function. In this or cycle is 360°.
4. The height of the graph P D (amplitude) in the sine curve is l.

3.1.2 Graph of y = cos θ for 0 ≤ θ ≤ 3600

The graph of y = cos is similar to the sine curve i.e. graph of y = sin.
Here again the table of values for and cos is shown below at the intervals of 300

Table 2: Table of Values for y = cos 0

	0^o	30^o	60^o	90^o	120^o	150°	180^o	210^o	240^o	270^o	300^o	330^o	360^o
Cos	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1

These points are plotted as in the sine curve and joined to give the cosine curve thus

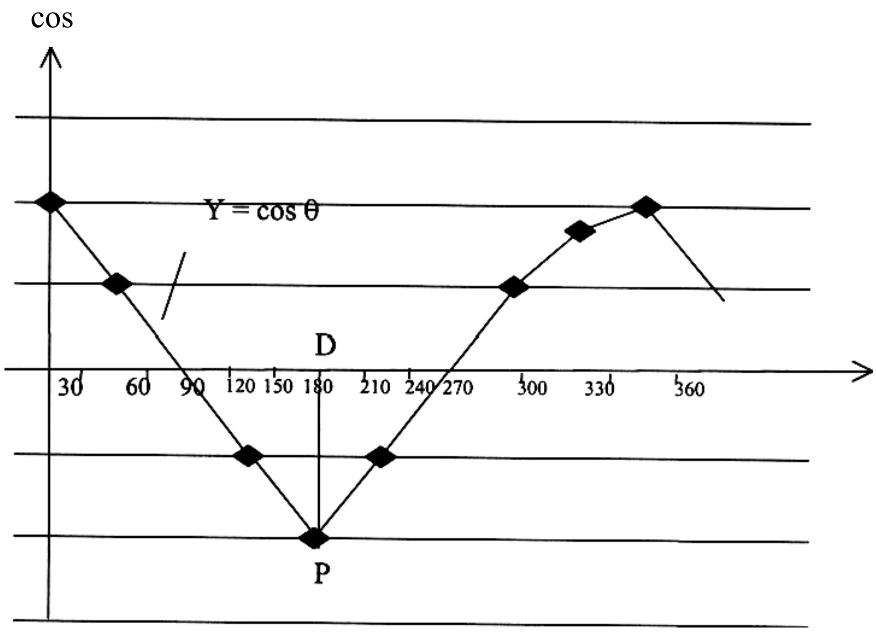


Fig. 4.3: Graph of y = cos θ

Method 2: Projection A

The graph of cos θ may be drawn in a similar way to that of sine. In this case the values of cos θ = sin (90 - θ).

The universally used method of plotting graph is the method by the use of table of values. So the cosine curve will not be shown by the projection method here properties of the cosine curve.

1. The cosine curve is continuous
2. The minimum value is continuous the minimum values i.e. - 1. So like the sine curve, it lies between -1 and 1.
3. The graph repeats itself at the interval of 360° and the function is also called a periodic function with periodical 360°
4. The length of graph of $y = \cos \theta$ (amplitude) is 1.

Note the curves of the sine and cosines are identical because they have the same wavelength. The differences are that:

(1) The sine curve goes from 0 to 1 while the cosine curve goes from 1 to 0 and
(2) Since $\cos \theta = \sin (90 - \theta)$, the difference between the curves is 90°

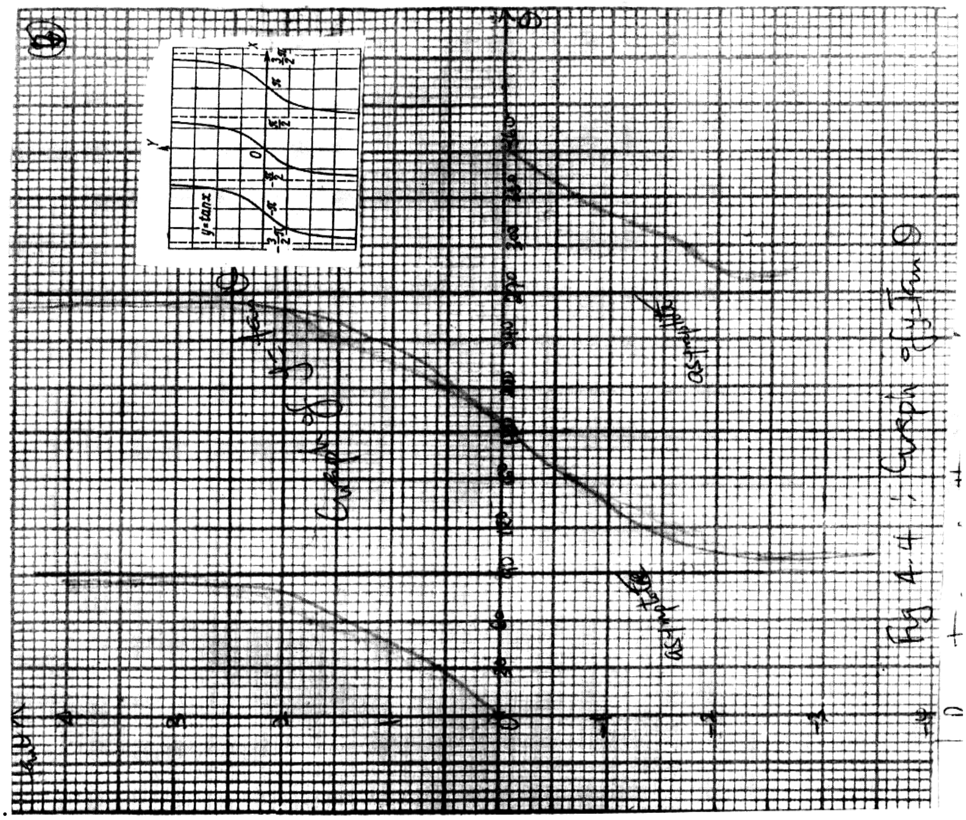
3.1.3 The graph of $y = \tan \theta$

The graph of y = tan 0 is treated as in the case of the sine and cosine curves thus the table of values is shown in Tables 3

Table 3: Table of values for y = tan θ, θ <_ 0<_ 360°

	0^o	30^o	60^o	90^o	120^o	150°	180^o	210^o	240^o	270^o	300^o	330^o	360^0
Tan	0	0.58	1.73	α	-1.73	-0.58	0	0.58	1.73	α	-1.73	-0.58	0

Scale:
Chose suitable scales: here the scales chosen are: 1 cm for I unit at the -axis (x-axis)
2cm for 1 unit at the tan axis (y-axis)
the graph of y = tan is shown in Figure 4.3 below



Properties of the graph of y = tanθ

1. The tangent curve is discontinues because tan is not defined at 90° and 270° respectively i.e. tan 90^o = tan 270^o = α
2. The graph of y = tan has 3 parts namely 0° → 90°, 90° 270^0, → 360^0
3. The tangent curve indefinitely approaches the vertical lines at 90° and 270° but never touches them. Such lines (at 90° and 270°) are called asymptotes(here the curve approaches straight line parallel to the y- axis and distance from it by ± 90°, ±270°, ±450° etc. but never reaches these straight lines. Put in another form, the lines at 90° and 270° are said to be asymptotic curves.

