MODULE 4

UNIT 1 SIMPLE HARMONIC MOTION I

CONTENTS

1.0 INTRODUCTION

We discussed linear, projectile and circular motions. Another common type of motion is the to- and fro motion which keeps repeating itself for ever if there are no frictional forces acting against it to dampen the motion. Such a motion we call a periodic, oscillatory or vibrational motion. Periodic or rythmic motion, we sense is an important feature in the physical world. You have only to think of the very concept of time, which arose from the observation of certain motions as we saw in Unit 1. Think of the cycling of the seasons. Do they not repeat themselves at regular intervals? Place your hand on your chest for about one minute. What do you sense? Your heart beat? That's an example of a rythmic motion. The most basic type of rythmic motion appears over and over again and this is what we call **simple harmonic motion**.

Examples of this vibratory or oscillatory motion are provided by the motion of a swinging pendulum, the balance wheel of a watch and by the motion of a man on the end of a spring.

In simple harmonic motion (s.h.m) the position of a point varies with time as a sine or a cosine function. Such motion occurs where we have restoring forces, (that is, forces that tend to bring an object back to a point), that vary linearly with a position variable. It is interesting to note that all stable equilibrium situations in nature involve a linear restoring force. This makes the study of simple harmonic motion very important.

In this unit we shall introduce the concept of s.h.m, show the connection between it and circular motion, and then derive expressions for the parameters used in solving problems of s.h.m. In the next unit, we shall study the s.h.m of say, a mass on a spring, the simple pendulum, and energy of a s.h.m.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe an experiment to demonstrate simple harmonic motion s.h.m
- define simple harmonic motion
- list at least seven examples of phenomena in which s.h.m. occurs
- show the connection between circular motion and s.h.m
- determine the acceleration, period, velocity and displacement of a s.h.m.

3.0 MAIN CONTENT

3.1 Definitions

3.1.1 What is simple Harmonic Motion (s.h.m)

Earlier in this course, we considered accelerations that were constant in magnitude and direction when we discussed linear motion. In circular motion, we saw that accelerations (centripetal) were constant in magnitude but not in direction. Now, in oscillatory motion, which is also called simple harmonic motion (s.h.m) we shall see that accelerations like displacements and velocities change periodically in both magnitude and direction. To aid our definition, let us consider a body N, oscillating in a straight line about a point O, say between A and B as shown in Figure 3.1 below.

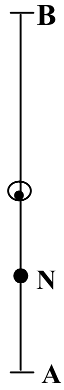


Fig 3.1

Let us also assume that N is a mass hanging from a spiral spring. We consider first its displacements and velocities. The displacement as measured from O to A is downwards when N is below O. While N moves away from O towards A, the velocity is directed downwards but upwards when N moves towards O. The velocity is zero at points A and B. When N is above O the displacement is upwards and the velocity is upwards or downwards depending on whether N is moving away from or towards O.

Thus we can look at the variation of the acceleration of the oscillating body on the spiral spring by studying the variation in its displacement. It is restricted to move about O and the magnitude of the elastic restoring force increases with displacement but always acts towards the equilibrium position O. We expect the resulting acceleration to behave likewise, increasing with displacement but being directed to O no matter what the displacement is. Thus, If N is below O, the displacement is downwards but the acceleration is upwards, but if the displacement is upwards the acceleration is downwards. Adopting the sign connection that quantities acting downwards are negative, and then we see that displacement and acceleration will always have opposite signs during an oscillatory motion.

The magnitude of the acceleration a is seen to be directly proportional to the magnitude of the displacement x. Such an oscillation is said to be a simple harmonic oscillation or motion (s.h.m) and is defined thus;

If the acceleration of a body is directly proportional to its distance from a fixed point and is always directed towards this point, the motion is simple harmonic.

The equation relating the acceleration and displacement in a S.H.M. is

$$a \propto x$$

$$\therefore \; a = (-\text{ constant})\, x \qquad 3.1$$

The negative sign indicates that acceleration is always in opposite direction to the displacement and directed to a fixed point.

SELF ASSESSMENT EXERCISE 3.1

What kind of motion would you expect equation (3.1) to represent if the negative term were positive?

Practically all mechanical motion are simple harmonic at small amplitudes or are combinations of such oscillations. Note that any system, which obeys Hook's law, can exhibit s.h.m. This equation of s.h.m. occurs in problems in other topics in physics like sound, optics, electrical circuits and atomic physics to mention but a few.

So you except to be discussing this topic a lot in your physics programme in this university. In calculus notation Eqn. (3.1) is written

$$\frac{d^2x}{dl^2} = -Cons\tan t.x \qquad 3.2$$

$$where \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

This is a second order differential equation and it could be solved to obtain the values of displacement and velocity.

Can you think of other phenomena exhibiting Simple Harmonic Motion?

3.1.2 Examples of Occurrence oOf Simple Harmonic Motion

We have seen that a repetitive to and fro motion about a mean position is known as an oscillatory or periodic or simple harmonic motion.

Examples of such a motion can be found in:

(i) The balance wheel of a watch
(ii) The pistons in a gasoline engine
(iii) The strings in the musical instruments
(iv) The molecules in a solid body vibrating about their mean positions in the crystal lattice
(v) The beating of the heart
(vi) Light waves and radio waves in space
(vii) Voltages, currents and electric charges etc.

Definitely, you see that the study of periodic motion can lay the foundation for future work in many different fields of physics.

SELF-ASSESSMENT EXERCISE 2

What do you understand by simple harmonic motion? List seven examples of phenomena where you expect s.h.m to occur.

3.2 Relating S.H.M. with Circular Motion

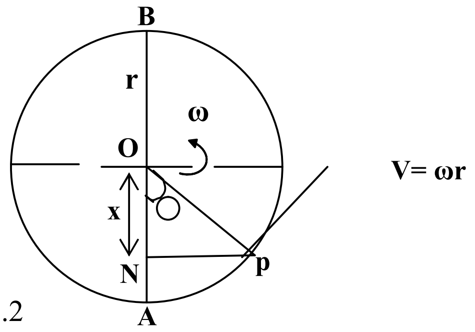


Fig 3.2

Recall what you learnt about circular motion in Unit 12. In the figure above let point P move round the circle of radius r and the centre O with uniform angular velocity ω. It will have a constant speed V round the circumference. The speed V is equal to ωr. Note that as P moves round the circle in the direction shown (that is anti clockwise), N the foot of the perpendicular from P on the diameter AOB moves from A to O to B and back to A through O. By the time N arrives back to point A, P also completes one cycle. Now, let initial positions of N and P be at A at time t =O. At a later time, t = t, N and P are now as indicated in the diagram with radius OP making angle O with OA. Let distance ON be x. We are now going to show that the motion of N from A to B and back to A is simple harmonic about O by describing the parameters that govern s.h.m.

3.2.1 Acceleration

The motion of N is due to that of P hence the acceleration of N is the component of the acceleration of P parallel to AB. We know that the acceleration of P is $\omega^2 r$ (or v^2/r) along PO. Hence the component of this parallel to AB is simply $\omega^2 rn \cos\theta$. Therefore the acceleration a of N is

$$a = -\ \omega^2 r \cos\theta \qquad 3.2$$

The negative sign, as already explained shows mathematically that acceleration is always directed towards O.
But, $x = r \cos\theta$ in the diagram

$$\therefore \quad a = -\ \omega^2 x \qquad 3.3$$

This equation (3.3) states that the acceleration of N towards O is directly proportional to its distance from O. We conclude that N describes a s.h.m. about O as P revolves round the circle-called the auxiliary circle – with constant speed.

For different values of x during the to and fro journey of N, corresponding values of acceleration of N can be got. Such representative values have been tabulated in Table 3.1

Table 3.1

x	**O**	**+ r**	**-r**
a	**O**	**$-\omega^2 r$**	**$+\omega^2 r$**

We see that at O displacement x is zero, acceleration a is zero. Acceleration a is maximum at the limit points A and B where the direction of motion changes.

Alternatively, one can use the arrangements in Figure 3.3 below to connect s.h.m with motion in a circle.

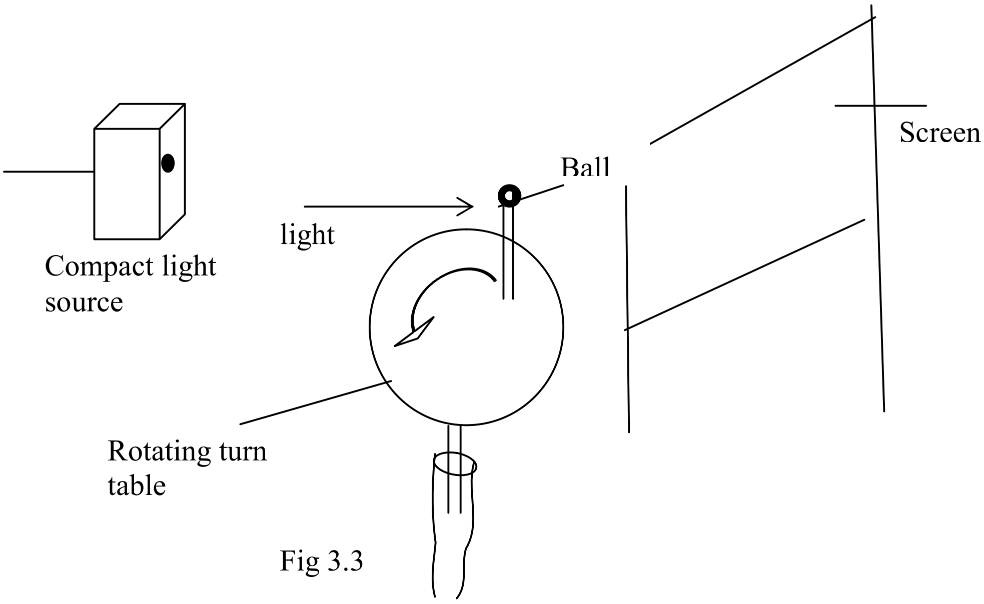


Fig 3.3

With the above set up, it is possible to view the shadow of the ball, rotating steadily in a circle, on the screen. The shadow moves with s.h.m and represents the projection of the ball on the screen.

3.2.2 Period

The period T of N is the time it takes N to do one complete to and fro motion ie to go from A to B and back to A in the Figure (3.2). In the same time, P will move round the auxiliary circles once. Therefore,

$$T = \frac{Circumfere\ \ nce\ \ of\ Auxiliary\ \ circle}{speed\ \ of\ \ p}$$

$$but\ \ V = wr$$

$$\therefore\ T = \frac{2\pi r}{v} = \frac{2\pi}{w} \qquad 34$$

For a particular s.h.m ω is constant and so T is constant and independent of the amplitude r of the oscillation. We note that if the amplitude increases, the body travels faster and so T remains unchanged. Know that a motion which, has a constat period whatever the amplitude, is said to be **isochronous**. This property is an important characteristic of s.h.m. The frequency f is the number of complete oscillations per unit time. That is f =I/T. An oscillation per second is a hertz.

3.2.3 Velocity

The velocity of N we have seen is the same as the component of P's is velocity parallel to AB which

$= - v \sin\theta$ from fig 3.2

$= - \omega r \sin\theta$ 3.5

Since $\sin\theta$ is positive when $0^o < \theta < 180^o$, that is, N moving upwards, and negative when $180^o < \theta < 360^o$, ie. N moving downwards, the negative sign ensures acting upwards and positive when acting downwards. The variation of the velocity of N with time (assuming P, and so N, start from A at time zero)

$= - \omega r \text{ Sin } \omega t$ (since $\theta = \omega t$) 3.6a

The variation of velocity of N with displacement

$x = - \omega r \text{ Sin}\theta$ 3.6b

$= \pm \omega r \sqrt{1 - Cos^2\theta}.$ $(Since \quad Sin^2\theta + Cos^2\theta = 1)$ 3.7

$= \pm \omega r \sqrt{1 - (x/r)^2}$

$= \pm \omega \sqrt{r^2 - x^2}$ 3.8

Hence the velocity of N is

$\pm \omega r$ (a maximum) when x = O

zero when $x = \pm r$

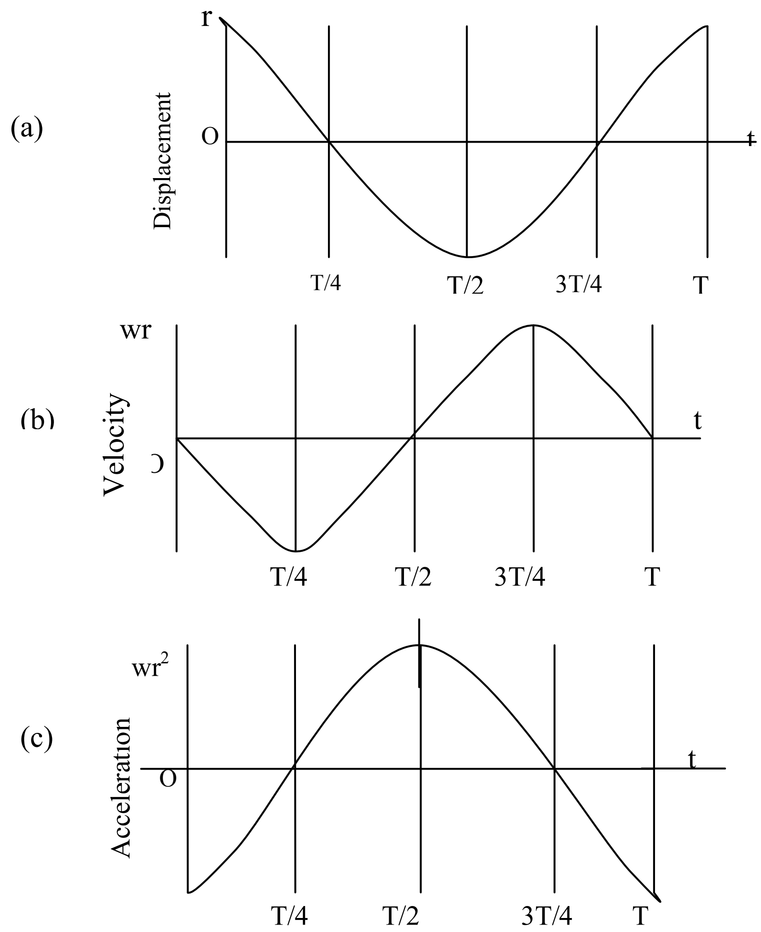
3.2.4 Displacement

This is given by:

$x = r \text{ Cos}\theta = r \text{ Cos } \omega t$ 3.9

The maximum displacement OA or OB is called the amplitude of the oscillation Fig. (3.2).

The graph of the variation of the displacement of N with time is displayed in Figure (3.4). It is a sinusoidal pattern just as the graphs of velocity and acceleration with time Fig. (3.4b&c).



Observe that when velocity is zero, the acceleration is a maximum and vice versa. We say that there exists a phase difference of a quarter of a period (ie. T/4) between the velocity and the acceleration.
I would like you to find out the phase difference between the displacement and the acceleration.

3.2.5 Expression for ω

We shall now discover what quantity ω is equivalent to in a s.h.m.
Recall that

$$a = -\omega^2 x$$

Ignoring the sign we can write

$$\omega^2 = \frac{a}{x} = \frac{ma}{mx} = \frac{ma/x}{m} \qquad 3.10$$

where m is the mass of the system.
The force causing the acceleration a at displacement x is ma, therefore ma/x is the force per unit displacement. Hence,

$$w = \sqrt{\frac{force\ per\ unit\ displacement}{mass\ of\ ocillating\ system}} \qquad 3.11$$

The period T of the s.h.m is given by

$$T = \frac{2\pi}{\omega}$$
$$= 2\pi\sqrt{\frac{mass\ of\ oscillating\ system}{force\ per\ unit\ displacement}} \qquad 3.12$$

This expression reveals that T increases if (1) the mass of the oscillating system increases and (2) the force per unit displacement decreases i.e. if the elasticity factor decreases.

