MODULE 2

UNIT 1 COLLINEAR VECTORS

CONTENTS

1.0 INTRODUCTION

The issue of vectors being collinear or non-collinear, cannot be overemphasized. The consequence of this concept is what this unit will bring out to you.

Problems in plane geometry and co-ordinate geometry are simplified for you, once you consider them under vector algebra.

You should take note of every diagram- the directions of the arrows showing the vectors, and how a simple statement given to you is eventually used to solve an example.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use the position vectors of two given points to express the line joining them (Relative Vectors)
- calculate correctly the magnitude of the given vectors.
- use the given vectors to show if they are linearly dependent or linearly independent.
- prove theorems based on collinearity or non-collinearity of vectors.

3.1.1. Definitions of collinear vectors

You will recall that when the triangle law of vector addition was discussed in the previous units, you learnt that if, $\vec{\mathbf{AB}}$, and $\vec{\mathbf{BC}}$ are not on a Straight line the resultant **AC** is the vector closing the triangle $\vec{\mathbf{ABC}}$. The Statement $\vec{\mathbf{AB}} + \vec{\mathbf{BC}} = \vec{\mathbf{AC}}$ the very obvious, if they are on the same line.

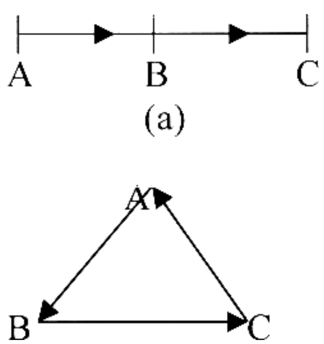
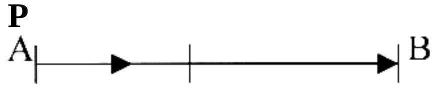


Fig V13

In the first case (a) you have collinear vectors A and B, but in the second case (b) A, B, and C are said to be non-collinear vectors.

Also, if vectors AP and PB are collinear, P being a point on AB, such that the ratio AP: PB=m: n or nAP=mPB.

P



If **r** is the position vector of P, and **a** and **b** are the position vectors of A and B respectively, then you can say n ($\mathbf{r}$-$\mathbf{a}$) = m ($\mathbf{b}$-$\mathbf{r}$), which transforms into $\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m + n}$ or n$\mathbf{a}$+m$\mathbf{b}$=(m+n)$\mathbf{r}$.

3.1.2 Mid-point theorem

Example 4

Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and have one half of its magnitude.

Solution

Draw figure V 14.

Note the directions **AB, BC, CA** of the vectors **c, a,** and **b** respectively for the triangle to be in equilibrium.

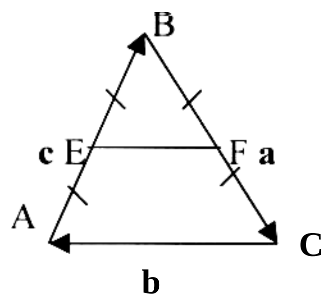


Fig V 14

$\vec{\mathbf{EF}} = \vec{\mathbf{EB}} + \vec{\mathbf{BF}}$

½**c** - ½**a** = ½ (**c** - **a**)

$\vec{\mathbf{AC}} = -\mathbf{b} = \vec{\mathbf{AB}} + \vec{\mathbf{BC}} = \mathbf{c}-\mathbf{a}$

$\vec{\mathbf{EP}}$ = ½ **AC** which proved that $\vec{\mathbf{EF}}$ is parallel to $\vec{\mathbf{AR}}$ and is ½ of its magnitude

This is the midpoint Theorem.

3.1.4 Division of a line Segment (Collinear Vector)

Let the position vector of points P and Q relative to an origin be given by p and q respectively. If R is a point which divides line PQ into segments, which are in the ratio m: n, then you will write the position vector of R as

$\mathbf{r} = \frac{m\mathbf{p} + n\mathbf{q}}{m + n}$ which is a familiar statement in geometry.

This means (m + n) $\mathbf{r}$ = m$\mathbf{p}$ + n$\mathbf{q}$, which gives you a rule for collinearity of points.

3.2.1 Centroid

If $\mathbf{r}_1$. $\mathbf{r}_2$ ... $\mathbf{r}_n$ are the position vectors of masses m_1, $m_2 \ldots m_n$ respectively relative to an origin O.

Then the position vector of the Centroid can be proved as

$$\mathbf{r} = \frac{m_1 r_1 + m_2 r_2 + \ldots m_n r_n}{m_I + m_2 \ldots m_n}$$

Simply put, the Centroid represents the weighed average or mean of several vectors.

SELF-ASSESSMENT EXERCISE 2

A quadrilateral PQRS has masses l, 2, 3 and 4 units located respectively at its vertices A (-l, -2,. 2), B (3, 2, -1), C(1, -2, 4) and D (3, 1, 2).

Find the coordinates of the Centroid.

Solution

The position vector of the Centroid is

$$\mathbf{C} = \frac{\mathbf{p} + 2\mathbf{q} + 3\,\mathbf{r} + 4\mathbf{s}}{1 + 2 + 3 + 4}$$

Where, $\mathbf{p} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{s} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$$\mathbf{c} = \frac{(lx{-}1+2x3+3x1+4x3)\mathbf{i} + (lx{-}2+2x2+3x{-}2+4\text{ x }1)\mathbf{j} + (1x2+2x(-1)+3x4+4\text{ x}2)\mathbf{k}}{10}$$

$$= \frac{20\mathbf{i} + 0\,\mathbf{j} + 20\,\mathbf{k}}{10}$$

$\therefore$ $\mathbf{c} = 2\mathbf{i} + 2\mathbf{k}$ which will give the coordinate as (2, 0, 2).

3.2.2 Linearly Dependent and Independent Vectors

If you can represent the sum of given vectors r_1, r_2 and r_3 by a single vector r_4 i. e. $r_4 = r_1 + r_2 + r_3$.

