MODULE 1

UNIT 1 BASIC PROPERTIES OF REAL NUMBERS

CONTENTS

1.0 INTRODUCTION

In this unit you will be introduced to the basic concepts of real numbers. Basics properties of real numbers is the first topic or concept you are required to study in this course. There are reasons among others why real numbers should be the first topic to study in this course.

Firstly numbers are very important in all calculations, a fact you are already familiar with

Secondly, the properties of numbers is very essential to the development of calculus.

Lastly, all the topics in mathematics that you will be required to study during your programme will involve the use of some properties of real numbers.

In view of the above you should endeavor to carefully study all the topics covered in this unit and as well as complete all assignment

Materials learnt in this unit will help you in understanding all other topics you will learn throughout this course.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- list correctly all types of numbers
- recall 3 basic axioms of a real number system
- identify all types of intervals
- define 4 properties of absolute value of a real number
- define a bounded set.

3.0 MAIN CONTENT

3.1 Sets

When you collect items of similar characteristics or functions together, you could say that you have a 'set of such items'. For example, you can have a set of books, a set of furniture, a set of dissecting instruments etc.

Therefore you could define a set as a collection of distinct (definite distinguishable) objects which are selected by means of certain rules or description. Hence a set is a mathematical concept used to describe a list, collection or a class of objects, figures etc.

The objects in the list or collections are called elements or members of the set

Example of Sets

1) The students of national Open University
2) {1,2,3,4...... }Set of Natural Numbers
3) The set of stalls in a market

Sets are denoted by single capital letters A, B, C etc. or by the use of braces, for example, {a, b, c} denotes the set having a, b, and c as members or elements.

You will now be introduced to some specific sets, and symbols associated with them which you will likely use throughout this course

Specific Sets	**Symbols**	**Statements**
1.Null or Empty set	ø	It is a Set which has no member.
2. An element of Set	α∈A	a is an element of A
3. Not an element of	b∉A	b is not an element of A
4. Universal Set	Ч	largest Set containing all elements under considerations (largest Set containing all Sets).
5. Subset	A⊂B or B⊂A	A is a subset of B (each element of A also belong to B).

Specific Sets	**Symbols**	**Statements**
6. Proper Subset	A⊂B	A is a proper subset of B (A is subset of B)
	A\B	B has at least one element Which is not in A.
7. Union of Sets	A∪B	This is the set of all Elements which belong to A or B i.e: A∪B = {x : x∈A or x∈B}
8. Intersection of Sets	A∩B	This is the set of all element which belong to A and B both i.e; A∩B = {x: x∈A and x∈B}

SELF ASSESSMENT EXERCISE 1

Look around where you are right now identify 10 different objects. Group each one that could be used for:

i. 1. Eating
ii. 2. Sleeping
iii. 3. Cooking
iv. 4. Sitting
v. 5. Decoration etc.

Classify each of the identified objects.

3.2 Real Numbers

You will continue the introduction to the course for differential calculus with the study of real numbers. You are already familiar with the following types of real numbers.

i. Natural Numbers

The set of positive whole number/1,2,3,4,....., are called natural numbers. The letter N is used to denote the set of natural numbers. These numbers are used extensively for counting processes. For example, they are used in counting elements of a set. You can represent this numbers as N= {X:X =1,2,3,...) set of.

ii. Integers

Next in line to the set of Natural Numbers is another set that makes subtraction possible i.e, it allows you to find the solution to a simple equation as x + 2 = 6.

This set is derived by adding the set of negative numbers and zero to the set of Natural numbers.

It is called the set of integers and it is denoted by the letter Ƶ or I. Hence the set of integers is given as

Ƶ = {x:x = ..., -3 -2, -1, 0, 1, 2, 3, ---}

iii. Rational Numbers

You can see a gradual building up process in the various stages of development of numbers. That is, in order for you to be able to carry out division and multiplication correctly, you need to enrich or add a new set of numbers to the set of integers. So that you will be able to find a solution to equations like 2x = 3.

Therefore if we add the set of Negative and positive fractions to the set of integers we get a new set of numbers called the set of Rational Numbers.

The word `rational' is from the word ratio. Since 2:1 = 2/1 and 1:2 = ½. Set of rational numbers could be given as that number that can be expressed as the ratio of two integers of the form p/q, where p and q are relatively prime integers, i.e; p and q have no common division other than 1. The set of rational number is denoted by the letter Q.

iv. Irrational Numbers

A number which is not rational is irrational. Irrational numbers are not expressible as p/q. The set of irrational is denoted by the letter IQ

Examples of such numbers are $\sqrt{2}$,$\sqrt{3}$, $\sqrt{7}$ log, π etc. The above number can be written as infinite decimal i.e; or decimal that does not repeat itself.

Rules Governing Addition of Number

Given that a,b, and c belong to the set R of real numbers then;

A1. R is closed under addition

$a+b \in R$ (This implies that the sum of any two real number must be a real number)

A2. Addition is commutative
$a+b = b+a$

A3. Addition is associative
$(a+b\}+c= a+(b+c)$

A4. Existence of additive identity
$0+a= a+0 = a$ (i.e; 0 is the additive identity)

A5. Existence of additive inverse
$a+b = b+a = 0$ (i.e; corresponding to each $a\in R$ there is $b\in R$ such that $a+b = 0 \rightarrow b = -a$)

similar to the rules for addition you have those for multiplication, still assuming that a, b and c R the.

MI. R is closed under multiplication.
If as $\in$ R (i.e the product of any two real numbers is a real numbers)

M2. Multiplication is commutative
$cb = bc$

M3. Multiplication is associative
$(bc) = (ab)$ c

M4. Existence of multiplicative identity
-a. 1 = 1.a = a (1 is the multiplication identity)

M5. Existence of multiplication inverse
ab = ba = 1 (i.e; corresponding to each a∈R a 0, there is b∈R
b=a-' such that $b=a^{-1}$

D1 Multiplication is distributive over addition
a(b+c) = a.b + a.c

The set of real numbers combined by means of the two band operations namely addition (+) and multiplication (.) as expressed above forms a field. The above rule Al-A5, M1M5 and D1 are known as the field axioms. Because of the field axioms satisfied by elements of the set of real number, the set R is a field.

Question: Is the set of rational numbers a field?

The third axiom possessed by the set of real numbers is the axiom of order. Thus there exist an ordering relation between any two elements of the set of real number. The relation is denoted by the symbol > or < which is read as `greater than' or `less than'.

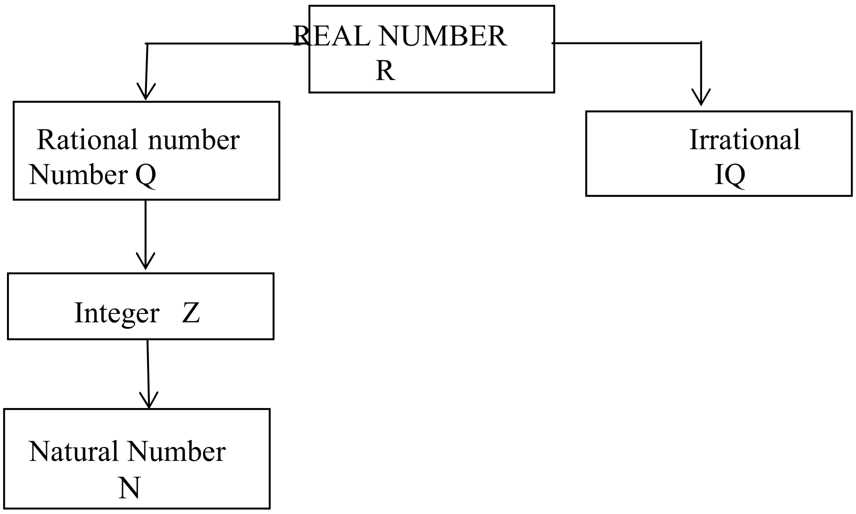
If a-b =0 then a=b or b=a. If a-b then a>b or b<a.

The properties of the order axiom will be stated based on `>' (the ones based on < are implied)

v. Real Numbers

The union of the set of rational (Q) and irrational (IQ) form the set of real number. It is denoted by the letter R.

You can visualize the development of real numbers system in a flow chart below:



In symbols we have this relationship

$N \subset Z$ - CQ, $Q \cap I Q = O$ and $Q \cup I Q = R$
$N \subset Z \subset Q$, $Q \cap I Q = \emptyset$, Q U $I Q = R$

From the above relationship what can you say about the following statements?

1. All integers are natural numbers
2. All rational numbers are real numbers
3. Some rational numbers are natural numbers
4. Not all real numbers are rational numbers
5. All natural numbers are irrational numbers.

3.3 Basic Axioms of Real Numbers

You are already familiar with the four arithmetic operations of addition, subtraction, multiplication and division of real numbers. From the last section you noticed that each arithmetic operation is directly or indirectly involved in the stages of the built-up of the structure of real numbers. This built up is derived from a set of fundamental axioms or truths which is turn are used to deduce other mathematical results or formulation. Such axioms are categorized into the following.

For example; the extend axiom says that the set of real numbers has at least two distinct elements

Next is how any two or more elements of the set of real numbers could be added. You must be familiar with addition. You will now see that addition of two or more real numbers is carried out under some specific rules.

i. If a, b and c belong to R then the law of trichotomy holds. Either a>b, a=b or b>a

ii. If a>b and b>c then a>c
(i.e.; `>' is transctive)

iii. If a>b then a+c > b+c
(i.e; addition is monotone)

iv. If a>b and c>o then ac>bc
(i.e.; multiplication is monotone)

Remark: If a∈R and a>o then a is said to be positive. If on the other hand a<o then a is said to be negative. If a = 0 then a is to be non-negative.

**** So far you have studied.***

Open Interval If a < b

3.4 Interval and Absolute Value

In this section you will continue the study of properties of real number by reviewing the concept of the real number line. After which you will be introduced to what an interval is and how a solution set of an inequality could be represented as a set of point in an interval.

Real numbers can be represented as points on a line called the real axis or number line. There is one-to-one correspondence between the members of the set of real numbers and the set of points on the number line. Commonly known to you is the fact that the set of real numbers to the right of 0 is called the set of positive numbers, while the set of real numbers to the left of 0 is called the set of negative numbers. 0 is neither positive nor negative.

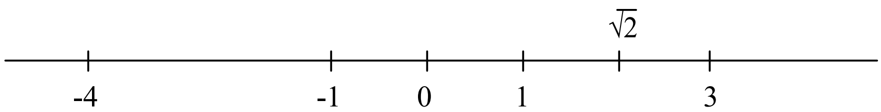


Figure 1: Showing the number line.

Remark: The one-to-one correspondence between the real numbers and the points of the number line makes it possible for us to use point and members interchangeably.

Definition of an Interval

Let $ab \in R$ and $a<b$ then the set of all real numbers contained between a and b is called an interval, these two real numbers a and b, Bare, referred to as the end points of the interval .

Open Interval

If $a<b$ then the set of real numbers specified by the inequalities $\{x: a<x <b\}$ is called an open interval and is denoted by (a, b) have a and b are not member of this set of real number.

Closed Interval

If $a<b$, then the set of real numbers specified by the inequalities (x: $a\leq x\leq b\}$ is called a closed interval and is denoted by [a,b]. All points between a and b as well as a and b belong to this set [â;b].

Half Open or Half Closed Interval

The set specified by the inequalities.

> $\{x:a\leq x<b\}$ or $\{x:a<x\leq b\}$ is called half open or half closed interval and is denoted by [a,b) or (a,b].

Infinite Interval

The set of all numbers less than or equal to a given number c or the set of numbers greater than or equal to a given number c is called an infinite interval

i.e the set $\{x : a \geq c\} = [c, \alpha)$

$\{x : x \leq c\}= [-\alpha, c)$

$\{x : x \in R\}= [-\alpha, \alpha)$

$\{x : x > c\}= [c, \alpha)$

$\{x:x<c\}= [-\alpha, c)$

3.5 Bounded Sets

You will now learn about bounded sets and use it to identify intervals that are bounded or unbounded Upper Bounds.

Upper Bound

Lets S be a non-empty subset of R. If there exist a number K∈R such that x ≤ k, for all x∈S then the set S is said to be bounded above. And K is known as an Upper bound.

Supremum

If there is a least member among the set of Upper bounds of the set S, this member is called Least Upper Bound (LUB) or Supremum of the set S and is denoted as Sup. S.

Example: Given that, S={1, 3, -1, -2, 4, 10}

i) List 4 Upper bounds for S
ii) Identify the Supremum for S

Solution

From the definition above any number K such that xc-s and x ≤ k, is an Upper bound i.e. K= {10,11,12, 13,)

The least among the k's is 10 therefore the Sup S = 10

Example let (i) S={x:x = 2} Then Sup. S = 2 why?