A vibration is simple harmonic if it's equation of motion can be written in the form

$$a = -\,(\text{positive constant})\quad x \qquad 3.13$$

and we, by convention, represent this positive constant by ω^2 since $T=2\pi/\omega$. Hence, ω is the square root of the positive constant in the acceleration –displacement equation.

We have thus defined and explained the important parameters that we use in describing a s.h.m.

Example.
A cork floating on a pond moves in s.h.m, bobbing up and down over a range of 4cm. The period of the motion is T =1.0s and a clock is started at t = Os when the cork is at its minimum height. What are the height and velocity of the cork at t = 10.5s?

Solution:
Let us suppose that the cork moves along the z – axis and we set the origin z = O to be the mid point of the motion. Thus the maximum value of Z is
Z_{max} = 2cm, and the minimum value is Z_{min} = -2cm. The motion takes the general form $z(t) = A \sin(\omega t + \delta)$. We know the period T and from the equation $\omega = 2\pi/T$. The constants A and δ must be determined from other information namely, the initial conditions. The amplitude, A is the maximum excursion from equilibrium and is given by $A = Z_{max} = |Z_{min}|$. The phase δ, is then determined by the initial condition that the height is a minimum when t =0 s. Thus the equation determining δ is

$Z_{min} = A \sin \omega t + \delta /_{t=0} = A \sin \delta$

When we substitute A = $|Z_{min}|$, this equation becomes $Z_{min} = |Z_{min}| \text{ Sin } \delta$.
Because Z_{min} is negative, this result implies Sin δ = -1

When the sign function is –1,its argument is –π/2 or 3π/2. In fact, any integer multiple of 2π can be added to or subtracted from – π/2, and it is just a matter of convenience to choose the phase to be –π/2. When a simple phase such as this occurs, it is often worthwhile to expand the sine function with trigonometric identities

$$\text{Sin }(\omega\, t + \delta) = \text{Sin }\left(\omega t - \frac{\pi}{2}\right)$$

$$= Sin\,(\omega t)\,Cos\left(\frac{\pi}{2}\right) - Cos(\omega t)\,Sin\left(\frac{\pi}{2}\right)$$

$$= -Cos\,\omega t$$

We have used the fact that Cos (π/2) =0 and Sin (π/2) = 1. Then instead of Sin (ωt +δ), We have –Cos (ωt) appearing in the expression for z(t). We gather our results.

$$z = -\text{ A Cos }\left(\frac{2\pi t}{T}\right)$$

where A = 2cm and T = 1s

The velocity is the time derivative of the expression, that is

$$V = \frac{dz}{dt} = -A\left(\frac{-2\pi}{T}\right)Sin\left(\frac{2\pi t}{T}\right)$$

$$= \frac{2\pi A}{T}\,Sin\left(\frac{2\pi t}{T}\right)$$

We now evaluate z and V at t = 10.5 s or 10.5 period. Both z and V repeat themselves every period, so the values of z and V at 10.5 s are the same as at 0.5 s (or 0.5 period):

$$\text{Cos }\left[\frac{(2\pi)(10.5s)}{1.0s}\right] = Cos(2\pi)(10.5)$$

$$= Cos\;\left[2\pi(10) + 2\pi(0.5)\right]$$

$$= Cos\left[2\pi(0.5)\right]$$

$$= -1 \qquad ;$$

$$Sin\left[\frac{(2\pi)(10.5s)}{1.0s}\right] = Sin(2\pi)(10.5)$$
$$= Sin\ [2\pi(10) + 2\pi(0.5)]$$
$$= Sin[2\pi(0.5)]$$
$$= 0$$

Thus, for t = 10.5 s

Z = - A (-1)
= A = + 2cm

and

$$v = \frac{2\pi A}{T}(0) = 0cm\,s^{-1}$$

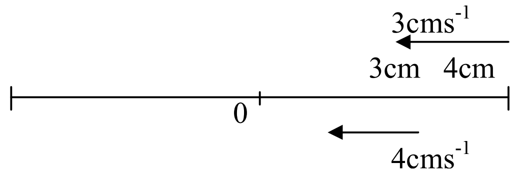
It is simple to deduce this result from physical reasoning. We are interested in where the cork is after exactly one half a period. So we look at it this way-because the cork starts at its minimum height, half a period later, it is at its maximum height, + 2cm in this case. That is a point where the cork stops momentarily and starts moving back down wards, so the velocity there is zero.

SELF-ASSESSMENT EXERCISE 3

A particle moving with s.h.m. has velocities of 4 cm s^{-1} and $3cms^{-1}$ at distances of 3cm and 4 cm respectively from its equilibrium position.

Find (a) the amplitude of the oscillation,
(b) the period
(c) the velocity of the particle as it passes through the equilibrium position.

Solution:



Let the above Figure represent the problem. Recall that

$$v = -\omega\sqrt{r^2 - x^2}$$

Assuming that velocities and displacements to the right are positive and those to the left are negative, we see that when x = + 3cm, velocity = $-4cms^{-1}$; therefore

$$-4 = -\omega\sqrt{r^2 - 9}$$

When x = + 4 cm, velocity = -3cm s^{-1}; therefore

$-3 = -\omega\sqrt{r^2 - 16}$

Squaring and dividing these equations we get $\frac{16}{9} = \frac{r^2 - 9}{r^2 - 16}$

Hence, r = ±5

(b) We substitute for r in one of the velocity equations to get

$\omega = 1s^{-1}$

$\therefore T = \frac{2\pi}{\omega} = 2\pi \ s$

(c) At the equilibrium position x =0

:. Velocity = $\pm\ \omega\sqrt{r^2 - x^2}$

= $\pm\ \omega$r

= ± 5 cm s^{-1}

4.0 CONCLUSION

In this unit, you have learnt the preliminary concepts of simple harmonic motion (s.h.m.) equation.

- that s.h.m is a periodic vibration of a body whose acceleration is directly proportional to its distance from a fixed point and is always directed towards this point i.e. a = - constant x
- at least seven phenomena where s.h.m. occurs
- that s.h.m is connected to circular motion where

  $a = -\omega^2 x$
- that the period of a s.h.m is the same as the time it takes a particle to move round on auxiliary circle.
- that the velocity of a s.h.m is given by –ωr Sinθ and the displacement by r Cosθ
- that when the velocity of a s.h.m is zero, the acceleration is maximum and vice versa.
- that the motion of a particle undergoing a s.h.m could be represented by a sinusoidal function.

5.0 SUMMARY

What you have learnt in this unit concerns the phenomenon of simple harmonic motion. You have learnt that:

- s.h.m is a to-and-fro motion under the influence of an elastic restoring force proportional to displacement and in the absence of all friction. That is a = - k x

- a complete vibration or complete cycle is one to –and fro motion regarded as one round trip.
- examplesofpriodic motion include seasons of the year, beating of the heart, lattice vibrations, the simple pendulum, electrical oscillations etc.
- s.h.m is intimately related to circular motion
- te periodic time, T of a s.h.m is the time required for one complete revolution or vibration, $T= 2\pi/\omega$
- The frequency f of vibration is $f = 1/T$
- In s.h.m the velocity and acceleration are also sinusoidal.
- $\alpha = - \omega^2 x$
- velocity $v = - \omega r \text{Sin}\theta$
- displacement $x = r \text{Cos}\theta = r \cos \omega t$
- amplitude is the maximum displacement.
- simple Harmonic motion is a special class of oscillation where the period T is the same for all amplitudes, be they large or small.

6.0 TUTOR-MARKED ASSIGNMENT

1a. Complete the following sentences:
When a particle oscillates in a straight line with simple harmonic motion, the period of the oscillation is independent of ---

b. The force towards the centre in a circular motion is called -------------- force.

2. Use a force displacement graph to represent the way in which the force F acting on a particle depends on the displacement r? (By convention, a force acting in the direction of +r is taken to be positive force).

3. What expression is ω in a s.h.m. Derive it from first principles. Use it to determine the expression for the period T of oscillation of the vibrating system.

4. What is a simple harmonic motion?

7.0 REFERENCES/FURTHER READING

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UNIT 2 SIMPLE HARMONIC MOTION II

CONTENTS

1.0 INTRODUCTION

In this Unit we shall study about harmonic motion as exhibited by a mass hanging from a coiled spring and the simple pendulum. This will bring us to the study of the energy of a simple harmonic motion. The rest of the introductory part is as covered in Unit 16. In the next Unit, we shall conclude our discussion on simple harmonic motion by studying damped oscillations, forced oscillations and resonance.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine the period of oscillation of a mass hanging from a coiled spring undergoing s. h. m
- determine the length of such a spring undergoing s. h. m. and also the effective mass of the spring
- explain what a simple pendulum is and how to determine its period of oscillation
- describe an experiment to use the simple pendulum to calculate acceleration due to gravity, g.
- determine the energy of s. h. m.

3.0 MAIN CONTENT

3.1 Mass Hanging from a Coiled Spring

3.1.1 Period of Oscillation

From Hooke's law, we know that the extension of a coiled spring is directly proportional to the force causing it.

In the diagram below Figure 3.1 you expect the mass hanging from a coiled spring to exert a downward tension mg on the spring. This is exactly

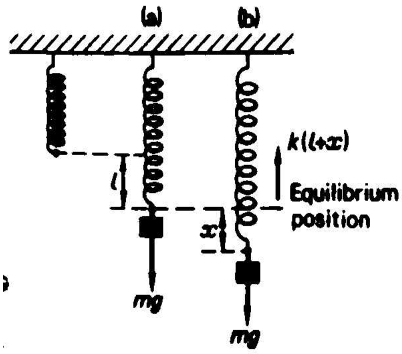


Fig. 3.1

what happens. Let the extension produced by this downward tension be l, and if k is the tension required to produce a unit length of the spring than the stretching tension is also kl. (k is also known as the spring constant and is measured in Nm^{-1}). This means that,

$$mg = kl. \qquad 3.1$$

When we now pull down the mass below its equilibrium position as shown, a distance x, the stretching tension becomes k(l + x). this is the same as the tension in the spring acting upwards as shown in Figure 3.1(b). Thus we can represent the resultant restoring force upwards on the mass as

$$K(l + x) – mg$$

$$= Kl + kx – mg \qquad 3.2$$

but mg = kl

$\therefore$ The resultant restoring force = kx 3.3

Note that when we then release the mass after extension it starts moving up and down continuously in what we call oscillatory motion. If at an extension x it has acceleration a, then its equation of motion will be

$$ma = -kx \qquad 3.4$$

The minus sign shows that at the instant while displacement x is downwards (i.e positive) acceleration a, is directed upwards the equilibrium position (i.e negative).

$$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x \qquad 3.5$$

What ω^2 = k/m. Because m and k are positive constants we see that ω^2 also is a positive constant. Consequently acceleration a is constant and this is a condition for a motion to be simple harmonic. We therefore conclude that the motion of the mass is simple harmonic as long as Hooke's law is obeyed.

The period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad 3.6$$

Squaring both sides we have that

$$T^2 = 4\pi m/k \qquad 3.7$$

If in an experiment, we vary the mass m and record the square of the corresponding periodic time, T on plotting the graph of T versus m, a straight line graph will be expected. This type of experiment has actually been done many times over. It was seen that the straight line graph did not pass through the origin. And explanation was sought by scientists and it was discovered that it was because the mass of the spring itself was not taken into consideration. So it was essential to determine the effective mass undergoing simple harmonic motion and this is done as follows together with a method of determining the value of g in the next session.

Example:
A light spiral spring is loaded with a mass of 50g and it extends by 10cm. Calculate the period of small vertical oscillations. Take g = $10ms^{-2}$

Solution:
Recall that the expression for the period of oscillation of a mass hanging on a spiral spring is,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

When k is the force per unit displacement. Substituting values given we have

$$\therefore \quad K = \frac{50\,x\,10^{-3}\;x\,10N}{10\,x\,10^{-2}\,m} = 5.0Nm^{-1}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{50\,x\,10^{-3}}{5}} = 2\pi\sqrt{10^{-2}s}$$

$$= 2\pi\,x\,10^{-1}s$$
$$= 0.63s$$

3.1.2 Measurement of g and Effective Mass of Spring.

If m is the effective mass of the spring then

$$T = 2\pi\sqrt{\frac{m + m_s}{k}} \qquad 3.8$$

Let us recall that

Kl = mg $\therefore$m = kl/g

So substituting this value for m in Eqn. (3. 8) we have

$$T = 2\pi\sqrt{\frac{K\,{}^{l}\!/_{g} + m_s}{k}}$$

Squaring both sides of equation (3.9) gives 3.9

$$T^2 = \frac{4\pi^2}{k}\left(\frac{kl}{g} + m_s\right)$$

$$\therefore l = \frac{g}{4\pi^2}T^2 - \frac{gm_s}{k} \qquad 3.10$$

When the static extension l (i.e the extension of spring before the vibration of the spring sets in) are used and their corresponding periods, T noted, then, a graph of l versus T^2 can be drawn. The result gives a straight line with intercept gm_s/k on the negative axis. The slope of the line is given by $g/4\pi^2$. This has been shown in figure 3.2.

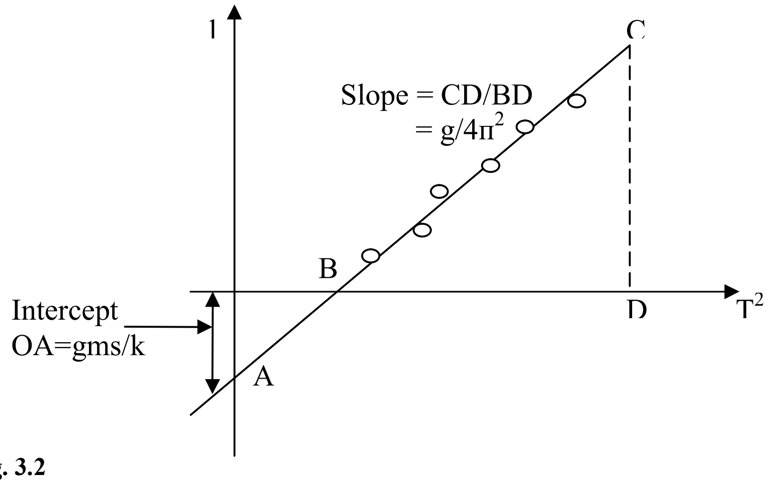


Fig. 3.2

It is estimated that theoretically the effective mass of a spring is about one third of its actual mass.

The Simple Pendulum

What is a Simple Pendulum?

As we stated in Unit16, simple harmonic motion occurs throughout nature and an example of such a motion is the swinging pendulum in some clocks. Such clocks served as accurate, time pieces for many centuries. You may ask - what does a pendulum consist of? But I tell you it is not far fetched. You can even construct one yourself. If you get the fruit of a gmelina tree, for example, (you know it is a tiny fruit) and using needle and thread, you pass the thread of about 20cm long through its centre and suspend the thread and fruit (now called the bob) from a ceiling or clamp as shown in figure 3.3 below. That constitutes a pendulum. Thus, we say that the simple pendulum consists of a small bob referred to as a particle of mass m suspended by a light inextensible thread of length l from a fixed point B say. Fig.(3.3). below

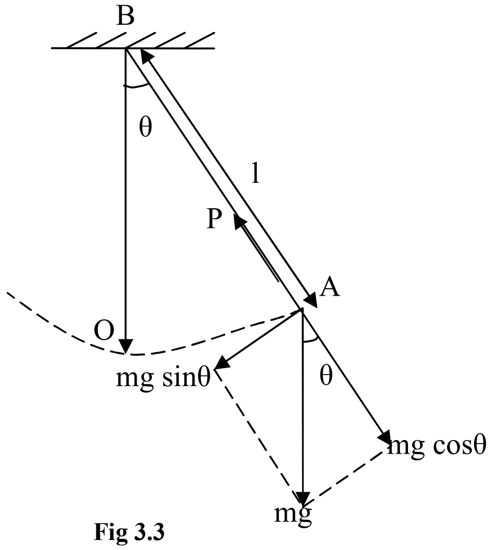


Fig 3.3

When the bob is displaced vertically to point A through a very small angle θ as shown and then released, it oscillates to and fro, in a vertical plane, about the equilibrium position 0. The motion of the bob is seen to trace an arc of a circle with radius l (assuming the bob is a point man). We shall see that this motion is simple harmonic about O.