3.2. Graphs of Reciprocal Of Trigonometric Functions

3.2.1 Graph of y = cot θ

This is the reciprocal of the graph of y = tan θ and is shown below.

Table 4; Table 3: Table of values for y = cot θ , $0^0 \leq \theta \leq 360^0$

	0^o	30^o	60^o	90^o	120^o	150°	180^o	210^o	240^o	270^o	300^o	330^o	360^o
Cot	α	1.73	0.58	0	-0.58	-1.73	α	1.73	0.58	0	-0.58	-1.73	α

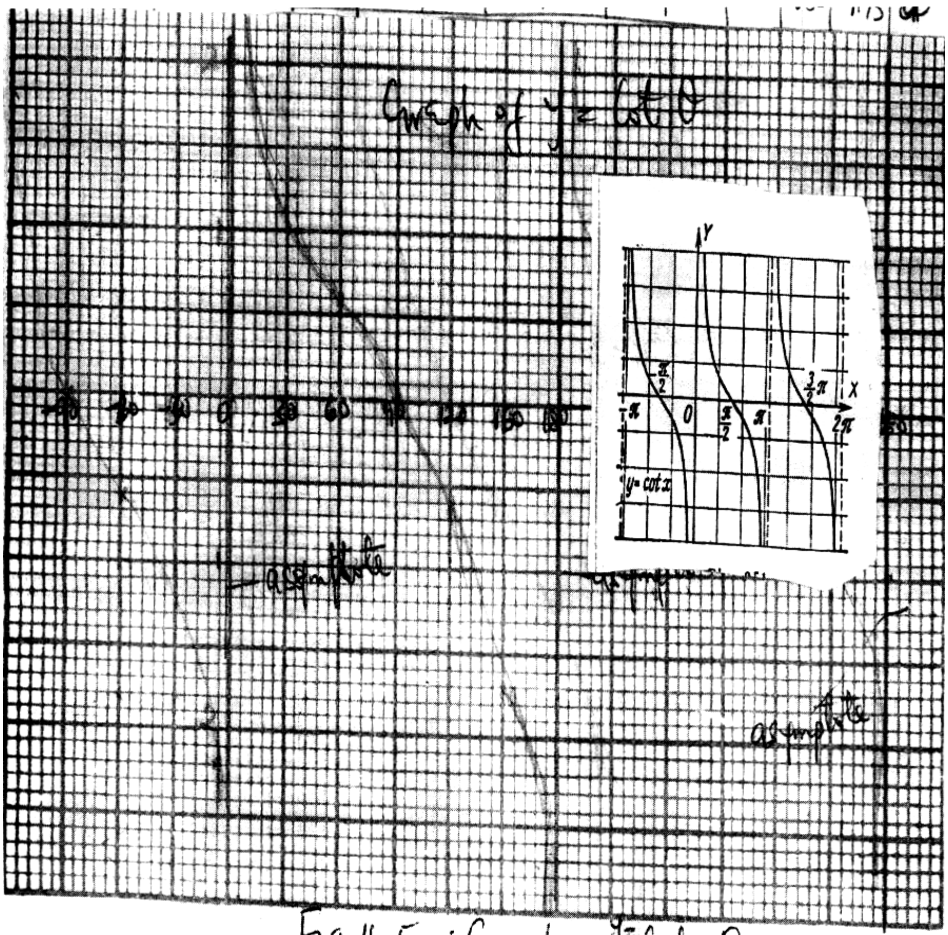
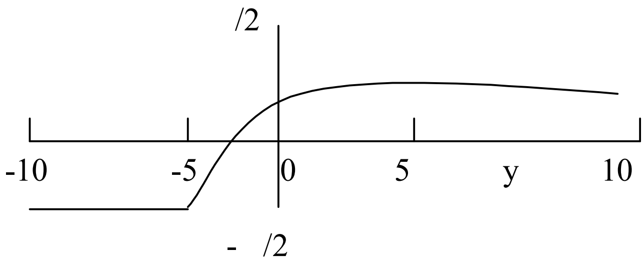
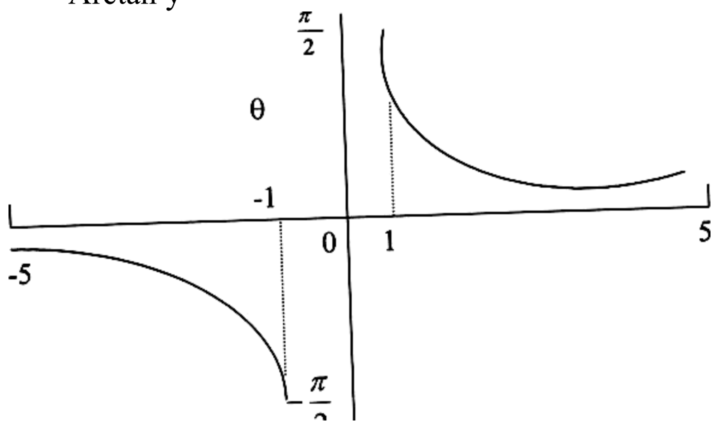


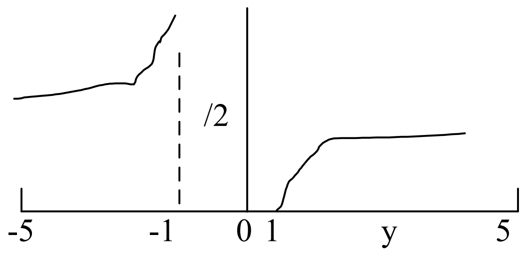
Fig 4.5 : Graph of y = Cot θ



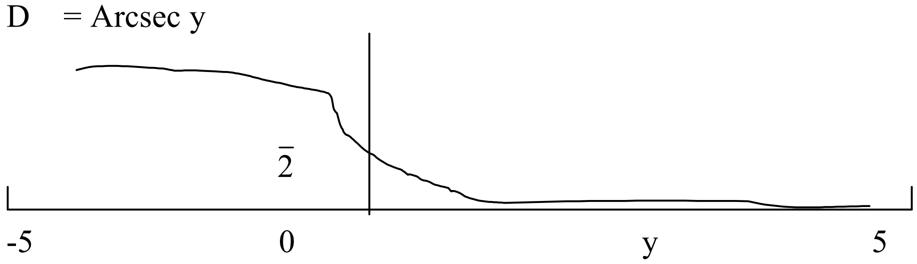
B. = Arctan y



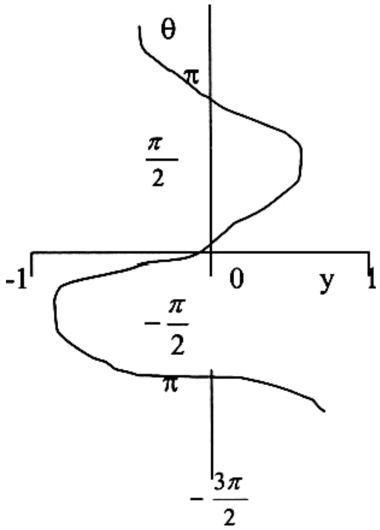
C 0 = Arccosec y



D = Arcsec y



E . = Arcot y



F. 0 = arcsin y. Range: Unrestricted.

3.2.3 Applications of Graphs of Trigonometric

Functions: Composite functions.