Lower Bounds

Let S be non-empty subset R. if length of Interval

The number (b-a) is called the length of the interval (a,b), and [a,b]

You are familiar with inequalities and you could recall that to solve an inequality is to find the set of numbers that satisfy it. Inequalities play such an important role in calculus, that is imperative that you know how to use the concept of interval to represent the set of solutions that satisfy a given inequalities.

Example:
Solve the inequality $2 - 2x \geq 4$

Solution:
$2(1-x) \geq 4$ (divide by 2)

$1\text{-}x \geq 2$ (subtracted 1)

$X \geq -1$ (multiplied by - 1)

Solution in set is given in the interval [-l, α)

SELF ASSESSMENT EXERCISE 2

Solve the following inequalities:

i. $3x - 3 \geq 9$
ii. $4x - 8 \geq 10$
iii. $4x - 7 \geq -10$

Absolute Value

You are familiar with the distance between zero and a point on the number line. You are equally aware that length or distance cannot have a negative value.

Let the distance between 0 and a be denoted by the symbol. $|a|$
$|x|$ = distance between 0 and x
$|a\text{-}b|$ = distance between a and b or b and a
therefore $|a| > 0$. You can define the absolute value of a number x as the distance between the point x and zero which satisfy the following conditions:

i. $|x| = x$ if $x > 0$.
ii. $|x| = x$ if $x < 0$
iii. $|x| = 0$ if $x = 0$

Example:

$|-3| = 3$

$|3| = 3$

$|0| = 0$

The relation

$|k| = x, x > 0$

is equivalent to the relation
$-x \leq k \leq x$

$\Rightarrow$ k$\in$[-x, x]
Some properties of Absolute value:

l. $|a| = a = |-a|$

2. $|ab| = |a| \ |b|$

3. $|a|^2 = a^2$

4. $|a+b| \leq |a| + |b|$ triangle inequality

5. $|a- b| \geq| 1a| - |b1|$

*Remark Properties 3 and 4 imply that the sum of the length of two sides of a triangle is always greater than the length off the third side.

Example: Show that:

i. $|a+b| \leq |a| + |b|$
ii. $|a+b| \geq |1a| - |b1|$

Solution:

l. $(|a| + |b|)^2 = |a|^2 + 2|a| \ |b| + |b|^2$

$a^2 + b^2 \ 2ab$ (since $|a| > a$)

$\Rightarrow$ $(|a| + |b|)^2 \ (a+b)$ (taking square root of both sides)

$|a| + |b| \ |a+b|$ (sine $|a|^2 = a^2$)

Hence the required result

ii. $|a-b| \ |1a| - |b1|$

Let c = a-b then a = c+b

$|a| = |c+b| \ |c| + |b|$

$= |a-b| + |b|$
$|a| - |b| \ |a-b|$ which is the required proof.

SELF ASSESSMENT EXERCISE 3

Show that $|a-b|$ c $\left||a| - |b|\right|$

There exists a number k∈R such that x∈k for all x∈S then the set S is said to be bounded below and k is known as a lower bound of S.

Infimum

If there is a greatest member among the lower bounds of the set S, then that member is called Greatest Lower Bound (GLB) or infimum of the set.

Remark: The supremum of a set if it exists is unique. The same applies to infimum of a set in other wards there cannot two distinct elements called the Sup S.

Examples: given the set s = (-4, -3, -1, 0, 2)

i. list 5 lower bounds of set S
ii. identify the infimum

Solution:

i. let k be the set of lower bound of s then k = (-10, -6, -8, -5, -4)
ii. the greatest member of set k is -4. Therefore inf S =-4.

Bounded Set

Let S be a non-empty subset of R. if there exist a number k∈R such that $|x|$=x for all x ∈s then the set S is said to be bounded.

In other words a set S is said to be bounded if it is bounded below and above

Example:

Given the following sets of number identify

i. bounded sets
ii. unbounded sets
iii. infimum
iv. supremum

Given the following sets of numbers

A = {-1,-2,0,1,2,3,4,)
B = {x : -2 < x 5}
C = {x : x > -1}
D = {x : x ∈ (-α,α)}
E = {x : x ∈ (-α, 0}

Identify (i) bounded sets

Determine which of the sets that are bounded or unbounded, for the bounded set, identify the supremum and infimum

Solution:

Set A is bounded
Sup A=4, and Inf A = -2
Set B is bounded
Sup B = 5 and Inf B = -2
Set C is unbounded
Set D is unbounded
Set E is unbounded

4.0 CONCLUSION

In this unit you have been able to learn about properties of real numbers and the development of real number system. You have observed that using the axioms you have studied you see a gradual and logical build up of the set of real numbers starting from the set of natural numbers.

You have studied how a set of real numbers could be represented using:

i) the concept of interval
ii) the concept of absolute value
iii) Inequalities

You have studied that a set of numbers represented by an interval can be bounded or unbounded.

5.0 SUMMARY

In this unit you have studied fundamental concepts of a set:

1. Extend axiom, Field axiom and order axiom of a set of real numbers

2. The gradual extension of the set of natural numbers to the Real number
3. The definition of absolute value of a real number as:

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

4. Types of intervals of set of real numbers namely
   a. Open interval (a,b) = {x : a < x < b }
   b. Close interval [a,b] = {x : a ≤ x ≤ b }
   c. Half-closed or half open interval
      [a,b)={x:a ≤ x < b}
      (a,b] = {x : a < x ≤ b }
      where $a_1b \in R$
5. That a bounded set is that set that is bounded below and above i.e. there is a number of K∈R such that
   $|x| \in k$ for x∈S then set S is said to be bounded

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR-MARKED ASSIGNMENT

1. Define a bounded set.

2. Determine if the following sets are bounded:
   i. S=(x : x∈= [0,I]}
   ii. S=(x : x∈ (0,1) }
   iii. S=(x : x∈ (0,1]j

iv. S=(x : x∈ [0,1)}
v. S=(x : x∈ (-α,1]}
vi. S=(x : x∈ [-2, α)}

3. Given examples to illustrate the following:
   a. A set of real numbers having a supremum
   b. A set of real numbers having an infimum
   c. A set of real numbers that is bounded

4. Determine if the set of Natural numbers is bounded below. What is the infimum if any

5. List elements of the following sets of integer:
   i. S=(x : x∈- (-4,I)}
   ii. S=(x : x∈ [-2,4]}
   iii. S=(x : x∈ (-1,3]}

6. State whether the following are true or false in the set of real numbers:
   a. 2 ∈ (- 2, 2)
   b. -1 ∈ (-00,0)
   c. 4 ∈ [4, 00)

7. Show that $|x| - |y| \leq |x-y|$

8. Give a precise definition of:
   i. The supremum of a bounded set
   ii. The infimum of a bounded set

UNIT 2 BASIC PROPERTIES OF REAL NUMBERS

CONTENTS

1.0 INTRODUCTION

The concept of functions and its corresponding definition as well as its properties are very crucial to the study of calculus. Simple observation of any physical phenomena has made it imperative for us to be interested in how variable objects are related. For example, you are familiar with how distance traveled by a body freely falling in a vacuum is related to the time of the fall or how the concentration of a medicine in the blood stream is related to the length of time between doses, or how the area of a circle is related to the to the radius of the circle.

The types of relationship between two variables in this unit will be considered, also the study of the concept of a function is very important since the properties of functions -are what you will use whenever you want to find the derivative of a function. It is important you study carefully and diligently all the various types of functions and their characterization.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a function
- identity all types of functions
- state the domain and range of a function
- combine functions to form a new function.

3.0 MAIN CONTENT

3.1 Definitions of a Function

You will start the study of this unit with the definitions of a function and its various forms of representation.

Definition 2.1

A function is a rule which establishes a relationship between two sets. Suppose X and Y are two sets, a function f from X to Y is a rule which attributes to every member $x \in X$ a unique member $y \in Y$ and it is written as

$f : X \rightarrow Y$ (which reads `f is a function from X to Y)

The set X is called the domain of the function, while the set Y is called the co-domain of the function.

Another definition of a function is given below as:

Definition 2.2

A variable y= f (x) (in words f (x) reads f' of x) is to function of a variable x in the domain X of the function if to each value of $x \in X$ there corresponds a definite value of the variable $y \in Y$.

Basically every function is determine by two things:

(1) the domain of the first variable x and
(2) the rule or condition the set of ordered pairs (x,y) must satisfy to belong to the function.

You will have a better understanding of the definitions above after going through the following examples.

Example 1

Let the domain of x be the set X={-2, -1, 0, 1, 2}
Assign to each value of X the number;

Y= 2x The function so defined is the set of pairs (-2, -4), (-1,-2), (0,0), (1,2) and (2,4)

Example 2

f : N→Z, defined by f (x) =1-x is a function since the rule f (x)=1-x assigns a every member $x \in N$ to a unique member of the set Z.
Z is a set of integers.

Example 3

If to each number in the set $x \in (-1, 2)$ we associates a number $y = x^2 + 1$ then the correspondences between x and $x^2 + 1$ defines a function.

3.2 Representations of Function

In the above definition of a function you were introduced to the concept of a domain .From the definition of a function ,the domain of a function could be defined as the set of value for which a function is defined. The independent variable x is a member of the domain. The dependant variable y that corresponds to a particular x-value is called the image of the x-value. The set of value taken by the independent variable y is called the range of the function. The range is the image of the domain.

Any method of representation of function must indicate the domain of the function and the rule that the ordered pairs (x,y) must satisfy in order to belong to the function> In this unit you will study two basic methods of representing a function namely:

1. Analytical method (i.e.; representation by means of a mathematics formula)
2. Graphical method

1. Analytical Representation

This is given by a formula which shows you how the value of the function corresponding to any given value of the independent value can be determined.

Example refer to example 5.5. in Unit 5.

The formulas $y=x^2 +1$ and y = 3-x specify y as a function of x. In the above example the domain of the function is assumed to be a subset of R for which the formula, representing the function makes sense.

2. Graphical Representation

A function is easily sketched by studying the graph of the function. In unit .... you would be required to plot the graphs of certain function so materials of this section will be useful to you then. Let us define what a graph of a function is

Definition 4. The graph of the function define by y= f (x) is the set of points in a rectangular plane whose co-ordinate pairs are also the ordered pairs (x,y) or [x, f(x)] of the function.

Another way you can view the above definition is to look at the steps of describing or drawing the graph of the function y = f (x) . To do this you choose a system of coordinate axes in the x-y plane. For each $x \in X$, the ordered pair [x, f (x)] determines a point in the plane (see fig. 1)

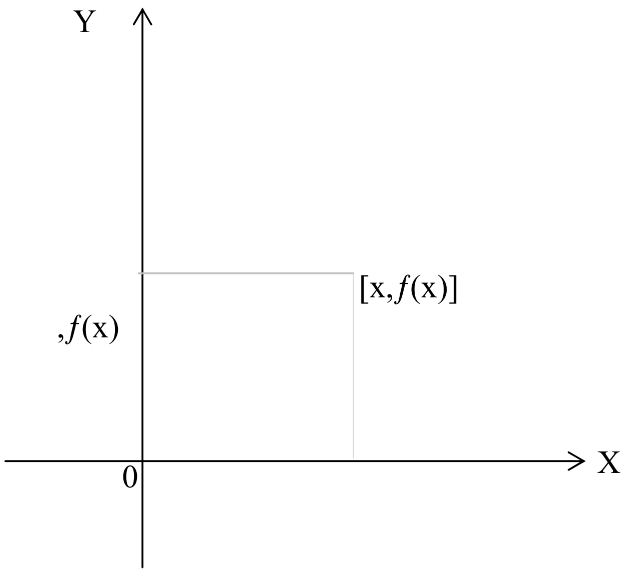


Fig. 1.

You will come across graphs of each type function that will be considered in this unit, the role each graph plays in understanding their respective functions will then become clearer to you.

3.3 Basic Elementary Functions

You will continue the study of function by considering the various types of functions and their graphs

1. Constant Functions

The simplest function to study is the constant functions. A constant function have only one constant value y for all values of belonging to the domain. i.e. ,f(x)= a for all $x \in X$ where X is the domain of the function (see fig 2)

You noticed that the graph of a constant function is a straight line parallel to the x- axis.

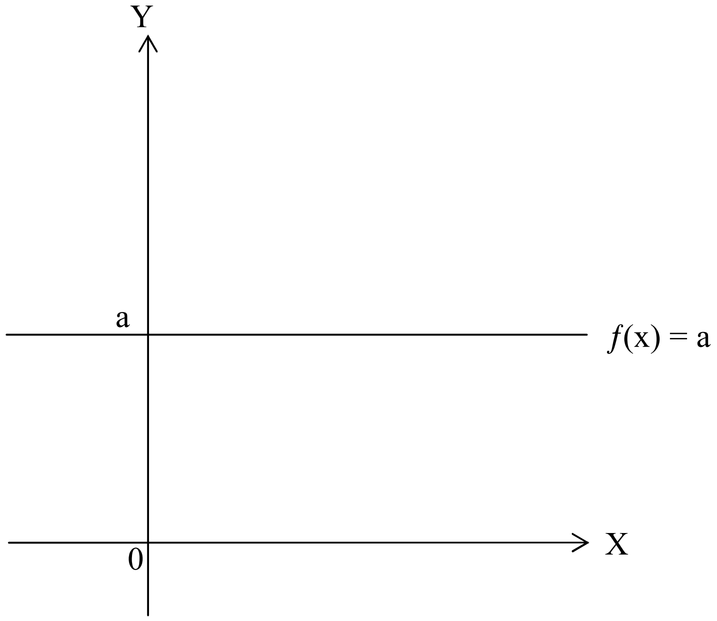


Fig. 2

In fig.2. ,f(x) =a is a graph parallel to the x-axis at a distance $|a|$ units from the x-axis.