Now, let the arc thread by the bob be OA = x and the angle of displacement OBA = θ at some instant of time when the bob is at point A. At that instant, the forces on the bob are the weight of the bob mg acting vertically downwards as shown and P the tension in the string (or thread). But mg has tangential component mgSin θ which acts as the balancing restoring force towards O and the radial component mgCosθ balancing the tension P in the string. If a is the acceleration of the bob along the arc at A due to mgsinθ then from Newtons law of motion we have,

$$ma = --- mg\sin\theta \qquad 3.11a$$

The displacement x is measured from 0 towards A. along arc OA whereas the negative sign shows that the restoring force is acting opposite to the direction of displacement that is towards θ. For very small angleθ, mathematics permits us to assume that sin θ = θ in radians (for example, if $\theta = 5^o$, Sin θ = 0.0872 and θ = 0.0873 rad.) and x = lθ. Therefore θ = x/l

Hence,

$$ma = mg\theta \quad = \quad -mg\frac{x}{l} \qquad 3.16b$$

$$\therefore \quad a \quad = -\frac{g}{l}x \qquad 3.12$$

setting $g/l = \omega^2$

we have

$$a = -\omega^2 x \qquad 3.13$$

We can then calculate that the motion of the bob is simple harmonic if the oscillations are of small amplitude θ as we assumed. In short θ should not exceed 10^0. The period T for the simple pendulum, is given by

$$T \quad = \quad \frac{2\pi}{w} \quad = \quad \frac{2\pi}{\sqrt{g/l}} \qquad 3.14$$

$$= \quad 2\pi\sqrt{\frac{l}{g}} \qquad 3.15$$

We notice that T does not depend on the amplitude of the oscillations. For a particular location on the surface of the earth where g is constant, the period of oscillation of a simple pendulum is seen to depend only on the length of the pendulum.

3.1.2 Measurement of g. With a Simple Pendulum

The simple pendulum method provides a fairly accurate means of determining acceleration due to gravity g. When the periodic time T for a simple pendulum is measured and recorded for corresponding different values of the length, l of the string supporting the pendulum bob, a plot of l versus T^2 gives a straight line so drawn so that the points on the graph are evenly distributed about the line. An example of such a result is shown in figure 3.4.

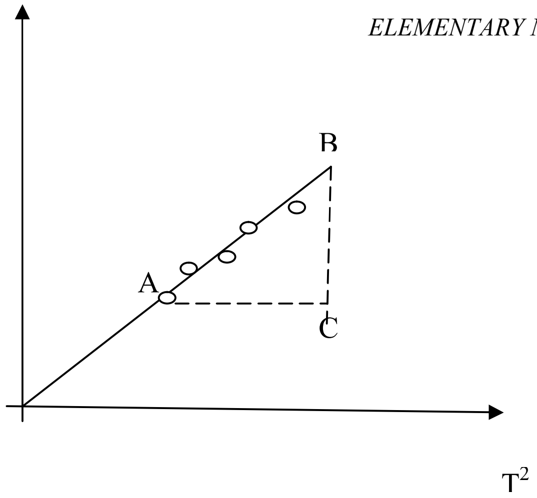


Fig 3.4

From the diagram, you see that such a line OB must pass through the origin. The slope of the line OB is given by BC/AC = l/T^2. From this, we determine the value of g thus

$$T = 2\pi \sqrt{l/g} \qquad 3.16$$

$$\therefore T^2 = 4\pi^2 \frac{l}{g}$$

$$\therefore g = 4\pi^2 \frac{l}{T^2} \qquad 3.17a$$

$$= 4\pi^2 \frac{BC}{AC} \qquad 3.17b$$

The necessary precautions you need to take in performing this experiment to achieve good results are that you must time at least fifty oscillations for each length of the pendulum; that you do not let the angle of swing to exceed 10^o; that the length of your string is measured from the support to the centre of the pendulum bob and that you count the oscillations as the bob passes the equilibrium position O on a round trip. I suppose you can now try to perform this kind of experiment at home even before you go to the Study Centre for it. It is easy and interesting to do it and get the expected result. That's where physics is stimulating. Wish you luck!

Example:
A simple pendulum has a period of 2.0s and an amplitude of swing 5.0cm. Calculate the maximum magnitudes of (i) the velocity of the bob (ii) the acceleration of the bob.

$$T = 2\pi / \omega$$

Solution:
Recall that,

$$\therefore \omega = 2\pi / T = 2\pi / 2.0\, s$$
$$= \pi\, s^{-1}$$

The velocity is a maximum at the equilibrium position where x displacement = 0

Recall the expression for the variation of velocity with displacement x which

$$= \pm\, \omega \sqrt{r^2 - x^2}$$

$$\therefore when\ x = 0 \qquad i.e\ \max imum\ velocity$$
$$v = \pm\, \pi \sqrt{25cm^2}$$
$$= \pm\ 5\pi\ cm\ s^{-1}$$
$$= \pm\ 16\, cm\, s^{-1}$$

(b) The acceleration is maximum at the limits of the swing where x = r = ± 5.0cm

$$\therefore a = -\omega^2 r$$
$$= -\pi \text{ x } 5\text{cm s}^{-2}$$
$$= -50\text{cm s}^{-2}$$

SELF-ASSESSMENT EXERCISE 1

A simple pendulum 2.0m long is suspended in a region where g = 9.81m s^{-2}. The point mass at the end is displaced from the vertical and given a small push, so its maximum speed is 0.11m s^{-1}. What is the maximum horizontal displacement of the mass from the vertical line it makes when at rest? Assume that all the motion take place at small angles.

Solution:
The angle that the string makes the vertical varies harmonically, $\theta = \theta_o \cos(\omega t + \delta)$, where ω is the angular frequency. The horizontal displacement from the vertical is x = lθ (where l is the strings length) as long as θ remains small. Thus x also varies harmonically.

$x = A \cos(\omega t + \delta)$ where $A = l\,\theta_o$.

This is the quantity we want to find.

Another good small angle approximation is that the vertical component of the velocity is small, so v = dx/dt. Thus we have

$$v = \frac{d}{dt}[A\cos(\omega t + \delta) = A\frac{d}{dt}[\cos(\omega t + \delta)]$$

$$\therefore \; v = -A\omega\, Sin(\omega t + \delta)$$

From this expression we see that v varies harmonically with amplitude Aω. The maximum value of v occurs when θ = 0. That is when the pendulum is passing through its equilibrium position. This is given by $V_{max} = A\omega = 0.11 m\, s^{-1}$ as given.

From the following equation,

$$\theta = 0_o\, Sin\,(\omega t + \delta) \text{ with } \omega = \sqrt{\frac{g}{l}}$$

we see that ω is

$$\omega = \sqrt{g/l} = \sqrt{\frac{9.81 m\, s^{-2}}{2.0 m}} = 2.21\, rad\, s^{-1}$$

so, from $V_{max} = A\omega$

$$A = \frac{V_{max}}{\omega} = \frac{0.11 m\, s^{-1}}{2.21\, rad\, s^{-1}}$$
$$= 0.05 m$$

We observe that this horizontal displacement 5.0cm is indeed small compared to the length of the pendulum so our small angle approximations are good. The figures below illustrate the motion.

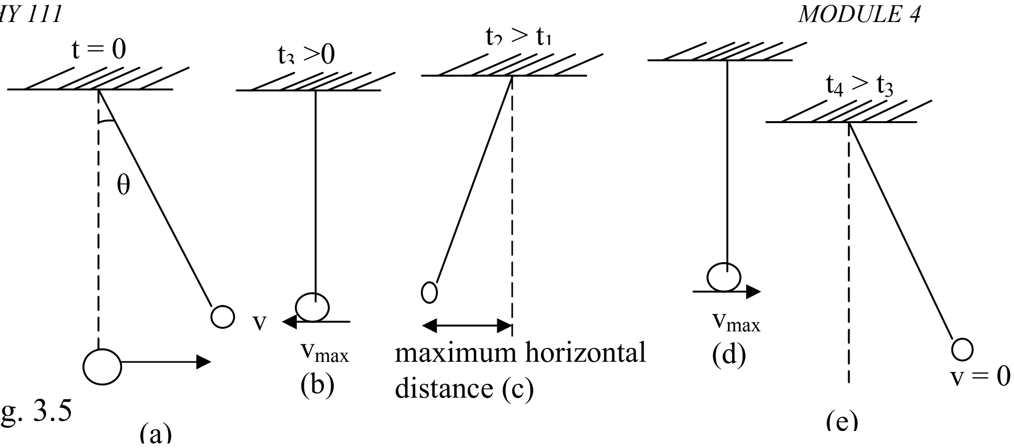


Fig. 3.5

3.2 Energy of Simple Harmonic Motion

During simple harmonic motion of an object, there is a constant interchange of energy of the object between its kinetic and potential forms. Note that if there is no influence of resistive forces (i.e. damping forces) on the object, its total energy E = (K. E. + P. E) is constant.

3.2.1 Kinetic Energy, K. E.

The velocity of a particle N of mass m at a distance x from ist centre of oscillation O is given by:

$$v = + w\sqrt{r^2 - x^2} \quad \textit{as shown in } (\textit{fig}. 3.5)$$

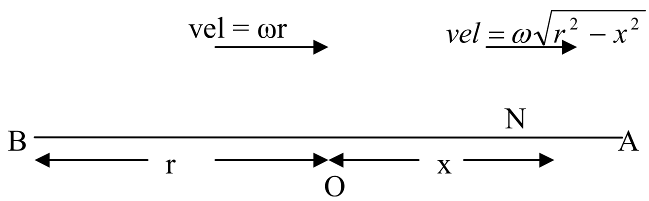


Fig 3.6

The kinetic energy K. E. at x, say, is

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2}mw^2(r^2 - x^2) \qquad 3.18$$

3.2.2 The Potential Energy, P. E.

During the motion of the particle N from O towards A or B, work is done against the force trying to restore it to O. Therefore, the particle loses some K. E. but gains some

P. E. When x = 0, the restoring force is zero. But at any displacement, say, x the force is $m\omega^2 x$ because the acceleration at that point has magnitude $\omega^2 x$.

Thus, average force on N while moving to displacement x

$$= \frac{0 + m\omega^2 x}{2} \quad = \frac{1}{2} m\omega^2 x$$

$$\therefore \textit{work done} \quad = \quad \textit{average force x displacement in the direction of force}$$

$$= \frac{1}{2} m\omega^2 x \quad \times x$$

$$= \frac{1}{2} m\omega^2 x^2 \qquad 3.20$$

$$\therefore P.E. \textit{ at displacement } x \quad = \frac{1}{2} m\omega^2 x^2$$

3.3 Total Energy, E

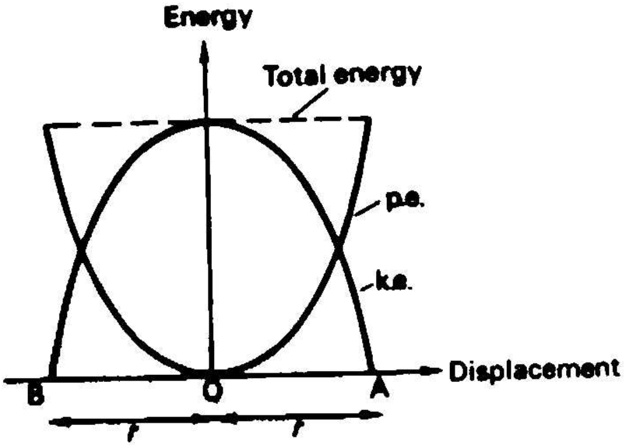
The total energy at displacement x is then given by K. E + P. E

$$\therefore \textit{Total Energy } E \quad = \frac{1}{2} m\omega^2 (r^2 - x^2) \quad + \quad \frac{1}{2} m\omega^2 x^2$$

$$= \frac{1}{2} m\omega^2 r^2 \qquad 3.21$$

We see that this value is constant and does not depend on x. It is also directly proportional to the product of (i) mass (ii) the square of the frequency (iii) the square of the amplitude.

We represent the variation of K. E. and P. E. for a simple harmonic motion in Figure 3.7 below:



In the case of the simple pendulum we note that all the energy is kinetic when the pendulum bob passes through the centre of oscillation. But at the maximum point of displacement when velocity is momentarily zero, the total energy is Potential.

SELF-ASSESSMENT EXERCISE 2

A small bob of mass 20g oscillates as a simple pendulum with amplitude 5cm and period 2 seconds. Find the velocity of the bob and the tension in the supporting thread, when the velocity of the bob is maximum.

Solution:
The velocity v of the bob is a maximum when it passes through its original position given by

$$v/_{x=0} \quad = \quad \omega\sqrt{r^2 - x^2}\,/_{x=0} \quad = \quad \omega\, r$$

Where r is the amplitude $= \quad 0.05m$

$$Since \quad T \quad = \quad \frac{2\pi}{\omega}$$

$$\therefore \quad \omega \quad = \quad \frac{2\pi}{T} \quad = \quad \frac{2\pi}{2} \quad = \quad \pi$$

Hence we have that $v_{\max}$ *is*

$$v_m \quad = \quad \omega\, r - \pi \ \times 0.05\, ms^{-1}$$
$$= \quad 0.16 ms^{-1}$$

Suppose P is the tension in the thread. The net force acting towards the centre of the circle along which the bob moves is given by (P – mg). The acceleration towards the centre of the circle, which is the point of suspension, is v^2_{max}/l where l is the length of the pendulum.

$$\therefore P - mg = \frac{mV_m^2}{l}$$

$$\therefore P = mg + \frac{mV_m^2}{l}$$

$$\text{Re}call \quad T = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore \quad l = \frac{gT^2}{4\pi^2} = \frac{9.8 \times 2^2}{4\pi^2}$$

$$\therefore P = 19.65 \times 10^2 N$$

4.0 CONCLUSION

In this Unit, you have learnt

- about the period of oscillation of a mass hanging from a coiled spring.
- how to measure the acceleration due to gravity g and the effective mass of the spring.
- to determine the period of oscillation of a simple pendulum and g also.
- to determine the kinetic energy, potential energy and total energy of a simple harmonic motion.

5.0 SUMMARY

What you have learnt in the unit concerns simple harmonic motion as it relates to a mass hanging from a coiled spring and a simple pendulum.

