MODULE 4

UNIT 1 CURVE SKETCHING

CONTENTS

1.0 INTRODUCTION

Most polynomial and some rational functions could be sketch with the knowledge of the signs of the first derivative dx/dy and the second derivative d^2y/d^2xThe signs of the first derivative can give an idea of the behaviour of the curve within a given interval. The second derivative is used to determine points at which the curve is concave upward or concave downwards or information could then be used to sketch the curve of a given functions. In this unit you will study how to use both the first and second derivative to sketch the graph of a function at every points of the graph.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Use the first to determine

  (i) points at which the given curves is increasing i.e. $\frac{dy}{dx} > 0$.

  (ii) Points at which the a given curve is decreasing i.e. $\frac{dy}{dx} < 0$

  and

(iii) Points at which a given curve is stationary i.e. $\frac{dy}{dx} = 0$

(2) Use the second derivative to determine points at which a graph is concave upwards or concave downwards.

3.1 Significance of The First Derivative to Curve Sketching

You will now consider the application of differentiation to curve sketching some curve could easily be sketched with the knowledge of the first and second derivatives of the function and the points where the first derivatives vanish i.e. equal to zero. You have already studied functions that are monotonic increasing or decreasing within an interval (see unit 2, section 3.2). You could determine whether a function is monotonic decreasing or increasing in a given interval. This is done by checking if the value of the first derivative within the given interval is positive or negative. This is stated as follows:

A function $y = f(x)$ is monotonic

(i) increasing in $x \varepsilon [a, b]$ if $\frac{dy}{dx} > 0 \ \forall \ x \varepsilon \ [a, b]$

(ii) decreasing in $x \varepsilon [a, b]$ if $\frac{dy}{dx} < 0 \ \forall \ x \varepsilon [a, b]$

(iii) constant (stationary) in $x \varepsilon [a, b]$ if $\frac{dy}{dx} = 0$

Example

Use the above stated facts to sketch the curve
$y = x^2$ for $x \varepsilon [-10, 10]$.

Solution

$y = x^2$

$\frac{dy}{dx} = 2x$ when $x = 0$ $\frac{dy}{dx} = 0$

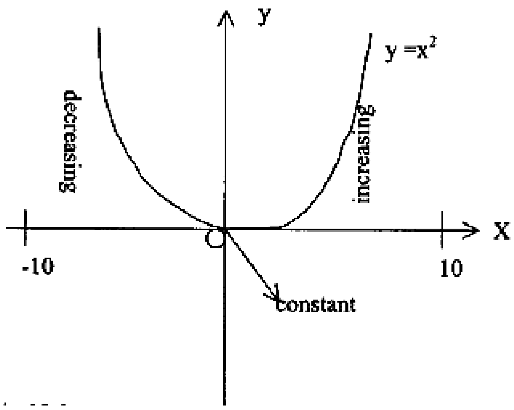
Start by first considering the values of dy/dx = 2x at points on the left of x = 0.

For $x \varepsilon [-10, 10]$. $\frac{dy}{dx} = 2x \leq 0$ (decreasing)

For the values of $\frac{dy}{dx} = 2x$ at points on the right of x = 0.

i.e. x [-10,10]; $\frac{dy}{dx} = 2x \geq 0$ (increasing)

(see fig: 10. 1)



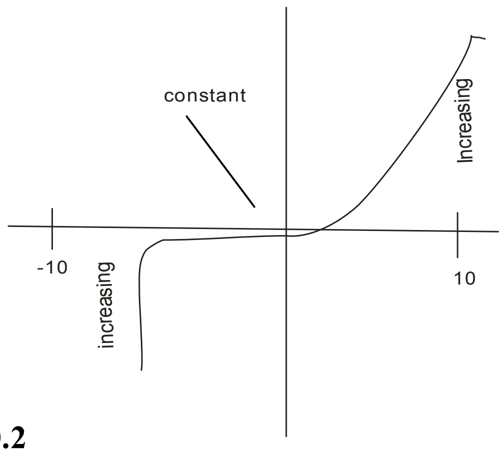


Fig 10.2

Example

Let $y = x^3$

$\frac{dy}{dx} = 3x^2$

when x = 0, $\frac{dy}{dx} = 0$

The values of dx/dy at points to the left and right of the point x = 0 is given as

(i) for. x [-10, 0]; dy $=3x^2 \geq$ o (increasing)

(ii) for. x [0, 10]; $\frac{dy}{dx} = 3x^2$ >- o (increasing) see fig 10.2.

Example

Given $y = \frac{1}{3}x^3 + x^2 - 8x + 1$

$\frac{dy}{dx} = x^2 + 2x - 8$

$= (x - 2) \times (x + 4) = 0$

$x = 2$ or $- 4$.

As before you will consider values of $\frac{dy}{dx}$

respectively. i.e. at points to the left and right of 2 and – 4 respectively. i.e.

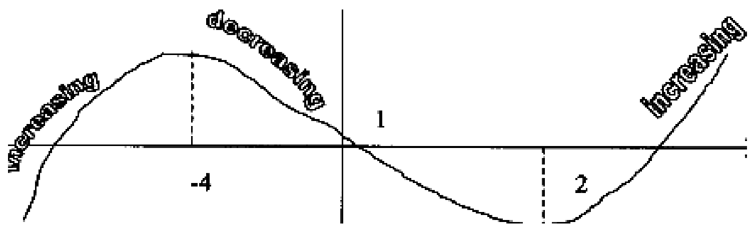
At $x = 2$

2. $\frac{dy}{dx} > 0$ for all $x \varepsilon (2, \alpha)$ i.e. increasing at the right side of

$\frac{dy}{dx} < 0$ for all $x \varepsilon (-4, 2)$ between - 4 and 2

$\frac{dy}{dx} > 0$ for all $x \varepsilon (-\alpha, -4)$

(see Fig 10.3)



3.2 Significance of the Sign of Seemed Derivative Curve Sketching

The above sketch could be improved if you apply the information you get by taking the second derivative of the function under investigation. A quick look at the graph shown in Fig. 10 3. shows that within the interval x ε (- 4, 1o) the graph is concave upward. Within the interval x x ε (-10, - 4) the curve is concave downward. Once you find the point at which dy/dx = 0 (i.e. the turning points along the curve y = f(x). Then by finding the second derivative d^2y/dx^2 you can determine which of the turning points (i.e. points at dx/dy = 0) is the concave upwards or downwards.

Definition:

If $\frac{d^2y}{dx^2}$ exists and $\frac{d^2y}{dx^2} > 0$,

for all x in a specified interval 1, then dy/dx is said to be increasing in I and the graph of f(x). is said to be concave upwards. If $\frac{d^2y}{dx^2} < 0$ for all x ε I, the $\frac{dy}{dx}$ is decreasing in I.

So that the graph of y = f(x). is said to be concave downwards.

3.3 Curve Sketching

Definition of Points of Inflection

A point where the curve changes its concavity from downwards to upwards or vice versa is called a point of inflection this occurs where

$$\frac{d^2y}{dx^2} = 0 \text{ or where } \frac{d^2y}{dx^2} \text{ does not exists.}$$

Example

$y = 3x^{1/3}$

$\frac{dy}{dx} = x^{-2/3}$

at x= 0 $\frac{dy}{dx}$ is not defined.

So the point x = 0 is a point of inflection for the $y = 3x^{1/3}$

Example

Given that y = f(x).

Let the following explain the behaviour of the curve.

(1) At point x = a

$$\frac{dy^1}{dx} = 0 \text{ if } x = \pm a$$

$$\frac{dy}{dx} > 0 \quad if \quad x^2 > a$$

$$\frac{dy}{dx} < 0 \quad if \quad x^2 > a$$

The curve increases or decreases as indicated below.

is flat

decrease

- X $\frac{dy}{dx}$ - a increases $\frac{dy}{dx}$

is flat

increase

a $\frac{dy}{dx} > 0$

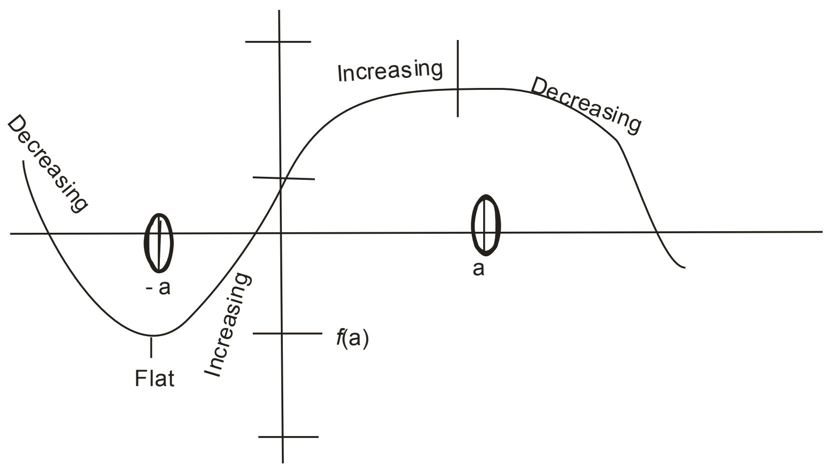
Let the point x = - a be the lowest point of the curve and the point x = a be the highest point of the curve.

Let the curve cut the y - axis at the point y = b, x = 0.

Let the curve cut the x- axis at the points $-x_l$ and x_2 sketch the given curve.