2. Polynomial Function

1. Any function that can be expressed as

   . ,$f(x)) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \text{-----}+$

   $x_{n-1} + a_n$ ——————————— (A)

   where a_1, ,,, a_1........ a_n, are constant coefficients is called a polynomial function of n degree.

   You can derive various forms of functions with different graphs by varying the value of n.

Example 1 - If you substitute n =1 into expression (A) above you get a linear function i.e., . $f(x) = a_o x + a_1$

The graph is in Fig. 3a.

2. If you put n=2 in expression a you get a quadratic functions.

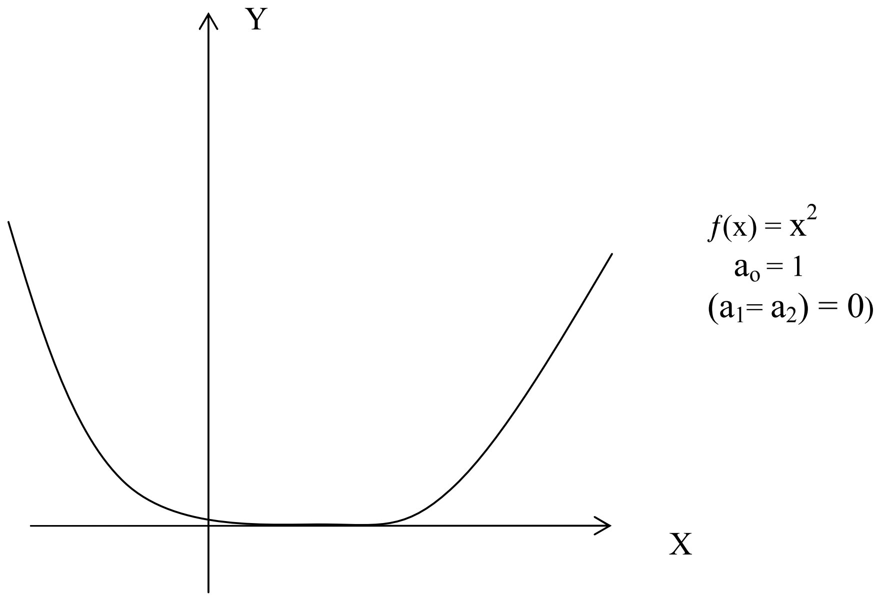
i.e. $f(x) = a_o x^2 + a_1 x + a_2$

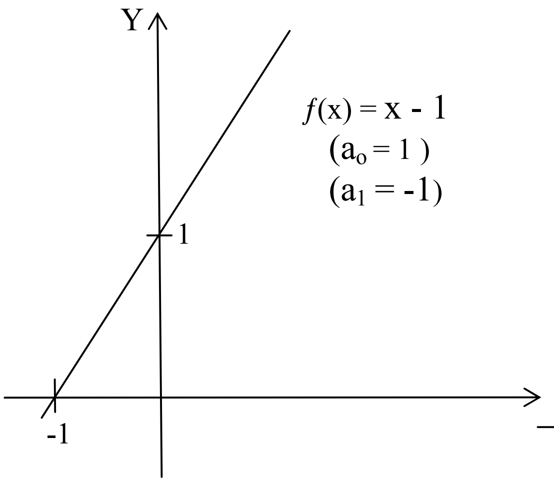
See Fig 3b.

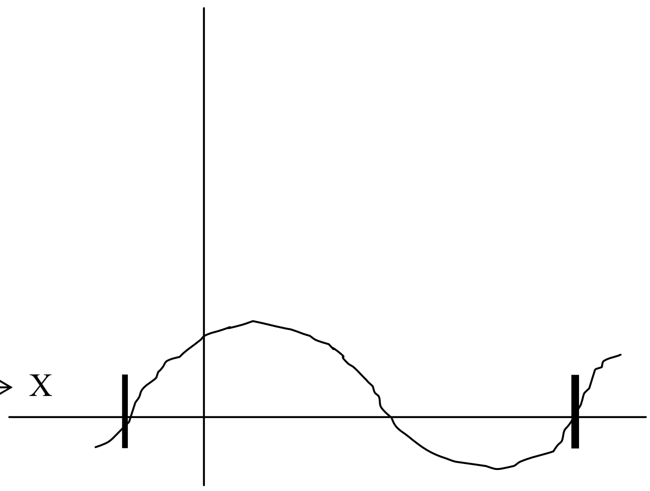
3. If you substitute n=3 into expression (A) you get a cubic function

i.e. $f(x) = a_o x^3 + a_1 x^2 + a_2 x + a_3$

You could continue this process as much as you want.







get equate for MathCAD

3. Identify Function

There is a function that assigns every member of the set domain to itself.

Let X be domain of the function then ,$f(x) = x$ for all $x \in X$. In some other textbooks identify function are denoted as Ix. The graph of an identity function is a straight line passing through the origin (see Fig 4)

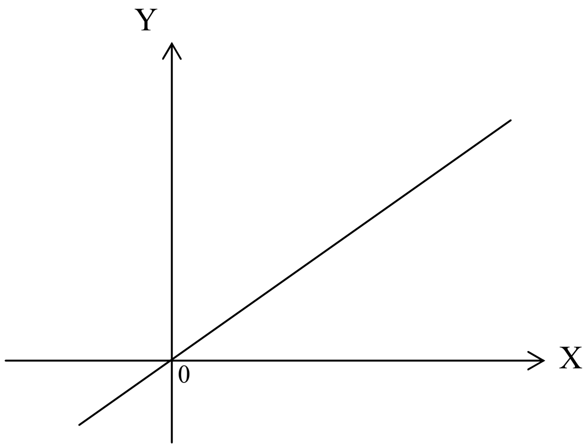


Fig 4

4. Algebraic Function

When two polynomial are combined together to construct a function of this form.

$$\frac{P(x)}{Q(x)} = \frac{a_o x^n + \ldots + a_n}{b_o x^m + \ldots + b_m}$$

The above function is called a rational algebraic function.

Example $,f(x) = \frac{2x}{x - 1}$

3.4 Transcendental Function

This is a. class of functions that do not belong to the class of algebraic functions discussed above. They are very useful in describing or modeling physical phonemic. Therefore you need to study them because they will be needed in the subsequent units.

1. The Exponential Function

A function $,f(x) = a^x$ where $a > 0$ and $a^{\pm 1}$ is called an exponential function.

A special case of an exponential function is where a = e i.e. $,f(x) = e^x$ this function is know as the natural exponential function. Its graph is shown in Fig. 5

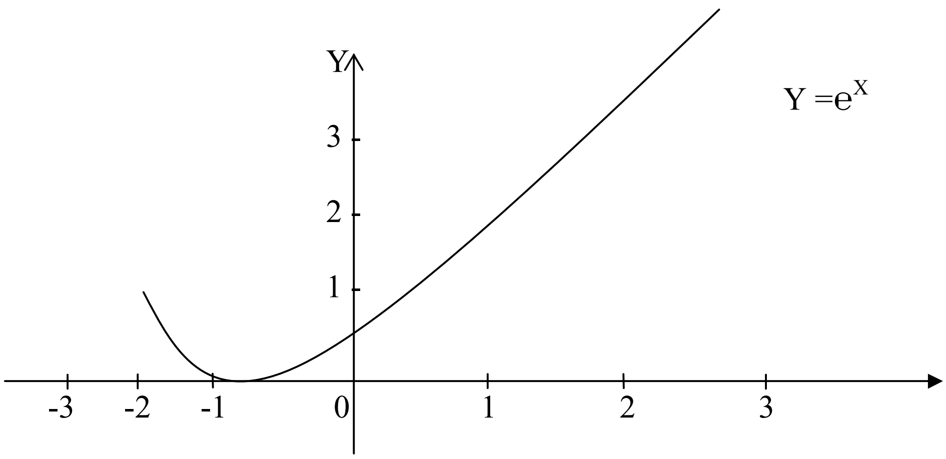


Fig. 5.

2. Logarithm Function

Any function ,$f(x)$ which has the property that:

ff(x, y) =,f(x) + ,f(y) for all x, y > 0

is called a logarithm function

Example: Let ,f(x) be a logarithm function then

$f(1) = f(1 . 1) = f$ (I) + f (1)= 2f (1)
$\Rightarrow$ f (1) = 2f (I)
$\Rightarrow$ f (I)=0

for x > 0 f (l) = f (x .1/x) = f (x) + f (1/x) = 0
$\Rightarrow$ f (1/x) = - f (x).

Let x > 0 and y > 0 then
,f(y/x)= ,f(y-1/x)= f (y) + f (1/x)

Since ,f(1/x) = - ,f(x) then
,f(y/x) = ,f(y) - f(x)

Let ,f(x) = log x.

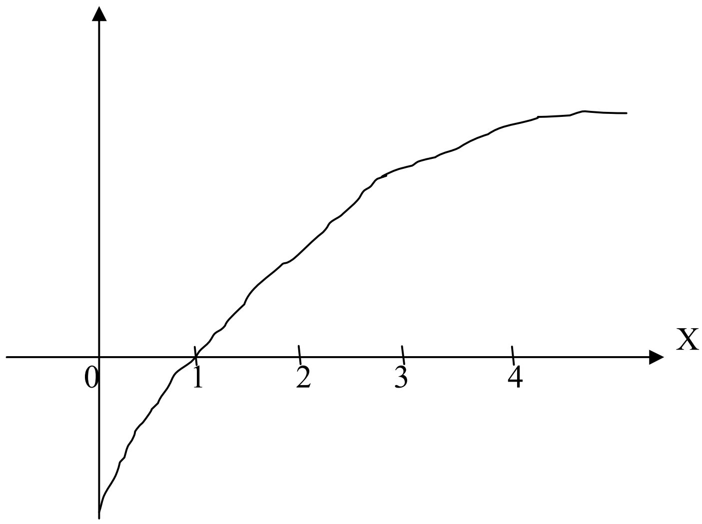
Then log (x/y) = log x - log y.
And log (y/x) = log y - log x.

Log (1/x) = - log x.

2. The Natural Logarithmic Function

The ,f(x) = Ln (x) where x ± 0
Is called the natural logarithmic function.

Its graph is shown in fig. 6. (this function derived its definition from calculus see unit...)



3. The Trigonometric Function

The function define as:

,f(x) = Sinx, ,f(x) = Cos x,,f(x) = tanx.
,f(x) =Cot x, ,f(x) = Cosec x, and f(x) =Sec x.

are called trigonometric functions.

4. Inverse Trigonometric Functions

The function define as
$f(x) = \text{Sin}^{-1-}x$, ,$f(x) = \text{Cos}^{-1}x$, ,$f(x) = \tan^{-1}x$.

are called inverse trigonometric functions.

5. Hyperbolic Functions

There are classes of function that can be form by combing the exponential function.

For example:

$$,f(x) = \frac{e^{X} + e^{-X}}{2} = \text{Cosh } x$$

$$,f(x) = \frac{e^{X} - e^{-X}}{2} = \text{Sinh } x$$

These functions are very useful in computing the tension at any point in high-tension cables you see in some of the highways across the country. They are also important in solving some classes of problems in calculus. The rules governing them are like that of the trigonometric functions.

The functions: Cosh x = $1/2(e^{X} + e^{-X})$ (cash reads gosh x) and Sinh x = $1/2(e^{X} + e^{-X})$ (sinh reads cinch x)

may be identified with coordinates of point (x,y) on the unit hyperbola $x^2 - y^2 = A$

Recalled that the functions sin x and cos x with the point (x, y) on the unit circle $x^3 + y^2 = 1$ in some text trigonometric functions are called circular functions. So the name hyperbolic is formed from the word hyperbola. Other hyperbolic functions like tanh x, Coth x, Sech x, and Cosech x can be derived from cosh x and sinh x.

6. Inverse Hyperbolic Functions

The functions Y= Sinh^{-1} x, Y = Cosh^{-1}x, are called inverse hyperbolic functions.

Others are Y= $\tanh^{-1}$ x, Y= Sech^{-1}x, Y=Cosech^{-1}x,

4.0 CONCLUSION

In this unit you have studied the definitions of a function. You have studied two ways a function can be represented. You have studied types of functions - elementary and transcendental functions.