Example:

(i) (a) Draw the graph of y = sin 20 for values of between and 360° (b) use your graph to find the value of the following when

(ii) 25^{o} (iii) 35^{o} (iv) 50°

Solution:

(i) make a table of values thus for sin 2 °

	0^{o}	30^{o}	45^{o}	60^{o}	90^{o}	120^{o}	135^{o}	150^{o}	180^{o}
SIN 2	0	0.87	1.0	0.87	0	-0.87	-1	-0.87	0

Note when θ = 30° sin 20 = sin (2x 30) = sin 60° = 0.87 etc. (ii) Chose a suitable scale for clarity

Here the scale of 1 cm to 30° on the 0 axis and 4 cm to 1 unit on the sin 20, axis since no value of sin 20 exceeded 1.

(iii) Plot the points. Here use graph sheet for a clearer picture of the graph.

(a) the angles being sort for are then marked out on the θ (x-axis) and a vertical line drawn from it to the graphs wherever it touches the graph, draw a horizontal line to the y -axis (sin 2θ) axis then read off the values or its approximations

(b) θ = 25° means that sin 2θ = sin 2 x25° = sin 50°. Then 50° lies between 30° and 60° so its value will be between the values of 30°(0.5) and 60° (0.87). This value is approximately 0.85 (see graph below)

(c) θ = 35° means that sin 20 = sin 2x 35° = sin 70° 70° lies between 60° and 90°. So its value will be between the values of 60° and 90° i.e. (0.87 and 0). From the graph it is approximately 0.49

(d) θ = 50° -----> 2θ = sin 2x50 = 100°
.'. sin 100 is approximately -0.25 from the graph

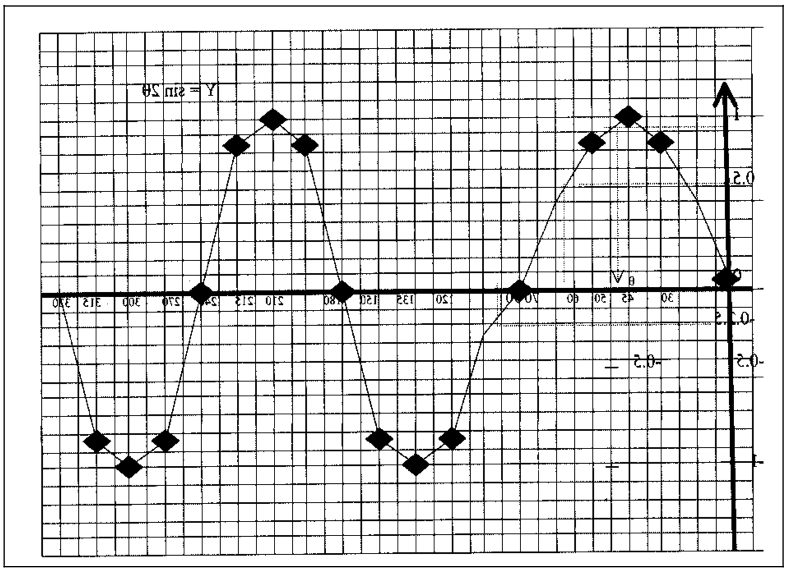


Fig. 4.6

2. Draw the graph of y = 3 - 25sin x for values of x between 0 and 360°

Solution:

The table below shows values of y = 3, y = sin x and y = 25sin x and finally y = 3 - 2 sinx

Table of values for x from 0° to 360°

x^0	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
Sinx	0	0.5	0.87	1.0	0.87	0.5	0	-0.5	-0.57	-1	-0.87	-0.5	0
3	3	3	3	3	3	3	3	3	3	3	3	3	3
2sinx	0	1.0	1.74	2.0	1.74	1.0	0	-1.0	-1.74	-2.0	-1.74	-1.0	0
Y=3-2 sinx	3	2.0	1.26	1.0	1.26	2.0	3	4.0	4.74	5.0	-4.74	-4.0	3

Observe that we first found the values of 2sin x (for the given values of x) before subtracting them from 3 as seen in the lastly 6 row of the table of values above.

With suitable scales the values of x i.e. plotted against the values of y = 3 - 2sin x as other graphs.

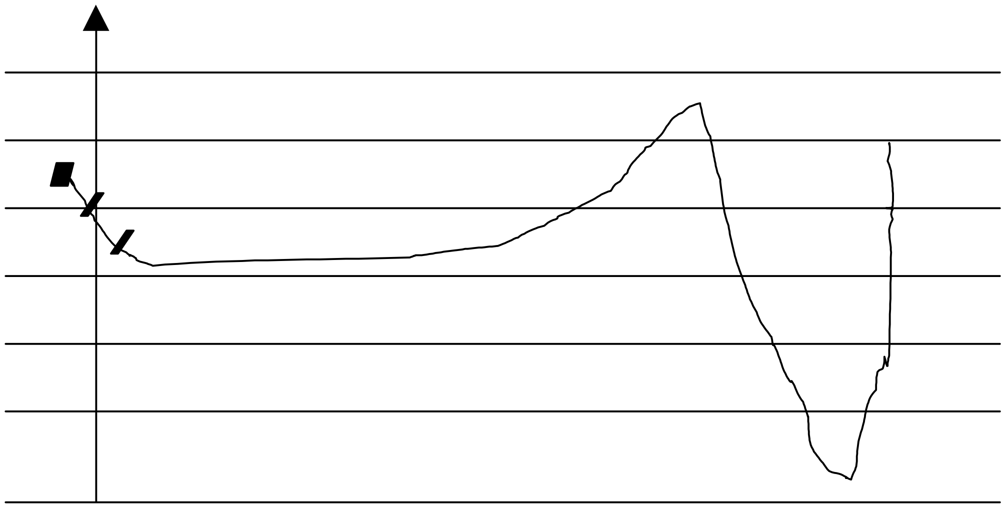


Fig. 4.7

Try the following exercises.
Exercise: 4.2

(1) (a) Construct a table for y = cos x - 3 sin x for values of x from 0° to 180° at 39° interval

(b) use a scale of 23cm to 30° on the x -axis and 2cm to 1 unit on the y-axis to draw your graph.