Then you can say $\mathbf{r}_4$ is linearly dependent on $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ and that $\mathbf{r}_1$, r_2, $\mathbf{r}_3$ and $\mathbf{r}_4$ constitute a linearly dependent set of vectors. On the other hand r_1, r_2 and r_3 are linearly independent vectors.

To understand this concept, think of the graph of y = 2x + 3 where you choose **independent** values of x to calculate y for a table of values. So the values of y **depend** on your choice of values of x and you can say (rightly so) that y is the dependant variable, or simply put, the subject of the formula is the dependent variable.

So in general - the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$,.... are called linearly dependent if you can find a set of scalars a, b, c.... not all zero, so that $a\vec{\mathbf{A}} + b\vec{\mathbf{B}} + c\vec{\mathbf{C}} + ... = 0$

Otherwise they are linearly independent.

SELF-ASSESSMENT EXERCISE 3

Given vectors **u**, **v** and **w** below, determine whether the vectors are linearly independent or linearly dependent.

(a) **u** = 2**i** + **j** - 3**k**, **v** = **i** - 4**k** and **w** = 4**i** + 3**j** – **k**

Solution

Let a, b, c, be scalar such that a(2, 1,-3)+b(1 0 -4) +c(4 3 -])=O. If a, b, and c exist, then **u, v** and **w** are linearly dependent.

$\therefore$ 2a + b +4c = 0 ... (1)
a + 3 c = 0 ... (2)
-3a - 4b - c=0 ... (3)

From (2) a = - 3c. Substituting in (1) you will have -6c+b+4c = 0 i.e. -2c+b=0. $\therefore$ b = 2c. $\therefore$ a:b:cis-3c :2c :c =-3 :2 : 1

$\therefore$ a=-3, b=2, and c= 1.

To check, in (1), (2) and (3) above,

2(-3) + 3 + 4(1) = -6 + 2 + 4 = -6 + 6 =0
-3 + 3 = 0. and

+9 – 8 – 1 = 9 – 9 = 0.

.'. - 3 **u** + 2**v** + **w** = 0 and so **u, v** and **w** are linearly dependant since

3**u** = 2**v** + **w**

4.0 CONCLUSION

Vectors could be collinear or non-collinear.

- Collinear vectors can be proved by the equation $r=(nr_1,+mr_2)/(m+n)$, or $(m + n)\mathbf{r}=nr_1+mr_2$, where **r** is the position vector of a point on the line joining r_l and r_2, dividing it in the ratio m : n

- You can use the knowledge of Collinearity and non-Collinearity of points to prove the mid-point theorems.

- The vectors $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$, ... are called linearly dependent, if you can find a set of scalar a, b, c .... not all zero, such that $a\vec{\mathbf{A}} + b\vec{\mathbf{B}} + c\vec{\mathbf{C}} + ... = 0$. Otherwise they are linearly independent.

5.0 SUMMARY

1. The position vectors r of R which divides the line $\vec{\mathbf{PQ}}$ with position vector **p** and **q** respectively for point pad $\vec{\mathbf{Q}}$ in the ratio n:m is given by
$$r_2 = \frac{m\mathbf{p}+ n\mathbf{q}}{m + n}$$

2. The position vectors of the Centroids of $\mathbf{r}_1$, $\mathbf{r}_2$ ... $\mathbf{r}_n$ position vectors of masses $m_1, m_2 ... m_n$ is
$$r = \frac{M_1r_1 , + m_2 r_2 + . m_nr_n}{m_1+m_2 + ... + m_n}$$

3. If $a\vec{\mathbf{A}} + b\vec{\mathbf{B}} + c\vec{\mathbf{C}} + …. = 0$ then $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$ are linearly dependent, otherwise they are linearly independent.

6.0 TUTOR - MARKED ASSIGNMENT

1. The position vectors of points P and Q are given by
   r, = 2**i** + 3 **j** - **k**, **r** = 4**i** - 3 **j** + 2**k**
   (a) determine PQ in terms of **i, j,** and **k.**
   (b) find its magnitudes.

2. Prove that the vectors **u** = 3**i** + **j** - 2**k**
   v = **-i** + 3**j** + 4**k**, and **w** = 4**i** - 2**j** - 6**k** can form the sides of a triangle.

3 Determine whether the vectors u, v and w given below are linearly independent or dependent where u, v and w are non collinear vectors such that **u** = 2**a** - 3**b** + **c**, **v** = 3**a** - 5**b** + 2**c**, and **w** = 4a-5**b**+**c**

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 2 NON-COLLINEAR VECTORS

CONTENTS

1.0 INTRODUCTION

Having gone through collinear points in the last unit, you will now have a look at non-collinear vectors.

Once again the consequence of this concept will be treated. You will prove theorems in geometry based on the on the non-collinearity of points.

You will also learn about linearly dependent vectors.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define non-collinear vectors.
- prove the condition for non-collinear vectors.
- prove theorems in geometry using the idea of non-collinear vectors.
- prove that given vectors are linearly dependent.

3.0 MAIN CONTENT

3.1.1 Non-collinear and collinear vectors

You will recall that when the triangle law of vector addition was discussed in the previous units, you learnt that if, $\overrightarrow{\mathbf{AB}}$, and $\overrightarrow{\mathbf{BC}}$ are not on a straight line the resultant $\overrightarrow{\mathbf{AC}}$ is the vector closing the triangle ABC. The statement $\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} = \overrightarrow{\mathbf{AC}}$ is the very obvious, if they are on the same line.

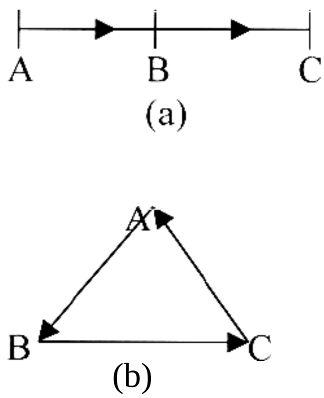


Fig V13

In the first case Fig V13 (a) you have collinear vectors A and B, but in the second case Fig V 13 (b) A, B, and C are said to be non-collinear vectors.