Step 1. $\frac{dy}{dx} = 0$ at $x \pm a$

Locate point x = ± a



Step 2. Find the values of $y = f(a)$ and $y = f(-a)$

Step 3. Find the point $y = f(o)$

Step 4. Find $\frac{d^2y}{dx^2}$ at $x = a^o$, $x = -a$

For the above $\frac{d^2y}{dx^2} > o$ at $x = -a$ and $\frac{d^2y}{dx^2} < 0$ at $x = a$..

Step 5: Use the above sketch the curve.

On step 5 use the information about the curve increasing and decreasing

i.e. left of $x = -a$ it is decreasing. and right it is increasing etc. see fig. 10.3.

SELF ASSESSMENT EXERCISE 1

Given the function

(a) $y = -1 + 3x - x^3$ (b) $x^3 - 3x + 1$

Find the following:

(i) Find $\frac{dy}{dx}$ (ii) solve the equation $\frac{dy}{dx} = 0$

(iii) Find $\frac{d^2y}{dx^2}$ (iv) solve $\frac{d^2y}{dx^2} = 0$

(v) Find y for which $x = 0$

Solutions

(a) (i) $3(1 - x^2)$ (ii) ± 1

(iii) $-6x$. (iv) -1

(b) (i) $3(x^2 - 1)$ (ii) ± 1

(iii) $6x$ (iv) 1

Hints For Sketching Curves

Below are seven useful hints for curve sketching

Hint 1 Find dy/dx

Hint 2 Find the turning points by solving the equation $\frac{dy}{dx} = 0$

Hint 3 Evaluate $y = f(x)$ at the turning points

Hint 4 Evaluate d^2y/dx^2 at the turning points to determine which of them is the maximum or minimum points.

Hint 5 Investigate the behaviour of the curve as x-> turning points either from left or right.

Hint 6 Investigate the behaviour of the curves as
(1) $x \to \infty$ (II) $x \to -\infty$ (III) $x = 0$

Hint 7 Sketch the graph with information gathered from hint 1 to hint 6 above.

Example: sketch the curve

$y = \frac{1}{3}x^3 - 4x + 2$ using

Hint 1; $y = \frac{1}{3}x^3 - 4x + 2$

$\frac{dy}{dx} = x^2 - 4$

Hint 2: $\frac{dy}{dx} = 0 \; (x^2 - 4) = 0$

$X = \pm 2.$

Hint 3: $x = 2$ $y = \frac{1}{3}(2)^2 - 4.\,2 + 2 = \frac{-10}{3}$

$x = -2$ $y = \frac{1}{3}(-2)^2 - 4(-2) + \frac{22}{3}$

Hint 4: $y = \frac{d^2y}{dx^2} = 2x$

at x = -2, $\frac{d^2y}{dx^2} = 2(-2) = -4 < 0$

at x = 2 $\frac{d^2y}{dx^2} = 2.2 = 4 > 0$

Hint 5:

Increasing - 2 decreasing 2 increasing

e.g.: x = - 6
y = -46
x = -3
y = 5

Hint 6: x --> ∞, y-> ∞
x--> ∞, y-->- ∞
x = 0, y = 2

Hint 7: (see graph below.)

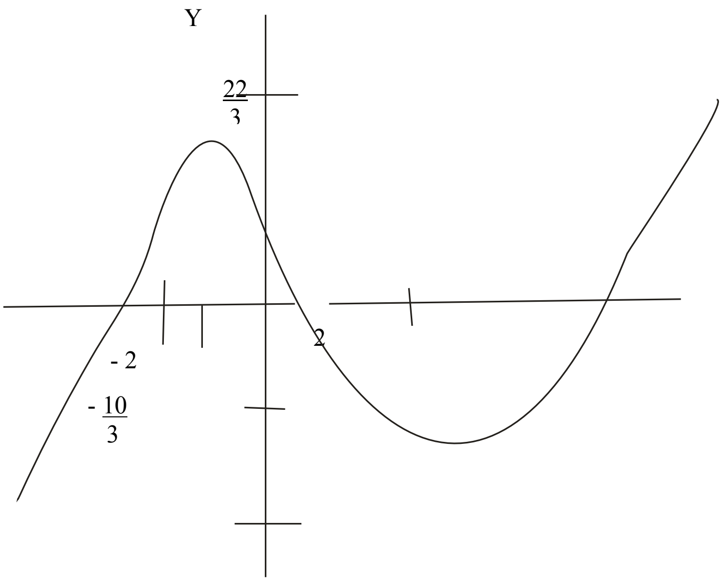


Fig 10.4

Example: $y = x + 9/x$

Hint 1: $\frac{dy}{dx} = 1 - \frac{9}{x^2} = \frac{x - 9}{x^2}$

Hint 2: $\frac{dy}{dx} = 0, \implies \frac{x^2 - 9}{x^2} = 0$

$x = \pm 3.$

Hint 3 $x = 3, \; y = 3 + \frac{9}{3} = 6$

$X = -3 \; y = -3 + \frac{9}{-3} = -3 - 3 = -6$

Hint 4:

Y = -10, x =

Hint 5:

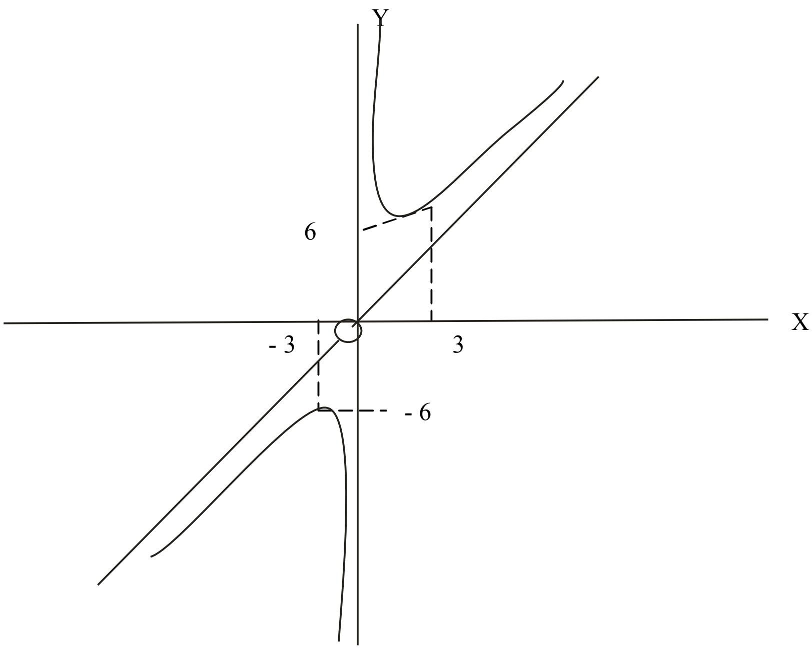
x ----> -3, y ---> 6, x ----> 3^+, y ---> -6

x ----> -∞, y ---> - ∞,

x ---> 3x y ----> 6, ----> 3^+, y ---> -6

x ----> ∞ y ---> ∞

Hint 6: Sketch



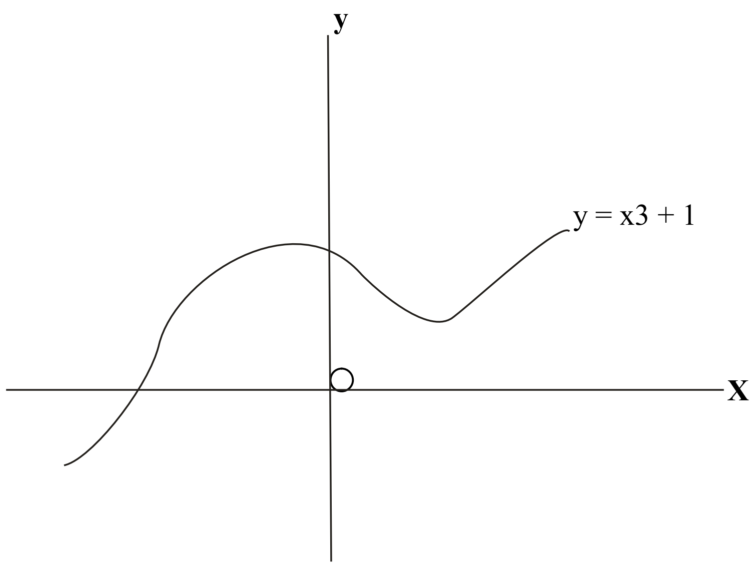
SELF ASSESSMENT EXERCISES 2

Use the hints given above to sketch the graph of

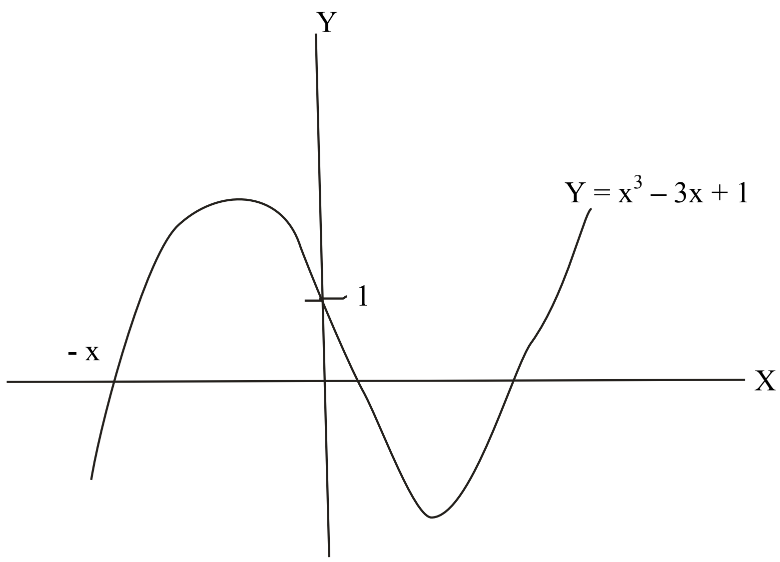
(1) $y = x^3 + 1$ (II) $y = x^3 - 3x + 1$