5.0 SUMMARY

You have studied to:

State the definition of a function of one independent variable.

- Use graph to describe types of functions, quadratic, sin etc.
- Recall various types of function.

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR MARKED ASSIGNMENT

Give precise definitions of the following:

1. Domain of a function
2. Function of an independent variable.
3. Exponential functions.
4. Logarithmic functions
5. Give two ways a functions can be represented.

UNIT 3 CHARACTERISTIC OF FUNCTIONS

CONTENTS

1.0 INTRODUCTION

Investigation of function are carried out by observing the graph of the function or the value of the function as the independent variable changes within a given intervals. In other words a function is investigated by characterization of its variation (or its behaviour) as the independent variable changes. The classification of the variety of function is very vast. The following types defined in this unit are by no means this unit you continue the study of functions by considering special features that` characterize a function.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify basic characteristics of functions such as monotonic property boundedness etc
- define an inverse function
- define a composite function
- combine functions, to form a new function
- determine whether a given function has an inverse or not.

3.0 MAIN CONTENT

3.1 Types of Functions

Zero of a function: The value of x for which a function vanishes, that is for which f(x) = 0 is called **the zero** (or root) of the function.

Example la.

The function $f(x) = x^2 - 3x + 2$ has two roots i.e.; x = 2 or 1.

One of the roots of the function.

$f(x) = x^3 - 2x^2 - 5x + 6$ is 1.
i.e., $f(1) = 1^3 - 2.1^2 - 5.1 + 6 = 0$

SELF ASSESSMENT EXERCISE 1

Find other roots of the above function.

1. Even and Odd Functions

A function y =$f(x)$ is said to be even, if the changes of the sign of any value of the independent variable does not affect the value of the function.

F(-x) _ F(x) .
i. e, $f(-x) = f(x)\ \forall x \in X$

A function y = $f(x)$ is said to be odd if the change of sign of any value of the independent variable results in the change of the sign of the function

i.e. ; $f(-x) = -f(x)$

Example

The function $y = x^2$ is an even function while the function y = sin x and $y = x^3$ are odd functions.

Remark: Arbitrary functions such as y = x +1, y = 2 sin x + 3 Cos x can of course be neither even nor odd.

2. Periodic Function

A function y = $f(x)$ is said to be periodic if there exists a number n $\neq 0$ such that for any x belonging to the domain of the function the values x + n of the independent variable also belonging to the domain of the function and the identity.

$f(x + n) = f(x)$ holds where n is called
the period of the function. 1

Example

If $f(x)$ is a periodic function with
period n the $f(x + n) = f(x + 2n) = f(x)$

Generally for any periodic function $f(x)$ with period n.
$f(x + nk) = f(x)$ for any $x \in Z$, $k \in N$,

A simple example of a periodic function is the function $f(x)$ =s i nx or $f(x)$ Cosx.

See Fig. 9.

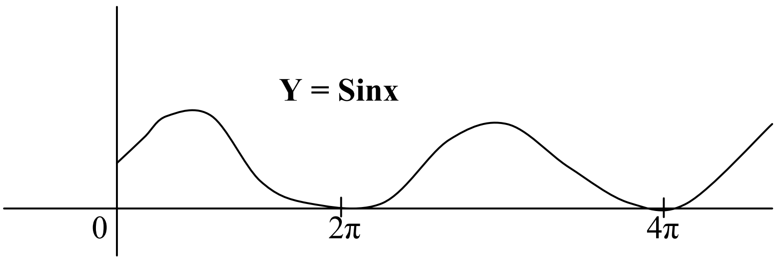


Fig. 9a.

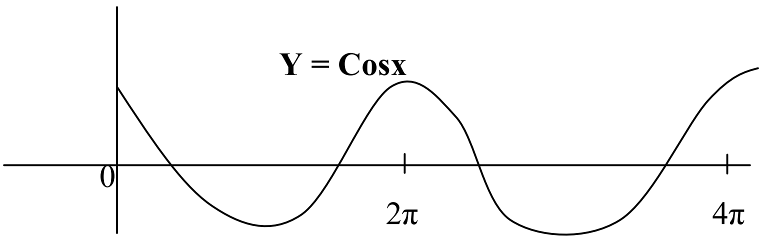


Fig. 9b

3. Monotonic Functions

A function is said to be monotonic if it is either increasing or decreasing within a given interval.

The study of monotonic function is an important concept in the application of calculus, this will be treated in the last two units of this course.

You will now consider explicit definitions of a monotonic increasing function and monotonic decreasing function within a given interval.

Definition l: A function $f(x)$ is said to be monotonic increasing in an interval.

If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
for any two points x1, $x_2 \in I$,
If $f(x_1) < f(x_2)$ then the function $f(x)$ is said to be strictly increasing.

Definition 2: A function $f(x)$ is said to be monotonic decreasing in an interval I

If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
for any two points x1, $x_2 \in I$

If $f(x_1) > f(x_2)$ then the function $f(x)$ is said to be strictly increasing.

Example

The function $y = x^2$ is monotonic decreasing in the interval $(-\infty, 0]$ and monotonic increasing in the interval $[0, \infty)$.
See fig. 10

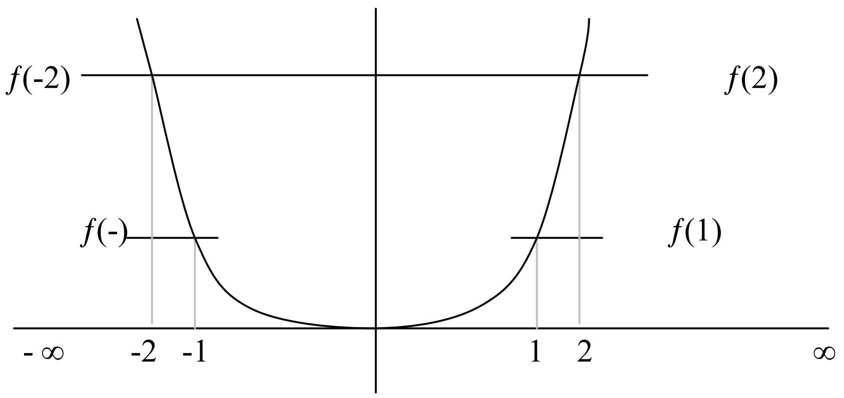


Fig. 10.
$-1, -2 \in (-\infty, o]$ and $-2 < -1$ but $f(-2) > f(-1)$
$1, 2 \in\in [o, \infty)$, $1<2$ and $f(1) < f(2)$

SELF ASSESSMENT EXERCISE 2

Determine whether the function f (x) = 2^x is monotonic increasing or decreasing in the interval I = (-∞, ∞).

Determine whether the following functions are monotonic increasing or decreasing in the interval (o, ∞):

i. f (x) = 2^x

ii. f (x) = 2^{-x}

iii. f (x) = 2^3

iv. f (x) = 2

4. Bounded Functions

Recall the definition of a bounded set defined in Unit 2. You will now use the same concept to define a bounded function. If a function f(x) assumed on a given interval I a value M which is greater than all other values (i.e.; f(x) < M for all x∈I) then the function f (x) is said to be bounded above. The M is called the greatest value of the function f(x) at that interval I. Similarly, if there is a constant M such that all other values of the functions is greater than (i.e.; f (x) >M for all x∈1) then we say that f (x) is bounded below and the value M is called the least value of the function f (x) in I.

Definition of a Bounded Function: A function f (x) is said to be bounded in an interval 1. If there exists a number k∈R such that

f(x) < K for all x∈ f(x) is bounded above

alternatively, if given M, f(x) > M in the interval, we say f(x) in bounded below

Example 1

1. The function f(x) = 2x+1 is bounded in the interval [-2, 2] i.e f(-2) < f(2) is bounded in the interval (-2, 2).

2. The function f(x) = $2x^2$-3x+2 is bounded in the interval x∈ [0,2]

SELF ASSESSMENT EXERCISE 3

Determine whether the following function are bounded in the given intervals.

i. $f(x) = x^2 - 4x + 4 \quad x \in (-\infty, \infty)$.

ii. $f(x) = x^2 - 4x + 4 \quad x \in (2, 10)$.

iii. $f(x) = 2 + x + x^2 \quad x \in (-1, 2)$.

3.2 Inverse Function

Domain and Range: since the domain and range will be useful in the study of inverse of a function you have to briefly review the concept as you have studied the fact that one of the ways a function can be determined is through the domain of the function i.e. the set containing the first variable for which a function makes sense. You shall consider some few examples of domain of a given function.

Example

i. Given the function
$f(x) = X^2$, x is a real number.

Here the domain of f is the set of all real numbers. The range is therefore $R^+ = [0, \infty)$. In symbols you write.

D= $\{x: x \in R\}$ and R = $[y;\ y \in R^+\}$.

ii. Given the function

$f(x) = x - 1$, x is a real number.

Here the domain of f is the set of all real numbers greater than 1. i.e.; D = $\{x: x > 1\}$ Since any other value of x will result to the square root of a negative number which does not make sense in the set of real numbers. The range R = $\{y : y \in R^+\}$

iii. Given the function

$f(x) = \frac{1}{X^2 - 1} = \frac{1}{(x-1)(x+1)}$ the domain

D = { x : x∈R, x ≠ - 1 or 1}. If x = -1 or 1 the value of the function will be meaningless.

SELF ASSESSMENT EXERCISE 4

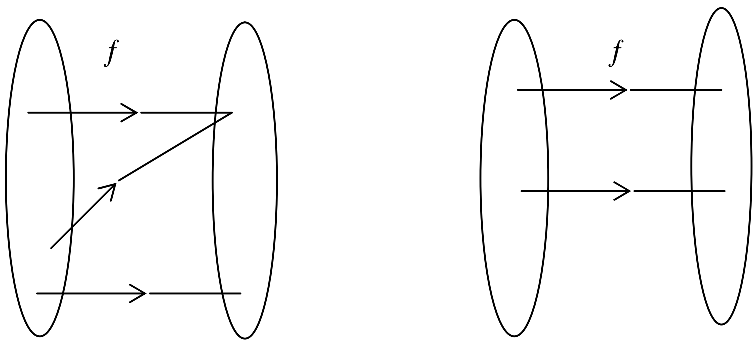
1. Find the range of example (iii) above.
2. Let the function f assign to each state in Nigeria its capital city. State the domain of f and its range.

You will continue the study in this section by giving definitions of certain features of functions. (there have been kept purposely for this moment.)

1. Onto Functions

Let the function y = *f* (x) with domain of definition X {i.e. the admissible set of values of x) and the range Y (the set of the corresponding values of y). Then a the function y = *f*(x) is an Onto function if to each point or element of set there corresponds a uniquely determined point (or element) of the set Y, i.e., if every point in set Y is the image of at least one point in set X.

Example: consider the function shown in fig 11



(A) (B)

Fig 11.

The function Fig. (A) is an Onto function. The function in Fig. (B) is not an onto function

Example: The function $f(x) = x^2$ is an Onto function

SELF ASSESSMENT EXERCISE 5

Give reason why the function in the Fig. (a) above is an onto function and the other one in Fig(b) is not.

2. One-to-One Function

Let the function y = f(x) be an onto function. If in addition each point (or element) of set X corresponds to one and only one point (or element) of set Y then the function y = f(x) is said to be one to one function.

Example

The function $y = x^2$ is an onto function and not a one to one function. Whereas the function $y = x^3$ is an onto function as well as a one to one function
(see fig 12)

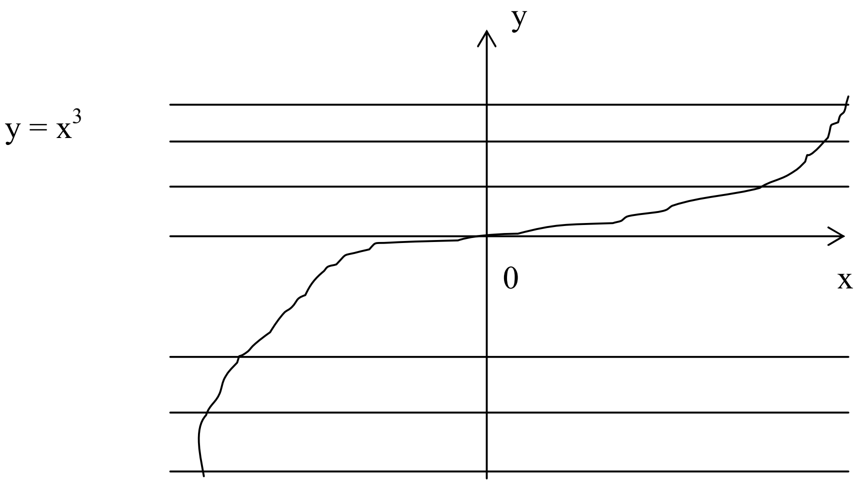


Fig 12. (a)

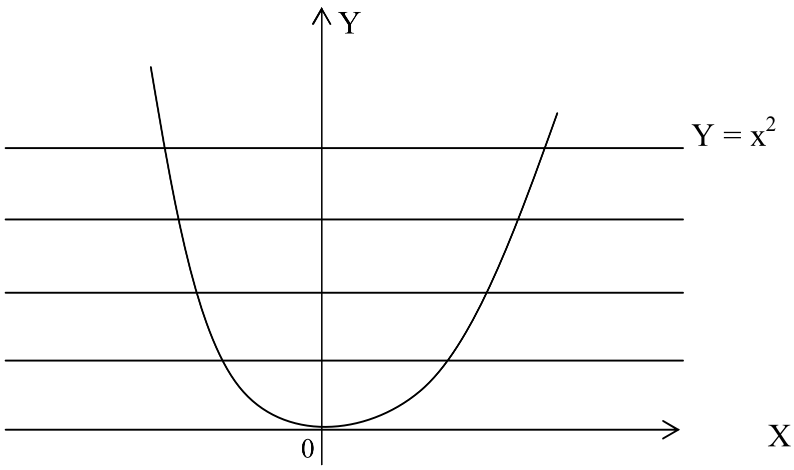


Fig. 12. (b)

In fig. 12 (a) no horizontal line intersect to the graph more than once thus the function.