3.0 MAIN CONTENT

3.1 Definition of non-collinear vectors

Non-collinear vectors are vectors that are not parallel to the same line. Hence when their initial points coincide, they determine a plane.

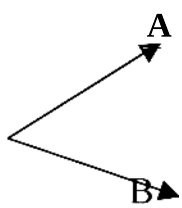


Fig. V14

See Figure V 14

3.2 Proof of a condition for non-collinear vectors

Example 1

Prove that if **a** and **b** are non-collinear that
X**a** + y**b**=0 implies x = y = 0.

Solutions:

Suppose x $\neq 0$, then you will have x**a** - y**b** = 0 implying x**a** = -y**b**
And **a** = $\frac{-y\mathbf{b}}{}$.

Which implies that **a** is a scalar multiple of **b** and so must be parallel to the same line (collinear). You can see this is contrary to the hypothesis **a** and **b** are non-collinear. .'. x = 0, them y**b** = 0, from which y = 0.

Example 2

If x_1**a** + y_1**b** = x_2**a** + y_2**b** where **a** and **b** are non-collinear, then $x_1 = x_2$, and $y_1 = y_2$.

Solution

You can write x_1**a** + y_1**b** = x_2**a** + y_2**b** as x_1**a** + y_1 **b** - (x_2**a** + y_2**b**) = 0 and so,
(x_1 - x_2) **a** + (y_1 - y_2) **b** = 0.

Hence, from example 1, $x_1 - x_2 = 0$, $y_I - y_2 = 0$
Or $x_1 = x_2$, $y_1 = y_2$

This you can interpret in words as two vectors are equal if their corresponding (scalar) coefficient are equal or their sum is zero.

The above result can be extended to the three dimensional situation. If **a, b** and **c** are **non-coplanar** (note here you are talking about planes rather than lines) then x**a** + y**b** + z**c** = 0, implies x = y = z = o
And further that if x_1**a** + y_1**b** + z_1**c** = x_2**a** + y_2**b** + z_2**c**, where **a, b** and **c** are non-coplanar, then $x_1= x_2$, $y_1 = y_2$ and $z_1 = z_2$

3.3 To prove that the diagonals of a parallelogram bisect each other

Prove that the diagonals of a parallelogram bisect each other.

Solution

Let ABCD be the given parallelogram with diagonals intersecting at P. From Figure

V 15

.'. $\overrightarrow{\mathbf{BD}}$= **b - a** and $\overrightarrow{\mathbf{BP}}$ = x (**b - a**) because $\overrightarrow{\mathbf{BD}}$ and $\overrightarrow{\mathbf{BP}}$ are collinear

You also have $\overrightarrow{\mathbf{AC}}$ = **a + b**, and so $\overrightarrow{\mathbf{AP}}$ = y (**a + b**)

From triangle ABP, $\overrightarrow{\mathbf{AB}}$ = $\overrightarrow{\mathbf{AP}}$ + $\overrightarrow{\mathbf{PB}}$ = $\overrightarrow{\mathbf{AP}}$ – $\overrightarrow{\mathbf{BP}}$ and you then have **a** = y (**a + b**) **-** x(**b – a**) = (x + y)**a** + (y - x)**b**

Since **a** and **b** are non-collinear, from Example 1, x + y = 1 and y **-** x = 0i.e. x = y = ½ and so P is the mid-point of both diagonals.

In conclusion you say that the diagonals of a parallelogram bisect each other. B

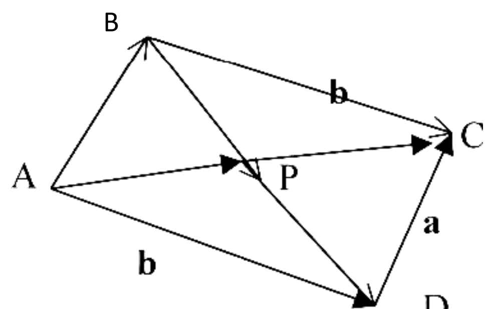


Fig. 15

3.4 To prove that the lines joining the mid-point of a quadrilateral form a parallelogram

If the mid points of the consecutive sides of any quadrilateral are connected by straight lines, prove that the resulting quadrilaterals is a parallelogram.

Solution:

Draw figure V 16
Note that in naming figures you must go A ---> B --> C -+ D without a break, clockwise or anti-clockwise.

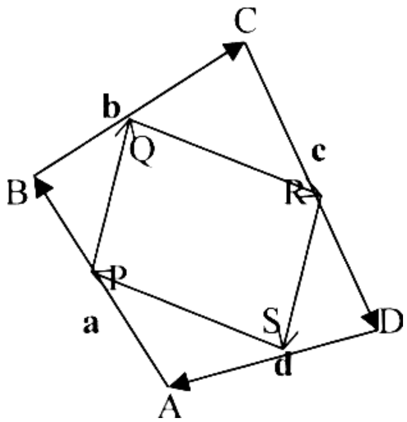


Fig V16

In the Fig. V16, you are having the sides as vectors **a, b, c**, and **d**. and the midpoints P, Q, R, S. of its sides.
Then you will have that,

$\overrightarrow{\mathbf{PQ}} = ½ (\mathbf{a} + \mathbf{b})$, $\overrightarrow{\mathbf{QR}} = ½(\mathbf{b} + \mathbf{c})$, $\overrightarrow{\mathbf{RS}} = ½(\mathbf{c} + \mathbf{d})$ and

$\overrightarrow{\mathbf{SP}} = ½(\mathbf{d} + \mathbf{a})$.