Solutions:

(i)



(ii)



4.0 CONCLUSION

In this unit you have studied how to use the sign of the first derivative of a function to determine a function that is monotonic increasing and monotonic decreasing. You also studied how to use the first derivative to determine stationary point. You have used the signs of the second derivative to determine a curve that is concave upwards or concave downward. You have studied how to use the information above with other information to sketch the graph of a function within the interval
[a, b] or (-∞,∞)

5.0 SUMMARY

In this unit you have studied how to:

(I) Investigate the behaviour of a function y = *f* (x) when

(a) $\frac{dy}{dx} < 0$ (b) $\frac{dy}{dx} > 0$

(c) $\frac{dy}{dx} = 0$ (d) $\frac{d^2y}{dx^2} < 0$

(e) $\frac{d^2y}{dx^2} > 0$

(II) Use the information in (10 above with other relevant ones such as the behaviour of y as x -> ∞ or ∞ to sketch the graph of y = *f*(x)

6.0 REFERENCES/FURTHER READINGS

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Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TUTOR-MARKED ASSIGNMENT

If $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$

(1) Find $\frac{dy}{dx}$ (2) solve the equation $\frac{dy}{dx} = 0$

(3) Find $\frac{d^2y}{dx^2}$ (4) solve $\frac{d^2y}{dx^2} = 0$

(5) Find y for which $x = 0$

(6) Find y when (i) $x \to \infty$ (ii) $x \to -\infty$

(7) Sketch the curve of $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$

(8) Locate the global minimum and maximum points on the graph in (7) above and the point of inflection.

UNIT 2 MAXIMUM – MINIMUM AND RATE PROBLEMS

CONTENTS

1.0 INTRODUCTION

In this unit you will study how to use first and second derivative of a function to solve optimization problems in social sciences, physics, chemistry, engineering etc. That is any problem where the information on how small or how big a given quantity should be is needed. It assumed that such problem should be able to be modelled by any mathematical formula. By differentiating such a function you can determine it minimum or maximum value. Differentiation could be applied to problems where it necessary to determine the rate at which a quantity is changing with respect to another quantity. There are various classes of problems that could be solved by appropriate application of differential calculus the ones enumerated in this unit and some where else is the course is by no mean exhausture.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

1) use first and second differentiation of a function to solve problems where the minimum amount of resources or material is required.
2) Use first and second differentiation of a function to solve problems where maximum value of a resources or material needed or should be attained
3) Use differentiation of a function to determine the rate at which a given quantity is changing with respect to another quantity.

3.0 MAIN CONTENT

3.1 Definition of Global and Local Minimum ad Maximum Value

In this section you will be able to use both first derivative and where necessary the second derivative to solve problems of classical optimization in economics, engineering, medicine and physics etc. The words minimum and maximum give the impression of a problem where you may wish to determine how small (minimum) or how large (maximum) a variable quantity may attain. You will start by considering the following important definitions.

Definition: A function $f(x)$ is said to have a local (or relative) maximum at point $x = x_o$ if

$$f(x) \geq f(x_o + h)$$

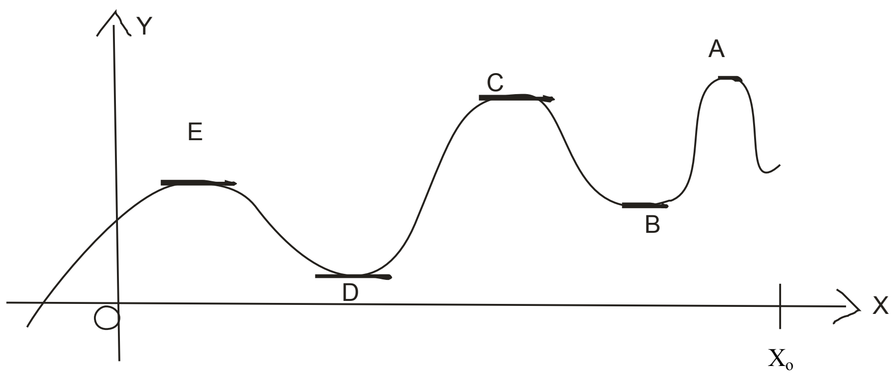
For all positive and negative value of h however small It is said to have a local (or relative minimum at point $x = x_o$ if

$f(x) \leq f(x_o + h)$ for all values of h however small.

Definition: A function $f(x)$ is said to have an absolute (global) maximum at $x = x_o$ if $f(x) \geq f(x)$ for all values of x in the domain of definition. It is said to have an absolute (global minimum at x = xo if $f(x_o) \leq f(x)$ for all values of x in the domain of definition.

Example

In Fig 10.5, you will notice that there are five turning points i.e. stationary points the points at which $\frac{dy}{dx} = 0$



Points E and C are points where the curve attains maximum or local maximum, while at point A the curve attains an absolute maximum or global maximum within the interval x ε [0, x_o] point B is a local maximum point while point D is a global minimum point.

Example: Given that y = x + 9/x

$$\frac{dy}{dx} = (1 - \frac{9}{x^2}) = (x^2 - 9) = 0, x = \pm 3$$

The point x = -3 is a relative maximum and not global maximum (see Fig 10.4) The point x = -3 is a relative minimum point.

SELF ASSESSMENT EXERCISE 1

Explain why the point x = 3 is not a global minimum for the curve given as

$$y = x + \frac{9}{x}$$

(Hint consider points near x = 3)

Example if y - $3x^4$- $16x^3$ +$24x^2$ + 1

Determine whether the function has a maximum or minimum points.

Solution

$$Y = 3x^4 - 16x^3 - 24x^2 - 1$$

$$\frac{dy}{dx} = 4.3x^3 - 3.16x^2 - 2.24x$$

$$= 12x^3 - 48x^2 - 48x$$

$$= 12x(x^2 - 4x + 4)$$

$$= 12x(x -2)(x - 2) =0$$

x = 0,or 2.

To determine which of the two points is a maximum or minimum the second derivative of these points will have to be evaluated

i. e. $$\frac{d^2y}{dx^2} = 3.\ 12x^2 - 2.48x + 48$$

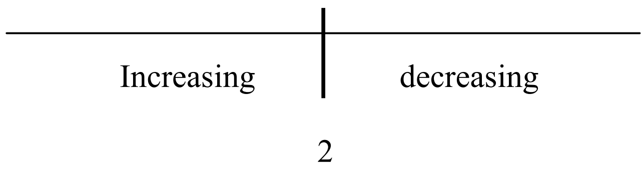
$$= 36x^2 - 96x + 48$$

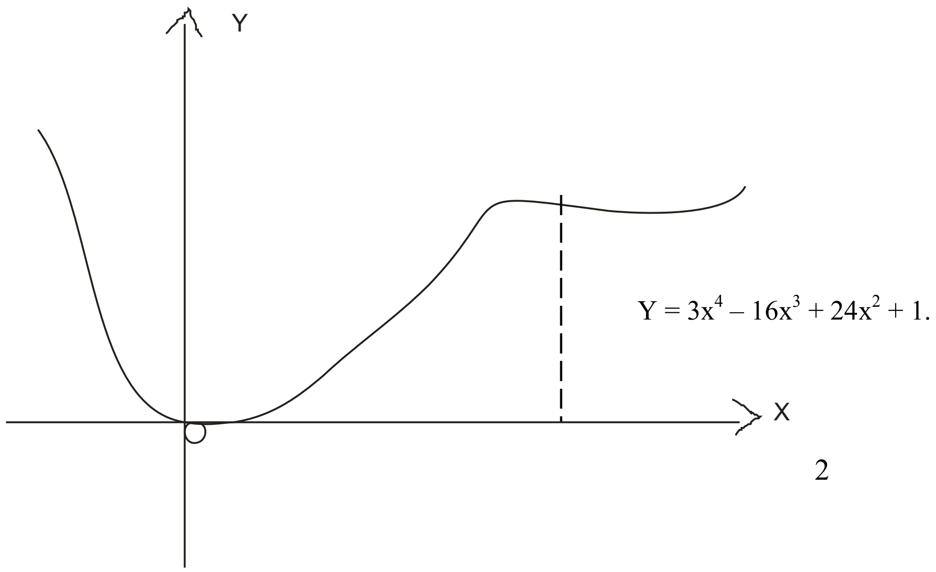
At x = 0, $$\frac{d^2y}{dx^2} = 48$$

$\frac{d^2y}{dx^2} > 0$ is a point of global minimum.

At x = 2 $\frac{d^2y}{dx^2} = 36(2)^2 - 96(6) + 48 = 0$

Look at points near 2.