(2) Draw the graph of y = sin x + cos x for the interval $0° \leq x \leq 360°$ use your graph to find

(a) the maximum values of y = sin x + cos x
(b) the minimum values of y = sin x + cos x

Exercise 4.3

1. Draw the graph of the following for values of 0° from 0° to 360° inclusive.

(a)	y = cos θ	(b)	y = -sin θ
(c)	v = 1-cos θ	(d)	v = -2sin 2θ

2. without plotting the graph, find the;

(i)	amplitude functions.	(ii)	periodicity of the following
(a)	y =5 sin 7	(b)	y = 5 sin (+ 360°)

(c) y = cos5β (d) y = -2 cos 2x

Solution:

(a)	i.	5	ii.	360/7
(b)	i.	5	ii.	360°
(c)	i	1	ii.	360/5
(d)	i	2	ii.	180

3 Copy and complete the table below for y = cos 2 + 2sin for 0° ≤θ ≤ 360 in the interval of 300

Table: y = cos 2θ + 2sin θ for 0° ≤ θ ≤ 360 in the interval of 30°

θ	00	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
sinθ	0	0.5	0.57	1.0	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
2sinθ	0	1	1.73	2	1.74	1.0	0	-1	-1.73	2.4	-1.73	-1.0	0
Cos2θ	1	0.5	-0.5	-1	-0.5	0.5	1.0	0.5	-0.5	-1	-0.5	0.5	1
Y=2sinθ +cos2θ	1	1.5	1.223	1.0	1.24	1.5	1.0	-0.5	-2.23	-3	-2.24	-0.5	1

The periodicity of the cosine, secant and cosecant of an angel x are also 360° but the periodicity of the tangent and cotangent of an angle x are both 180° (this is because tan (x ± R180°) = tan x.

Note the table below shows the (1) amplitude (height)
(ii) periodicity of some functions.

Angle	Function	Amplitude	Periodicity
θ	Y = sinθ	1	360°
	Y = 2sinθ	2	360°
	Y= 5 sinθ	5	360
2θ	Y = sin2θ	1	$360/2=180^0$
	Y =2sinθ	2	180°
	Y = 5sinθ	5	180°
nθ	Y = sin nθ	1	360/ n
	Y = 2sin nθ	2	360/ n
	Y = 5 sin nθ	5	360/ n

Note that for any graph of y = A sin θ , the amplitude is /A / i.e. where A is any constant (coefficient of sin θ) and a periodicity of 360°, while if the graph is that of y = A sin θ the amplitude is still /A/, provided A is any constant and its periodicity is 360/n where n is any constant. This also applies to cosine, secant and cosecant. Amplitude is always a positive number.

4.0 CONCLUSION

Having treated the graph of trigonometric functions and their reciprocals and also in this unit, you have seen that the treatment of graphs here are the same with the treatment of graphs of algebraic functions, the only difference is in the values assigned to the
independent variable (x) which in this case are angles. The processes are the same thus

(1) table of values
(2) choice of scales
(3) plotting of the points and joining it is believed that the treatment of graphs of trigonometric functions, will enable you see the interrelatedness of function waves, motions etc.

5.0 SUMMARY

In this unit, we have attempted to draw the graphs of trigonometric functions, their reciprocals and inverse functions. The properties of these graphs of trigonometric functions were highlighted such as:

(i) The sine and cosine curves are continuous functions while the tangent and cotangent are discontinuous functions.
(ii) The periodicity of a function is the interval at which the graph repeats itself and such functions are called periodic functions example, the sine, cosine, tangent etc are periodic functions.
(iii) The periodicity for the sin, cos, sec and cosec is 360° while that of the tan and cot is 180°
(iv) The amplitude or the length of a graph is the distance between the highest point and the x-axis of the function
(v) The sine and cosine curves lies between -1 and 1 and they have the similar shape because cos θ - sin (90 -θ)

7.0 REFERENCE/FURTHER READING

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UNIT 2 TRIGONOMETRIC IDENTITIES AND EQUATIONS

CONTENTS

1.0 INTRODUCTION

In the earlier units, you learnt about trigonometric ratios, their reciprocals and inverse trigonometric functions. There are lots of important relations between trigonometric functions. For example;

$\frac{\sin \theta}{\cos \theta} = \tan \theta$; $\frac{1}{\sin \theta} = \text{cosec } \theta$

$\frac{\cos \theta}{\sin \theta} = \cot \theta$; $\frac{1}{\cos \theta} = \sec \theta$

If these relations are true for any given value of such relations are called trigonometric identities, provided the functions are defined.

This unit will focus on trigonometric identities, which should form the basis for proving other identities, compound angles, difference and product formulae, multiple and half angles and finally trigonometric equations, which are embedded in them.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define trigonometric identities correctly
- prove given trigonometric identities correctly
- simply and solve problems involving trigonometric identities and equations

- express sum and difference of two given angles in trigonometric identities
- express multiple and half angles of given identities
- factorise trigonometric expressions.

3.0 MAIN CONTENT

3.1 Trigonometric Identities (Fundamental Identities)

3.1.1 Trigonometric Identities (Right-Angled Triangle)

Trigonometric identities are relations, which are true for any given value of given a right-angled triangle ABC, right-angled at B and angle C = with the usual notations see Fig 5.1.

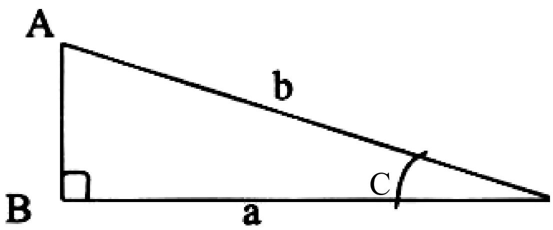


Fig: 5.1.

$\text{Sin } C = \frac{c}{b} \rightarrow b \sin c$ ______________________ (1)

$\text{Cos } C = \frac{a}{B}$ -------> $b \cos C = a$ _________________ (2)

By Pythagoras theorem $a^2 + c^2 = b^2$, so substituting the values of a and c from (1) and (2) we obtain;

$(b \cos C)^2 + (b \sin C)^2 = b^2$, simplifying

$b^2 \text{ COS}^2 C + b^2 \sin C = b^2$,

dividing through by b^2, we have

$$\text{Sin}^2 c + \cos^2 c = 1 \text{ OR } \cos^2 c + \sin^2 c = 1$$

Also

$$\text{Sin}^2 c = 1 - \cos^2 C \quad \text{and}$$

$$\text{Cos}^2 C = 1 - \sin^2 C$$

Example:

1. Without tables or calculators find

(a) $\sin^2 60° + \cos^2 60°$

(b) $\sin^2 330° + \cos^2 330°$

Solution:

(a) $\sin^2 60^o$ = 3/2 and $\cos^2 60°$ = ½ and substituting into $\sin^2 60° + \cos^2 60°$, gives;

$$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} + \frac{1}{4}$$

$$= \frac{4}{4} = 1$$

(b) sin 330° = -sin 30° = -1/2 and cos 330° = cos 30 = 3/2 substituting into the given expression $\sin^2 330° + \cos^2 330°$ to obtain;

since this relation $\sin^2$ + $\cos^2$ =1, holds true for all values of , it is then a trigonometric identity.