But from the directions of the vectors on the parallelogram, it is in equilibrium, which implies $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + = 0$

∴ $\overrightarrow{\mathbf{PQ}} = ½(\mathbf{a} + \mathbf{b}) = -½(\mathbf{c} + \mathbf{d}) = \overrightarrow{\mathbf{SR}}$ and

$\overrightarrow{\mathbf{QR}} = ½ (\mathbf{b} + \mathbf{c}) = ½ (\mathbf{d} + \mathbf{a}) = \overrightarrow{\mathbf{PS}}$

This implies that the opposite sides are equal and parallel (equal vectors) and so PQRS is a parallelogram (definition of a parallelogram).

4.0 CONCLUSION

Vector could be collinear or non-collinear.

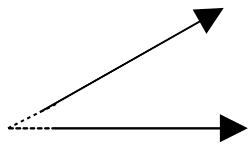
- For non-collinear vectors **a** and **b**, $x\mathbf{a} + y\mathbf{b} = 0 \Rightarrow x = y = 0$.
- You can use the knowledge of Collinearity and non-Collinearity of points to prove the following theorems amongst others:

(i) That the diagonals of a parallelogram bisect each other.

(ii) If straight lines connect the midpoints of the consecutive sides of any quadrilateral, the resulting quadrilateral is a parallelogram.

5.0 SUMMARY

- Non-collinear vectors are vectors that are not parallel to the same line. When their initial points coincides they determine a plane.



- If $x_1\mathbf{a} + y_1\mathbf{b} = x_2\mathbf{a} + y_2$ where **a** and **b** are non-collinear, then $x_1 = x_2$ and $y_1 = y_2$
- 3. If **a, b** and **c** are non-coplanar vectors, then $x\mathbf{a} + y\mathbf{b} + 2\mathbf{c} = 0$ implies $x = y = z = 0$.

- If $x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$ where **a, b** and **c** are non-coplanar then $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

6.0 TUTOR-MARKED ASSIGNMENT

If **u** and **v** are non-collinear vectors and $\vec{\mathbf{P}} = (x+4y)\,\mathbf{u} + (2x+y+1)\,\mathbf{v}$ and $\vec{\mathbf{Q}} = (y - 2x + 2)\mathbf{u} + (2x-3y-1)\,\mathbf{v}$.

Find x and y such that $3\vec{\mathbf{P}} = 2\vec{\mathbf{Q}}$

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

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UNIT 3 RECTANGULAR RESOLUTION OF VECTORS

CONTENTS

1.0 INTRODUCTION

The Rectangular resolution of the vectors is necessary because you have seen that vectors can be easily added if it is written in its components form.

If all that you are given, therefore is the magnitude and direction, you will be required to **resolve** (breakdown) the vectors into its components. This unit will help you to achieve this aim.

In other to understand the concepts in this unit easily, you will be restricted to the 2 dimensional (coplanar) vectors using the x - y-axis at first.

2.0. OBJECTIVES

At the end of this unit, you should be able to:

- resolve vectors successfully into its components
- find the component of Vectors in a particular direction
- find components of the sum of several vectors
- find the sum or resultant of several vectors.

3.0 MAIN CONTENT

3.1 Rectangular Resolution of Vectors

In the last Unit, you learnt that all vectors of its unit vector and its magnitude. You will resolve a given vector into its components.

Example 1:

Find the components of the following vectors in the direction of Ox and Oy.

(a) **u** of length 3cm direction 060°. (b) **v** of length 5cm direction 240°
(c) **w** of length 8cm direction 330°.

Solution

Make a rough sketch of each vectors.

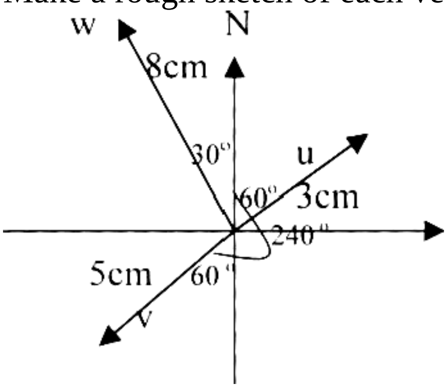


Fig. V17

(a) The component of **u** in the direction of Ox (x - axis) is x=3 Cos 30° = $3 \times \frac{\sqrt{3}}{3} = \sqrt{3}$

The component of **u** in the direction of Oy (y - axis) is
$3\cos 60° = 3 \times ½ = \frac{3}{2}$

$\therefore$ $\mathbf{u} = \sqrt{3}\,\mathbf{i} + \frac{3}{2}\mathbf{j}$

(b) The component of **v** along the x - axis is negative, $-5 \cos 30° = -5 \times \frac{\sqrt{3}}{2}$

$= \frac{5\sqrt{3}}{2}$

The component of **v** along the y - axis is negative, $-5\cos 60° = -5 \times ½$

$= \frac{5}{2}$

∴ $\mathbf{v} = -\frac{5\sqrt{3}\mathbf{i}}{2} - \frac{5\mathbf{j}}{2}$

(c) The component of **w** along the x - axis is negative; you will have $-8\cos 60' = -8 \times ½$

$= -4$

Along the y - axis, the component of **w** is positive,

$8\cos 30° = 8 \times \frac{\sqrt{3}}{2}$

$\mathbf{w} = -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$

You may wish to know that the choice of the angle to use in this case is the angle inclined to the required axis, which is the cosine multiplied by the magnitude. If you choose to use the angle inclined to the perpendicular axis then it will be the sine multiplied by the magnitude. This is easily done since the two angles are complementary. See fig. V18.