Therefore the point x = 2 is a point of inflection.

3.2 Application to Minimum and Maximum Problems

It time to use the theory explained above to solve practical problems that call for the minimization or maximization of values of a function.

Example: Find two possible positive numbers whose product is 36 and whose sum could be made relatively small

Solution

You could start by making a guess based on the fact that the factors of 36 that are positive are:

(6, 6), (9, 4), (36, 1), (12, 3), (2, 18).

Taking their sum you see that

$12 < 13 < 15 < 20 < 37.$

So the number could is 6. It could be cumbersome to solve such a problem if the number is very large, find factors of a possible short cut is to apply differentiation. We need to note, that x and 36/x are two numbers whose product is 36.
Their sum is given as a function $y = f(x)$

$$\text{i.e.} \quad y = x + \frac{36}{x}$$

The issue at stake is to see the positive values of x that will give the problem is now reduced to the problem of minimization of the value of the function
$y = x + \frac{36}{x}$

Since $y = x + \frac{36}{x}$

$\frac{d}{dx} = 1 + \frac{36}{x^2} = 0$

$\Longrightarrow \quad x^2 - 36 = 0$

$\Rightarrow \quad x = \pm\ 6$

$\frac{d^2y}{dx^2} = + \frac{72}{x^3}$

if $x = -6$ $\quad \frac{d^2y}{dx^2} < 0$ maximum point

if $x = 6$ $\quad \frac{d^2y}{dx^2} > 0$ minimum point.

Therefore the answer is $x = 6$.

Example

The management of a manufacturing company found out that their profit was not enough as that wish it should be. A mathematical formulation gives the cost of production and distribution as ₦ a. The selling price is ₦ x. The number sold at a given period is put at $n = K/\ (x^2 - a) + C(100 - x^2)$ where K and C are certain constants. What selling price will bring a maximum profit to the company.

Solution

The first thing to do is to find or formulate an equation that gives the total profit for the company.

Profit = Selling Price - Total Cost

i.e. $P = N\ (n\,x - n\,a).$

$= (x - a)\ (K/(x - a) + C\ (100 - x)$

$= K + C\ (100-x)\ (x-a)$

$$\frac{dp}{dx} = C[(100-x)\ 1 + (x-a)-1]$$

$$\frac{dp}{dx} = 0 =:> 100 - x = x - a$$

$2x = 100 - a$

$$x = \frac{100 + a}{2}$$

₦ x is the selling price therefore $x > 0$. Having this in mind you find

$$\frac{d^2P}{dx^2} = -\ 23C \text{ hence } C > 0.$$

To guarantee that $\frac{d^2P}{dx^2} < 0.$

The selling price that give the maximum profit is given as $\frac{100+a}{x} = 2.$

Hence the above shows that the selling price is determined by the cost of production and distribution.

Example

An open storage tank with square base and vertical sides is to be constructed from a given amount of plastic material calculate the dimensions that will produce a maximum volume.

Solution

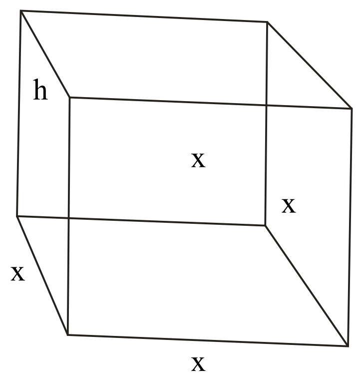
The formula for the volume of a rectangular box is given as

V = base x height (see Fig. 10.6)

Let base = x^2 (square base of side = x)

$\therefore \quad V = x^2 h$

Total surface area of tank
= Total amount of material



Therefore

$A = 4xh + x^2$

Solving for h you get

$\Rightarrow \quad h = \frac{a - x^2}{4^2}$

but $V = x^2 h = \frac{x^2 (A - x^2)}{4x} = \frac{x}{4} (A - x^2$

$$\therefore \quad \frac{dv}{dx} = \frac{1}{4}(A - x^2) + \frac{x}{4}(-2x)$$

$= \frac{1}{4} \left[(A - x^2 - 2x^2)\right]$

Let $\frac{dv}{dx} = 0 =:> [(A - x^2 - 2x^2)] = 0$

$\Rightarrow \quad A = 3x^2 \quad => x = \pm \sqrt{A/3}$

Since x represent sides of the tank definitely x > 0 therefore $x = \pm \sqrt{A/3}$
To check if this value of V you evaluate second derivative at

$$x = \sqrt{A/3} \text{ i.e. } \frac{d^2y}{dx^2} = \frac{1}{4}(-6x) < 0$$

$$\text{for } x > 0 \quad \frac{d^2v}{dx^2} < 0. \text{ Thus } x = \sqrt{A/3}$$

is a maximum point for the function V = x2 h.

Therefore the dimension that will yield maximum volume is given as $x = \sqrt{A/3}$and

$$h = \frac{A - x^2}{4x} = \frac{A - A/3}{4\sqrt{A/3}} = \frac{1}{2}\sqrt{A/3}$$

$$\text{Thus } x = x = \sqrt{A/3},\ h = x = \tfrac{1}{2}\sqrt{A/3}$$

This is dependent on the surface area of tank chosen.

In the next example you will consider the case where you will seek to minimize the amount of material to be used.

Example

Given a storage rectangular tank with a square base can contain $32m^2$ of water.

Find the dimensions that require the least amount of material to construct such a tank. (Neglect the thickness of the material and the waste in construction.

Solution

Let $V = x^2 h$ be the volume of the rectangular tank (see Fig 10. 6)

But $V = -32m^3$ therefore $32 = x^2 h$

$\Rightarrow \quad \frac{32}{x^2} = h$

Since the amount required to build the tank is equal to surface area of the tank which is given as

$A = 4 x h + x$

$$= 4x\left(\frac{32}{x^2}\right) + x^2 = \frac{128}{x} + x^2$$

$$\therefore \quad \frac{dA}{dx} = \frac{-128}{x^2} + 2x$$

equating to zero you get

$$\frac{-128}{x^2} + 2x = 0$$

$$\therefore \quad -128 + 2x^3 = 0$$

$$x^3 = 64.$$

$$x = 4$$

For $x = 4$ $\frac{dA}{dx} = 0$.

To check if $x = 4$ is a minimum point you evaluate d^2A/dx^2 at $x = 4$

$$\text{i.e.} \quad \frac{d^2A}{dx^2} = \frac{256}{x^3} + 2$$

Since $x > 0$ =:> $\frac{d^2A}{dx^2} = 4 + 2 = 6 > 0$

Thus at $x = 4$ the function has a minimum value.

Therefore the dimension of the tank that will minimize amount of material is given as 4, 4, 2.

Example

A milk can is to be made in the form of a right circular cylinder from a fixed amount of plastic sheet Find the radios and height that will produce the maximum volume.

Solution (see fig, 10.7)

$$V = \pi r^2 h \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \qquad (1)$$

The idea is find the values of r and h that will Maximize V.

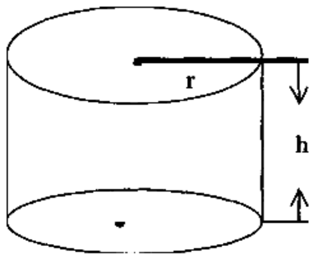


Fig 10.7

Let A be the surface area of a right circular cylinder

$$A = 2\pi r^2 + 2\pi r h \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \qquad (ii)$$

Thus $h = \frac{A - 2\pi r^2}{2\pi r}$

Substituting h into equation (i) you have that:

$$V = \pi r^2 \left(\frac{a - 2\pi r^2}{2\pi r}\right)$$

$$= \tfrac{1}{2}(A - 2\pi r^2)$$

Taking Differentiations

$$\frac{dv}{dr} \quad \frac{A}{2} - 3\pi r^2$$

Equating to 0 and solving for r you get

$3\pi r^2 = A/2$

$r^2 = A/6\pi$

$\therefore\ r = \frac{\sqrt{a}}{6\pi}$

$$h = \frac{A - 2\pi.\frac{A}{6\pi}}{2\pi\sqrt{A/6\pi}} = 2\sqrt{A/6\pi}$$

Now $\frac{d^2N}{} = -6\pi r\ . < 0\ r = +\sqrt{A/6\pi}$

dr^2

The radius $r = \sqrt{A/6\pi}$ $h = 2\pi$ that will produce maximum volume

Example: In Fig 10 .7 let $V = \pi r^2 h = 8$ and $h = 8\pi r^2$

Find the value of r that fail use the teart amount to produce a volume of 8 cm^3

In other words minimize the surface area. Given that the surface area

$$A=2\pi r + 2\pi r.\frac{8}{m_2} = 2\pi r^2 + \frac{16}{r}$$

$$=2\pi r^2 + \frac{16}{r^2}$$

Differentiations

$$\frac{dA}{dr} = 4\pi r - \frac{16}{r^2}$$

equating to zero and solving for r you get

$$4\pi r - \frac{16}{r^2} = 0$$

$$\pi r^3 - 4 = 0$$

$$\Longrightarrow \quad r^2 = 4/\pi$$

$$r = \left(\frac{4}{\pi}\right)^{1/3}$$

$$\frac{d^2A}{dx^2} = 4\pi + \frac{32}{r^3} > 0 \text{ for } r = \frac{4}{\pi}^{1/3}$$

Therefore $r = \left(\frac{4}{\pi}\right)^{1/3}$

SELF ASSESSMENT EXERCISE 2

Given the graph of the $f(x)$ in Fig 10.8.