- that the vibration of mass hanging from a coiled spring is in the vertical plane with mg = kl
  K is the spring constant
  With the restoring force given by kx
  Hence with an acceleration of a, for an extension x the equation of motion for the mass is

$$ma = -kx.$$

For $\omega^2 = k/m$, ω = angular velocity

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

- the length of the spring is given by

$$l = \frac{g}{4\pi^2} T = \frac{gm_s}{k}$$

Where m_s is the mass of the spring

- that the period T of the simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

From where g the acceleration due to gravity could be determined more accurately as

$$g = 4\pi^2 \frac{l}{T^2}$$

- that the kinetic energy of a simple harmonic motion is

$$K.E. = \frac{1}{2}m\omega^2(r^2 - x^2)$$

- that Potential energy of s. h. m. is

$$P.E = \frac{1}{2}m\omega^2 x^2$$

at displacement x

- that total Energy of s. h. m. is given by

$$K.E. + P.E. = \frac{1}{2}m\omega^2 r^2$$

6.0 TUTOR-MARKED ASSIGNMENT

1a. Define simple harmonic motion and state the relation between displacement from its mean position and the restoring force when a body executes simple harmonic motion.

b. A body is supported by a spiral spring and causes a stretch of 1.5cm in the spring. If the mass is now set in vertical oscillation of small amplitude, what is the periodic time of oscillation?.

2. A flat steel strip is mounted on a support. By attaching a spring balance to the free end and pulling side-ways, we determine that the force is proportional to the displacement, a force of 4N causing a displacement of 0.02m. Then a 2kg body is attached to the end, and pulled aside, a distance 0.04m and released.
   (a) Find the force constant of the spring
   (b) Find the frequency and period of vibration.
   (c) Compute the maximum velocity attained by the vibrating body.
   (d) Compute the maximum acceleration
   (e) Compute the velocity and acceleration when the body has moved half way toward the centre from its initial position.
   (f) How long a time is it required for the body to move half way in to the centre from its initial position?

7.0 REFERENCES/FURTHER READING

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UNIT 3 SIMPLE HARMONIC MOTION III

CONTENTS

1.0 INTRODUCTION

See introduction of Units 16 and 17.
In this unit you will study about damped simple harmonic motion, s. h. m., forced oscillation and resonance. These will lead us to see that s. h. m. is a mathematical model. We shall conclude by considering a physical pendulum where we observe that in reality the pendulum string and bob can have dimensions and some mass. The importance of s. h. m. to life is also emphasized in this unit. After which we shall move on to the motion of rigid bodies in the next unit because this is what we experience in real life situations. There, you will learn about translational and rotational motions of rigid bodies.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain damped oscillations-stating the conditions under which a physical oscillator can experience it
- draw the wave patterns of the effects of different types of damping phenomenon
- state some applications of damping phenomenon
- define resonance and give examples of its occurrence
- state the importance of resonance
- show that the period of a physical pendulum is given by

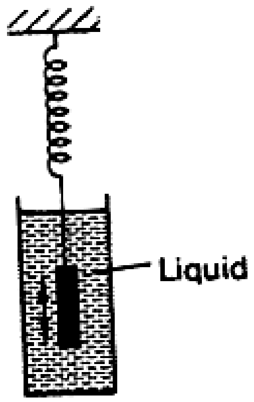
$$T = 2\pi\sqrt{I/mgh}$$

3.0 MAIN CONTENT

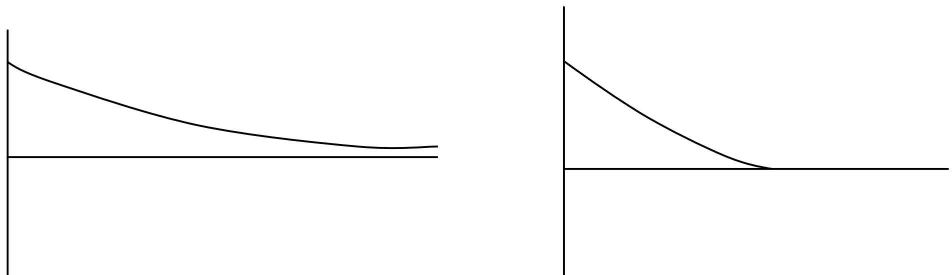
3.1 Damped Oscillations

In Units 16 and 17 we discoursed simple harmonic motion as vibrations that continue perpetually without diminishing in amplitude. I want to let you know that in reality, this does not obtain. The amplitude of the oscillations of, for example, a simple pendulum, gradually decreases to zero over time as a result of resistive force arising from the surrounding air in this case. In other forms of s. h. m. it will arise from the surrounding medium (e. g. liquid or gas). The motion for such oscillations is not therefore a perfect s. h. m. It is said to be damped by air resistance, that is, there is steady loss of energy as the energy is converted to other forms. Usually it will be internal energy through friction but energy may also be radiated away. For example, a vibrating turning fork loses energy by sound radiation.

The behaviour of a mechanical system, we know, depends on the extent of the damping. For example, the mass hanging from a coiled spring and immersed in a liquid as shown in Figure 3.1, when set to vibrate, experiences more damping than when it is in air. Know that undamped oscillations are said to be free. Fig. 3.2a shows a graph of its



displacement against time. Figure 3.2b depicts the case of slightly damped oscillations with decreasing amplitude. When the



(C) heavily

(d) critically

Fig 3.2

vibrating system is heavily damped, no oscillations occur. The system just gradually returns to its equilibrium position as shown in Figure 3.2c. Now, when the time taken for the displacement to be zero is very small, the vibrating system is said to be critically damped as in Fig. 3.2d.

When the damping forces are proportional to the velocity, v, the period remain constant as the amplitude diminishes and the oscillator is said to be isochronous. The dotted line in Fig. 3.2b is an exponentially diminishing curve.

It will interest you to know that the motion of some devices is critically damped on purpose to achieve a certain desired objective. For example, the shock absorbers on a car critically damp the suspension of the vehicle and so resist the setting up of vibration, which could make control difficult or cause damage. In the shock absorber shown in Figure (3.3) the motion of the suspension up or down is opposed by viscous forces when the liquid passes through the transfer tube from one side of the piston to the other. You can test the damping of a car by applying your weight momentarily on the car. You will notice that the car will rapidly return to its original position without vibrating.

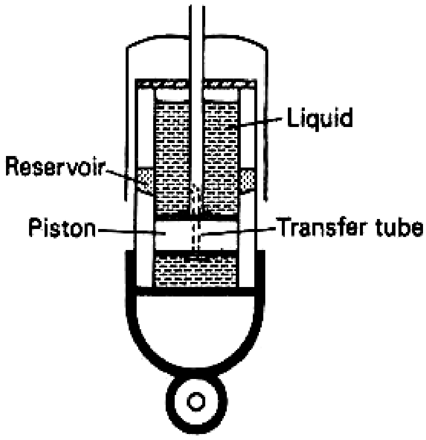


Fig. 3.3

Instruments such as balances and electrical meters are critically damped so that the pointer moves quickly to the correct position without oscillating. The damping is often produced by electro-magnetic forces.

SELF-ASSESSMENT EXERCISE 1

Describe some examples of simple harmonic motion that are not discussed in this unit.
What do you understand by damped oscillation?

3.2 Forced Oscillation and Resonance

Barton's Pendulums

A number of paper coned pendulums of length varying from ¼ m to ¾ m, each loaded with a plastic curtain ring are suspended from the same string as a 'driver' pendulum which has a heavy bob and a length of ½ m. this is shown in Figure 3.4 below:

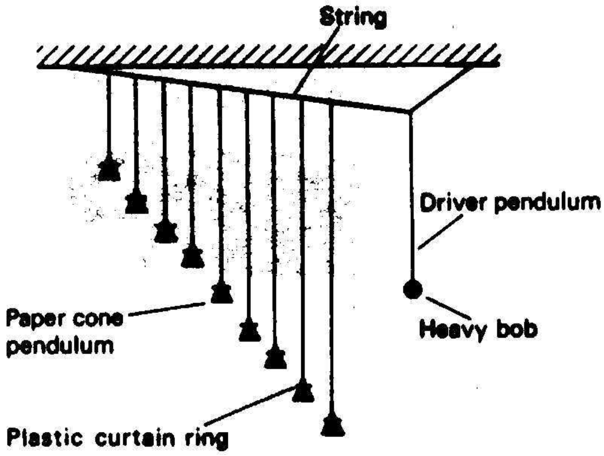


Fig. 3.4

When the driver pendulum is pulled well aside and then released, it oscillates in a plane perpendicular to the plane of the diagram. After a short time, the motion settles down and all the other pendulums oscillate with very nearly the same frequency as that of the driver though with different amplitudes. This is an example of forced oscillation. Out of the set of pendulums, the one whose length equals that of the driver pendulum has the greatest amplitude of vibration. Thus, its natural frequency of oscillation is the same as the frequency of the driving pendulum. This is an example of resonance and the driving oscillator passes on its energy most easily to the other system, that is, the proper cone pendulum of the same length.

I would like you to note that the amplitudes of oscillations also depend on the extent to which the system is damped. Thus, when the rings on the paper cone pendulums are removed, their masses reduce and so the damping increases. All amplitudes are then found to be reduced and that of the resonance frequency being less pronounced. The results are summarized in Figure (3.5). It is shown that the sharpest resonance is given by a lightly damped system.

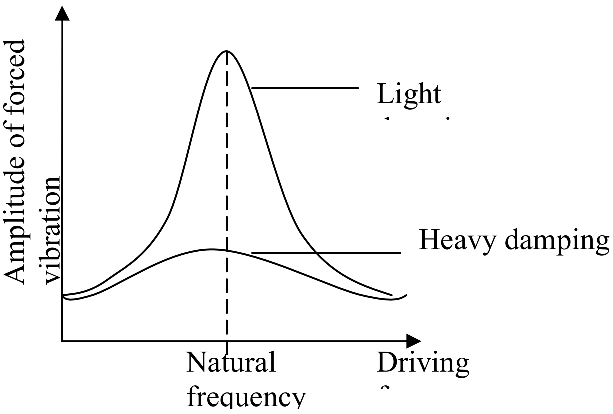


Fig 3.5

3.2.3 Examples of Resonance

These are common throughout science and are generally useful. Resonance occurs in the production of musical sounds from air columns in wind instruments. In many cases it occurs between the vibrations of air columns and of small vibrating reeds. Electrical resonance occurs when a radio circuit is tuned by making its natural frequency for electrical oscillations equal to that of the incoming radio signal. I am sure you have experienced this a lot in your home while turning your radio.

Resonance effect is also used to obtain information about the strength of chemical bonds between ions in a crystal. Taking light of infrared radiation as a kind of oscillating electrical disturbance and irradiating it on a crystal, the ions of the crystal will start oscillating. Then, with the radiation of the correct frequency, the ions could be set into vibration by resonance. The crystal would absorb energy from the radiation and the absorbed frequency could be found using a suitable instrument called the spectrometer. For example, sodium chloride would absorb infrared radiation and resonance could be observed in such crystals.

In mechanical system, resonance can constitute a menace to engineers. For examples, resonance occurring in bridges can lead to the breaking of such bridges. A life example is the breaking of the Tacoma Narrows Suspension Bridge in America in 1940. This resulted when a moderate gale (wind) set the bridge oscillating and producing an oscillating resultant force in resonance with a natural frequency of the bridge. An oscillation of large amplitude was thus built up and it destroyed the structure. To avoid destruction due to resonance, materials for building constructions, aircraft etc are subjected under sever resonance test in the factories before they are put to use. You see that resonance phenomenon aids science in some respects but constitutes a nuisance in other respects. I want you to find out and list more examples of resonance phenomena. They are many in literature.

SELF-ASSESSMENT EXERCISE 2

What is resonance? Does resonance constitute a menace to science? Discuss.

Solution:
See text above.

3.2.3 Energy Considerations

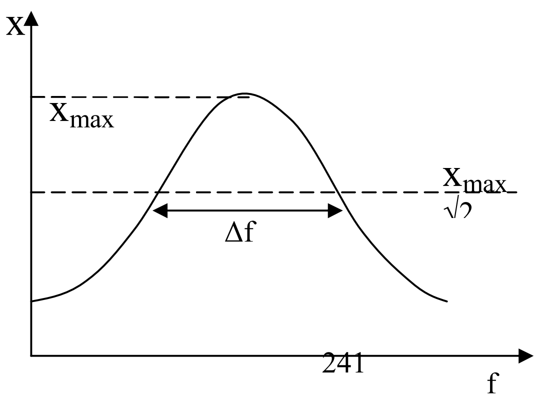


Fig 3.6 Finding the Q factor of a Resonant system.

Whether or not a body is at or close to resonance, the oscillator settles down in a steady state where the energy supplied from the driver per cycle is equal to the energy dissipated per cycle. The sharpness of the resonance, called the Q-factor (Fig. 3.6) is equal to:

$$\frac{\textit{energy lost per cycle}}{\textit{energy at the start of the cycle}}$$

It is also given by

$$Q = \frac{f_o}{\Delta f} \qquad 3.1$$

Where Δf is the width of the resonance curve

When

$$x = \frac{x_{\max}}{\sqrt{2}} \qquad 3.2$$

X_{max} being the maximum value of displacement x and where f_o is the resonant frequency.

3.3.4 Phase

At resonance, an oscillator lags behind the driver by 90^o ie it is 90^o out of phase with the driver. When the driver is at a much lower frequency than the oscillator's natural frequency ($f_d < f_N$) the oscillator is in step with the driver. When the driver frequency is much higher than the natural frequency ($f_d > f_N$), the driver and the oscillator are 180^o out of phase (Fig. 3.7).

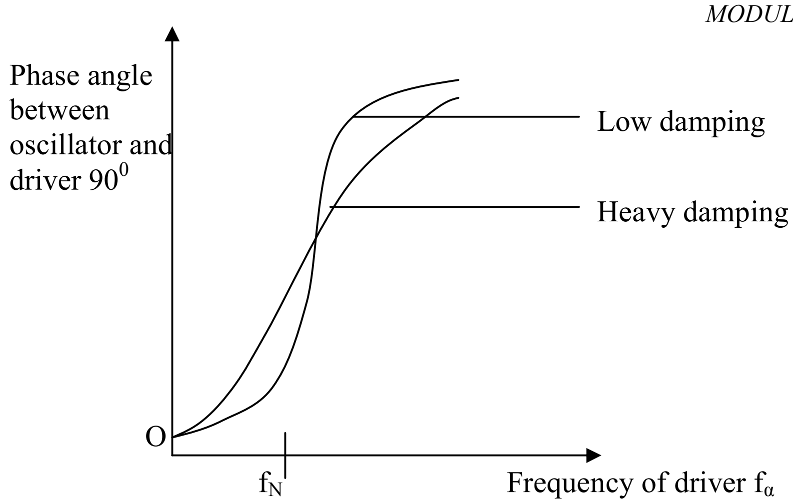


Fig 3.7 Phase relationship between driver and oscillator for different amounts of damping

SELF-ASSESSMENT EXERCISE 3

What is the phenomenon that allows you to increase the amplitude of your motion when you swing on a swing?

3.3 S. H. M. – A Mathematical Model

We want to emphasize here that s. h. m. is purely an idealized situation that does not exist in nature or in the practical world. Real oscillators such as a motor cycle on its suspension, a tall chimney swaying in the wind, atoms or ions vibrating in a crystal etc only approximate to the ideal type of motion we call s.h.m.

Simple harmonic motion is a mathematical model, useful because it represents many real oscillations due to its simplicity. It does not have complications such as damping, variable mass and stiffer (elastic modulus). The only condition it (s. h. m.) has to satisfy is that the restoring force should be directed towards the centre of motion and be proportional to the displacement.

A more complex model might, for example, take damping into consideration and hence may be a better description of a particular oscillator. Such may probably not be widely applicable. On the other hand, if a model is too simple, it may be of little use for dealing with real systems. Hence, a model must have just the correct degree of complexity. The mathematical s. h. m. has this and so is useful in practice.

3.4 The Physical Pendulum

It is not always that a pendulum consists of a massless string with a pointlike mass at the end of it. At times a pendulum can consist of a suspended swinging object of some form. We call this a physical pendulum. Any object can be suspended from any point on the object and act as physical pendulum. This illustrates the fact that s. h. m. is a general characteristic of motion about a stable equilibrium. You can even set up a physical pendulum, with your measuring ruler in your room.