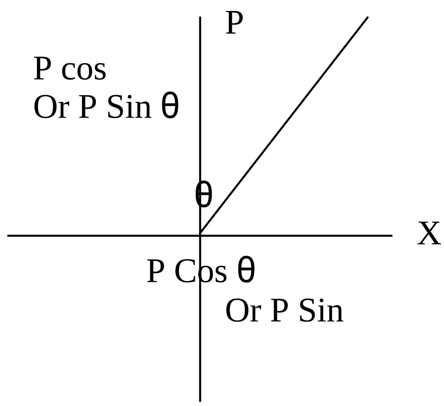


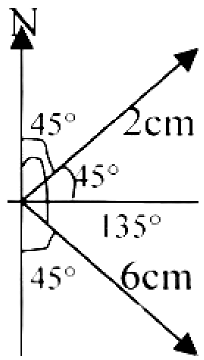
Fig. V18

SELF-ASSESSMENT EXERCISE 1

Find the component of the following vectors in the direction **i** and **j.**

(a) length 2cm direction 045°

(b) length 6cm direction 135°



(a) $2\text{Cos}45°\mathbf{i} + 2\cos 45°\mathbf{j}$

$= 2\text{x}\frac{\sqrt{2}}{2}\,\mathbf{i} + 2\text{x}\frac{\sqrt{2}}{2}\mathbf{j}$

$\sqrt{2}i + \sqrt{2}\ j$

(b) $6 \text{ x } \frac{\sqrt{2}}{2}\mathbf{i} - 6 \text{ x } \frac{\sqrt{2}}{2}\mathbf{j}$

$= 3\sqrt{2}\mathbf{i} - 3\sqrt{2}\mathbf{j}$

3.2 Resolution of two or more vectors.

Suppose **u** and **v** are vectors which are coplanar with **i** and **j** as in figure V20

Let the components of **u** be

u = u**i** + u**j** and for **v** = v**i** + v**j**

The resultant of this two vectors

is **u + v** = (u + v)**i** + (u + v)**j** which gives us the rules you had in the last unit.

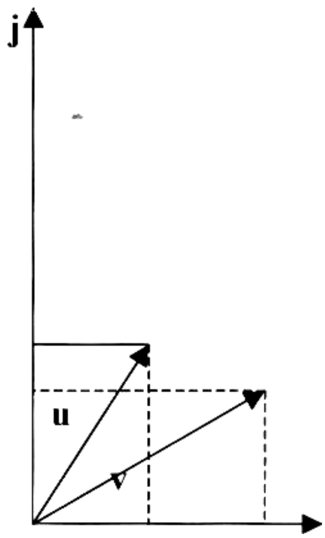


Fig. V20

3.3 Magnitudes and directions of resultants of vectors

Example 2

Find the magnitudes and direction of a simple vector that can represent the two vectors **u** and **v**.

u = 5 units direction 060° **v** = 4 units direction 240°

$= \frac{5\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$

v = -4Cos30° - 4Cos60°

$= -4 \times \frac{\sqrt{3}}{2}\mathbf{i} - 4 \times \frac{1}{2}$

$= -2\sqrt{3}\mathbf{i} - 2\mathbf{j}$

$\therefore \mathbf{u} + \mathbf{v} = (\frac{5\sqrt{3}}{2} - \frac{2\sqrt{3}}{2})\mathbf{i} + (\frac{5 - 2}{2})\mathbf{j}$

u + v = 0.866**i** + 0.5**j**

∴ The magnitude of the resultant is
0.866**i** + 0.5**j**

$\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$\sqrt{\frac{3}{4} + \frac{1}{4}}$

direction is $\tan^{-1} \frac{1\frac{1}{2}}{\sqrt{3}/2}$

$= \tan^{-1} \frac{\mathbf{1}}{4} \times \frac{2}{\sqrt{3}}$

$= \tan^{-1} \frac{1}{\sqrt{3}}$

= 30°

SELF-ASSESSMENT EXERCISE 2

If you have vectors **u, v,** and **w** **u** = 3**i** + 4**j**;
v = 2**i** - 3**j**;
w = -I + **j**

Find the direction of the following vectors. (a) **u** + **v** (b) **u - v** (c) **v** + **w** (d) **w - u**
Solution

(a) **u** + **v** = (3 + 2)**I** + (4 - 3)**j**
= 5**i** + **j**
.'. $\theta = \tan^{-1} \frac{I}{5}$
= 11.3°

You will state,' the direction of **u** + **v** as **u** is inclined at 11 ° to **v**. or in the direction 079°.

Example 3

Vectors of magnitude 4, 10, and 6 units lie in the direction of 045°, 090° and 135° respectively. Find

(a) the component of their sum in the direction of the Unit vector **i**
(b) the component of their sum in the direction of the unit vector **j**.
(c) the magnitude and direction of the single vector which is equal to their sum.

(a) Let the vectors be **u, v, w**, respectively

$\mathbf{u} = 4 \times \frac{\sqrt{2}}{2}\mathbf{i} + 4 \times \frac{\sqrt{2}}{2}\mathbf{j}$

$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}$

$\mathbf{v} = 10\mathbf{i} + 0\mathbf{j} = 10\mathbf{i}$

$\mathbf{w} = 6 \times \frac{\sqrt{2}}{2}\mathbf{i} - 6 \times \frac{\sqrt{2}}{2}\mathbf{j}$

$= +3\sqrt{2}\,\mathbf{i} - 3\sqrt{2}\,\mathbf{j}$

.'. sum of their component in the direction of **i** is $(2\sqrt{2} + 10 + 3\sqrt{2})\mathbf{i}$

= 10 +5~2 = 7.07**i**

(b) In the direction of **j** = $(2\sqrt{2} - 3\sqrt{2})j$

$= -\sqrt{2}j$

(c) $|\mathbf{u} + \mathbf{v}| = \sqrt{7.07^2 + (\sqrt{2})^2}$

$= \sqrt{50 + 2} = \sqrt{52} = 7.2$

$\theta = \tan^{-1} \frac{-\sqrt{2}}{10 + 5\sqrt{2}}$

4.0 CONCLUSION

A vector can be resolved into its components, given its magnitude and directions.

You only need to know that the adjacent side to the given angle takes the cosine, while the opposite side takes the Sine, to be able to write the vectors in its components.