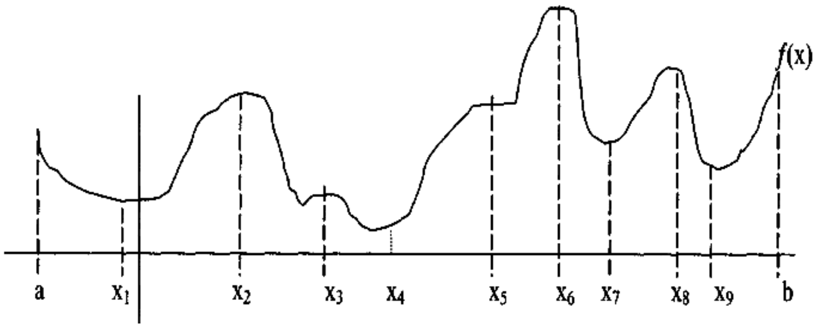


Fig 10.8.

(i) Define a global maximum of $f(x)$ and indication at which point in Fig(10.8) is this maximum attained if x ε [a, b].

(ü) Define a relative minimum of a function. Indicate at which points did the function $f(x)$ shown in Fig (10.8) is relatively minimized.

(iii) Define point of inflection, Indicate them.

Remark: To find the global a maximum and minimum of f (x) in the interval $a \leq x \leq b$. Locate all points where $f^1(x) = 0$. Call these points x_l, x_L, x_3,..., x_7 The global maximum is the largest of the numbers $f(x^2)$ =, $f(x^6)$, $f(x_8)$

Solution:

(i) $f(x_6)$ is the largest value therefore at $x = x_6$ = global maximum is attained.

(ii) Points of relative minimum are at $x = x_l$, $x = x_4$, $x = x_7$ and $x = x_9$

(iii) points of inflection are at $x = x_3$ and $x = x_5$.

Remark: Definitions are given in the beginning of this section

SELF ASSESSMENT EXERCISES 3

i. Find the minimum value of the function $f(x) = ½ x - \sin x$ in the interval x ε [0, 4 7π]
Solution

$X = \pi/3$, $X = 7\pi/3$.

ii. Find the largest value of $f(x) = 108x - x^4$

Solution

f (3) = 243.

3.3 Application to Rate Problem

You have already study that the derivative of a function gives its rates of change. In other words suppose you have a quality y which varies with another quality x, them the rate of change of y with respect to x is given as dy/dx.

You would have study this type of problem under the topic variation during your course of study in your preparation for the GCE 0 level or SSCE examination. That is y is increasing or decreasing with respect to x as according as dy/dx is positive or negative. This situation was described under the section on application to curve sketching, i.e. dy/dx > 0 implies that the function y = f (x) is increasing while dy/dx == f (x) is decreasing

Example: Suppose the amount of petrol y litres in a car tank after traveling x kilometers is given as

$$y = 30 - 0.02x$$

Then $\frac{dy}{dx} = -0.02$

The negative sign means that y decreases as x increases. Hence the amount of petrol in the car's tank is decreasing at the rate of 0.02 lt/km.

Example: A circular sheet of material has a radius of 4cm. At what rate is the area increasing with respect to the radius? If the radius increases to 4.2cm, what is the approximate increase in the area.

Solution

Let radius of circular sheet be r cm and the area be represented as A cm 2

Therefore $A = \pi r^2$ (area of a circle with radius r)

The rate of increase of A with respect to r is given as

$$\frac{dA}{dr} = 2\pi r$$

Thus if r = 4cm them $\frac{dA}{dr} = 2\pi (4) = 8\pi$ cm

But $\frac{dA}{dr} = 2\pi r$

Let ΔA and Δr be small changes in area and radius respectively

The $\Delta A = 2\pi r \ \Delta r$

Since $r = 4$ and $\Delta r = 0.2 = (4.2 - 4)$ cm

Then $\Delta A = 2\pi (4) 0.2 = 1.4\pi$

Therefore the approximate increase in area is $1.4\pi\ cm^2$.

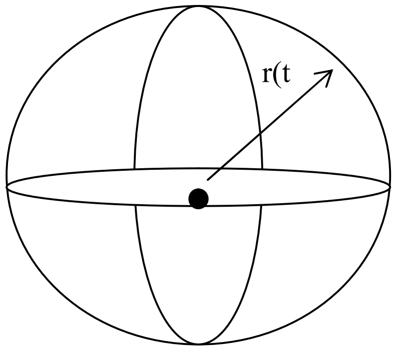
Example: A spherical balloon is inflated with gas. If its radius is increasing at the rate of 2cm per second, how fast is the volume increasing when the radius is 8cm (see Fig 10.9)

Solution

The volume at time t is expressed in terms of the radius at time t by the formula

$$v(t) = \frac{4}{3} \pi (r(t))^3$$

$$\frac{dv(t)}{dt} = 4\pi (r(t))^2 \frac{dr}{dt}$$



If at time t the radius is 8 cm

Then $\frac{dv(t)}{dt} = 4\pi (8)^2 .2 = 512\pi.$

This implies that the volume is increasing at the rate of $512\pi\ cm^3$ per second.

Example: water ns into a large concrete conical storage tank of radius 5 and its height is 1om (see Fig 10.10) at a constant rate of $3m^3$ per minute. How fast is the water level rising when the water is 5m deep

Solution:

In this case the list of variable quantities will be given

Quantities that are changing is

V = the volume (m) of water in the tank at time (t)

x = the radius (m3) of the Surface of the water at time t.

y = the depth (m) of water in the tank at time t.

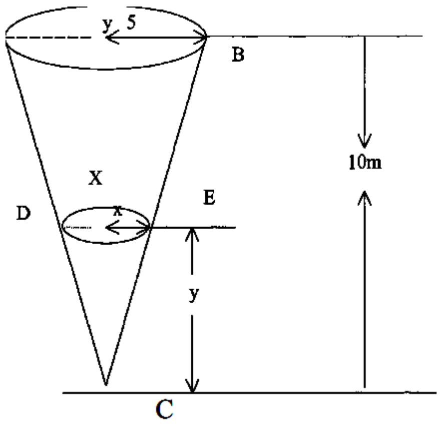


Fig. 10.10

The rate at water flow into the tank is constant and is given as dv/dt = $3m^3$ 1 min.

The function establishing a relation between the variable quantities is given as

1. $v = \frac{1}{3}\pi x^2 y$ (this is volume of cone $= \frac{1}{3}\pi r^2 h$.)

The function is better expressed in terms of only one variable x or y . You can expressed it in terms of y alone. By noting that in Fig 10.10 Δ ABC and ΔDEC are similar triangles.;

Therefore $\frac{DE}{Ab} = \frac{CX}{CY} = \frac{X}{5} = \frac{y}{10}$

$\Rightarrow \quad x = ½\, y$

Hence equation (1) becomes

$V \frac{1}{3} = \pi(½\, y)^2 y = \frac{\pi y^3}{12}$

Therefore $\frac{dv}{} = \frac{\pi y^2}{} \frac{dy}{}$

dt 4 dt

Given that $\frac{dv}{dt} = 3$ and y = 5.

Therefore. $\frac{dy}{dt} = \frac{4}{\pi y^2} \cdot \frac{dv}{dt} = \frac{4}{\pi r}\left(\frac{1}{5}\right)^2, \quad 3$

$\Rightarrow \quad \frac{12}{25}\ m^3\ 1min$

Example

A boat is pulling into a dockyard, for repairs by means of a rope with one and attached to the tip of the boat the other end passing through a ring attached to the dockyard at a pint 5m higher than the tip in at the rate of 3m/sec, how fast is the boat approaching the dockyard when 13m of the rope are out? (see Fig 10.11)

Solution

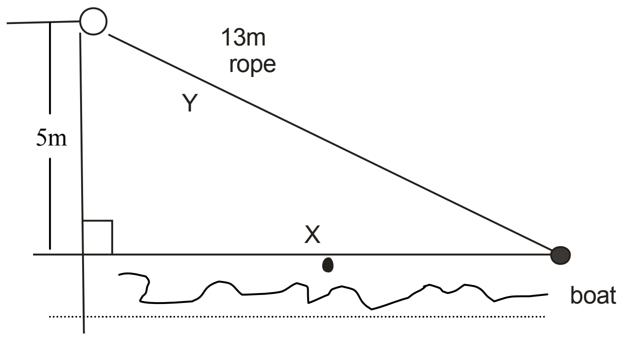


Fig 10.11.

There are two quantities dockyard that are changing. They are as follows:

(i) x = distance of boat to dockyard

(ii) y = the length of the rope.

The quantities that are fixed are:

(i) height of the dockyard = 5m

(ii) rate of change of rope to time t, $\frac{dy}{du}$

the formula connecting these quantities is given as (see Fig 11.11)

$$x^2 + (5)^2 = y^2$$

$\Rightarrow \quad x^2 + 25 = y^2$

Differentiating you get;

$$2x \frac{dx}{dt} = 2y \frac{dy}{dt.}$$

$$\therefore \quad \frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt.}$$

given that y = 13 and h = 5 (constant)

and x =
$\sqrt{13^2 - 5^2} = 12$

$$\frac{dx}{dt} = \frac{13}{12} .3 = \frac{13}{4} \text{m/sc.}$$

That is the boat is approaching the dockyard at the rate of 13/4 m/sec.