$Y = x^3$ is one to one function.

In Fig. 12. (b) the horizontal lines intersects the graph in more than one point thus the

$f(x) = x^2$ is not a one to one function.

3.3 Composite Functions

Generally functions with a common domain can be added and subtracted. That is, if the functions f (x) and g(x) have the same domain. Then: $(f \pm g)(x) = f(x) \pm g(x)$

Example:

Let $f(x) = x^2$ and $g(x) = 3x - 2$
Then $f(x) + g(x) = x^2 + 3x - 2$

The above concept can be extended to the case of multiplication. i.e.; given that f (x) and g(x) have the same domain then

$fg(x) = f(x)\,g(x)$.

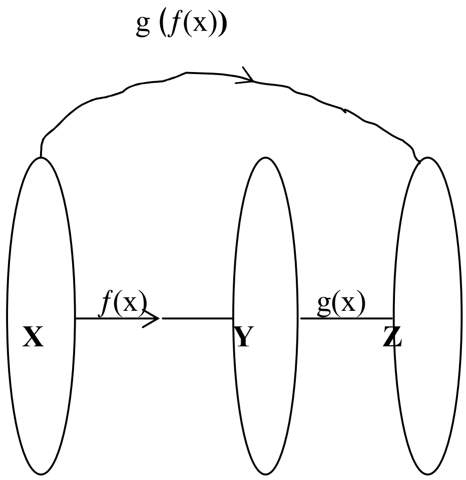
Using the above example we have that:
$f(x)\,g(x) = x^2(3x - 2) = 3x^3 - 2x^2$

Division is also allowed between functions having the same domain.

Let f(x) = 2x and g(x) = x – 1 Then; $\frac{f(x)}{g(x)} = \frac{2x}{x-1}$

There is another way function can be combined which is quite different from the ones described above. In this case two function f(x) and g(x) are combined by first finding the range of f(x) and making it the domain of g(x).

This idea is shown in fig 13.



The function you get by first applying f to x and then applying g to f(x) is given as g (f(x) and called the composition of g and f and is denoted by the symbol

go$f$ (which reads g circle f) i.e.; (go f) (x) = g (f(x))

Example

1. Given that f (x) = $\frac{1}{x}$ and g(x) = $x^2 + 1$

   $f$og= f (g(x)) = $\frac{1}{x^2+1}$

   go$f$= g o (f (x)) = $\frac{1}{x^2} + 1 = \frac{x^2+1}{x^2}$

2. Given that f(x) = x^2 and g(x) = x + 1

   go$f$ = g(f(x)) = $x^2 + 1$

   $f$og = f(g (x)) = $(x+1)^2 = x^2 + 2x + 1$

In the two examples above you can easily conclude that go$f$ ≠ $f$og
The composition of functions can be extended to three or more functions.

Example

Let f (x) = x - 1, g(x) = x^2 + 1, h(x) = 2x.

Then hogo f = h (go f) = h (g (f(x)))
=2 ($(x- 1)^2$ + 1) = $2x^2 - 4x + 4$

SELF ASSESSMENT EXERCISE 6

Give that f(x) = x, g(x) = x-1, h(x) = √(x -1)
Find the following composite functions.

1. $f$og
2. go$f$
3. ho$f$
4. hog
5. $f$ogoh

You will now use materials discussed above in this section to study and define the inverse for any given function. A function that will have an inverse must fulfill the function, since the inverse function is a unique function in respect of the original function.

Definition of Inverse. of. a Function: If a function y= f(x) is a one to one function, then there is one and only one function x = g(y) whose domain of definition is the range of the function y = f(x). such that;

$$f g (f(x)) = x \text{ and } g(x) = f^{-1}(x)$$

Examples

1. If given that f (x) = x^3 then f^{-1}(x) =3√ x
2. Use the above and illustrate the fact that f^{-1} o f = f^{-1} o f

Given that f^{-1} (x) = g (x) = $\sqrt[3]{x} = x^{1/3}$ And f(x) = x^3

$f^{-1} \circ f = g \circ f = g\,(f(x)) = (x^3)^{1/3} = x$
And $f \circ f^{-1} = f \circ g = f\,(g(x)) = ((x^{1/3})^3 = x$

Find the inverse of the following function.
1. $2x-4$
2. $6x-5$
3. $f(x) = x^5$
4. $2x^3 - 1$

Solutions:

1. Let $y = 2x - 4$
Then $y + 4 = 2x$
$\Rightarrow x = \frac{y+4}{2}$ (solving for x)

then $f^{-1}(x)\ \frac{x+4}{2}$ (interchanging x and y)

2. Let $y = 6x - 5$
Then $y + 5 = 6x$ (solving for x)
$x = \frac{y+5}{6}$
$f^{-1}(x) = \frac{x+5}{6}$ (interchanging x and y)

3. Let $y = x^5$

then $x = \sqrt[5]{y}$ (solving for x)

$f^{-1}(x) = \sqrt[5]{x}$ (interchanging x and y)

4. Let $y = 2x^3 - 1$
$y + 1 = 2x^3$
$\frac{y+1}{2} = x^3$
$x = \sqrt[3]{\frac{y+1}{2}}$ (solving for x)

$f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}}$ (interchanging x and y)

SELF ASSESSMENT EXERCISE 7

1. Show that $f^{-1} \circ f = f \circ f^{-1} = x$ in example 1 to 4 above.
2. Given the following functions
   a. $f(x) = 6x - 3$
   b. $f(x) = x^7$
   c. $f(x) = mx = b$
   d. $f(x) = \frac{1}{1-x}$
   e. $f(x) = \frac{1}{x^3 - 1}$
   d. $f(x) = \frac{1}{1+x}$
      i. State the domain of each function.
      ii. Derive the inverse of each function.

4.0 CONCLUSION

In this unit, you have studied characteristics of functions, you have used graphs to represent functions and identity some characteristics exhibited by these functions. You have studied how to form a new function by combining two or more functions.

Furthermore, you have studied how to determine whether a function has an inverse or not.

5.0 SUMMARY

In this unit you have:

Defined a function

- Discussed various types of functions
- Use graphs to describe the characteristics of functions such as periodic, monotonic, one to one onto and transcendental functions.
- Defined domain and range of a function
- Formed new functions by combining two or more functions - composition of functions.
- Discussed the inverse of a one to one function.

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR-MARKED ASSIGNMENT

1. Give a precise definition of the following unit examples
    a. domain of a function
    b. inverse of a function
    c. composition of functions
    d. bounced function
    e. an even function
    f. a periodic function
    g. a monotonic decreasing function in an interval
    h. maximum value of a function is an interval.

2. Given the following functions.

    a. $f(x) = \frac{2x}{x-5}$

    b. $f(x) = \frac{1}{x^3 - 1}$

    c. $f(x) = 27x^3 - 2$

    d. $f(x) = \frac{x}{(x-1)(x+2)}$

    1. State the domain of definition for each function

    2. Find the inverse of each function if it exists.

3 Given the following function $f(x) = x^2$, $g(x) = 2x-1$, $h(x) = x+1$
Find the:
a. fg
b. f/g
c. fog
d. $fogoh$
e. $(g-h)\ of$

UNIT 4 LIMITS

CONTENTS

1.0 INTRODUCTION

In the last units, you have been adequately introduced to the concept of a function In this unit you will be indorsed to the concept of the limit of a function. This is one of the most important concept in the study of this course calculus. Generally, it is believed that calculus begins with the idea of a limitations process. The history behind the study of limits of function is an interesting one and it will be nice if you hear some of the story.

> A French mathematician by name Joseph Lious Lag range (1736 - 1813) was among the first mathematicians that initiated the concept of limits. This was followed by another French mathematician Augustine-Lious Cauchy (1789-1837) and a Czech priest by name Bernhard Bolzono (1781-1848).

However the present day definition of limit is largely due to the work of Heinrich Edward Heine and Karl Weiestrass. In this unit an attempt to be a bit expansive in the study of the limit of functions will be made. Therefore you should be more patient when studying the materials of this unit. Bear in mind that discussions on the concept of limit of a function will easily be re-introduced into the concept of continuity of function in unit 4 and 5.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a limit of a function
- show that the limit of a function is unique
- evaluate the limit of a function
- to evaluate the right and left hand of a function.
- use the "∈, δ" method to prove that a number ℓ is the limit of a function at a given point.

3.0 MAIN CONTENT

3.1 Definition of Limit of a Function

In this Section; you will begin the; concept of limit shall be studied by first presenting it in an informal and intuitive manner. You are familiar with the word "limit". It gives you the picture of a restriction or boundary. For example consider a regular polygon with n sides inscribed in a circle. As you increase the sides of the regular polygon then each side of the n-side regular polygon gets closer to the circumference of the circle. Here if you consider sides of the polygon as the independent variable denoted by n and the shape of the regular polygon as the dependent variable then the shape of the n-sided regular polygon approaches the shape of the circle as n approaches infinity (see fig 13)

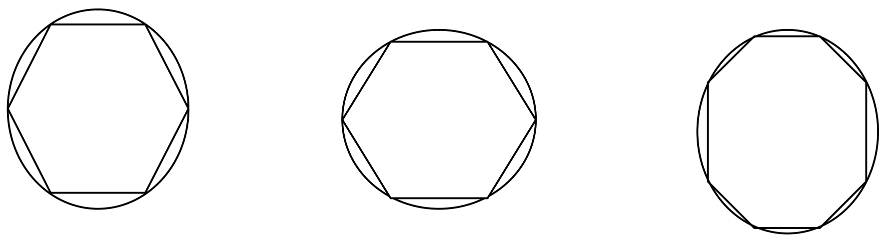


Fig. 13.

In this case we say that the limit of inscribed n-sided regular polygon is the circle as n tends to ∞.

Now consider the function $f(x) = x^2 - 1$ what is the value of $f(x)$ when x is near 1 ? In table 2.

X	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	1.	.99	.96	.91	.84	.75	.64	.51	.36	0.19	0

Table 1

X	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
f (x)	0	.21	.44	.69	.96	1.25	1.56	1.89	2.24	2.61	3

Table 2

You can see that as x gets closer and closer to 1, *f*(x) approaches 0. So the value of *f*(x) can be made to get closer to 0 by making x get closer to 1. This is expressed by saying that as x tends to l, the limit of *f*(x) is 0.

Another way is to start by noting that a function *f*(x) could be observed to approach a given number L as x approaches a known value x_o. That is once a number ℓ is identified as x approaches x, without insisting that *f*(x) must be defined at xo then we can write that

Lim *f*(x) = ℓ

In other words we say that the limit of the function *f*(x) as x approaches or tends to x,, is the number ℓ or as x tends to xo , *f*(x) tends to ℓ or for x approximately equal to xo *f*(x) is approximately equal to ℓ

In the above definition you will observed that there are two important things to note namely.

1. the existence of the unique number ℓ , and
2. the fact that the function need not be defined at the point x_o.

What is more important is that the function is defined near the number x_o

Consider the following example that will further explain the concept of limit.

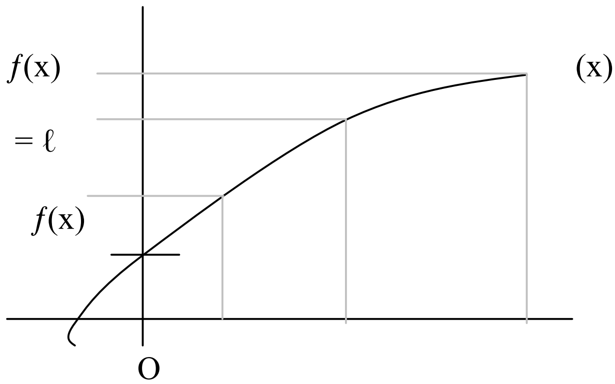


Fig. 14.