If the two or more vectors are given, they are resolved and their resultant gives the vectors in components form.

5.0 SUMMARY

If the magnitude and direction of a vector is given, you can resolve it into its components parts e.g.

In figure V21

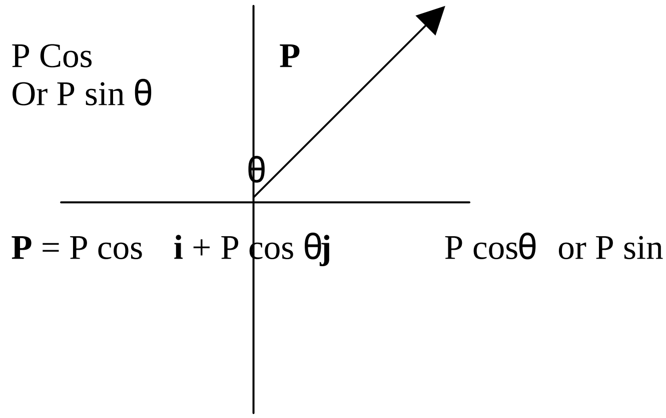


Fig. V21

6.0 TUTOR-MARKED ASSIGNMENT

l. Vectors of magnitude 4, 3, 2 and 1 unit respectively are directed along $\overrightarrow{\mathbf{AB}}$, $\overrightarrow{\mathbf{AC}}$, $\overrightarrow{\mathbf{AD}}$, and $\overrightarrow{\mathbf{AE}}$. $\overrightarrow{\mathbf{AB}}$ is in the same direction as the Unit Vector **j** and BAC = 30°, CAD = 30°, DAC = 90°, Find

a. the components of $\overrightarrow{\mathbf{AB}}$, $\overrightarrow{\mathbf{AC}}$, $\overrightarrow{\mathbf{AD}}$, $\overrightarrow{\mathbf{AE}}$

b. the magnitude and direction of the sum of the vectors.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 4 THE SCALAR OR DOT PRODUCTS

CONTENTS

1.0 INTRODUCTION

In this unit you will be introduced to the first product of vectors; the scalar or dot product.

This product gives a scalar result and so the term `dot' product.

The algebraic laws on this product will give you an easy way to calculate these products, so pay attention to the laws.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define scalar or dot product
- state the algebraic laws on dot product
- calculate easily, without diagrams, the dot product of two given vectors.

3.0 MAIN CONTENT

3.1 Product of Vector

There are two ways a vector can be multiplied as opposed to the scalar multiplication in the previous unit.

These are the Dot or Scalar Product, and the vector or cross product. In this unit you are learning about the Dot or Scalar Product.

3.2 The Dot or Scalar Products.

Definition

The Dot or Scalar Product denoted **u.v** (u dot v) is defined as the product of the magnitude of a and v and cosine of the angle θ^2 between them you write in symbols.

u.v = uv cos $\theta, 0^o \leq \theta \leq 2\Pi$

This product gives a scalar, Since u, v, and Cosθ are scalar, and so the term scalar product.

3.2.1 Algebraic Laws on Dot Product

1. **u.v** = **v.u** (commutative)
2. **u.** (**v + w**) = **u.v** + **u.w** (distributive)
3. m (u.v) =(mu)v = u.(m**v**)
4. **i.i** = **j.j** = **k.k** = l x l cos 0 = l x lx l = l
   but **i.j** = **j.k** + **k.**j = 1 x 1 x cos 90° = 1 x 1 x 0 = 0

From this you once again have a very easy way to find the dot product of the vectors.

If **u** = x_1 **i** + y_1 **j** + z_1 **k** and **v** = x_2**j** + y_2**j** + z_2**k**
Then **u.v** = (x_1 **i** + y_1**j** + z_1**k**) (x_1**i** + y_2**j** + z_2 **k**)
=(x_1, x_2 + y_1 y_2 + z_1, z_2)

Since **i.j** = **j.k** = **j.i** = and only same unit vectors multiply to give 1

Example 1

Prove that **u.v** = **v.u** the commutative law.

Solution

u.v = uv cos = **v. u** Since uv = vu are scalar. And is the angle between **u** and **v**

3.2.2 The dot product of perpendicular vectors

The dot product of perpendicular vectors is zero.

Proof

u.v = uvcos90° = uv x 0= 0

3.2.3 Dot product of vectors in component form

SELF-ASSESSMENT EXERCISE 1

Given that $\mathbf{r}_1 = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{r}_2 = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$ $\mathbf{r}_3 = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Calculate

a. $\mathbf{r}_1 . \mathbf{r}_2$ b. $\mathbf{r}_2 . \mathbf{r}_3$ c. $\mathbf{r}_2 . \mathbf{r}_1$
d. $\mathbf{r}_1 . (\mathbf{r}_2 + \mathbf{r}_3)$

Solutions

(a) $\mathbf{r}_1.\mathbf{r}_2$ = (3 x 2) + (-2 x - 4) + (1 x -3)
= 6 + 8 - 3
= 14 - 3
= 11

(b) $\mathbf{r}_2.\mathbf{r}_3$ = (2 x -1) + (-4 x 2) + (-3 x +2)
= -2 - 8 - 6
= -16

(c) $\mathbf{r}_3.\mathbf{r}_1$ = (-1 x 3) + (2 x -2) + (2 x 1)
= –3 – 4 + 2
= -7+2
= -5

(d) $\mathbf{r_1}$ ($\mathbf{r_2}$+$\mathbf{r_3}$) = $\mathbf{r}_1$ ($\mathbf{r}_2$+$\mathbf{r}_3$)
= (3**i** - 2**j** + **k**) (2 - 1)**i** + (-y +2) **j** + (-3 + 2) **k**

= (3**i** - 2**j** + **k**) (**I** - 2**j** - **k**)
= (3 x 1) + (-2 x -2) + (l x -1)
= 3 + 4 - 1
= 7 -1
= 6