Hence, the so-called 'physical' pendulum is any real pendulum in which all the mass is taken to be concentrated at a point. Figure 3.8 represents a body with irregular shape pivoted about a horizontal frictionless axis O and displaced from the vertical by an angleθ. The distance from the pivot to the centre of gravity is h, the moment of inertia of the pendulum about an axis through the pivot is I and the mass of the pendulum is m. The weight mg causes a restoring torque Γ of value given by

$$\Gamma = -mgh\sin\theta$$

When released, the body oscillates about its equilibrium position. Note that, unlike the s.h.m., the motion of the physical pendulum is not simple harmonic since the torque Γ is proportional not to θ but to $\sin\theta$. However, if θ is small, we can again approximate $\sin\theta$ by θ so that the motion becomes approximately harmonic.

Assuming this approximation then,

$$\Gamma = -(mgh)\theta \qquad 3.4$$

The effective torque constant is

$$K^1 = -\frac{\Gamma}{\theta} = mgh \qquad 3.5$$

Hence, the period of the physical pendulum is

$$T = 2\pi\sqrt{I/K^1} = 2\pi\sqrt{I/mgh} \qquad 3.6$$

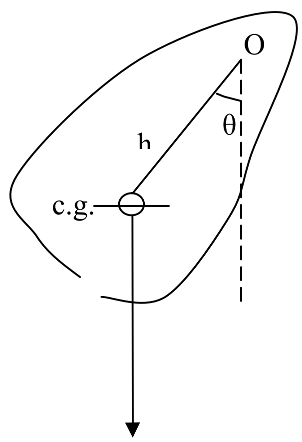


Fig 3.8: A Physical Pendulum

Example:

Let the body in Fig.3 be a meter-stick pivoted at one end. Then, if L is the total length = 1m, then the moment of inertia I is

$$I = \frac{1}{3} mL^2 \ \ And\ if\ \ h = \frac{L}{2}\ and\ \ g = 9.8ms^{-2}$$

$$Then \quad T \quad = \quad 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mgL/2}}$$

$$= \quad 2\pi\sqrt{{}^{2}\!/_{3}\,\frac{L}{g}}$$

$$= \qquad 1.65s$$

$$= \quad 2\pi\sqrt{\frac{2}{3}\,\frac{(1m)}{9.8m5^{-2}}}$$

SELF-ASSESSMENT EXERCISE 4

Find the moment of inertia of the complex shape – a connecting rod pivoted about a horizontal knife edge. The rod has mass 2kg at its centre of gravity (c.g) is at 0.2 below the knife edge.

Solution: we apply the principle that period

$$T \quad = \quad 2\pi\sqrt{\frac{I}{mgh}}$$

So, we set the system into vibration and using 100 complete vibration in 120s the period was found to be 1.2.5

$$\therefore T = 1.2s \quad = 2\pi\sqrt{\frac{I}{2kg\ x\ 9.8ms^{-2}\ \ x\ 0.2}}$$

Rearranging, we have

$$I = \frac{(1.2)^2\ (2kg)\left(9.8ms^{-2}\right)(0.2m)}{4\pi^2}$$

$$= \qquad 0.143\ \text{kg m}^2$$

4.0 CONCLUSION

In this unit you have learnt that,

a) most real oscillators are damped, that is, there is steady loss of energy as it is converted to their forms
b) damping of oscillators is due to the presence of additional velocity - dependent drag, or resistive forces causing the amplitude of the vibrating particle to decrease.
c) when a system that, by itself, would move in simple harmonic motion is driven by a force with sinusoidal time dependence, the system moves with the frequency of the driving force. The amplitude of the resulting motion of the system shows resonant behaviour when the frequency of the driving force equals the natural frequency of the system.
d) the width of the resonance peak is inversely related to the exponential rate of fall off of the undriven system due to damping.
e) s. h.m is a mathematical model.

5.0 SUMMARY

What you have learnt in this unit concerns damped simple harmonic oscillations, forced oscillations and resonance and the physical pendulum. You have learnt that

- the amplitude of oscillations of a particle in s.h.m. is damped by resistive forces due to the surrounding medium.
- when the amplitude is reduced to zero in minimal time the system is said to be critically damped
- when the damping forces are proportional to velocity, the period remain constant as the amplitude diminishes the oscillator is said to be isochronous
- resonance occurs when the driving frequency is the same as the natural frequency of the oscillator resulting in a maximum amplitude of oscillation.
- the sharpness of the resonance curve is called the Q-factor and is given by

  $$Q = f_o/\Delta f$$

  where Δf is the width of the resonance curve when

  $$x = x_{\max} / \sqrt{2}$$

  X_{max} is the maximum displacement and fo is the resonant frequency.
- the period of a physical pendulum is

  $$T = 2\pi\sqrt{I / mgh}$$

6.0 TUTOR-MARKED ASSIGNMENT

1. A light helical spring is suspended from a beam, and a mass m, is attached at its lower end, causing the spring to extend through a distance a. The mass is now caused to execute vertical oscillations of amplitude a. When the mass is at its lowest point, what is the energy stored in the spring?

2. A wire of mass per unit length 5.0 g m^{-1} is stretched between two points 30 cm apart. The tension in the wire is 70N. Calculate the frequency of the sound emitted by the wire when it oscillates in its fundamental mode.

b. Explain, with reference to this example, the term damped harmonic motion.

3. A thin rod of mass M and length L swings from its end as a physical pendulum. What is the period of the oscillatory motion for small angles? Find the length L of the simple pendulum that has the same period as the swinging rod.

$$T_{Simple\ p.} = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{2}{3}\frac{L}{g}}$$

$$\therefore\ l = \frac{2}{3}L$$

7.0 REFERENCE/FURTHER READING

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UNIT 4 RIGID BODY DYNAMICS 1

CONTENTS

1.0 INTRODUCTION

So far in this course, we have been concerned primarily with the motion of point masses. We have also treated different objects such as boxes and planet as if they were point objects or particles. But we know that in nature, we hardly come across an ideal point mass. We have to deal with motion of bodies, which have finite dimensions. So we have to develop a technique for studying the motion of such bodies.

A special class of such bodies is known as rigid bodies. In this Unit, you will first learn what a rigid body is. You will see that the definition of a rigid body provides a model for studying the motion of various kinds of physical bodies. You will then study about the different kinds of motion of a rigid body. A rigid body can execute both translational and rotational motion. We shall see that the general motion of a rigid body is a combination of both translation and rotation.

You will find that the translational motion of a rigid body can be described in terms of the motion of its centre of mass. So, we shall be able to apply the dynamics of point masses for description of translational motion. Hence, our chief concern will be the study of dynamics of rotational motion of rigid bodies.

To aid our understanding of the dynamics of rigid body we shall also study moment of inertia, radius of gyration, moments and couples and recapitulate equilibrium of coplanar forces. These will put us on a sound footing for studying angular momentum and its conservation, torque and kinetic energy of a rotating body in the next Unit which will be the last Unit of this course.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify a rigid body
- distinguish between the features of translational and rotational motion of a rigid body
- outline the features of the general motion of a rigid body
- explain the significance of moment of inertia of a rigid body about a certain axis
- solve problems on the concept of rotational dynamics of rigid bodies.

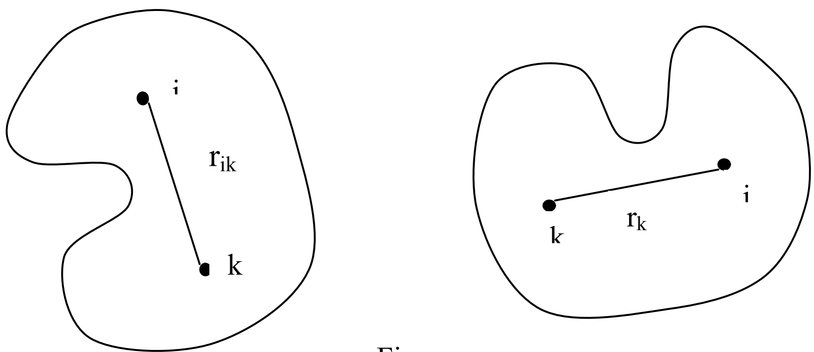
3.0 MAIN CONTENT

3.1 A Rigid Body and Its Motion

3.1.1 What is a Rigid Body

To attempt to answer this, just think of the wheel of a car rotating about its axle. Let us consider any two points on the wheel. You will see that the relative separation between them does not change when the wheel is in motion. This is an example of a rigid body. Can you think of objects in your room you can refer to as rigid bodies? Is the Bic ball pen you use in writing a rigid body?

Technically speaking, a rigid body is defined as an aggregate of point masses such that the relative separation between any two of these always remains invariant, that is, for any position of the body,
r_{ik} = a constant as shown in Figure 3.1. below.



Fig

In short, a rigid body is one which has a definite shape. It does not change even when a deforming force is applied. But we know that in nature there is no perfectly rigid body as all real bodies experience some deformation when forces are exerted. So, a perfectly rigid body can only be idealised. We shall see that this model is quite useful in cases where such deformation can be ignored. For example, the deformation of a lawn tennis ball as it bounces off the ground can be ignored.

You know that if a heavy block is dragged along a plane, frictional force acts on it.

But its deformation due to the frictional force can be neglected. However, you cannot neglect the deformation of a railway track due to the weight of the train. So, the model of a rigid body cannot be applied in the last case.

SELF-ASSESSMENT EXERCISE 1

Which of the following can be considered as rigid bodies?
(a) A top (b) A rubber (c) A ballet (d) a balloon (e) The earth.
Let us now study the motion of a rigid body.

3.1.2 Translational Motion of a Rigid Body

Suppose you are traveling in a bus, then, during a certain interval of time, your displacement will be exactly equal to that of your co – passenger provided both of you do not move with respect to the bus. This will also be true for any two objects attached to the body of the bus, say a bulb and a switch. This is the characteristic of translational motion. A rigid body is said to execute pure translational motion if each particle in it undergoes the same displacement as every other particle in a given interval of time. Translational motion of a rigid body is shown in Figure 3.2

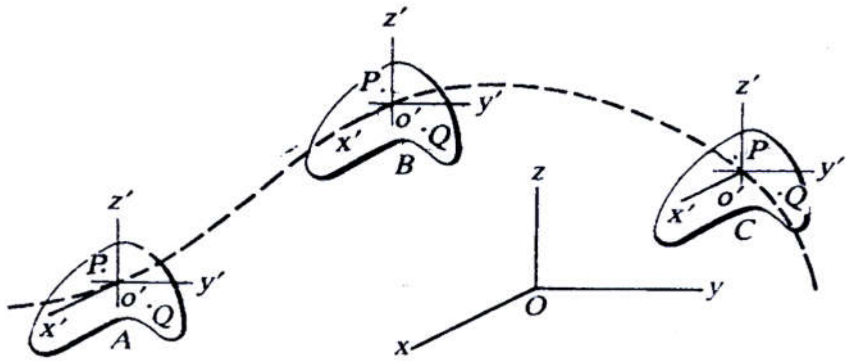


Fig 3.2 Translational Motion Of A Rigid Body

You must have noticed that the path taken is not necessarily a straight line. The magnitudes of the distance between P O^1 and Q should always be the same.

SELF-ASSESSMENT EXERCISE 2

Give two examples of translational motion
Now that you have worked out exercise 3.2 you can see that if we are able to describe the motion of a single particle in the body, we can describe the motion of the body as

a whole. We have done this exercise a number of times before. However, you may like to consolidate your understanding by working out the following exercise.

SELF-ASSESSMENT EXERCISE 3

A rigid body of mass M is executing a translational motion under the influence of an external force F_e. Suggest a suitable differential equation of motion of the body.

What does the answer to exercise 3.3 signify? We know that the relative separation between any two points of a rigid body does not change. That is,

$$\frac{dr_{ik}}{dt} = O$$

So all the points follow the same trajectory on as the centre of mass. Hence for studying translational motion, the body may be treated as a particle of mass M located at its centre of mass (C.M). You may recall that we had treated the sun and a planet as particles in Units 11, 12, and 13. They were treated as particles as their sizes are negligible compared to the distances between them and also because the shapes of these bodies were insignificant. But here we are considering a rigid body as a particle for another reason as explained above. Thus we can represent the translational motion of its C.M. It becomes easier to describe the translational motion in this way. Recall that we had applied the above idea when we studied cases like a body falling or sliding down an inclined plane in Unit 14. Let us now discuss the rotational motion of a rigid body.

3.1.3 Rotational Motion of a Rigid Body

Let us consider the motion of the earth. Every point on it moves in a circle (the corresponding latitude), the centres of which lie on the polar axis. Such a motion is an example of a rotational motion. A rigid body is said to execute rotational motion if all the particles in it move in circles, the centres of which lie on a straight line called the axis of rotation. When a rigid body rotates about an axis every particle in it remains at a fixed distance from the axis. So, each point in the body, such as P describes a circle about this axis. See Fig. 3.3.

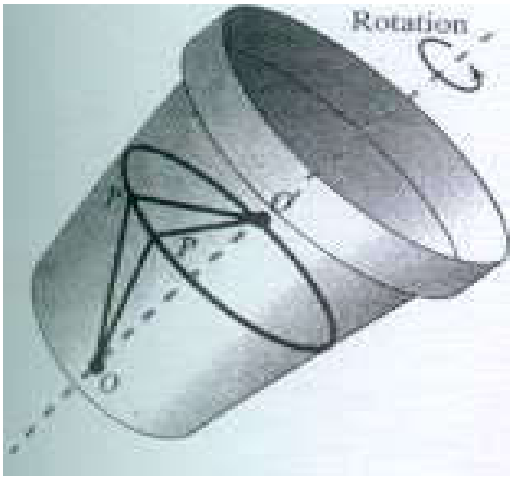


Fig. 3.3

You must have realised that perpendiculars drawn from any point in the body to the axis will sweep through the same angle as any other such line in any given interval of time.

3.1.4 General Motion of a Rigid Body

The general motion of a rigid body is the combination of translation and rotation. This can be understood by considering the example of a moving car. If you look at the tyres of the moving car you observe that the wheel is turning round as well as moving forward or backwards as the case may be. So the car changes position as the wheels rotate.

You may perform an activity for the sake of better understanding of the motion of a rigid body.

SELF-ASSESSMENT EXERCISE 4

Take a beer bottle or a pencil and roll it on its side on a table.
What do you observe?

You would have observed that the bottle or pencil, besides rotating round also changed location as it rolled down the table. That gives you a feel of what we are talking about. Now think of more examples.

We shall now move on to study moment of inertia because it will play an important role in the determination of the angular momentum of a rotating rigid body.

In dealing with circular motions, we have all these while considered particles in motion with the result that a particle revolved round a circle of the same radius. But now we are going to consider the rotation of a system of connected "particles" moving in circles of different radii. The spatial distribution of the mass of the body affects the behaviour of the body. We note also that the mass of a body is a measure of its in – built resistance to any change of linear motion. Thus we say that mass measures inertia. The corresponding property for rotational motion is called the moment of inertia. The more difficult it is to change the velocity of a body rotating about a particular axis, the greater is its moment of inertia about that axis. From experiments, it was seen that a wheel with most of its mass concentrated in the rim is more difficult to start and stop than a uniform disc of equal mass - spread rotating about the same axis. The former has a greater moment of inertia. Take note of this important point – **that moment of inertia is a property** of a body rotating about a particular **axis.** If the axis **changes,** the value of the moment of inertia **also changes**.

3.2 Moment of Inertia

We need now to measure the moment of inertia which takes into account the mass distribution of the body about the axis of rotation and which plays a role in rotational motion. This is analogous to that played by mass in linear motion.