SELF ASSESSMENT EXERCISE 4

(1) A small spherical balloon is inflated by a young boy that injects air into 10 mm^3/sec. At the instant the balloon contains 288mm^3 of air, how fast is its radius moving.

Solution:

$$\frac{5\pi}{72\pi} \text{ mm/sec.}$$

(2) A company that sells vegetable oil has a conical distribution tank of radius 3m and height of 6m. If vegetable oil is poured into the tank at a constant rate of 0.05m^3 per second. How fast is the oil level rising when the oil is 2m deep.

Solution:

$$\frac{0.05m^3}{\pi} \text{/sec.}$$

4.0 CONCLUSION

In this unit you have studied the applications of differentiations to minimum and maximum problem. You have also applied differentiation to determine the rate a given quantity changes with respect to another.

You have specifically studied how to use the first and second derivatives of a to solve problem where a minimum amount of material or resource is needed and also where certain maximum value is to be attained. You have applied rules for differentiation to solve these problems

5.0 SUMMARY

In this unit you have studied how to:

(1) Calculate the minimum amount of material or resources needed in a project by using first and second derivatives of a function.

(2) Calculate the maximum value of a quantity or commodity using the first and second derivatives of a function.

(3) Determine the rate at which a quantity changes with respect to another quantity i.e. rate problems.

6.0 REFERENCES/FURTHER READING

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Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TUTOR-MARKED ASSIGNMENT

l. Given x = sin h t and y = -cos h t

(a) find dx (b) dy (c) $\frac{dx}{dy}$

2. An auto catalyst reaction R is defined as

$R(x) = 2x (a - x)$, a = the amount of substance
X = product of reaction

(i) At what values of a and x will the reaction attain its maximum?

(ii) Find the maximum value of R.

(Remark: An auto catalyst reaction is a chemical reaction where the product of the reaction acts as a catalyst).

3. A closed rectangular dish with a square base can contain a maximum of $20cm^3$ of liquid. Find the dimensions that require the least amount of materials to construct such a tank (Neglect the thickness of the material and waste in construction).

4. A state owned water corporation department has a large plastic conical storage tank of radius r = 6m and height h = 9m. Because of water scarcity water is allowed to run into the tank at a cons tank rate of $4m^3$ per minute;

(i) How fast is the water level rising when the water is 6m deep?

(ii) How deep will water when the water level is rising at the rate of lm^3 /min.

UNIT 3 APPROXIMATION, VELOCITY AND ACCELERATION

CONTENTS

1.0 INTRODUCTION

One of the areas of application of differentiation is the approximation of a function y = f(x). In this unit you will study how to estimate the changes produced in a function y = f(x). when x changes by a small amount Δx. In other words if there is a change in x by a very small amount Δx, then there will be a corresponding changes Δy in y. The approximate estimate. An important rate of change of distance with respect to time. Which corresponds to the speed if a body in motion. Of moving object at a given instant of time could be computed using the derivative of the distance function. The second derivative of the distance function will produce the acceleration of the moving body. The objective to achieve in the study of this unit is hereby studied.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the approximate value of the change in y = (x) if there is a small change in x.
- to compute the velocity of a moving body at a given instant of time.
- compute the acceleration of a moving body at a given instant of time.

3.0 MAIN CONTENT

3.1 Application to Approximation

3.1.1 Differential

You will start the study of the application of differentiation by examining the concept of "differential" of y and x where $f(x) = y$.

Definition: If x is the independent variable and $y = f(x)$. has a derivative at xo, say, define dx to be an independent variable with domain = R and define dy to be $\frac{dy}{dx} = f^1(x)$.

You will now consider two types of changes that can take place within a specified domain of a function.

Let $y = f(x)$. then an increment Δx in x produces a corresponding increment Δy in y .

Given that:

$$\lim_{\Delta x \to 0} f(x + \Delta x) - f(x) = f^1(x)$$

$$\text{but} \quad f^1(x) = \lim_{\Delta x \to 0} \frac{f^1(x)\,\Delta x}{\Delta x}$$

$$\text{then} \quad \lim \frac{f(x + \Delta x) - f(x) - f^1(x)\,\Delta x}{\Delta x} = 0$$

The difference $f(x + \Delta x) - f(x)$ is called the increment of f from x it $x + \Delta x$ and is denoted by the symbol:

$$\Delta f = f(x + \Delta x) - f(x)$$

The product $f^1(x)\,\Delta x$ is called the differential at x with increment Δx and is usually denoted by d f:

$$\text{i.e.} \quad d f = f^1(x)\,\Delta x.$$

Given that:

$$f(x + \Delta x) - f(x) = f^1(x)\,\Delta x. + y(x)$$

for a very small Δx.

$$\Delta f \approx d f .$$

where lim $\underline{g(x)} = 0$

$$\Delta x$$
$$\Delta x \rightarrow 0$$

since $y = f(x)$ you could re-write the above as:

$\Delta y \Rightarrow dy$

then $dy = f^1(x)\, dx$

let $x = f(t)$ and $y = g(t)$

then $dx = f^1(t)dt$ and $dy = g^1(t)dt$,

if $dt \neq 0$ and $f^1(t) \neq 0$

then $dy = \dfrac{g^1(t)}{f^1(t)}$

from the above you can say that a function $y = f(x)$ is the product of the derivative of the function and dx

Example:

The dimension of a square is x cm.

(i) how do small changes is x affect the area given as $A = x^2$

Solution

$$\Delta f = f(x + \Delta x) - f(x)$$

$$= f(x + \Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2$$

$$\Rightarrow \quad f\,d \quad = f^1(x)\,\Delta x = 2x\,\Delta x$$

The error of the estimate is the difference between the actual change and the estimated change that is

$$\Delta f - d f = (\Delta x)^2$$

If $\Delta x \rightarrow 0$ then this error $\rightarrow 0$.

Example: Use differentials to determine how much the function.
$Y = x^{1/3}$ changes when

(1) x is increased from 8 to 11

(2) x is decreased from 1 to 0.5

Solution

(1) Let $x_o = 8$ and $\Delta x = 11 - 8 = 3$

Given that; $y = x^{1/3}$

$$\frac{dy}{dx} = 1\, x^{-1/3} \Rightarrow \Delta y \cong \frac{1}{3}(x^{-1/3})dx$$

$$\Delta y = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}\ \frac{1}{8^{2/3}} = \frac{1}{3}(^3\sqrt{8})^{-2} = \frac{1}{3}(2)^{-2}$$

$$\Delta y = \frac{1}{3} \cdot \frac{1}{4} \cdot \Delta x = \frac{1}{3} \cdot \frac{1}{4} \cdot 3 = ¼ = 0.25$$

A change in the value of x from 8 to 11 increased the value of y by 0.25.

(2) let $x_o = 1$ and $\Delta x = (0.5 - 1) = -0.5$.

$$\Delta y = 1/3(1)^{-2/3} \cdot (-0.5) = -1/6 = -0.167$$

Example

If $Q = \frac{9}{x}$ and x is decreased from 1 to 0.85, what is the approximate change in

the value of Q.

Solution:

Let $x_o = 1$

Then $x - x_o = \Delta x = 1 - 0.85 = -.15$.

$$\frac{dQ}{dx} = \frac{-9}{x^2}$$

when $x = 1$, $\frac{dQ}{dx} = -9$

$\Delta Q = -9. (-.15) = 1.35.$

Example

If $x = \sin t$ and $y = \cos t$

$0 < t < \pi$

Find (1) dx and dy (II) dy/dx

Solution

(1) $dx = \cos t$ and $dy = \sin t$

(2) $\frac{dy}{dx} = \frac{-\sin t}{\cos t} = \frac{-x}{y.}$

Example

Use differential to approximate the value of $\sqrt{125}$

Solution

Let $y = \sqrt{x}$

You are aware that $\sqrt{121} = 11$

So the problem reduces to finding an approximate change in y as x increases from 121 to 125.

$x_o = 121, \quad \Delta x = 125 - 121 = 4$

$$\Delta y \frac{dy}{dx} \Delta x .$$

since $y = x^{½}$ ==> $\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{x}}$

if $x = 121$ $\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{121}} = \frac{1}{22}$

$\Delta y = \frac{dy}{dx}\Delta x \frac{1}{22} .4 = \frac{2}{11}$

then $\sqrt{125} \cong \sqrt{121} + 2 = 11 + 0.1182$

$\sqrt{125} \cong 11.182$

Example

Given that $P = (3q^2 - 2)^2$

When q = 3 it is increased by 0.8%. Find the approximate percentage change in P

Solution

Let $q_o - 3$, $\Delta q = .9\%$ of q_o

Given $\Delta P = \frac{dP}{dq} \Delta q$

$P = (3q^2 - 2)^2$, $\frac{dP}{dq} = 2(3q^2 – 2) . 6q.$

$\Delta q = 0.008 \times 3 = 0.024$

$\Delta P = (2(3(3)^2 - 2)63) \times 0.024.$

$= (50 \times 6 = 3) \times 0.024 = 21.6$

% change in $p \cong \frac{\Delta P}{P} \times 100$

$P = (3q^2 - 2)^2 = (3(3)^2 - 2)^2 = (25)^2 = 125.$

$\therefore$ change in $P = \frac{21.6}{125} \times 100 = 17.28\%$

SELF ASSESSMENT EXERCISE 1

Use differential to estimate the following:

(1) $\sqrt{l04}$ (2) $(1020)^{1/3}$ (3) $(24)^{-¼}$

(4) If $y = \sqrt{x}$, find the approximate increase in the value of y if x is increased from 2 to 2.05.