Example 2

Find the dot product of the following vectors. r_2 = 2**i** + 3**j** - 5**k**
r_2 = i - 2**j** + 4**k**

Solution

$r_1.r_2$ = (2**i** + 3**j** - 5**k**) (**I** - 2**j** + 4**k**)
= (2 x 1) + (3 x - 2) + (-5 x 4)
= 2 - 6 -20
= 2 – 26
= - 24

SELF-ASSESSMENT EXERCISE 2

The position vectors of P and Q are r_2 = 2**i** + 3 **j** - **k**
r_2 = 4**i** - 3**j** + 2**k**

(a) Determine **PQ** (b) Find $\mathbf{r_1.r_2}$

Solution

(a) **PQ** = q – p
= $r_2 - r_1$
= (4 - 2)**i** + (-3 - 3)**j** + (3-(-1
= 2**i** - 6**j** + 3**k**

(b) $\mathbf{r_1.r_2}$ = (2 x 4) + (3 x -3) + (-1 x 2)
= 8 - 9 - 3
= 8- 12
= -4

4.0 CONCLUSION

You have just learnt a very important product of vectors, the Dot Scalar Product.

The word dot comes from the way you write this product **a.b** and the result is always an ordinary number or scalar, hence the term scalar product.

Despite the definition **a.b** = ab cos, you have seen how easy it really is to get the Dot product by simply adding the products of the coefficients of **i, j** and **k**. This is due to the Algebraic laws of vector Algebra.

You are now ready for the next Units, which will use the consequence of the Algebraic laws to derive some results.

5.0 SUMMARY

- The Scalar dot product of a and b is a.b = ab cos θ where θ is the angle between them.
- The algebraic laws holds in vector algebra
- If **a**=x_1**i** + y_l**j** + 2**k** and **b** = x_2**i** + y_2**j** + 2_2k_2 then **a.b** = (x_1x_2 + y_Iy_2 + $z_1z_{2)}$

6.0 TUTOR-MARKED ASSIGNMENT

1. Determine the value of a so that

u = 2**i** + a**j** + **k** and **v** = 4**i** - 2**j** - 2**k** are perpendicular.

2. If **u**= **I** + 3**j** - 2**k** and **v** = 4**i** - 2**j** - 4**k**

Find (a) **u.v**, (b) u (c) v (d) 3**u** + **v** I (e) (2**u** + **v**) . (**u** - 2**v**)

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 5 PROPERTIES OF SCALAR OR. DOT PRODUCTS

CONTENTS

1.0 INTRODUCTION

In this unit, you will be looking at some properties of scalar or dot products. These are due to algebraic laws on dot products.

You will be able to calculate the angle between two vectors.

You will be able to find the projection of a vector along another vector, a very useful conception in mechanics.

The proof of the popular cosine rule in trigonometry will also express the equation of a plane using the dot products.

You should concentrate on the use of dot product in these calculations, as there are other approaches to them.

2.0 OBJECTIVE

At the end of this Unit, you should be able to:

- calculate easily the angles between two vectors without diagrams
- express and calculate the projection of a vector along another vector.

3.0 MAIN CONTENT

3.1 The definition of Scalar or Dot Product.

The Dot Product of vector **u** and **v** is **u.v** = **uv** cos where cos is the angle between them.

The algebraic laws on Scalar or Dot Products are:

1. **u.v** = **v.u** (commutative laws.)

2. **u**. (**v** + **w**) = **u.v** + **u.w** (Distributive)

3. m (**u.v**) = (m**u**).**v** = (**u.v**)m where m is scalar

4. **i.i** = **i.i** = **k.k** = **i** x **i** cos **0** = **l x I x 1**=**1**
   but **i.i** = **i.k** =**i** x **i** cos 90°=**1 x l x 0** =**0**

5. The dot product of $\mathbf{x_1 i}$ + **y j** + 2 **k** and $\mathbf{x_2}$ **i** + $\mathbf{y_2}$ **j**+ $\mathbf{z_2 k}$
   is (x_I x_2 + y_1 y_2 + z_I z_2).

6. If **u.v** = **0**, and **u** and **v** are not null vectors, then **u** and **v** are perpendicular vectors.

7. 1 **u** 1=√**u.u**

8. **u.u** > 0 for any non **-** zero vector **u**

9. **u.u** = 0 only if **u** = 0.

3.2 Angle between two vectors.

From the definition of dot product and with the method of calculating the dot product, you can find θ from

$$\cos\theta = \frac{\mathbf{u.v}}{uv} \quad \text{i.e.} \quad \frac{\text{The Dot Product}}{\text{the product of their magnitude}}$$

Example 1

Find the angle between the vectors **u**= 2**i** + 2**i** - **k** and **v** = 6 **i-** 3**j** + 2**k** **u.v** = (2 x 6 +

(2x - 3) + (-1 x +2))
=12 - 6 - 2
=12 – 8 = 4
u = |u| = $\sqrt{2^2+2^2+(1)^2}$
=$\sqrt{4 + 4 + 1}$
=$\sqrt{9} = 3$

v = |v| =$\sqrt{6^2 + (-3)^2 + 2^2}$
=$\sqrt{36 + 9 + 4}$
=$\sqrt{49} = 7$

$\therefore \cos\theta = \frac{\mathbf{u.v}}{uv} = \frac{4}{3 \times 7}$

$\therefore \theta = \cos^{-1}\frac{4}{21} = 79°$

Therefore, the angle between **v** and **v** is 79°

SELF-ASSESSMENT EXERCISE 1

Find the angle between
u = 4**i** - 2**j** + 4**k** and **v** = 3**i** – 6**i** - 2**k**

Solution

u.v = (4 x 3) + (-2 x -6) + (4 x -2)
= 12+ 12 -8
=24 - 8
=16

|u| = $\sqrt{4^2 + (-2)^2 + 4^2}$
= $\sqrt{16 + 4 + 16}$
= $\sqrt{36} = 6$

|v| = $\sqrt{32 = (-6)^2 (-2)^2}$
=$\sqrt{9 + 36 + 4}$
= $\sqrt{49}$
= -7

.. $\theta = \text{Cos}^{-1}\frac{16}{6 \times 7}$

= $\text{Cos}^{-1}\frac{8}{21}$

= 67.6°
...The angle between **u** and **v** is 67.6°

3.3 Projection of a vector along another vector.

You can define the projection of a vector (the component) **u** along any other vector **v** using the concept of Scalar product.