Consider a rigid body rotating about a fixed axis through O with constant angular

Velocity ω, as shown in Figure (3.4) below. A particle A, of mass m_1, at a distance r_1 from O describes its own circular path and v_1 is its linear velocity along the tangent of the path at the instant shown, then

$$v_1 = r_1\omega \qquad 3.1$$

and

$$\text{the Kinetic Energy of } A = \frac{1}{2}m_1v_1^2$$

$$= \frac{1}{2}m_1r^2\omega^2 \qquad 3.2$$

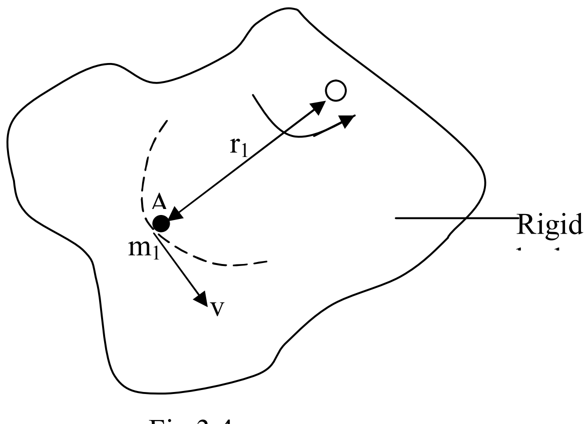


Fig 3.4

The kinetic energy of the whole body is the sum of the kinetic energies of its component particles. Assuming these have masses m_1, m_2, m_3, ….. m_n and are at distances

r_1, r_2, r_3, ……r_n from O, then, since all the particles have the same angular velocity ω, we have

Total K.E for the whole body =

$$\frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ...+$$

$$= \sum_{i=1}^{n}\frac{1}{2}\omega^2 m_i r_i^2 \qquad 3.3a$$

i.e. Total K.E =

$$\frac{1}{2}\omega^2\left(\sum_{i=1}^{n}\frac{1}{2}m_i r_i^2\right) \qquad 33b$$

$$\sum_{i=1}^{n} m_i r_i$$

Where represents the sum of the $m_ir_i^2$ values for all the particles of the body. Note that the quantity $\sum m_ir_i^2$ depends on the mass and its distribution and it is a measure of the moment of inertia I of the body about the axis in question.

So we define I as

$$I = \sum_{i=1}^{n} m_i r_i \qquad 3.4$$

We can then write the K.E. as
K.E of body =

$$\frac{1}{2}I\omega^2 \qquad 3.5$$

Comparing this with the kinetic energy for linear motion ½ mv^2 we see that mass m is replaced by the moment of inertia I and the velocity v is replaced by the angular velocity ω.

The Unit of moment of inertia I is kg m^2.

Values of I for regular shapes of bodies can be determined using calculus. For example, That for a uniform rod of mass m and length L about an axis through its centre is $mL^2/12$.

When the rotation is about an axis at one of its ends it becomes $mL^2/13$. Fig. 3.5 say

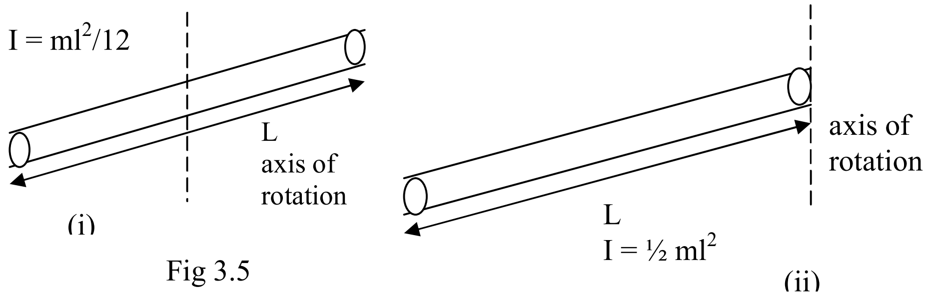
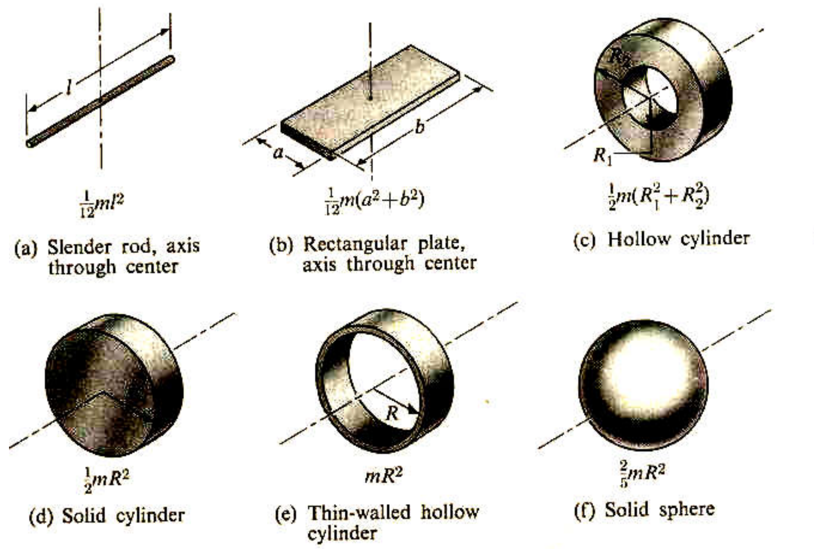


Fig 3.5

Do you think rotational kinetic energy $\frac{1}{2} I\omega^2$ is a new kind of energy? Not at all. It is simply the sum of the linear kinetic energies of all the particles making up the body, and is a convenient way of representing the K.E of a rotating rigid body.

The mass of a flywheel is concentrated in the rim, thereby giving it a large moment of inertia. When it rotates, it possesses large K.E. This explains why it is able to keep an engine (e.g in a car) running at a fairly steady speed despite the fact that energy is applied only intermittently to it. You may do well to know that some toy cars have a small lead flywheel which is set into rapid rotation by a brief push across a solid surface. The K.E of the flywheel will then keep the car in motion for some distance.

Values of moments of inertia for other regular shapes are shown in Figure 3.6



9–7 Moments of inertia.

Fig 3.6

Example:
Three small bodies, which can be considered as particles are connected by light rigid rods, as in the Figure 3.7 below.

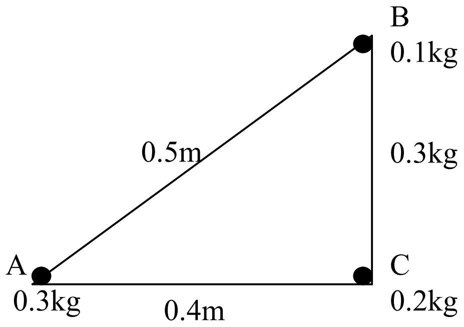


Fig 3.7

What is the moment of inertia of the system (a) about an axis through A, perpendicular to the plane of the diagram?
(b) about an axis coinciding with the rod BC?

Solution:
Since particle A lies on the axis, it does not contribute to the moment of inertia because the distance from the axis of rotation is zero.

Hence,

$$I = \sum m r^2 = (0.1kg)\,(0.5m)^2 + (0.2kg)\,(0.4m)^2$$
$$= 0.057 kg\ m^2$$

(b) The particles B and C both lie in the axis and so they too contribute nothing
Hence,

$$I = \sum m r^2 = (0.3kg)\,(0.4m)^2$$
$$= 0.048\ kg\ m^2$$

SELF-ASSESSMENT EXERCISE 5

If in Figure 3.4 above the body rotates about an axis through A and perpendicular to the plane of the diagram, with an angular velocity $\omega = 4$ rad s^1, what is the rotational kinetic energy?

Solution.

$$K.E. = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.057kg)(4rads^{-1})$$
$$= 0.456J$$

3.2.1 Raduis of Gyration

No matter what the shape of a body is, it is always possible to find a radial distance from any given axis at which the mass of the body could be concentrated without changing the moment of inertia of the body about that axis. This distance is known as the radius of gyration of the body about the given axis. It is denoted by K. So if mass m of the body actually were concentrated at this distance, the moment of inertia would be that of a particle of mass m at a distance k from an axis, or mk^2. But we see that this is equal to actual moment of inertia I, therefore

$$mk^2 = I \qquad 3.6$$

SELF-ASSESSMENT EXERCISE 6

What is the radius of gyration of a slender rod of mass m and length L about an axis perpendicular to its length and passing through the centre?

Solution:

The moment of inertia about an axis through the centre is $I = mL^2 / 12$

Therefore,

$$\sqrt{mL^2/12m}$$

$$= L/2\sqrt{3}$$

$$= 0.289L$$

$K_0 =$

We therefore note that the radius of gyration, like the moment of inertia depends on the location of the axis.

3.2.2 The Dumbbell.

The simplest rotating object that we can contemplate is a dumbbell. It consists of two point masses m_1 and m_2 connected by a massless rigid rod of length L as shown in Figure 3.8 below.

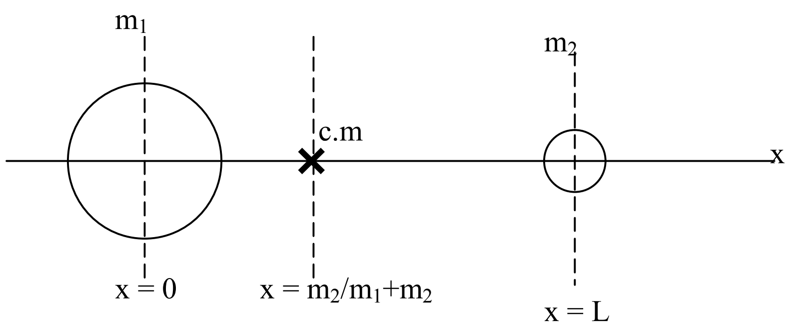


Fig 3.8

Let the total mass M be $m_1 + m_2$. Positioning mass m_1 at the origin of the x – axis and man m_2 at x = L. it could be shown that the centre of mass is at x where

$$x = \frac{(m_1)(0) + (m_2)(L)}{m_1 + m_2}$$

$$= \frac{m_2 L}{M}$$

If we consider the case in which the axis of rotation goes through the centre of mass (C.M) (i.e through point x = m_2 L/M, then the axis is taken perpendicular to the rod. So, measuring from the C.M, the coordinates of m_1 and m_2 will be $-m_2$ L/M and L-(m_2/m) = m_1L/M respectively.

Now, the rotational inertia about an axis passing through the centre of mass and perpendicular to the axis of the dumbbell is given by

$$I = m_1\left(\frac{m_{2L}}{M}\right)^2 + m_2\left(\frac{m_{1L}}{M}\right)^2$$

$$= L^2\left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\right) = \left(\frac{m_1 m_2 L^2 (m_2 + m_1)}{(m_1 + m_2)^2}\right)$$

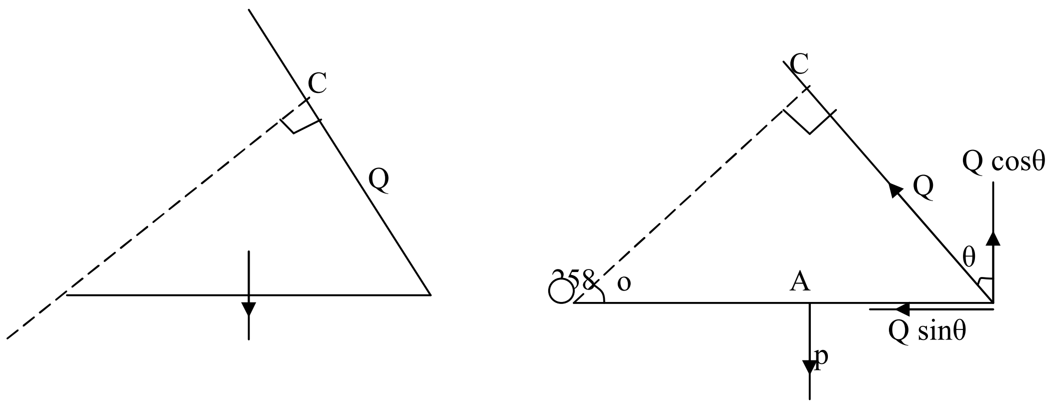
$$= \frac{m_1 m_2}{m_1 + m_2} L^2$$

3.3 Moments and Couples

Knowledge of moments and couples will aid our understanding of the next section of this unit which will deal on torques and angular momentum.

"A force applied to a hinged or pivoted body changes its rotation about the hinge or pivot. Experience shows that the turning effect or moment or torque of the force is greater, the greater the magnitude of the force and the greater the distance of its point of application from the pivot.The moment or torque of a force about a point is measured by the product of the force and the perpendicular distance from the line of action of the force to the point.

Thus in Figure, 3.9 if OAB is a trapdoor hinged at O and acted on by forces P and Q as shown, then,



O
Hinge p B

Fig 3.9

Moment of P about

O = P x OA 3.7

and moment of Q about

O = Q x OC 3.8

Note that the particular distance must be taken. Alternatively we can resolve Q into components Q cosθ perpendicular to OB and Q sin θ along OB as shown in Figure 3.b.

The moment of the latter about O is zero, its line of action passes through O. for the former, we have

Moment of Q cos θ about O = Q cosθ x OB 3.9a
= Q x OC 3.9b

(since cosθ = OC / OB), we see that this result is as we had before.

Note that moments are measured in Newton metres (Nm) and are given a positive sign if they tend to produce clockwise rotation.

A couple consists of two equal and opposite parallel forces whose lines of action do not coincide. It always tends to change rotation. A couple is applied to a water tap to open it. Figure 3.10 shows a diagrammatic representation of a couple. We can say that the moment or torque of the couple P – P about O

= P x OA + P x OB (both are clockwise)
= P x AB 3.10

Hence, moment of couple = one force x perpendicular distance between forces.

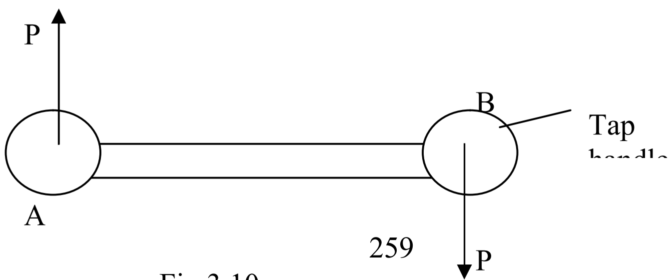


Fig 3.10

3.3.1 Equilibrium of Coplanar Forces

General conditions for equilibrium. If a body is acted on by a number of coplanar forces (that is, forces in the same plane) and is in equilibrium (i.e. there is rest or motion under constant speed) then

(i) The components of the forces in both of any two directions (usually taken at right angles) must balance.

(ii) The sum of the clockwise moments about a point equals the sum of the anticlockwise moments about the same period.

The first statement is a consequence of there being no translational motion in any direction and the second follows since there is no rotation of the body. In brief, if a body is in equilibrium the forces and the moment must both balance. The following worked example shows how the conditions for equilibrium are used to solve problems.

Example:
A sign of mass 5.0kg is hung from the end B of a uniform bar AB of mass 2.0kg. The bar is hinged to a wall at A and held horizontally by a wire joining B to a point C which is on the wall vertically above A. If angle ABC = 30^0, find the force in the wire and that exerted by the hinge ($g = 10ms^{-2}$).

Solution:
The weight of the sign will be 50N and that of the bar 20N (since w = mg). The arrangement is as shown in Figure 3.11a. Let P be the force in the wire and suppose Q, the force exerted by hinge, makes angle θ with the bar. The bar is uniform and so its weight acts vertically downwards at its centre G. Let the length of the bar be 2^L.