In fig(14), the curve represent the graph of *f* (x) . The number xo appears in the x-axis, the limit ℓ appears in the y-axis. As x approaches x_o from either side (i.e.; along the x-axis). *f* (x) approached ℓ along the y-axis.

Examples: Find the limit of the functions as x$\longrightarrow$1

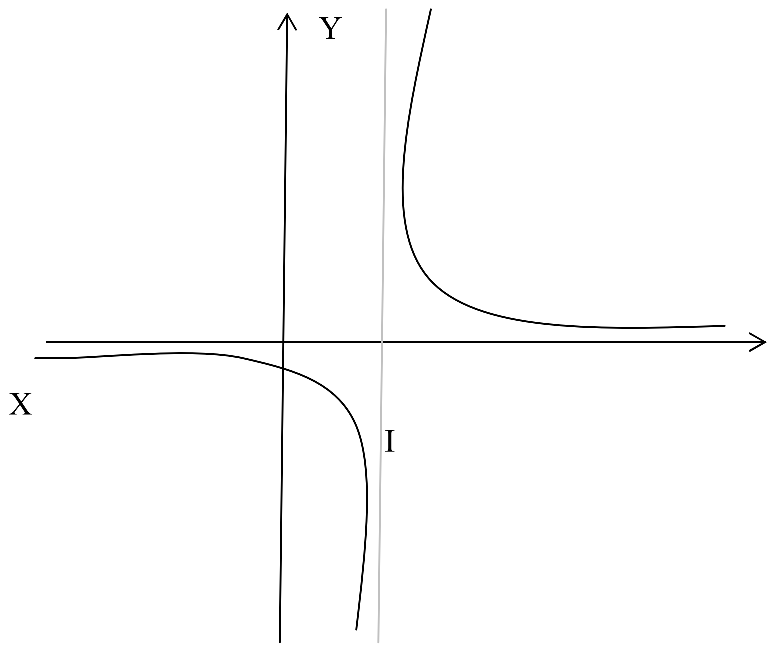
1. $f(x) = \dfrac{1}{x-1}$

2. $f(x) = \dfrac{1}{|x-1|}$

3. $f(x) = \dfrac{x^2-1}{x-1}$

Solutions:

1. $f(x) = \dfrac{1}{x-1}$ in the graph of $f(x)$ as x approaches 1. (see graph below)



From the right $f(x)$ becomes arbitrary large. Larger than any pre-assigned positive number. As x approaches curve from the left $f(x)$ becomes arbitrarily large negative-less than any pre-assigned negative number. In this case $f(x)$ cannot be said to approach any fixed number. The above gives a clear picture where the limit of function does not exist as x approaches a given point for a fuller understanding, you will consider two more examples.

1. $f(x) = \frac{1}{|x-1|}$ (see fig. 16)

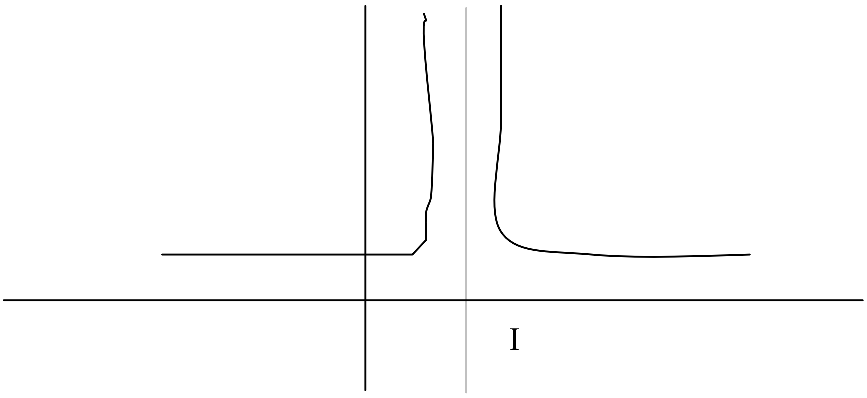


Fig. 16.

In fig. 16, as x approaches 1 from the left and the right 2. f (x) becomes arbitrary large. In this case 2. f(x) becoming arbitrary large cannot approach any fixed number ℓ .

Therefore $f(x) = \frac{1}{|x-1|}$ does not have a limit as x tends to 1.

2. $f(x) = \frac{x^2-1}{x-1}$ (see table A & B below)

X	0	.1	.2	.5	.4	.5	.6	.7	.8	.9	1
f(x)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2

Table A

X	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
f(x)	0	.21	.44	.69	.96	1.25	1.56	1.89	2.24	2.61	3

Table B

At a first glance f(x) is not defined at the point x = 1, since division by zero is impossible. Recall that in finding the limit of function at a given point xo it is not required that f(x) must be defined at xo. The above is a clear example of functions having limits at points where they are not defined. You will meet other examples of such function as you progress in this course.

In tables A & B the limit of the function as x tends to 1 is 2.
By direct evaluation you can simplify:

$$f(\text{x}) = \frac{x^2 - 1}{x - 1} \quad \text{as}$$

$$f(\text{x}) = \frac{(x-1)(x+1)}{x-1} = \text{x} + 1$$

Therefore lim. as $x \to 1$, f(x) = x + 1 is 1 + 1 = 2.

SELF ASSESSMENT EXERCISE 1

1. $f(\text{x}) = \dfrac{x^2 - 4}{x - 1}$ 2. $f(\text{x}) = \dfrac{1}{x - 2}$

Determine whether the function above have limits as x approaches 2. If so, find the limits.

3.3 Properties of a Limits of a Function

A formal definition of limit is hereby given.

Definition : The number ℓ is said to be the limit of the function y = f(x) as x tends to x_o if for any positive number $\in > o$ (however small) we can find some positive number δ such that:

$$|f\text{x} - \ell| < \in \text{ whenever } o < |\text{x} - \text{x}_o| < \delta$$

Using the above definition it can be shown that the limit of the function f(x) = 3x -1 is equal to 2 as x tends to 1.

To prove the above insufficient to show that for $\in > o$ you can find $\delta > o$ such that the inequalities:

$$|(3\text{x} - 1) - 2| < \in \Rightarrow o < |\text{x} - 1| < \delta \text{ is satisfied equivalently.}$$

$$|3\text{x} - 1 - 2| = |3\text{x} - 3| = 3|\text{x} - 1| < \in$$

$$\Rightarrow |\text{x} - 1| < \in/3$$

Since x must be near 1 as much as possible we chose $\delta = \in/3$. Hence:
x -1 < $\delta = \in/3$, which is the required proof.

Remark: The definition above implies that the distance between *f* (x) and L must be small as much as the distance between x and x_o is. Recall that the absolute value of a number 1.1 is a distance function (see unit 1) The method of the proof used above is called the "∈, δ proof" in this course you will get a better view about the definition if you go through another example when the graph of the function is shown with the limit indicated

Examples: show that the limit of *f* (x) = x^2 as x approaches (tends to) 2, is ℓ=4 (Use the ∈, δ proof)

Solution:

In finding the solution to the above you have to show this: For any positive number ∈ > o you look for another positive number δ > o such that the inequalities

$$|x^2 - 4| < \in \quad \text{whenever} \quad o < |x - 2| < \delta \quad \text{is satisfied}$$

Note that:

$$|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2| \; |x + 2|$$

is the product of a factor |x +2| that is near 4 and a factor |x - 2 | that is near 0 when x is near 2. If x is required to stay within, say, 0.2 of 2 then you will have a situation like this:

$$2 - 0.2 < x < 0.2 + 2$$

$$= 1.8 \; x < 2.2 \text{ and}$$

$$3.8 < x + 2 < 4.2.$$

As a result

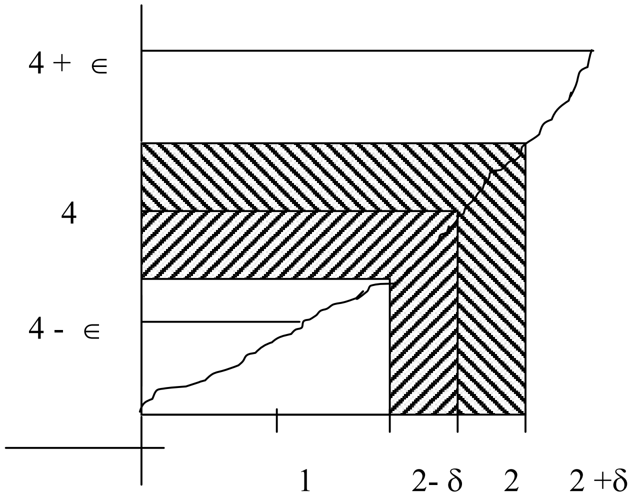
$$|x^2 - 4| \quad 4.2|x - 2|$$

Now 4.2|x -2| < ∈ proved |x-2 | < ∈/4.2

Therefore you could choose δ, as the min {0.2 ∈/4.2 } if you do then you will have that:

$$|x^2 - 4| < \in \text{ when } 0 < |x - 2| < \delta$$

See Fig. 17.



3.4 Right And Left Hand Limits

A function f (x): could have one limit as x approaches x_o from the right and another limit as x approaches xo from the left. Recall that in the above definitions of limits of function the word "arbitrarily close" was loosely used, to describe the approach of x to xo without indicating how x should approach xo how x should approach x_o. If x approaches xo from the right-hand side:

i.e., for values of $x > x_o$ you write that: $x - 3 \rightarrow x_0$

and for values of $x < x_o$ you write $x \rightarrow x_0$

and say that x approached x_o from the left hand side.

Definition: If lim $f(x) = \ell$ and $x \rightarrow x_0$

where $\ell +$ is called the right hand limit of the function f(x) and L^- is the left hand limit of the same function f(x) .

Remark: If the limit of a function exists as $x \rightarrow x_0$ then

$$\lim_{x \rightarrow x_0^+} f(x) = \lim_{X \rightarrow X_0^+} f(X)$$

Example: Investigate the limit of the function defined by

$$f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \quad \text{as x approaches 0.}$$

Solution

From the above f(x) = -1 for x < 0

$\Rightarrow$ $x \to 0_0$ $f(x)$ = -I and

$x \to 0_\infty$ $f(x)$ =1

Thus lim $f(x)$ does not exit.

Hence lim $f(x)$ does not exist $x \to 0_0$

An interesting function you would not like to miss when dealing with one-sided limits is the greatest - integer function defined as;

[x] = greatest integer than x_0

Example: Investigate the limit of the function F defined by:

f (x) = [x] as x approaches x_0

(see Fig. 18)

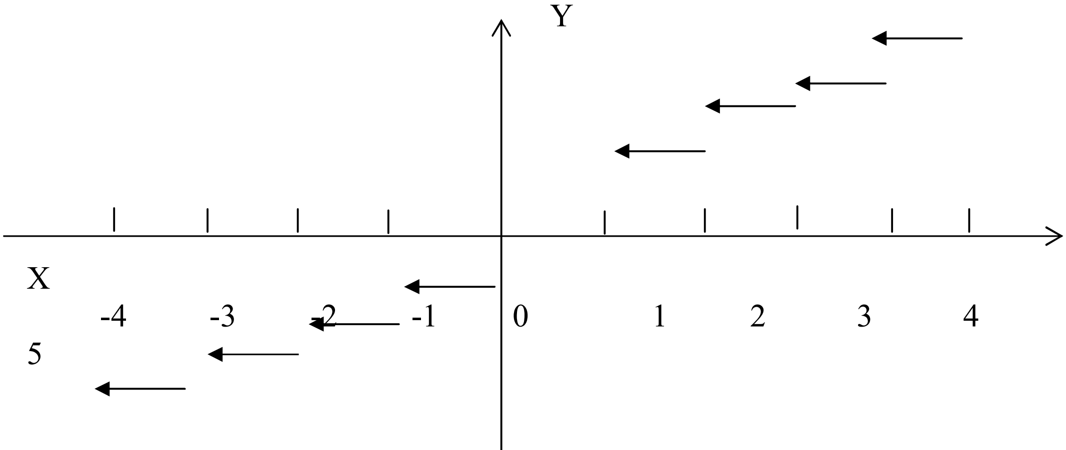


Fig. 16.

In fig 16, the function is 0 at 0 and remain 0 to 1 jumps to 1 remain 4 throughout the interval [1,2). At 2 the function jumps to 2 remain 2 in the interval [2,3) at 3 jumps to 3 and remains 3 in the interval [3,4) and so on.

To investigate the limit we take values less than 3 and values greater 3.