From the above trigonometric identity $\text{Sin}^2 + \cos^2 = 1$, the following trigonometric identities can be deduced:

$\text{Sin}^2 C + \cos^2 C = 1$

Divide through by cos θ, this becomes

$$\frac{\text{Sin}^2C}{\text{Cos}^2C} + \frac{\cos^2C}{\cos^2C} = \frac{1}{\cos^2C}$$

but $\frac{\sin^2C}{\cos^2C} = \tan C$ and $\frac{1}{\cos C} = \sec C$

$$\frac{\sin^2C}{\cos^2C} + 1 = \frac{1}{\cos C} \text{ gives } \left(\frac{\sin C}{\cos C}\right)^2 + 1 = \text{Sec}^2C$$

$= \text{Tan}^2C + 1 = \sec^2C$

$= \sec^2C - 1 = \tan^2C$

Again, if we divide $\sin^2 C + \cos^2 C = 1$ by $\sin^2 C$, it becomes

$$\frac{\text{Sin}^2 C}{\sin^2 C} + \frac{\cos C}{\sin^2 C} \quad \frac{1}{\sin^2 C}\text{, but}$$

$$\frac{\cos C}{\sin C} = \cot C \text{ and } \frac{1}{\sin C} = \text{cosec } C$$

$1 + \cot^2 C = \text{cosec}^2 C$
$\text{Cosec}^2\theta - 1 = \cot^2\theta$

Other relations which can be deduced are:

(1) tan C x cot C = 1
(2) cos C x sec C = 1
(3) sin C x cosec C = 1
(4)

$\frac{1 - \cos^2 C}{\sin^2 C}$ Note that $1 - \cos^2 C = \sin^2 C$

$$\frac{1 - \cos^2 C}{\sin^2 C} = \frac{\sin^2 C}{\sin^2 C}$$

$\therefore 1 - \cos^2 C = 1$

Hence from these $\sin^2$ examples it can be deduced that knowing the value one of the trigonometric functions of an acute angle is possible to find the value of the others.

3.1.2 Trigonometric Equations

Trigonometric equation is an equation involving an unknown quantity under the sign of a trigonometric function.

Techniques for solving trigonometric equations:

(1) take care to see that the transformed equation is equivalent to the original equation.
(2) reduce the given equation to an equation involving only one trigonometric ratio where involving only one trigonometric ratio where possible. This is about the simplest way of solving a trigonometric equation, example $3 + 2\cos\theta = 4\sin^2\theta$, it is convenient to express this equation in terms of $\cos\theta$ since $\sin^2\theta = 1 - \cos$ i.e.
$3 + 2\cos\theta = 4(1 - \cos^2\theta)$ the solve the equation as a quadratic equation in one variable $(\cos^2\theta)$

(3) when the terms of the equation have been squared or you have performed some transformed that do not guarantee equivalence, check all the solutions to avoid less of roots.

Example:

1. Solve the equation, giving values of θ from 0 to 360 inclusive.

(a) $3 - 3 \cos \theta = 2 \sin^2 \theta$(b) $\cos^2 \theta + \sin \theta + 1 = 0$

Solution

(a) $3\text{-}3\cos\theta = 2\sin^2 \theta$, here the members of the equation can be expressed as $\cos \theta$ since $\sin \theta = 1 - \cos^2 \theta$

$\therefore\ 3 - 3\cos \theta = 2(1 - \cos^2\theta)$
$= 3\text{-}3\cos\theta = 2\text{-}2\cos^2\theta$
$= 2\cos^2\theta - 3\cos \theta + 1 = 0.$

This is a quadratic equation in cos θ and thus can be solved by any of the methods of quadratic equation.

By factorization $2\cos \theta - 3\cos \theta + 1 = 0$ gives;
$(2 \cos\theta - 1) (\cos \theta - 1) = 0$
either $2\cos\theta - 1 = 0$ OR

$\cos\theta - 1 =$

if $2\cos\theta - 1 = 0$ =:> $\cos = ½$ and $\cos \theta$ is +ve
$\therefore$ $\theta = \cos^{-1} ½ = 60° \text{ or } 300°$

if $\cos \theta = 1$, $\theta = \cos^{-1} 1 = 0 \text{ or } 360°$
$\therefore$ the values of θ which satisfy the equation within the given range of $0 \leq \theta \leq 360°$ are $\theta = 0°$. 60°, 300° and 360°.

(b) $\text{Cos}^2 \theta + \sin\theta + 1 = 0$

It is easier to transform $\text{Cos}^2 \theta$ to $1\text{-}\sin \theta$ to form an equation of powers of a $\sin^2\theta$. Thus;

$(1 - \sin^2 \theta) + \sin \theta + 1 = 0$
$1\text{-}\sin^2 \theta + \sin \theta + 1 =0$
$=:> 1\text{-}\sin^2 \theta + \sin\theta + 2 = 0$

Factorizing: $(\sin \theta - 2)(\sin \theta+l) = 0$

.'. either $\sin\theta - 2 = 0$ or $\sin\theta + 1 = 0$
if $\sin\theta - 2 = 0 \Rightarrow \sin\theta = 2$ and $\theta = \sin^{-1} 2$

This value of θ does not satisfy the given equation because sinθ lies between -1 and 1 to satisfy the given equation. So $\theta = \sin^{-1} 2$ is not a solution if $\sin\theta + 1 = 0$.

Sin = - 1 => $\theta = \sin^{-1}(-1) = 270°$
∴ θ = 270° is the root of the equation becomes it falls within the range $0 \leq \theta \leq 360°$

2. Find all the solutions of the equation in the interval $0 < \theta \leq 360°$ $16\cos^2\theta + 2\sin\theta = 13$

Solution

$\text{Cos}^2\theta = 1 \sin^2\theta$, this will be substituted into the equation to give;
$16(1 - \sin^2\theta) + 2\sin\theta = 13$

$16 - 16\sin^2\theta + 2\sin\theta = 13 = 0 =:> 16\sin^2\theta - 2\sin\theta - 16 - 13 = 0 \Rightarrow 16\sin^2\theta - 2\sin - 3 = 0$

Factorising gives $(8\sin\theta + 3)(2\sin\theta - 1) = 0$

.'. either $8\sin\theta + 3 = 0$ OR
$2\sin\theta - 1 = 0$

so, if $8\sin 0 + 3 = 0 =:> \sin\theta = -3/8$
$\theta = \sin^{-1}(-3/8) = \sin^{-1}(-0.375)$
From the tables 0 = -22°. This lies either in the third or fourth quadrant since sinθ is negative

∴ θ = 180 + 22° 2' or 360° - 22° 2'
= 202° 2' or 337° - 58'
if $2\sin\theta = 1 ===> \sin\theta = ½ =:> \theta = \sin^{-1}(½)$
.'. θ = 30° since sinθ is positive, θ

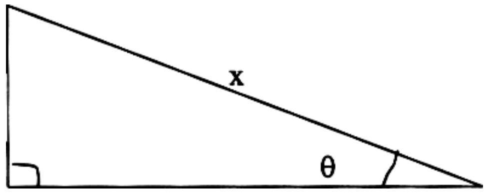
is either in the first or second quadrant
.'. θ = 30° or 180° - 30°
= 30 ° or 150°
∴ the solution of the equations for $0 \leq \theta \leq 360°$
is θ = 30, 150°, 202° 21' and 337° 58' .