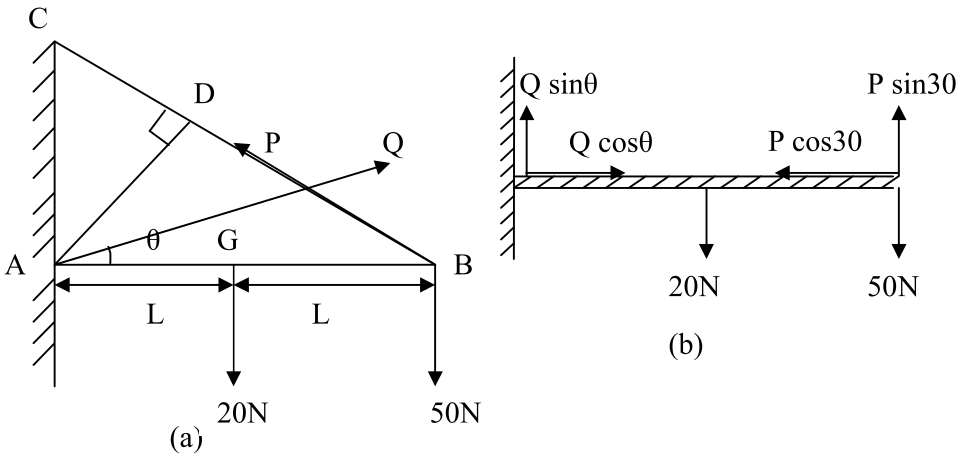


Fig 3.11

i. There is no rotational acceleration, therefore taking moments about A we have Clockwise moments = anticlockwise moments. i.e.:

$20 \text{ x } L + 50 \text{ x } 2L = P \text{ x } AD$ (AD $\perp^{law}$ to BC)

$\therefore 120L = P \text{ x } AB \sin 30$ since $30 = \frac{AD}{AB}$

$= P \text{ x } 2L \text{ x } 0.5$

$\therefore P = 1.2 \text{ x } 10^2$ N

Note: by taking moments about A there is no need to consider Q since it passes through A and so has zero moment.

ii. There is no translational acceleration, therefore the vertical components (and force) must balance, likewise the horizontal components. Hence resolving Q and P into vertical and horizontal components (which now replace them) shown in Fig. 11b, we have :

Vertically

$$Q \sin \theta + P \sin 30 = 20 + 50$$

$$\therefore Q \sin \theta = 70 - 120\left(\frac{1}{2}\right)$$

$$Q \sin \theta = 10 \qquad (1)$$

Horizontally. $Q \cos \theta = P \cos 30 = 120(\sqrt{\frac{3}{2}})$

$$\therefore \varphi \sin \theta = 60\sqrt{3} \qquad (2)$$

Dividing (1) by (2)

$$\tan \theta = 10/\left(60\sqrt{3}\right)$$

$$\therefore \theta = 5.5^0$$

Squaring (1) and (2) and adding

$Q^2 (\sin^2 \theta + \cos^2\theta) = 100 + 10800$

$\therefore Q = 10900$ $(\sin^2 \theta + \cos^2\theta = 1)$

and $Q = 1.0(4) \text{ x } 10^2$ N.

Structures: Forces act at a joint in many structures and if these are in equilibrium then so too are the joints. The joint O in the bridge structure of Fig. 3.12 is in equilibrium under the action of forces P and Q exerted by girders and the normal force

S exerted by the bride support at O. The components of the forces in two perpendicular directions at the joint must balance.

Hence,

$$S = Q \sin \theta \text{ and } P = Q \cos \theta$$

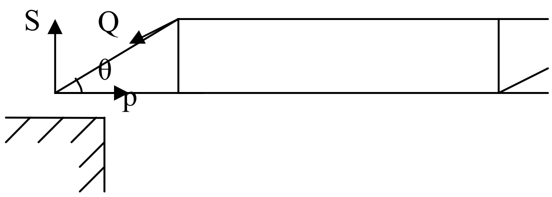


Fig 3.12

If θ and S are known (the latter from the weight and loading of the bridge) then P and Q (which the bridge designer may wish to know) can be found. Other points may be treated similarly." (Duncan, 1982)

4.0 CONCLUSION

In this Unit, you have learnt

- what a rigid body is.
- that a rigid body can undergo both rotational and translational motions at the same time.
- to distinguish between the features of translational and rotational motion of a rigid body.
- to define moment of inertia and state its significance.
- to determine the turning effect of a force.
- to state the conditions off equilibrium of coplanar forces.

5.0 SUMMARY

What you have learnt in this unit concerns rigid body dynamics.
You have learnt that:

- a rigid body is an aggregate of point masses such that the relative separation between any two of these always remains invariant.
- a rigid body can execute both translational and rotational motion.
- a rigid body executes pure translational motion if each particle in it undergoes the same distance as every other particle in a given interval of time.
- the total K.E for the whole rotating body is given by $\sum \omega m_i r_i^2$
- the moment of inertia for the rotating body is

$$I = \sum_{i=1}^{n} m_i \, r_i^2$$

where the symbols have their usual meanings.

- the moment or torque of a force about a point is measured by the product of the force and the perpendicular distance from the line of action of the force to the point. $\frac{1}{2}$
- a couple consists of two equal and opposite parallel forces whose lines of action do not coincide. It always tends to change rotation.
- if a body is acted on by a number of coplanar forces then for equilibrium
  - (i) The components of the forces in both of any two directions must balance.
  - (ii) The sum of the clockwise moments about a point equals the sum of the anticlockwise moments about the same point.

6.0 TUTOR-MARKED ASSIGNMENT

1. Two point-like masses are placed on a massless rod that is 1.5m long. The masses are placed as follows 1.6kg at the left end and 1.8kg 1.2m from the left end.
   - (a) What is the location of the centre of mass?
   - (b) By moving the 1.8kg mass, can you arrange to have the centre of mass in the middle of the rod?

2. A pulley is rotating at the rate of 32 rev/min. A motor speeds up the wheel so that 30.0s later it is turning at 82 rev/min.
   - (a) What is the average angular acceleration in radians per sec?
   - (b) How far will a point 0.30cm from the centre of the pulley travelled during the acceleration period, assuming that the acceleration is uniform?

3. The flywheel of a gasoline engine is required to give up 300 J of kinetic energy while its angular velocity decreased from 600 rev min^{-1} to 540 rev.min^{-1}. What moment of inertia is required?

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Indira Ghandi National Open University School Of Sciences, PHE – 0l, Elementary Mechanics Systems Of Particles

UNIT 5 RIGID BODY DYNAMICS 11

CONTENTS

1.0 INTRODUCTION

This unit is a continuation of the preceding unit; so, most of the introductory remarks are covered there.

Additionally, in this Unit you will study about the force that causes rotation; angular momentum and its conservation. We shall also see real physical systems, such as divers and figure skaters executing complex maneuvers, yet they are not rigid bodies showing that angular momentum and its conservation are very useful concepts. More examples of the applications of angular momentum and its conservation abound though they are beyond the scope of this course. You will definitely study about some of them in your future years.

We shall wrap up this course with the introduction of the concept of the top or gyroscope.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state what causes rotation of rigid bodies
- explain the concept of moment of a couple
- define the angular momentum of a rigid body
- apply the law of conservation of angular momentum
- solve problems based on the concept of rotational dynamics of rigid bodies.

3.0 MAIN CONTENT

3.1 Rotational Dynamics of a Rigid Body

3.1.1 Torque

It is time for us to ask what causes rotation. The analogies we have made between linear motion and rotational motion earlier in this course will be useful here. Recall that Newton's, second law describes the dynamics of linear motion whereby we have that a force causes linear motion given by an equation.

Here you will learn that rotational motion is caused by what we call a torque. You know that when we talk of a force, you intuitively think of a push or pull, so, in the case of torque. I would want you always to think of a twist. Know also that to increase the angular velocity of a rotating body, a torque of a couple must be applied. We see that torque is analogous to force.
where Torque,

$$F = \frac{Mdv}{dt} = ma. \qquad 3.1$$

Torque $$\Gamma = \frac{dI\omega}{dt} = I\frac{d\omega}{dt} \qquad 3.2$$

It is often necessary to find the work done by a couple so that the energy exchange that takes place as a result of its action on a body can be known.

Consider a wheel as represented in Figure 3.1. Let the radius of the wheel be r and two equal and opposite forces p act tangentially so that rotation occurs through angle θ.
Now, Work done by each force = force x distance

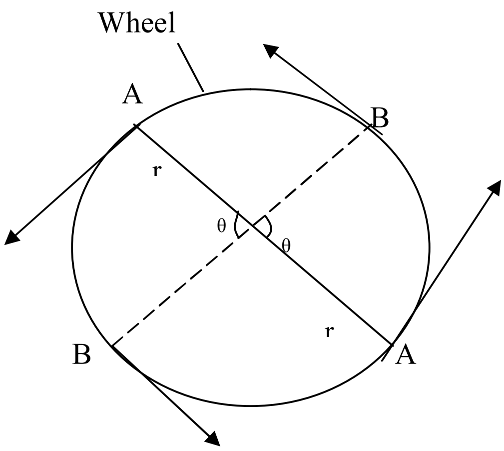


Fig 3.1

$\therefore$ Work done by each fo… c AB = $p_x r\theta$ 3.3

$\therefore$ Total work done by couple = $Pr\theta + Pr\theta = 2Pr\theta$ 3.4

But torque or moment of couple = Px 2r = 2Pr 3.5

Therefore, work done by couple = torque x angle of rotation
= Γθ 3.6

Example:
If P = 2.0N, r = 0.50m

And the wheel makes 10 revolutions, then,

θ = 10 x 2π ; and Γ = P x 2r
i.e Γ = 2.0N x 2 x 0.05m
= 2Nm

∴ work done by couple = Γθ = 2x20π = 1.3 x 10 ^{2}J.

In general if a couple of torque Γ about a certain axis acts on a body of moment of inertia, I, through an angle θ about the same axis and its angular velocity increases from O to ω, then,

Work done by couple = kinetic energy of rotation

$$i.e.\ \Gamma\theta = \frac{1}{2} 1 \omega^2$$

SELF-ASSESSMENT EXERCISE 3

A rope is wrapped several times around a uniform solid cylinder of radius 0.1m and mass 50 kg pivoted so it can rotate about its axis. What is the angular acceleration when the rope is pulled with a force of 20N?

Solution:
The torque is Γ = (0.1m) (20N)
= 2.0Nm

And the angular acceleration is

$$a = \frac{\Gamma}{I} = \frac{2.0\ Nm}{\frac{1}{2}(50kg)(0.1m)^2}$$
$$= 8rads^{-2}$$

3.1.2 Angular Momentum

3.1.2.1 Definition

We recall that in linear motion we talked of linear momentum. Now, in rotational motion we shall talk of angular momentum.

Let us consider a rigid body that is rotating about an axis O with an angular velocity ω at some instant of time. See figure 3.2 below.

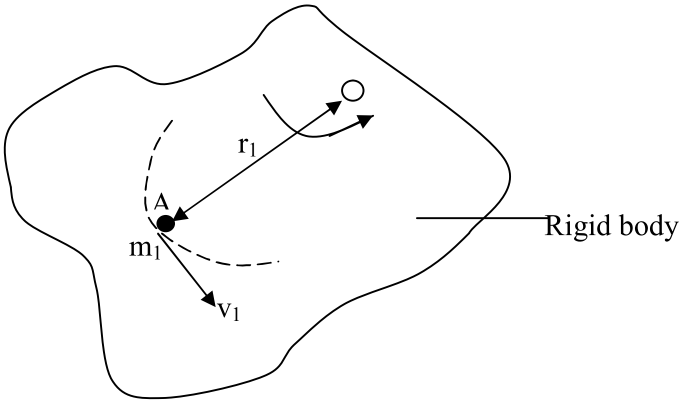


Fig 3.2

Let A be a particle of this body a distance r, from O the axis of rotation. If the particle has linear velocity V, as shown in the diagram then the linear momentum of A is m_1v_1 = m_1 ωr_1 (since $V_1 = \omega r_1$).

The angular momentum L of A_O about O is then defined as the moment of momentum about O.

Hence,
Angular momentum L of A = $r_{1\ x}\ m_1\omega r_1$

$$= \omega m_1 r_1^2 \qquad 3.7$$

∴ Total angular momentum = $\sum \omega mr^2$
of a rigid body

$$= \omega\sum mr^2 \qquad 3.8a$$
$$\therefore L = 1\omega \qquad 3.8b$$

Where we recognize 1 as the moment of inertia of the body about O. It is thus evident that angular momentum is the analogue of linear momentum (mv) where 1 is equivalent to mass m and ω replaces velocity V.

We can then state Newton's second law of rotational dynamics as follows.

A body rotates when it is acted on by a couple.

$$\therefore \Gamma = 1\alpha \qquad 3.9$$

where Γ is the torque of moment of the couple causing rotational acceleration α.

In terms of momentum we have that

Torque = rate of change of angular momentum

i.e

$$\Gamma = I\frac{d\omega}{dt} = \frac{dl}{dt} \qquad 3.10$$

This is analogons to force which is the rate of change of linear momentum

$$F = \frac{mdv}{dt} \qquad 3.11$$

3.1.3 Conservation of Angular Momentum and Its Applications

Angular momentum is a vector that points in the same direction as ω. For uniform rotational motion about an axis, the angular momentum does not change in either magnitude or direction. Just as in the case for linear momentum, angular momentum is independent of time for a system on which there is no torque due to external forces. Note that it is possible that the external torque is zero even when the external force is not zero. This will depend on where the external force is applied and on its direction. Similarly, a net torque could exist when a net force is zero. When the net torque is zero, the angular momentum is independent of time and is conserved. For rigid bodies, the rotational inertia is constant, and the conservation of angular momentum means that the angular velocity is constant in time. When the rotational inertia can vary because the system considered can vary its shape, then the conservation of angular momentum becomes a very important and useful principle.

Hence the principle of angular momentum states that:

The total angular momentum of a system remains constant provided no external torque acts on the system rigid or otherwise.

Mathematically we have that,

$$\frac{dL}{dt} = 0 \qquad 3.12$$

Ice skaters, ballet dancers, acrobats and divers use this principle of conservation of angular momentum. For example, the diver in the Figure 3.3 below leaves the high diving board with outstretched arms and legs and some initial angular velocity about his centre of gravity. His angular momentum Iω remains constant since no external torques act on him. To make a somersault he must increase his angular velocity. He does this by pulling in his legs and arms so that I decreases and ω therefore increases. By extending his arms and legs again, his angular velocity falls to its original value. Similarly a skater can whirl faster on ice by folding her arms.

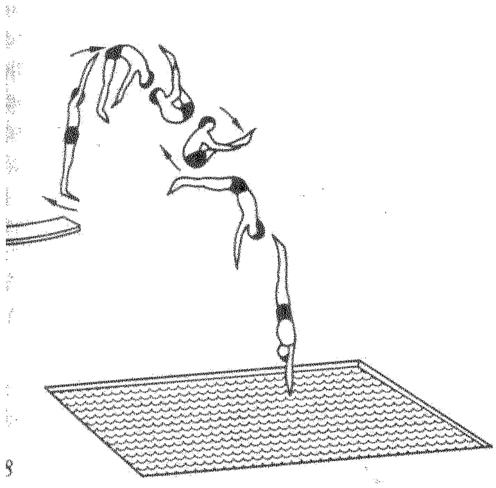


Fig 3.3

The principle of conservation of angular momentum is useful for dealing with large rotating bodies such as the earth, as well as tiny, spinning particles such as electrons, protons, neutrons.

The earth is an object which rotates about an axis passing through its geographic north and south poles with a period of 1 day . If it is struck by meteorites, then since action and reaction are equal, no external couple acts on the earth and meteorites. Their total angular momentum is thus conserved.
Neglecting the angular momentum of the meteorites about the earth's axis before collision compared with that of the earth. Then, Angular momentum of earth plus meteorites after collision = angular momentum of earth before collision.