(5) if $y = (x^3 + 1)^2$ when x = 2 it is increasing by 0.5% find the approximate percentage change in y.

Solutions

(1) $\sqrt{l04} \approx 10.2$
(2) $(1020)^{1/3} \approx 10 + 2/30 = 10.067$

(3) $(24)^{-½} = 1/24 \approx 21 - (0.063)$

(4) $y \approx 1.432$

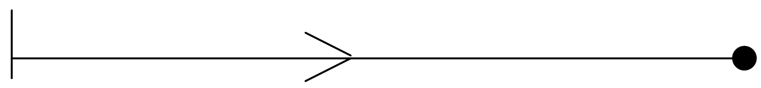
(5) 2.7%

3.2 Application of Differentiation to Velocity

You are familiar with the word speed. This is defined as the distance traveled by a body divided by the time it takes the body to arrive the distance.

i.e $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

In the above the motion must occur along a straight line. Such as



(motion of car along a straight road)

However it might be a tricky question if you want to know the speed of a body in at a given instant along a curve.

For example: The motion of a rocket fired into the air to be able to describe how fast the rocket is rising after say a few second it is fired might not be very clear as finding the speed of the car along a straight road. To be able to know the speed of the rocket one must know the function that described the motion of the rocket in the air. Again one must be able to know how fast the rocket is rising after it is fired for every single point in the curve describing the motion. In other words to be able to know how fast such a body is traveling one must know the velocity of the body at a given instant. You are aware that to find the velocity of a body all you need to do is to find the distance between the staring or initial point and point at which you want to find the velocity.

i.e. $\text{Average velocity} = \dfrac{\text{Dist } (t_o) - \text{Dist}(t_1)}{t_1 - t_o}$

where t_o = initial time and t_1 = time traveled The above formula cannot give you the value of instantaneous velocity which is the object of study in the section.

Let a body travel along a curve $f(t) = t^2$, if you sub divide the entire time it took to cover this curve in a given interval of time t $[t_o, t_1$ in a subinterval of length h . Then the average velocity between t_0 and $t_0 + h$ will be given as;

$$\frac{f(t_0 + h) - f(t_0)}{h}$$

The smaller h is the closer thus average velocity is to ($2t_o$+ h). so at the time t = t_o this average velocity is $2t_o$. Take a closer took again at equation (1) you will agree that the average velocity at a given instant in the same process used in deriving the slope of a curve y = f (t) at the point t = to thus the velocity of the body at time to is the same numerically with the slope of y = f (t) = t^2 at t = t_o.

Definition: If f (t) is the position of a moving body at time t, its velocity is defined as ;

$$\frac{d f(t)}{dt} = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$$

its speed is defined as $\left(\frac{d f(t)}{dt}\right)$

the above implies that you might have a negative velocity. This always happen in the case of falling bodies. In most textbooks the velocity of a body is represented as y.

i.e. if y = f (t) then $\frac{d f(t)}{dt}$ = y

Example

A shot is fired from a top of Abia Tower at Umuahia which is at a height 100 + 24t - $12t^2$ After t seconds. Find

(i) its velocity after 2 seconds

(ii) its maximum height

(iii) its velocity as the bullet hits the ground.

Solution

Let y (t) = 100 + 24 = - $2t^2$

Then $\frac{d f(t)}{dt}$ = y(t) = 24 - 4t

(i) After 2 saec t = 2.

∴ y(2) = 24 - 4.2 = 16 m^2/sec.

Since y(2) > 0 it implies that the bullet is rising.

Ans. = $16m^2$/sec.

(ii) The maximum height can only be attained at the maximum value of y(t) i.e.

y(t) = 0, => 24 - 4t = 0

t = 6.

$Y(12) = 100 + 24.6 - 2(6)^2 = 172$

Observe that when the velocity = 0.

Then the bullet will start to fall down. This occurs at the instant the bullet attain it highest point (see Fig 10.12)

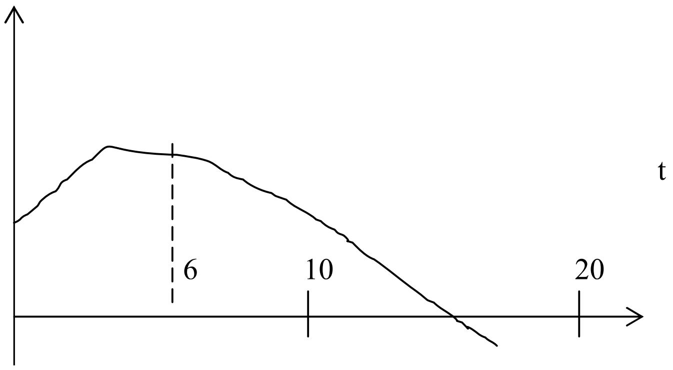


Fig 10.12.

At time t = 6 y(6) = 0

Y(16) = 172m. is the maximum height
Ans. =172m.

(iii) When the bullet hits ground y(t) = 0

.'. $100 + 24t - 2t^2 = 0$

==> $t^2 - 12t - 50 = 0$

$$\frac{-12 \pm \sqrt{144 + 4.50}}{2}$$

$$\frac{12 \pm \sqrt{344}}{2} = \frac{12 \pm 18.547}{2}$$

= 15.274 (see Fig 10.12)

at = t = 15 .274.

y(15.274) = 24 - 4 (15.274)

= -37.096.

Ans. -37.096 m/s. This should be expected

Since the bullet is falling down in practical terms the speed = 1 -37.096 m/s/ = 37.01

Compare to the rising speed of 16m/s at t = 2.

SELF ASSESSMENT EXERCISE 2

(a) A ball thrown straight up has a height $f(t) = -16t^2 + 160t$ after t seconds Find its

(i) maximum height
(ii) the velocity when it hits the ground

Solution

(i) 400m. (ii) -160.m/s.

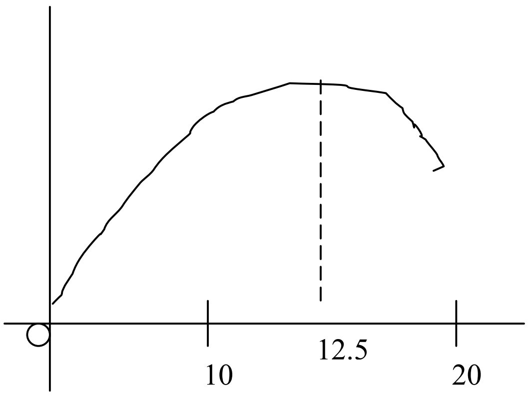
(b) a ball thrown upwards from a building attains a height of $f(t) = (-16t^2 + 400t + 8000)$ m after t seconds. Find

(i) the time it attains its maximum height
(ii) the maximum height
(iii) the velocity after 15 seconds.
(iv) Sketch the curve between t = 0 and t = 20

Solution

(i) 12.5 secs (ii) 10.500m (ii) - 80m/s.

(iv)



3.3 Application of Differentiation to Acceleration

If you applied brakes on a moving car it moves slower and slower it velocity decreases. This implies that velocity, of a moving body can either be increasing decreasing or constant.

Definition : If the velocity of a moving body at t is given as v(t) the acceleration of the body is given as

$$\frac{dv}{dt}(t) - v(t)$$

Put simply acceleration is the derivative of the velocity, i.e. if measures the rate of change of velocity during motion.

If f(t) represent the distance covered by a moving body after t second.

Then $\frac{d^2 \int(t)}{dt^2}$ = acceleration.

Simply put acceleration is the second derivative of the equation of the distance covered after t seconds.

Example
A stone thrown above the ground attains a height of 100 + 24t -8t after t seconds.

Find the acceleration at time t.

f (t) = 100+ 241 - 8t^2

f (t) = 24 - 16t

f (t) = -16 m/s.

Ans. = -16m/s.

Example

The distance covered by a moving body after t seconds is given as

f (t) = t^3 - 3t^3 + 2

Find the

(i) acceleration of the body at t = 2

(ii) At what time will the acceleration equal to zero.

Solution

f (t) = t^3 - $3t^2$ +2.
f (t) = $3t^2$ - 6t

f (t) = 6t - 6.

At t = 2 f (2) = 2.6-6 = $6m^2$ /see.

f (t) = 0 => t = 1.

Ans: (i) $6m^2$/sec. (ii) 1.

SELF ASSESSMENT EXERCISE 3

A small body is made to travel in a straight line so that at time t sec after a start in distance f(t) from a fixed point 0 on the straight line is given by

f (t) = t^3 - $4t^2$ + 12.

Find;

(i) How far has the body traveled starting from the point 0.
(ii) Evaluate the velocity after t = 2.

(iii) A what time is the acceleration zero.

(iv) What is the acceleration after 4 seconds.

Solution:

(i) 12m (ii) -4m/s. (ii) 1.33 secs. (iv) $16m^2$/S.