$f(x) = [x] = 3 \quad \forall x \in [3, 4)$

$f(x) = [x] = 2 \quad \forall x \in [2, 3)$

Therefore:

$\text{Lim} f(x) = 2 \quad x \to 3.$

$\text{Lim} f(x) = 3 \quad x \to 3^+.$

Since $\lim_{x\to 3} f(x) \neq \lim_{X\to 3^+} f(x)$ Then the $\lim_{x\to 3} f(x)$ does not exist

Generally for the greatest -integer function:
$\lim_{x\to x_0^+}[x] = x_0$ and $\lim_{x\to x_0^+}[X] = X_0 - 1$

Example: $g(x) \begin{cases} x^2, & x > 0 \\ \sqrt{x}, & x < 0 \end{cases}$ investigate the limit as $x \to 0$

Solution

$\lim_{x \to x_0^+} g(x) = 0$ and $\lim_{x \to x_0^+} g(x) = 0$, hence $\lim_{x \to x_0} g(x) = 0$

You shall now look at one of the most important properties of the limit of a function. This is the uniqueness property.
Uniqueness: If the limit of a function $f(x)$ exists as x approaches x_o it is unique.

The above property is a theorem which you will be required to give the proof.

Example: Proof that if the limit of a function f (x) as x approaches x_0 exists it is unique.

Proof:
Let $\lim_{x \to x_0} f(x) = \ell_1$

Another one for the same function $f(x)$ be given as:

$\lim f(x) = L_2$

You will be required to show that:

$L_2 = \ell_1$ by providing that the assumption $L_2 \neq L_1$ leads to absurd result that: $| L_2 - L_1 | = | L_2 - L_2 |$

By definition of limit: for any positive

$\in_1 > 0$ there is $\delta_1 > 0$

$| f(x) - L_1 | < \in_1$

when $0 < | x - x_o | < \delta_1$

and for $\in_2 > 0$ there is $\delta_2 > 0$

such that $| f(x) - L_1 | < \in_2$

when $0 < | x - x_o | < \delta_2$

Let $0 < | \ell_1 - \ell_o | = |\ell_2 - f(x) + f(x) - \ell L_2|$
$| \ell_2 - f(x)| + | f(x) - \ell L_2 | < \in_1 + \in_2)$
(By triangle inequalities) by definition above

$\in_1 = \frac{1}{2} | \ell_1 - \ell_2 |$ and $\in_2 = \frac{1}{2} | \ell_1 - \ell_2 |$

Then $\frac{1}{2} | \ell_1 - \ell_2 | < \in_1 + \in_2$

$= \frac{1}{2}| \ell_1 - \ell_2 | + \frac{1}{2} | \ell_1 - \ell_2 | = | L_1 - L_2 |$

$| L_1 - L_2 | < | \ell_1 - \ell_2 |$ which is absurd or contradictory. Hence he assumption that $\ell_1 \neq \ell_2$ is false. Therefore $\ell_1 = \ell_2$ which is the required result.

4.0 CONCLUSION

You have studied the informal and formal definitions of the limit of a function, which is a major starting point for the study of the subject called calculus. You have studied the important properties like uniqueness of the limit of a function. You have used the δ and $\in$ method to prove that a given number ℓ is the limit of a function as $x \longrightarrow x_o$ for a function $f(x)$.

5.0 SUMMARY

In this unit you have studied how to

1. State an informal definition of the limit of a function $f(x)$ as x tends to x_o

2. State the formal definition of the limit ℓ of a function $f(x)$. as $x \longrightarrow x_o$ using the δ and ∈ symbols. i.e.; If $|x - x - 1| < \delta > 0$ then $|f(x), \ell| < \in > 0$

3. To show that if $\lim_{x \to x_0} f(x) = \ell$ exists then ℓ is unique.

4. To determine whether the number ℓ is the limit of a function $f(x)$ as $x \longrightarrow x_o$

5. The left hand and right hand limits and thus
   The $\lim f(x) = x_o$ If $\lim f(x) = x_o$ if $f(x) =$ $x \longrightarrow x_o$

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR-MARKED ASSIGNMENT

1. Define a limit of a function
2. Show that the limit of a function is unique
3. Evaluate the limit of the following:
   (a) limit $f(x)=x^2+3x-6$,as $x \to 2$
   (b) limit$f(x)=3x^5+7xe^x-56$,as $x \to -3$

UNIT 5 ALGEBRA OF LIMITS

CONTENTS

1.0 INTRODUCTION

You have studied properties of a limit of a function in the previous unit. In this unit you will conclude the study of limit of a function with the following; Algebra of Limits i.e.; Sum and Difference of Limits as well as Products and Quotient of Limits. This has a direct link to the rules of differentiation that will be studied in unit 7 and 8.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the theorem on limits sum, product and quotients theorem
- evaluate limits of functions using the Sum, Product and Quotient theorem s on limits of a function.
- evaluate limits of function as $x \rightarrow \infty$ and $x \rightarrow -\infty$

3.0 MAIN CONTENT

3.1 Sum of Limits

In the last section we applied the "δ proof" to prove a more general cases involving the algebra of limits.

You will begin by considering the following theorems on limits of functions

Theorem 1: If $\lim_{x \to x_0} f(x) = f$ and $\lim_{x \to x_0} g(x) = g$ then

1. $\lim_{x \to x_0} [f(x) + g(x)] = f + g$
2. $\lim_{x \to x_0} [xf(x) = xf$

The proof of the (1) of the theorem will follow the pattern used in proving the uniqueness property.

Proof. Let $\in > 0$. To prove (1) above you must show that you can find $\delta > 0$ such that:

If $0 < |x - x_o| < \delta$ then $|[f(x) + g(x) – (f + g)|$

Note that:

$|f(x) + g(x) – (f + g)| = |\{f(x) - f) + (g(x) – g)| \leq$

$|f(x) - f + (g(x) – g)|$ (by triangle inequality)

You will make $|f(x) - g(x) – (f + g)|$ less than $\in$ by making

$|f(x)- f|$ and $|g(x)- g|$ each less than ½ $\in>0$

Since $\in> 0$ this implies that ½ $\in> 0$

Since by the statement of the theorem

$\lim_{x \to x_0} f(x) – f$ and $\lim_{x \to x_0} g(x) – g$

therefore there will exist two number $\delta_1>0$ and $\delta_2>0$ such that

If $0 < |x - x_o| < \delta_1$ then $|f(x)- f| <$ ½ $\in$

and If $0 < |x - x_o| < \delta_2$ then $|f(x) - f| <$ ½ $\in$

and $|g(x)- g| >$ ½ $\in$. Now set δf = minimum of δ_1 and δ_2

Therefore $|f(x) - g(x) – (f + g)| = |f(x)- f| + |g(x)- g|$
$< \in >$ ½ $\in + >$ ½ $\in$

Thus it is shown that if:
If $0 < |x - x_o| < \delta_2$ $|f(x) + g(x) - (f+g)| < \in$

Which is the required proof.

2. To prove that
$\lim_{x \to x_0} kf(x) = kf$

Let $\in > 0$. You must find $\delta > 0$ such that:

If 0<1 $|x\text{-}xo| < s$ then $|k f(x) = k f| < s$

There are two cases to consider:

1. when $k = 0$
2. $k \neq 0$

If k =0 then $|0 - 0| < \in$ when $0 < |x - x_o| < \delta$

From the above any value for $\delta > 0$ will do.

To prove the case $k \neq 0$.
Since $\lim_{x \to x_0} f(x) = f$
then there is $\delta > 0$ such that
If $0 < |x - x_o| < \delta$ then $|f(x) - f| < |k| - \in$

From the last inequalities you have that

$|k| - |f(x) - f| < \in$
$\Rightarrow$ $|k f(x) - fk| < \in$ which is the required proof.

SELF ASSESSMENT EXERCISE 1

If $\lim_{x \to x_0} f(x) = f$ and $\lim_{x \to x_0} g(x) = g$

Show that:
$\lim_{x \to x_0} f(x) - g(x) = f - g$

The result of the last two theorems can be extended to any finite number of function.

Example: If $\lim_{x \to x_0} f_1(x) = f_1$ $\lim_{x \to x_0} f_2(x) = f_2$

$\lim_{x \to x_0} f_n(x) = f_n$ then

$$\lim_{x \to x_0} (k_1, f_1(x) + k_2 f_2(x) + ... + k_n f_n (x)) = k_1 f_1 + ... + k_n f_n$$

3.3 Products and Quotients of Limits

You shall now consider further theorems on limits (the t proof are beyond the scope of this course).

Theorem2: If $\lim_{x \to x_0} f(x) = f$ and $\lim_{x \to x_0} g(x) = g$

Then (I) $\lim_{x \to x_0} f(x) = f$ and $\lim_{x \to x_0} g(x) = g$

(II) $\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{f}$ $f(x) \neq 0, \quad f \neq 0$

(III) $\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{g}$ $g(x) \neq 0, \quad g \neq 0$

Theorem3. Let $f(x)$, g(x) and h(x) be functions defined on an interval I containing (a, o), except possible the functions are not necessary defined at x=0, such that:
$f(x)=g(x)=h(x)$ for all $x \in I$ and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell$
then $= \lim_{x \to x_0} g(x) = \ell$

From the above theorems it easy to conclude that every polynomial

$f(x) = a_n x^n + a_1 x^n + a_1 x^{n-1} + ... + a_1 + a_o$

satisfies $\lim_{x \to x_0} f(x) = f(x_o)$.

Examples

Evaluate the following, limits:

i. $\lim_{x \to 2} (2x^2 - 5x + 1) = 2(2)^2 - 5(2) + 1 = 8 - 10 + 1 = -1$

ii. $\lim_{x \to 0} (3x^5 - 6x^4 - 3x^2 + x + 10) = 10$

iii. $\lim_{x \to 1} (x^5 - x^3 - 4x^2 + x + 1) = (1)^5 - (1)^3 - 4(1)^2 + |1| = -3$

As consequence of theorem 2 you can see that if P and Q are two polynomials and Q $(x_o) \neq 0$ then $\lim_{x \to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}$ if $Q(x_o)=0$

then $\lim_{x \to x_0} \frac{P(x)}{Q(x)}$ does not exist.

Examples

Find the limits of the following functions.

i. $\lim_{x \to 2} \frac{2x - 1}{x^2 - 3} = \frac{4 - 1}{4 - 3} = \frac{3}{1} = 3$

ii. $\lim_{x \to 1} \frac{x^2 + x + 1}{x^2 + 2x} = \frac{1 + 1 + 1}{1 + 2} = \frac{3}{3} = 1$

iii $\lim_{x \to 2} \frac{2 - x}{4x} = \frac{2 - 4}{8} = \frac{-2}{8} = \frac{-1}{4}$

iv. $\lim_{x \to 1} (3x - 6x + 1) = 3 - 6 + 1 = -2$

3.5 Infinite Limit

In this section you will be studying about functions whose limits tends to infinity as x approaches a given number.

If a function $f(x)$ increases or decreases without bound as x tends to certain point xo we say that $f(x)$ diverges. That is for a function $f(x)$ if corresponding to every number $K \in R$ there is $\delta > 0$ such that

If $o < |x - x_o| < \delta$ then $f(x) > k$

Then f (x) is said to approach + ∞ as x tend x_o in symbols you write it as $\lim_{x \to x_0} f(x) = +\infty$

It is possible to have a situation whereby point ℓ< ∞ as x increases or decreases without bound. In other words a function f(x) is said to tend to ℓ as x tends to +∞ as if to each there is a number k ∈ R such that:

If x > k then $|f(x) - \ell| < \in$

symbolically this could be represented as: $\lim_{x \to x_0} f(x) = \ell$

In a similar manner f(x) is said to tend to L as x tend to - ∞ , if each ∈>0 there is K∈R such that x <k ⇒ $|f(x) - L| < \in$

In symbols you write $\lim_{x \to x_0} f(x) = \ell$

Example

Take a look at the graph of the function $\frac{1}{f(x)} = x$, x > 0

(see fig.(3.3))

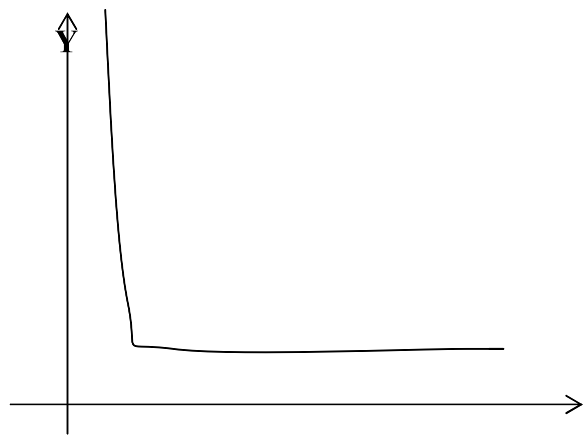


Fig. 17.