3. Find without table, the value of sec θ, sinθ if tan θ = -5/12

Solution;

Using a right angle triangle fix the sides of the triangle using the knowledge

5



12

Fig; 5.1

That tan θ is $\frac{Opposite}{adjacent}$. Finding x i.e. x ≈ the hypotenuse side by

Pythagoras theorem gives $5^2 + 12^2 = x^2$

$25+144=x^2 y=\cos^2\theta + 2\sin\theta$ for $0° \leq \theta \leq 360$ in the interval of 30°

$169 = x^2$

.'. $x = \sqrt{169} = \pm 13$

.'. sin θ = 5/±13 and cos θ = ± 12/±3, but θ is obtuse, hence sin θ = 5/13 and cosθ = -12/13

.'. sin θ = 5/13 and sec θ = $1/\cos\theta$

gives; $\sec\theta = \frac{-1}{\frac{12}{13}} - \frac{-13}{12}$

4. Prove the following identities:

$\sec^2\theta + \text{cosec}^2\,\theta = \sec\theta\,\text{cosec}^2\,\theta$

Solution:

In problems of this sort, start from whatever expressions (either left hand side or right hand side) to show that it is equal to the other (right hand side or left hand side) whichever is simpler Thus starting from the left hand side (LHS)

$\text{Sec}^2\,\theta + \text{cosec}^2\,\theta = \frac{1}{\text{Cos}^2\theta} + \frac{1}{\sin^2\theta}$

Simplifying gives $\frac{\sin^2\theta + \cos\theta}{\text{Cos}^2\theta\,\sin^2\theta}$

But $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \quad \frac{\sin^2\theta + \cos\theta}{\text{Cos}^2\theta \sin^2\theta} = \frac{1}{\text{Cos}^2\theta \sin^2\theta}$$

$$= \frac{1}{\text{Cos}^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta \text{ x } \text{cosec}^2\theta = \sec^2\theta \text{ x } \text{cosec}^2\theta = \text{RHS}$$

Note that in examples 1 and 2 we concentrated only on angles in the first revolution or basic angles. This is because in many applications of trigonometry they are the ones usually required.

3.2 Compound Angles

3.2.1 (A) Addition Formulae

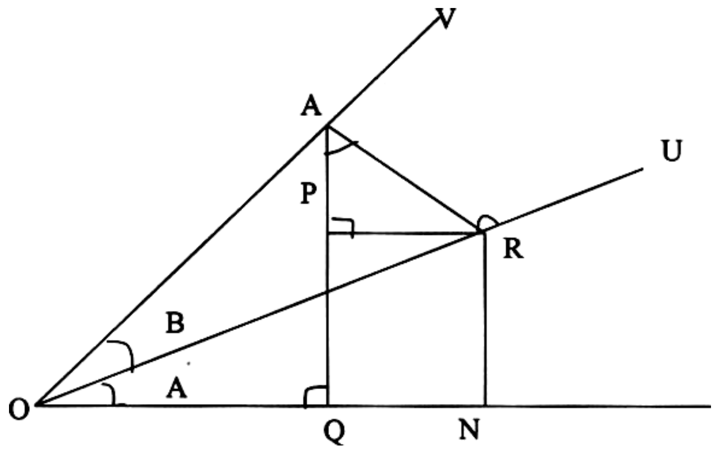


FIG 5.2.

In Figure 5.2 above

< PAR = 90° - <ARP

<PRO = <RON (alt <s PR| |ON)

$$\text{Sin}(A+B) = \frac{AQ}{\theta A} = \frac{PQ + AP}{\theta A}$$

$$\frac{RN + AP}{\theta A} = \frac{RN}{\theta A} + \frac{AP}{\theta A}$$

$$\frac{RN}{OR}\ \frac{OR}{OA} + \frac{AP}{AR}\ \frac{AR}{OA}$$

= Sin A Cos B + Cos A Sin B

$$\therefore \text{Sin}(A+B) = \text{Sin } A \text{ Cos } B + \text{Cos } A \text{ Sin } B$$

Similarly from the same fig 5.2

$$\text{Cos}(A + B) = \frac{0Q}{\theta A} = \frac{\theta N - QN}{\theta A}$$

$$= \frac{ON - PR}{OA} = \frac{ON}{OA} - \frac{PR}{OA}$$

$$\frac{ON}{OR}\frac{OR}{OA} - \frac{PR}{MR}\frac{MR}{OA}$$

= CosACosB - SinASinB

~

.'. Cos(A +B) = CosACosB - SinASinB

Tan (A+ B) = $\frac{\text{Sin (A+B)}}{\text{Cos(A + B)}}$ since tan θ = $\frac{\text{Sin θ}}{\text{Cos θ}}$

$$\frac{\text{Sin A Cos B + CosASin B}}{\text{CosACosB – SinASinB}}$$

Dividing both the numerator and denominator by CosACosB,

Tan(A + B)

$$= \frac{\frac{SinACosB}{CosACosB} + \frac{CosASinB}{CosACosB}}{\frac{CosACoB}{CosACosB} + \frac{SinASinB}{CosACosB}}$$

Simplifying gives; $\frac{\text{tan A + tan B}}{\text{1- tan A tan B}}$

.'. tan(A+ B) = $\frac{\text{tan A + tan B}}{\text{1 - tan A tan B}}$

3.2.1b. Difference Formulae

The difference formulae can be obtained from the addition formula for replacing B with (-B) in each case thus ;

(a) Sin(A - B) = SinACosB - CosASinB

(b) Cos(A - B) = CosACosB + SinASinB

(c) Tan(A -B) = $\frac{\text{tanA-tanB}}{\text{1+ tanA tan B}}$

Example:

Without using tables or calculators find the values of the following leaving your answers in surd form.

(i) Cos(45° - 30') (ii) sin(60 + 45°) (ii) tan 75°

Solutions;

(i) Cos (45° - 30^{o}) is in the form of cos (A -B) and by the addition/difference formula it is

(Cos(A - B) = CosACosB + SinASinB expanding Cos(45 - 30) thus, where A = 45 and B = 30 gives

Cos 45Cos30 + Sin45° Sin30 so substituting the values for

Cos 45 = 1/ 2   and sin 45 = 1/ 2

Cos 30 =   3/2 and Sin 30 = ½ in

Cos 45Cos30 Sin45Sin30   gives
(1/ 2) (3/2) + (1/ 2)(1/2)

=   3/2 2 + 1/2 2

$\frac{3 + 1}{2\ \ 2} = \frac{1 +\ \ 3}{2\ \ 2} = \frac{2\ (1 +\ \ 3)}{4}$

(ii) Sin (60 +45) = sin60 Cos45 + Cos60Sin45, here Sin(A +B) = SinACosB + CosASinB as applied If A = 60 and B = 45
and substituting the values of Sin 60° =   3/2 ,   Cos60 = ½ , Sin45° = Cos° = 1/ 2 to obtain;

(3/2)(1/2) + (1/2)(1/ 2)