4.0 CONCLUSION

In this unit you have studied the applications of differentiation to problem solving. You have applied the first derivative at a function to

(i) Approximate. Values of a variable quantity

(ii) Find the rate at which a quantity is changing with respect. To another.
(iii) Find the equation of a tangent and normal to a curve.

You have also studied how to use the first and second derivatives of a function to find the approximate value of the change in value of a given quantity with respects to a small change in another value. You have studied how to compute the instantaneous velocity and acceleration of moving body.

5.0 SUMMARY

In this unit you have studied how to

(1) Approximate a value of a function f by its differential d f i.e. $df \approx f^1 (x)\Delta x$.

(2) calculate the velocity and acceleration of a moving body i.e. v(t) $\frac{df}{dt}$ (t) and

$$\text{ace }/^n = v(t0 = \frac{d^2 f(t)}{dt^2}$$

where f (t) is the distance covered in any measurable unit after t second.

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR-MARKED ASSIGNMENT

1. If P = 4/x and x is decreased from 0.5 to 0.1 what is the approximate change in the value of P.

2. Given that f(x) x ½ estimate the change in f if
   (i) x is increased from 32 to 34.
   (ii) x is decreased from 1 to 9/10

3. Use the differential to estimate √1004

4. A ball thrown straight up from the top of a building at height of $180 + 64t - 16t^2$ after t sec. Compute;
   (a) It velocity after 1 sec.
   (b) Its maximum height
   (c) Its velocity as it hits the ground.

5. A ball is $180 + 64t - 16t^2$ meters above the ground at time t sec. Find its acceleration at time t.

6. A ball thrown above the ground attains a height of $f(x) = 20 = 4t - t^2$ after t seconds. Find;
   (a) The maximum height
   (b) The velocity after 3 sees.
   (c) The velocity when the ball hits the ground.
   (d) The acceleration of the ball

UNIT 4 NORMAL AND TANGENTS

CONTENTS

1.0 INTRODUCTION

In this unit you will apply the differentiation of a function y =f(x) to find the slope of a tangent to a curve at a point. You could recall that this idea was extensively discussed in unit 6.You then use the slope of the tangent to compute the slope of the normal to the curve at the same point. So it is necessary you review the slope f a curve studied in unit 6.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the slope of a tangent to a curve by method of differentiation
- find the slope of a normal to a curve by the method of differentiation
- derive the point-slope equation of a tangent at a given point of a curve
- derive the point - slope equation of a normal at a given point of curve.

3.0 MAIN CONTENT

3.1 The Point-Slope Equation of a Line

Let a line L pass through the point P (x_l, y_l) and let Q (x, y) any other pint on the curve.

The slope of the line L is given as:

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m (x - x_1) \quad \text{------ I}$$

The above equation I is known as the point-slope form of the equation of the line L. Since it givens the equation of the line is terms of a single point $P(x_1\ y_1)$ on the line and the slope m of the line. That is why it is called the point -slope equation of a line.

Therefore once you know the coordinate of just one point on the curve and you can determine the slope of the tangent-line at that point by method of differentiation. Then you can easily form the equation of the tangent -line by using equation I above.

Example: Given that the slope of a line is 2 and the line passes through the point P (2, - 2). Write the equation of the line

Solution

Using point-slope formula

$y - y_1 = m\ (x-x_1)$
and given that $m = 2_1\ x_1= 2_1$ and $y_1 = - 2$

then you have that:

$y- (-20) = 2(x-2)$
$y = 2 = 2x-4y = 2x -6$

which is the required equation of a line.

3.2 Equation of a Tangent to a Curve

In unit 5 you studied that the slope of the curve y = f (x) at any given point is given as:

$$m= \frac{(dy)}{dx} = f^l\ (x)$$

Which is also the slope of the tangent to curve y = f (x) at the given point $(x_1\ y_1)$.

Therefore the equation of a tangent - line to a given curve y = f(x) at a given point (x_1, y_1). On the curve can be written as;

$y -y_1 = m\ (x-x_1)$

$y - y_1 = \frac{dy}{dx}\ (x -x_1)$

(x_1, y_1)

$= f^l(x_1)\ (x-x_1).$

Therefore tangent is given as;
$y = f^l(x_1)(x - x_1) + y_1$

3.3 Equation of Normal to Curve

The normal is the line that is peculiar to the tangent. As such if m is the slope of the tangent passing through the point (xl, yl) then the slope of the normal passing through the [point (xl yl) is given as;

$$M_N = \frac{-1}{M_T}$$

Since $M_T = f^1 (X^1)$

Then $M_N = \frac{-1}{f\,(x_1)}$

Therefore the equation of the normal line at point (x_1, y_1) is given as;

$$y = -1 \frac{(x - x_1)}{f(x_1)} + y_1$$

Where y = f (x).

Examples

Write the equation of the tangent to the following curves at the given points.

(i) $y = x^2$ (-2, 4)

(ii) $y = x^3$ (-1,-1)

(iii) $y = {}^1/_x$ (1,1)

(iv) $y^2 = x^2$ (2,4)

Solution

(i) $\frac{dy}{dx} = 2x$ at x_1 _ - 2, $2x_1 = 2\,(-2)$

$m = -4$

$y = -4(x-(-2) = 4$

$y = -4x - 8 + 4 = -4x - 4$

$y = -4x - 4.$

(ii) $\frac{dy}{dx} = 3x^2$, $m = 3(x_1)^2 = 3\,(-1)^2 = 3.$

y = 3(x-(-2) = (-1) = 3x = 3 - 1
y = 3x = 2

(iii) $\frac{dy}{dx} = -\frac{1}{x^2}$ m = $\frac{-1}{(x_1)^2}$ = - 1

y = -(9x - 2) + 1 = -x + 2 +1
y = 3 - x

(iv) 2y = $\frac{dy}{dx}$ = 2x

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow m = \frac{2}{2} = 1$
y = (x-2) + 2 = x
y = x

(v) xy = 2x + xy = 1 (12)

(vi) x $\frac{dy}{dx}$ + y + 2 + 4 $\frac{dy}{dx}$ 0 (differentiating)

$\Rightarrow$ (x + 4) $\frac{dy}{dx}$ + y +2 = 0 (collecting like terms)

$\frac{dy}{dx} = -\frac{(y+2)}{x+4} = -\frac{(2 = 2)}{1 = 4} = \frac{-4}{5}$

Equation of tangent is

Y = $\frac{-4}{5}$ (x-1) = 2

$\Rightarrow$ 5y = -4x = 1 + 10 = -4x + 11.
5y = -4x + 11

Equation of normal is

Y = $\frac{5}{4}$ (x – 1) + 2
4y = sx – 5 =
4y = 5x -3.

(iii) y^2 - 2x - 4y + x^2 = 4 (-1, -2)

Differentiating

$$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} + 2x = 0$$

Collecting like terms

$$(2y - 4)\frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} - \frac{2-2x}{2y-4} = \frac{1-x}{y-2} = \frac{1-(-1)}{-2-2} = \frac{2}{-4}$$

m = - ½

Equation of tangent

Y= - ½ (x-(-1))+(-2)
2y=-x-1-4=-x-5
2y+x+5=0.

Equation of normal
Y=2(x+1)-2=2x+2-2
Y=2x

SELF ASSESSMENT EXERCISE 1

1. Find the equation of the tangent and normal to the curves at the given points. (for exercises a - d)

(a) $y = 2x^2 - 1 \quad x=1$
(b) $y = x^2 - 2x \quad x=2$
(c) $y = \frac{x - 1}{x + 1} \quad x = 1$
(d) $2x^2 - xy = 16 = y^2$ at (2,4)

2. The slope of the tangent at a point $P(8_1, y_1)$ on the curve $y = 2x^2 - 6x + 1$ is 10. Find x_1 and y_1.

3 The curve where equation $y = ax^2 - bx - 6$ passes through the point (-1, - 4) and the slope of the curve at the is 2 find the value a and b.

Solution

(1). (a) y=4x-3, 4y=3-x
(b) y=2x-4, 2y=xn-2
(c) 2y = x-1, y=2-2x
(d) 5y = 2x + 16, 2y = 5x -2.

(2). (4,9)
(3). a = 2, b = -6

4.0 CONCLUSION

In this unit you have reviewed the point - slope equation formula for the point slope equation of the tangent line by using differentiation to calculate the slope of the tangent at a given point. You have used the slope of a tangent to determine the slope of a normal and consequently derived the formula for the point slope equation of a the normal line to a curve at a given point. You have solved example on the above.

5.0. SUMMARY

You have studied in this unit how to:

- Determine point slope equation of a line $(y - y_1) = m (x - x_1)$
- Derive the point - slope equation of the tangent to a curve at a given point.
- Derive the point - slope equation of the normal to a curve at a given point.
- Determine the slope of the a tangent and normal to a curve by differentiation.

6.0 REFERENCES/FURTHER READING

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7.0 TUTOR MARKED ASSIGNMENT

Find the tangent and normal to the curves at the specified points.

1. $y^2+x^2-4x+3y=1$ x=1,y-1

2. $y^2=x+\underline{4}$ $x=1, y=\sqrt{5}$

3. $y = \frac{2-+ x}{3-x}$ (2,5)

4. $x^2 + y^2 = 25$ (3,4)

5. $y=x^3-x$ $x_0=-2$

6. Derive the point - slope from of equation of a tangent and normal to a curve y = f(x) at the point (x_1, y_1).