The function is a decreasing function. As x gets longer and larger f(x) gets smaller and smaller. This suggest that $\lim_{x \to a} \frac{1}{x} = 0$

also as gets smaller and smaller the function f(x) gets bigger and bigger the value 0 ~(x) takes arbitrary large value. In this case

$$\lim_{x \to a} \frac{1}{x} = +\infty$$

SELF ASSESSMENT EXERCISE 2

Draw the graph of $f(x) = \frac{1}{x}$, $x > 0$ investigate the limit as

i. x tends to 0.

ii. x tends to $-\infty$

Finally it could be possible that $f(x)$ increases or decreases without bound just as x also increases or decreases i.e. given an arbitrary number k, there exists $k_2 \in R$ such that $x > k, \Rightarrow f(x) > k_2$. In that case you write symbolically:

$$\lim_{x \to a} f(x) = \infty$$

For the case $x<k_l$ and $f(x) > k_2$ you write $\lim_{x \to a} f(x) = -\infty$ and for the case of $x < k_l$ and $f(x) < k_2$ you have $\lim_{x \to -\infty} f(x) = \infty$

For each of the following function.

Given the following function.

1 $f(x) = \frac{1}{x - 1}$

2 $f(x) = \frac{1}{x^2 - 1}$

3 $f(x) = \frac{1}{x^2 - 4}$

Find the limits as (1) $x \longrightarrow 1^+$, (ii) $x \longrightarrow -\infty$ (iii) $x \longrightarrow +\infty$

Sketch the graph in each case. Consider the following functions and their graphs and

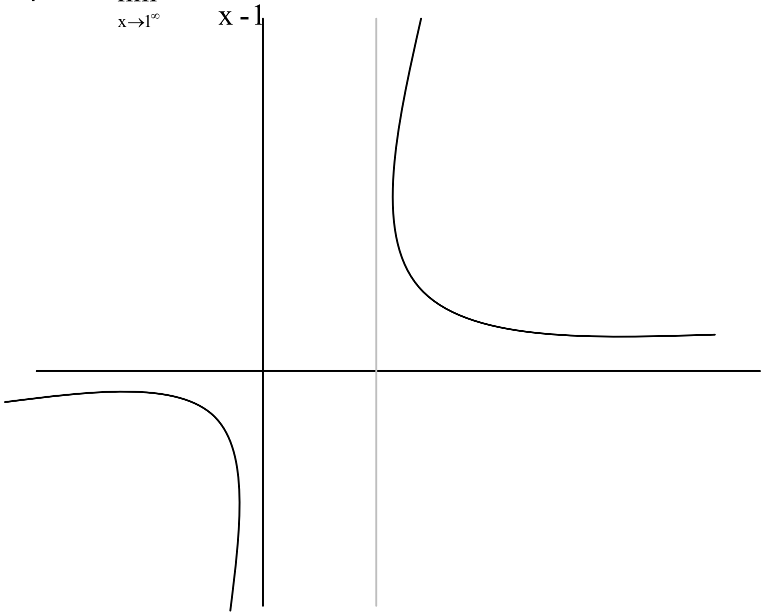
Limits of each:

1 $\lim_{x \to 1^+} = \frac{1}{x - 1} = +\infty$

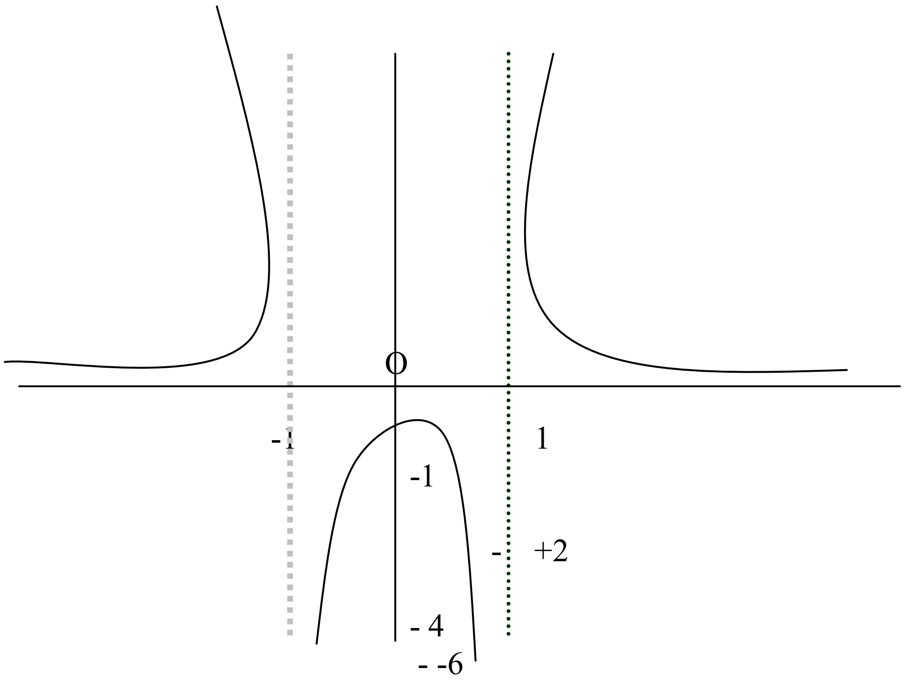
2 $\lim\limits_{x\to 1^-} = \dfrac{1}{x-1} = -\infty$

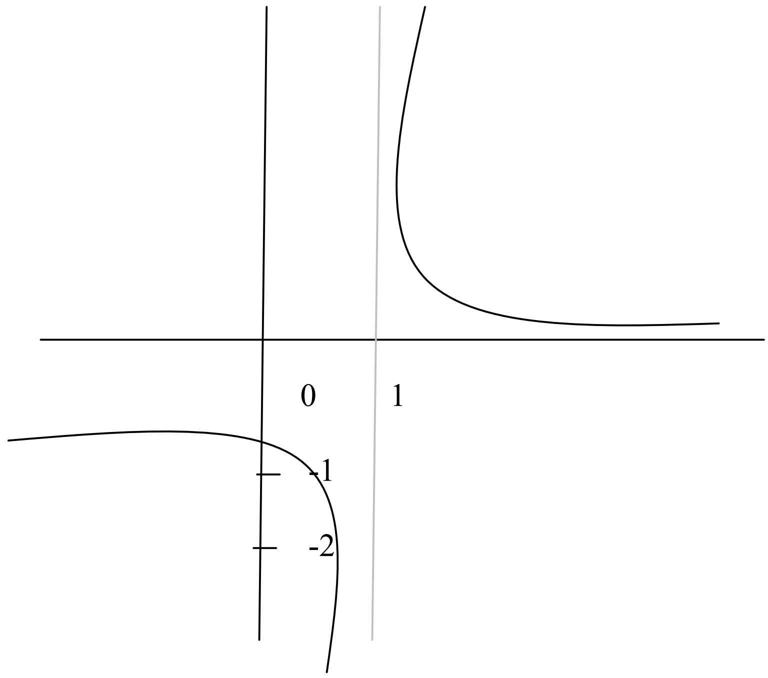
3 $\lim\limits_{x\to 1^\infty} = \dfrac{1}{x-1} = 0$

4 $\lim\limits_{x\to 1^\infty} = \dfrac{1}{x-1}$



$f(x) = \dfrac{1}{x^2 - 1}.$





a (I) $\lim_{x \to 1^+} f(x) = +\infty$ (II) $\lim_{x \to \infty} f(x) = 0$

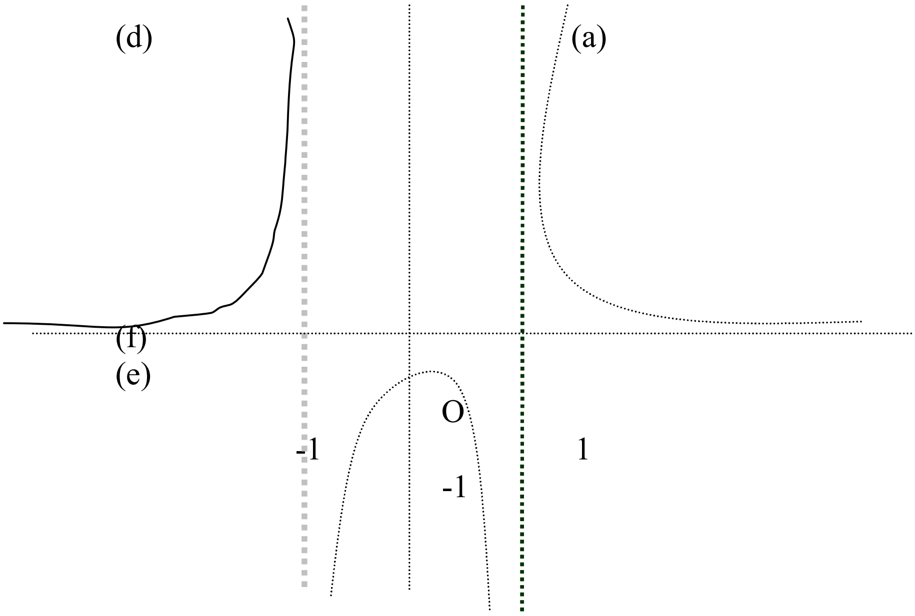
(III) $\lim_{x \to 1^2} f(x) = -\infty$ (IV) $\lim_{x \to \infty} f(x) = 0$

(ii) Investigate the limits of the function

$$f(x) = \frac{1}{x^2 - 1}$$

(I) $x \to 1^+$ (II) $x \to 1^-$ (III) $x \to -1^+$

(IV) $x \to -1^-$ (V) $x \to -\infty$ (VI) $x \to +\infty$



a (a) $\lim_{x \to 1^+} f(x) = +\infty$ (b) $\lim_{x \to 1^-} f(x) = -\infty$

(c) $\lim_{x \to -1^+} f(x) = +\infty$ (d) $\lim_{x \to -1^-} f(x) = 0$

(e) $\lim_{x \to \infty} f(x) = +\infty$ (d) $\lim_{x \to -\infty} f(x) = 0$

From the above you can easily see that: $\lim_{x \to 1} f(x)$ does not exist.

Since the left hand limit $\neq$ Right hand limit

i.e.: $\lim_{x \to 1^+} f(x) = +\infty \neq -\infty = \lim_{x \to 1^-} f(x)$

Also , $\lim_{x \to -1} f(x)$ does not exist

Because $\lim_{x \to 1^+} f(x) = -, \infty \neq -\infty = \lim_{x \to -1^-} f(x)$

4.0 CONCLUSION

You have seen how arithmetic operation on limits is used in evaluating limits of various functions especially polynomials. You have seen how the graph of a rational function could aid in evaluating infinite limits. You will see how the limiting process that we have studied in this unit will continue to serve as a reference point in subsequent units of this course

5.0 SUMMARY

In this unit you have studied:

1. That the limit of the sum of a finite number of functions is equal to the sum of their limits.

   i.e.; $\lim\limits_{x \to x_0} d_1, f_1, (x) + ... + \lim\limits_{x \to x_0} d_n f_n(x) = \lim\limits_{x \to x_0} ef_1(x) + ... + d_n f,)$

2. The limit of the product of a finite number of the product of their limits.

   i.e.; $\lim\limits_{x \to x_0} f_1 (x) .... \lim\limits_{x \to x_0} f_n (x) = \lim\limits_{x \to x_0} (f_1 (x) f_2 (x) .... f_n (x)$

3. The limit of the quotient of two functions is equal to the quotient of their limits

   $$\lim \frac{f(x)\ x \to x_o}{\lim g(x)\ x \to x_o} \qquad \lim \frac{f(x)\ x \to x_o}{\lim g(x) x \to x_o}$$

6.0 REFERENCES/FURTHER READING

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Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

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7.0 TUTOR-MARKED ASSIGNMENTS

Give a precise definition of the following with suitable example where necessary:

1. The limit of a function $f(x)$ as x tends to x_o.
2. The right-hand limit of a function as x tends x_o
3. The limit of a function $f(x)$ as (a) $x \longrightarrow -\infty$ (b) $x \longrightarrow +\infty$
4. State the definition of the left and right hand limits. Hence give examples of functions y- $f(x)$ possessing limits as $x \longrightarrow x_o^{+}$ and $x \longrightarrow x^{o-}$ and having no limits as $x \longrightarrow x_o$
5. Find the limits that exists:

   Limits of each:

   (i) $\lim_{x\to 1^+} = \dfrac{x^2}{x+1}$ (ii) $\lim_{x\to 1^-} = \dfrac{x^2}{x+1}$

   (iii) $\lim_{x\to 2} = \dfrac{x^2+2x}{x-1}$ (iv) $\lim_{x\to 1} = \dfrac{x^2+2x+-1}{3x^3+x+1}$

   (v) $\lim_{x\to -1} = \dfrac{x+x^2}{x+1}$ (vi) $\lim_{x\to 5} = \dfrac{2x-10}{x^2-8x^2+17x-10}$

6. Sketch the graph

   $f(x) = \dfrac{x}{x^2-9}$

   Hence find the limits that exist as

   (i) $x \longrightarrow -3^{+}$ (ii) $x \longrightarrow -3^{-}$

   (iii) $x \longrightarrow -3$ (iv) $x \longrightarrow 3^{-}$

   (v) $x \longrightarrow 3^{+}$ (vi) $x \longrightarrow -\infty$