$\frac{3}{2\ \ 2} + \frac{1}{2\ \ 2}$

$\frac{1 +\ \ 3}{2\ \ 2} = \frac{2\ (1 +\ \ 3}{4}$

(iii) tan 75° = tan (45^{o} + 30^{o}) Applying the formula

tan(A -B) = $\frac{\text{tanA-tanB}}{1+ \text{tan A tan B}}$, where

A = 45 and B = 30 and tan 45° = 1 and tan 30 = 1/ 3 gives

$$\frac{1 + \frac{1}{3}}{1 - 1\ \frac{1}{3}} = \frac{\frac{3+1}{3}}{\frac{1-1}{3}}$$

$$= \frac{3+1}{3\ -\ 1}$$

and simplifying by rationalising the denominator gives;

$$\frac{(\ 3+1)\ (3+1)}{(\ 3-1)\ (\ 3+1)} = \frac{3+2\ \ 3+1}{3-3+\ \ 3-1}$$

$$= \frac{4+2\ \ 3}{2} = 2+3$$

Exercise 5.1

1. Prove the following identities

(a) $\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}$

(b) $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

(c) $2\cos^2\theta - 1 = 1 - 2\sin^{20} = \cos^2\theta - \sin^2\theta$

(d) $\text{cosec}\,\theta + \tan\theta\sec\theta = \text{cosec}\,\theta\sec^2\theta$

2. Solve the following equations, giving values of θ from 0° to 360° inclusive

(a) $\sec^2\theta = 3\tan\theta - 1$ Ans: 45°, $63^0\ 26`$, 225°, $243^0\ 26'$

(b) $3\cos^2\theta = 7\cos\theta + 6$ Ans: 131° 49', 228° 12'

(c) $2\sin\theta = 1$ Ans: 30°, 150°.

3. Find without tables/calculators the values of

(a) Sin θ, tan θ, if cos θ = 45 and θ is acute
Ans: sin θ = 3/5 and tan θ = ¾

(b) cos θ, cot θ, if sin θ = 15/17 and θ is acute
Ans: cos θ = 8/17. cot θ = 8/15

(c) sen θ, sec θ, if cot θ = 20/21 and θ is reflex;
Ans: sin θ = - 21/29 sec θ = -29/20

4. If sinA = 4/5 and cosB = 12/13, where A is obtuse and B is acute, find without tables/calculators the values of

(a)	sin (A – B)	Ans: 63/65
(b)	tan (A – B)	Ans: -63/16
(c)	tan (A + B)	Ans: -33/56

3.2.2. Multiple and Half Angle

3.2.2a. Multiple Angles (Double Angle)

This is an extension of the addition formula; In each case, putting B = A we obtain for sin (A+ B) = sin (A + A) = sin 2A since sin(A + B) = sinAcosB + cosAsinB and replacing B with A gives:

sin(A + A) = sinAcosA + cosAsinA
sin 2A = 2sinAcosA and cos(A +A) = cosAcosA - SinAsinA
$\cos(2A) = \text{Cos}^2 A - \sin^2 A$ but $\sin^2 A = 1 - \cos^2 A$
substituting gives $\cos(2A) = \cos^2 A - 1 + \cos^2 A$
.'. $\cos^2 A = 2\cos^2 A - 1$ and $\cos^2 = 1 - \sin^2 A$ s so $\cos^2 A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin A$
.'. $\cos^2 A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\frac{2\tan A}{1 - \tan^2 A}$$

3.2.2b. Half Angles

By substituting half angles example A/2 or B/2 into the double angles, the formulae above become

(a) $\sin\left(\frac{A}{2} + \frac{A}{2}\right) = \sin A = 2\sin\frac{A}{2}\text{Cos}\frac{A}{2}$

(b) $\text{COS}\left(\frac{A}{2} + \frac{A}{2}\right) = \text{COSA} = \text{COS}^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1$

$= 1 - 2\sin^2\frac{A}{2}$

(c) $\tan\left(\frac{A}{2} + \frac{A}{2}\right) = \tan A = \frac{2\tan A/2}{1 - \tan^2 A/2}$

(d) $\sin^2\frac{A}{2} = \frac{1 - \cos A}{2}$, this comes from $\sin^2 A == \frac{1 - \cos A}{2}$,

(e) $\cos^2 \frac{A}{2} = \frac{1+\cos A}{2}$

(f) $\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}$

3.3. Sum and Difference Formulae (Factor Formulae)

(a) $\sin(A + B) = \sin A\cos B + \cos A\sin B$ ________________ (1)
$\sin(A- B) = \sin A\cos B - \cos A\sin B$ ________________ (2)
adding both (1) and (2)
$\sin(A +B) + \sin(A -B) = 2\sin A\cos B$
subtracting both (1) and (2)
$\sin(A + B) - \sin(A -B) = 2\cos A\sin B$

(b) $\cos(A +B) = \cos A\cos B - \sin A\sin B$ ________________ (3)
$\cos(A - B) = \cos A\cos B + \sin A\sin B$ ________________ (4) adding both (3) and (4)
$\cos(A+B) + \cos(A -B) = 2\cos A\cos B$
and substracting both (3) and (4) gives:
$\cos(A+ B) - \cos(A - B) = -2\sin A\sin B$
which can be rewritten as
$\cos(A -B) - \cos(A+B) = 2\sin A\sin B$
(this is to avoid the minus sign gotten in the first one).

3.3.1 Product Formulae

From the above sum and difference formulae, another interesting identities emerged $\sin(A+B) + \sin(A - B) = 2\sin A\cos B$ if A+B is equal to x i.e. A+B = x and A - B = y, this implies that adding both gives

$$\sin(A+B) + \sin(A -B) = 2\sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = 2\sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos y - \cos x = 2\sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

These formulae can also be stated in this form $\cos A\cos B = \frac{1}{2} \{\cos(A + B) + \cos(A - B)\}$ $\sin A\sin B = \frac{1}{2} \{\sin(A -B) - \cos(A+B)\}$ $\sin A\cos B = \frac{1}{2} \{\sin(A + B) + \sin(A -B)\}$ These are called the product formulae.

Example

Find the value of the following angles without tables or calculators.

(a) cos 75° cos l5° (b) sin75° + sin 15' (c) cos83° - cos 17°

Solution

(a) to solve the given problem, apply the product at formula which states that cosAcosB = ½ {cos(A + B) + cos(A -B) } so taking A = 75° and B = 15° substituting gives

cos 75°cosl5° = ½ { cos(75 +15) + cos(75 -15) }
= ½ { cos90 + cos 60}

and the values of cos90' and 60° without tables or calculator are:
cos 90 = 0 and 60= ½

and substituting

cos75°cos 15° = ½ (0 + 1/2)
= ½ x ½ = ¼

(b) For sin75° + sin 15° to solve this apply the sum and difference formula which states that sinx + siny = $\frac{2\sin x+y}{2}$ cos $\frac{x-y}{2}$

so taking x = 75' and y ==15^{o} and substituting into the formula gives:

sin $\frac{75°+\sin 15°}{2}$ = cos $\frac{75-15}{2}$

=2sin $\frac{90}{2}$ Cos $\frac{60}{2}$

= 2sin45°cos30°

the values of sin45° = 1/ 2 cos30° = 3/2 , are known, so substituting back

sin75 ° + sin 15^{o} = 2 (1/ 2)(3/2)
= 3/ 2
= $\frac{3 \text{ x } 2}{2}$ = $\frac{6}{2}$

(c) cos 83^{o} -cos 17, since cosA-cosB = -2sin $\frac{A+B}{}$ sin $\frac{A-B}{}$, then

2 2

substituting for

A = 83° and B = 17° into the above formula, we have

$$\text{Cos}83° - \cos 17° = -2\sin \frac{83+17}{2} \sin \frac{83-17}{2}$$

= -2 sin 100/2 sin 66/2

= $-2\sin 50^{o} \sin 33^{o}$.