Since the effective mass of the earth has increased after collision the moment of inertia has also increased. Hence, the earth will slow up slightly. Similarly, when a mass of object is dropped gently on to a turntable rotating freely at a steady speed, the conservation of angular momentum leads to a reduction in the speed of the turntable.

Example:
Calculate the angular momentum of earth's motion about its axis of rotation given that earth's mass is 6 x 10^{24}kg and its radius is 6.4 x $10^{6}$$_{m.}$ Assume that the mass density is uniform.

Solution :
arth makes one revolution about its axis in 24h. Thus, its period of rotation is

$$\mathrm{T} = 24h \times \frac{60\min}{1h} \times \frac{60\sec}{m}$$
$$= 86400s$$

Hence,

$$\omega = \frac{2\pi}{\mathrm{T}} = \frac{6.28rad}{86{,}400s}$$
$$= 7.3 \times 10^{-5}\, rad \quad s^{-1}$$

Now since the rotational inertia of a uniform sphere is

$$\mathrm{T} = \frac{2}{5}MR^2$$
$$= (0.4)\left(6 \times 10^{24}\, kg\right)\left(6.4 \times 10^{6}\, m\right)^2$$
$$= 10 \times 10^{37}\, kg.m^2$$
$$L = I\omega = \left(10 \times 10^{37}\, kg.m^2\right)\left(7.3 \times 10^{-5}\, rads\right)$$
$$= 7 \times 10^{33}\, kg.m^2 s^{-1}$$

We notice that our calculated value of I is some 20 percent larger than the correct value of 7.9 x 10^{37}kg.m^2. Why is it so

SELF-ASSESSMENT EXERCISE 4

The earth is suddenly condensed so that it's radius becomes half of its usual value without its mass being changed. How will the period of daily rotation change?

Solution: of (b) from the principle of conservation of angular momentum, we get

$$I_1\omega_1 = I_2\,\omega_{2.}$$

$$Here\, I_1 = \frac{2}{5}MR_1^2 \quad , I_2 = \frac{2}{5}MR_2^2$$

$$and \quad R_2 = \frac{R_1}{2}$$

$$\therefore \frac{2}{5}MR_1^2\omega_1 = \frac{2}{5}M\frac{R_1^2}{4}\omega_2$$

$$or\ \omega_2 = 4\omega_1$$

$$But \quad \omega_1 = \frac{2\pi}{T_1}\ and\ \omega_2 = \frac{2\pi}{T_2}$$

where T_2 and T_2 are the usual and changed time periods of daily rotation of earth

So the time period of daily rotation will become 6h.

3.1.4 Experiment on Conservation Of Angular Momentum.

A simple experiment to illustrate the principle of the conservation of angular momentum is illustrated below in Figure 3.4

$$\therefore \ T_2 = \frac{T_1}{4} = \frac{24h}{4}$$
$$= 6h$$

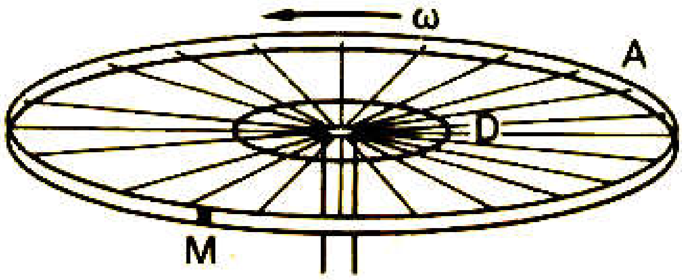


FIG. 3.10 Conservation of angular momentum

In the Figure, a bicycle wheel A without a tyre is set rotating in a horizontal plane and the time for three complete revolutions is taken with the aid of a tape maker M on the rim. A ring D of known moment of inertia, I is then gently placed on the wheel concentric with it, by dropping it from a small height. The time for the next three revolutions is then determined. This is repeated with several more rings of greater known moment of inertia.

If the principle of conservation of angular momentum is true, then

$$I_0\omega_0 = (I_0 + I_1)\,\omega_1 \qquad 3.13$$

Where I_0 is the moment of inertia of the wheel alone, ω_0 is the angular frequency of the wheel alone and ω_1 is the angular frequency with a ring. Thus if t_o, t_1 are the respective times for three revolutions,

$$\frac{I_0}{t_1} + I_1 \quad = \frac{I_0}{t_0} \qquad 3.1.4$$

Dividing through by I_0 *gives*

$$\therefore \quad \frac{I_1}{I_0} + 1 \quad = \frac{t_1}{t_0} \qquad 3.15$$

Thus a graph of $t_1/\ t_o$ against I_1 should be a straight line. Within the limits of experimental error, this is found to be the case.

Example:
Consider a disc Fig. 3.5 of mass 100g and radius 10cm is rotating freely about axis O through its contre at 40 r.p.m. Then, about O the moment of inertia I is

$$\mathrm{I} = \frac{m\alpha^2}{2} \quad = \frac{1}{2} \times 0.1kg \times 0.1^2 m^2$$
$$= 5 \times 10^{-4} kg.m^2$$

and

$$angular\ momentum = 1\omega = 5 \times 10^{-4}\omega$$

Where ω is the angular velocity corresponding to 40 r.p.m.

Suppose some wax, w of mass m 20g is dropped gently on to disc at a distance r of 8.0cm from the centre O.

The disc then slows down to another speed, corresponding to an angular velocity ω_1 say. The total angular momentum about 0 of disc plus wax.

$$= \quad 1\omega_1 + mr^2\omega_2 \quad = 5x10^{-4}\omega_1 + 0.02 \text{ x } 10.08^2\ \omega_1$$
$$= 6.28 \text{ x } 10^{-4}\omega_1$$

From the conservation of angular momentum for the disc and wax about O

$$6.28 \text{ x } 10^4\omega_1 \quad = \ 5 \text{ x } 10^{-4}\omega$$

$$\therefore \frac{\omega_1}{\omega} = \frac{500}{628} \quad = \frac{n}{40}$$

where n is the r.p.m of the disc

$$\therefore n = \frac{500}{628} \times 40$$
$$= 3.2\ (approx.)$$

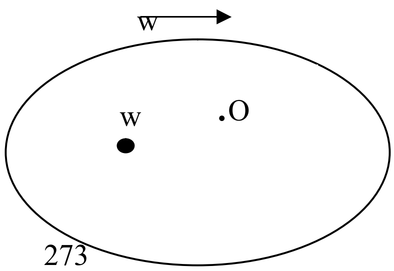


Fig 3.5

SELF-ASSESSMENT EXERCISE 3

(a) Define angular momentum.

(b) Describe how you would demonstrate (using a simple experiment) the principle of conservation of angular momentum.

Solution:

(a) See the text. It is useful to include in your definition Units of angular momentum, also mention that it is a conserved quantity in physics and is a vector.

(b) Remember to label the diagram you will use in the demonstration. Note that the question asks only for a demonstration not for a verification.

SELF-ASSESSMENT EXERCISE 4

The moment of inertia of a solid flywheel about its axis is 0.1kgm^3. It is set in rotation by applying a tangential force of 19.6 N with a rope wound round the circumference, the radius of the wheel is 10cm. Calculate the angular acceleration of the flywheel. What would be the acceleration if a mass of 2kg were hung from the end of the rope?

Solution

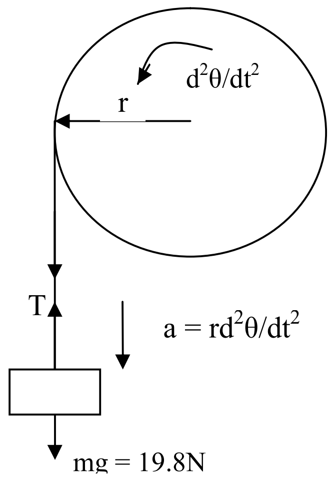


Fig 3.6

The couple C =

$$\frac{Id^2\theta}{dt^2} = momentum\ of\ inertia \times angular\ acceleration$$

$$\therefore\ (=19.6 \times 0.1) Nm$$

$$\therefore angular\ acceleration = \frac{196 \times 0.1}{0.1}$$

$$= 19.6\ rad\ 5^2$$

If a mass of 2kg is hung from the end of the rope, it moves down with an acceleration a. See the figure above. In this case, T is the tension in the rope.

$$mg - T = ma \qquad (i)$$

For the flywheel Tr = couple

$$= \frac{\mathrm{I}d^{2\theta}}{dt^2} \qquad (ii)$$

where r is the radius of the flywheel

Now, the mass of 2kg descends a distance given by rθ where θ is the angle the flywheel has turned. Hence the acceleration $a = rd^2\,\theta/\,dt^2$. Substituting we have

$$mg - T = mr\frac{d^{2\theta}}{dt^2}$$

$$mgr - Tr = mr^2\frac{d^2\theta}{dt^2} \qquad (iii)$$

$$adding \quad eqns \ (ii)\ and\ (iii)$$

$$mrg = (\mathrm{I} + mr)\frac{d^2\theta}{dt^2}$$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{mgr}{\mathrm{I} + mr^2}$$

$$= \frac{2 \times 10 \times 0.1}{0.1 + 2 \times (0.1)^2}$$

$$= 16.7\ rad\ S^{-2}$$

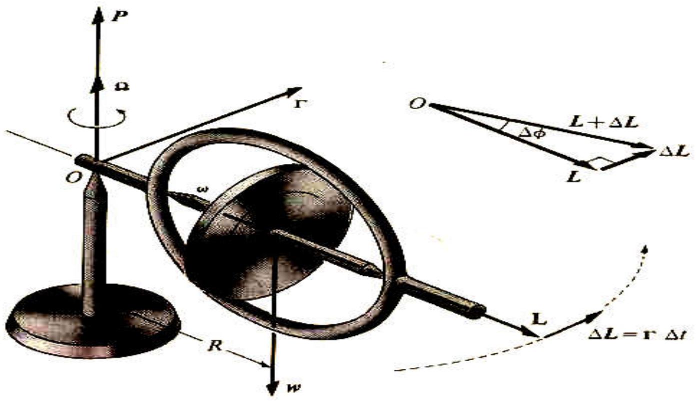
3.1.5 The Top and the Gyroscope

A symmetrical body rotating about an axis, one point of which is fixed is called a top. If the fixed point is at the centre of gravity, the body is called a gyroscope. We note that the axis of rotation of a top or gyroscope can itself rotate about the fixed point so the direction of the angular momentum vector can change.

An example of the mounting of a toy gyroscope is shown in figure 3.7 below

Fig 3.7: Vector ΔL is the change in Angular Momentum produced in time Δt by the moment Γ of the force w Vectors ΔL and Γ are in the same direction.

The top (since the fixed point is not at the centre of mass) is spinning about its axis of



symmetry and if the axis is initially set in motion in the direction shown, with the proper angular velocity, the system continues to rotate uniformly about the pivot at O. The spin axis remains horizontal.

The angular momentum of a top would equal the product of its moment of inertia about the axis and its angular velocity about the axis would point along it. If It's axis were fixed in space. But since the axis itself also rotates the angular momentum no longer lies on the axis. However, if the angular velocity of the axis itself is small compared to the angular velocity about the axis, then the component of the angular momentum arising from former effect is small and can be neglected. The angular momentum vector L, about the fixed point O, can then be drawn along the axis as shown and as the top rotates about O, its angular momentum vector rotates with it.

The upward force P at the pivot has no moment about O. The resultant external moment is that due to the weight w; its magnitude is

$$\Gamma = \omega R \qquad 3.14$$

The direction of Γ is perpendicular to the axis as shown. In a time Δt, this torque produces a change ΔL in the angular momentum, having the same direction as Γ and given by

$$\Delta L = \Gamma \Delta t \qquad 3.15$$

The angular momentum L + ΔL, after a time ΔL is perpendicular to L, the new angular momentum vector has the same magnitude as the old but a different direction. The top of the angular momentum vector moves as shown, and as time goes on it swings around a horizontal circle. Since the angular momentum vector lies along the gyroscope axis, the axis turns also, rotating in a horizontal plane about the point O. This motion of the axis of rotatiion is called precession". (Sears et at, 1975)

4.0 CONCLUSION.

In this unit, you have learnt that

- applied torque increases the angular velocity of a rotating body.

$$I \frac{d\,\omega}{dt}$$

- torque , Γ =

 where, I is the moment of inertia and ω is angular velocity.
- work done by a couple or torque is given by the kinetic energy of rotation. That is

$$\Gamma\,\theta = \frac{1}{2} 1\omega^2$$

- the angular momentum L of a rotating body is given by

$$L = I\omega$$

- the total angular momentum of a system remains constant provided no external torque acts on the system – rigid or otherwise.
- a symmetric body rotating about an axis, one point of which is fixed is called a top.

$$\frac{dL}{dt} = O$$

5.0 SUMMARY

What you have learnt in this unit concerns the angular momentum of a rigid body. You have learnt that:

- torque is the rotational analogue of force in linear motion.
- to increase the angular velocity of a rotating body a torque or a couple must be applied.
- torque is given by Γ where
- K.E of rotation is

$$\Gamma = \frac{Id\omega}{dt}$$

$$\Gamma\theta = \frac{1}{2} \mathrm{I}\omega^2$$

- the angular momentum L of a system about an axis is defined as the moment of its momentum about that axis.
  $L = I\omega$
- when the net torque on a system in rotational motion is zero, the angular momentum is independent of time and is conserved.
- using the formulas in this Unit and in the previous one you can solve problems relating to angular momentum.
- a gyroscope is a symmetrical body rotating about its centre of gravity

Summary of Equivalences Between Linear And Rotational Motion

Quantity or Formular in Linear Motion	**Equivalent in Rigid Body Rotation**
Displacement (s)	Angular displacement θ
Velocity (V)	Angular velocity (ω)
Acceleration $\left(a = \frac{dv}{dt}\right)$	Angular acceleration $\alpha_{\perp} = \frac{d\omega}{dt}$
Mass (m)	Moment of inertia (I)
Force (f)	Torque (Γ)
Kinetic energy $\left(\frac{1}{2}mv^2\right)$	**Kinetic energy** $\left(\frac{1}{2}I\omega^2\right)$
Work done (Fs)	Work done (Γθ)
$F = ma$	$\Gamma = I\alpha$
$m_1v_1 + m_2v_2 = \text{constant}$	$I_1\omega_1 + I_2\omega_2 = \text{constant}$
$V = u + at$	$\omega_{final} = \omega_{initial} + \alpha t$
etc	Etc

6.0 TUTOR-MARKED ASSIGNMENT

1. A shaft rotating at 3×10^3 revolutions per minute is transmitting a power of 10 kilowatts. Find the magnitude of the driving couple.

2. The turntable of a record player is a uniform disc of moment of inertia 1.2×10^{-2} kg m^2. When the motor is switched on it takes 2.5s for the turntable to accelerate uniformly from rest to 3.5 rad s^{-1} (33 1/3 r.p.m.)
   (a) What is the angular acceleration of the turntable?
   (b) What torque must the motor provide during this acceleration?

3. A stationary horizontal hoop of mantle, 0.04 kg and mean radius 0.15m is dropped from a small height centrally and symmetrically onto a gramophone turntable which is freely rotating at an angular velocity of 3.0 rad. 5^1. Eventually the combined turntable and hoop rotate together with an angular velocity of 2.0 rad.S^{-1}. Calculate

   (i) The moment of inertia of the turntable about its rotation axis
   (ii) The original kinetic energy of the turntable.
   (iii) The eventual kinetic energy of the combined hoop and turntable.

   Account for any change in kinetic energy which has occurred.

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