2. Solve the equation sin 5x + sin 3x - 0 for values of x from - 180° to 180° inclusive.

Solution

Applying the formula. $\text{Sin}A + \sin B = 2\sin \frac{A+B}{2} \cos \frac{A-B}{2}$, and substituting for A = 5x and B =3x gives;

$$\text{Sin}5x + \text{Sin}3x = 2\sin \frac{5x+3x}{2} \cos \frac{5x-3x}{2}$$

$$= 2\sin \frac{8x}{2} \cos \frac{2x}{2}$$

= 2sin 4x cox

but sin 5x + sin3x = 0,

this implies that 2sin 4xcosx = 0 but 2 cannot be zero either so, sin4x cosx = 0 or cos x = 0

since x lies in the range of -180° to 180°

4x will lie in the range of 4(-180) to 4(180) = -720 to 720 so if cosx = 0 => x = 90° or - 90°

x = - 90°, 90°

if sin 4x = 0 sin-10 = 4x

since these are values at which when sinx 4x = 0, -180 and 180 the 0 between - 180 and 180.

4x = -720°, -180°,0°, 180°, 720° (including the intervals)

x = - 180°, -45°, 0°, 45°, 180° (dividing through by 4)

so the value of x which satisfies the equation are;

x = -180°, -90°, -45°, 0°, 45°, $90^{o}1$, 180°.

Exercise 5.2

1. Solve the equation sin(x+ 17°) cos(x -12°) = 0. 7 for values of x from 0° to 360 inclusive.

Ans: x = 30° 37', 54^0 23', 210^0 37', 234^0 23'

2. Prove the identities

(a) $\frac{\cos B+\cos C}{\sin B - \sin C} = \cot \frac{B-C}{2}$

(b) sin x - sin(x+60) + sin (x+120° =0

(c) cosx + cos(x+120) + cos(x+240°) =0

3. Solve for the following equations, for values of x from 0° to 360° inclusive.

(a) cosx + cos 5x = 0

Ans: 30°, 90°, 150°, 240°, 270°, 330°, 45°, 135°, 225°, 315°. (b) sin3x + cos2x = 0 (hint cos2x = sin(90° - 2x)

Ans: 54°, 126°, 198°, 270^0, 342^0.

4. Express the following in factors.

(a) sin2y - sin2x

Ans: 2cos(y + x) siny -x)

(b) sin(x + 30) + sin(x -30°)

Ans 3sinx

(c) cos(0° - x) + cosy

Ans: 2cos(45° - ½ x – ½ y)

(d) sin2(x + 40°) + sin2(x -40°)

2sin2x cos80

4.0 CONCLUSION

In this unit, you have seen the beauty of the relations of trigonometric identities and how easy they are applied in solving trigonometric functions problems. From the addition formulae, we were able to define the sum and difference and product formulae by simple manipulation of one o f the angles and by the operations of addition and subtraction. This made trigonometric identities fun.

5.0 SUMMARY

In this unit, the following trigonometric functions identities were deduced from the fundamental identities i.e.

$\text{Sin}^2\theta + \cos^2\theta = 1$
$1 - \sin^2\theta = \cos^2\theta$
$1 - \cos^2\theta = \sin^2\theta$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \text{cosec}^2\theta$
$(\tan\theta)(\cot\theta) = 1$
$(\cos\theta)(\sec 0) = 1$
$(\sin\theta)(\text{cosec}\,\theta) = 1$

From the addition formulae, (addition and subtraction).

Sin(A+B) = sin AcosB + cosAsinB
Sin(A -B) = sinAcosB - cosAsinB
Cos(A+B) = cosAcosB - sinAsinB
Cos(A-B) = cosAcosB - sinAsinB

$\text{Tan}(A+B) = \frac{\tan A + \tan B}{1 \tan A \tan B}$ and

$\text{Tan}(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

The following multiple angles (double angles), half angles, sum and difference and product formulae were deduced.

$\text{Sin}2A = 2\sin A\cos A$
$= 1 - 2\sin^2 A$

$\text{Tan } 2A = \frac{2\tan A}{1 - \tan^2 A}$

$\text{Cos}2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

Half Angles

$\text{Sin}A = 2\sin \frac{A}{2} \text{Cos} \frac{A}{2}$

$\text{Cos}A = \text{Cos}^2 \frac{A}{2} - I$

$= 1 - 2\text{Sin}^2 \frac{A}{2}$

$\text{Tan } A = \frac{2 \tan A}{1-\tan^2 A}$

Sum and Difference formulae (factor formulae)

2cosAsinB = sin(A+B) - sin(A -B)

2sinAcosB= sin(A+B - sin(A -B)

2cosAcosB = Cos (A+B) + cos(A-B)

2sinAsinB = cos(A-B) - cos(A+B) And finally the

Product formulae

CosAcosB = ½ {cos(A + B) + cos(A -B) } SinAsinB = ½ {sin(A -B) - cos(A+B)}
SinAcosB = ½ { sin(A + B) + sin(A -B) }

6.0 TUTOR-MARKED ASSIGNMENT

1. find the values of the following without tables or calculators, leaving your answers in surd form.

(a) (i) tan 105° (ii) cos 15° (iii) cos 345°
(iv) sin 165°

Ans: (i) - 2 - 3 (ii) $\frac{3 + 1}{2\ \ 2}$ (iii) $\frac{3 + 1}{2\ \ 2}$ (iv) $\frac{3 - 1}{2\ \ 2}$

(b) if cosA 4/5 and cos B 12/13 (A and B are both acute) Find the values of
(i) sin (A+ B) Ans; 56/65
(ii) cos(A -B) Ans; 63/65
(iii) tan(A+B) Ans; 56/33

2. Solve the equations for $0 \leq \theta \leq 360°$
(a) sin2θ = tan θ Ans: θ = 0°, 45°, 135°, 180°, 225°, 340°, 360°
(b) cos2θ = 2cosθ Ans: θ = 11. 47° or 248. 53°

3. Find without tables or calculators, the values of

(a) $2\sin 15° \cos 15°$ Ans: ½

(b) $\frac{540°}{8} \cos \frac{540°}{8}$ Ans: $\frac{1}{2\ \ 2}$

(c) $\dfrac{2 \tan \frac{540^{o}}{8}}{1 - \tan^{2} \frac{540^{o}}{8}}$ Ans;-1

(d) $\sin^{2} 22\frac{1}{2}^{0} - \cos^{2} 22\frac{1}{2}°$ Ans: - -1/ 2

7.0 REFERENCES/FURTHER READING

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