Suppose u and v are non-zero vectors and the angle between them is θ Then the real number p is given from

$$\cos\theta = \frac{\mathbf{u.v}}{uv} \text{ as } p = \frac{\mathbf{u.v}}{v}$$

$$\text{or } p = \mathbf{u}\cos\theta = \frac{\mathbf{u\ .v}}{|\mathbf{v}|}$$

You refer to p as the projection or component of **u** in the direction of **v**.
If **u** = 0, then θ is undefined and we set p = 0, Similarly the projection of **v** on **u** is the number $q = v \text{ I } \cos\theta = \frac{\mathbf{u.v}}{|\mathbf{u}|}$

Example 2

Find the projection of the vector 2**i - 3i** + 6**k** on the vector **i** + 2**j** + 2**k**

Solution

Let **u** = 2**i - 3i** + 6**k** and **v** = **i** + 2**j** + 2**k** then the projection **u** in the direction of **v** is $P = \frac{\mathbf{u.v}}{|\mathbf{v}|}$

u.v = (2 x 1) +(-3 x 2) + (6 x 2)
= 2 – 6 + 12
= 14-6
= 8

$|\mathbf{v}| = \sqrt{1^2 + 2^2 + 2^2}$
$= \sqrt{9}$
= -3
$\therefore P = \frac{8}{3} = 2$

SELF-ASSESSMENT EXERCISE 2

Find the projection of the vector **u** = 4**i** - 3**j** + **k** on the line passing through the points P (2, 3, -1) and Q (-2, -4, 3).

Solution

The relative vector of PQ = **r** = (-2, -4, 3) - (2, 3, -1) = -4**i** - 7**j** + 4**k**.

.'. The projection of **u** along PQ = $\frac{\mathbf{u} \cdot \mathbf{r}}{r}$

$$= \frac{(4 \times -4) + (-3 \times -7) + (1 \times 4)}{\sqrt{(-4)^2 + (-7)^2 + 4^2}}$$

$$= \frac{25-16}{\sqrt{81}}$$

$$= \frac{9}{9} = 1$$

3.2.1 Cosine Rule Example 3

Prove the law of cosine for plane triangle.

Solution:

From the Figure V23

$\mathbf{q} + \mathbf{r} = \mathbf{P}$

$\mathbf{r} = \mathbf{P} - \mathbf{q}$

$\mathbf{r}.\mathbf{r} = (\mathbf{P} - \mathbf{q}).(\mathbf{P} - \mathbf{q})$

$\mathbf{r}^2 = p^2 + q^2 - 2\mathbf{q}.\mathbf{p}$

$= p + q^2 - 2q \,.\, p \cos \theta$

Which is the cosine formula.

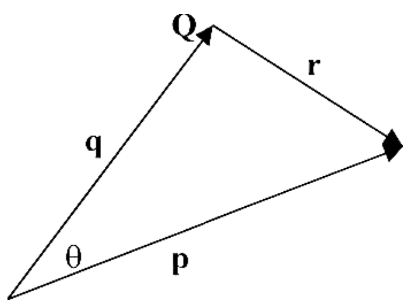


Fig. V23

3.2.2 Equation of a plane.

Example 4

Determine the equation for the plane perpendicular to the vector **P** = 2**i** + 3 **j** +6**k** and passing through the terminal point of the vector **q** = **i** + 5**j** + 3 **k**.

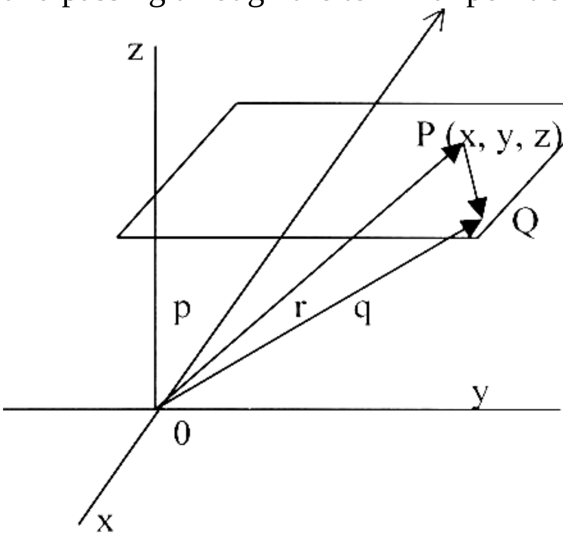


Fig. V24

Solution:

See Fig. V24
Let r be the position vector and Q the terminal point of **q**. Since PQ = **q** – **r** is perpendicular to **p**, then (**q - r**). P = 0
p=0 or **r.p**=**q.p** is the required equation of the plane in vector form. i.e. (x**i** + y**j** + 2**k**). (2**i** + 3**j** + 6**k**)
= (**i** + 5**j** + 3**k**). (2**i** + 3**j** + 6**k**)
2x + 3y + 6z = 2 + 15 +18
2x + 3y + 6z = 35 is the required equation of the plane.

3.2.3 Perpendicular Vectors Exercise 3

Determine the value of y so that
u = 2**i** + y**j** + **k** and **v** = 4**i - 2j -2k** are perpendicular.

Solutions

You have read from the algebraic laws on dot product that **u** and **v** are perpendicular

If **u.v** = o
... **u.v** = (2**i** + y**j** + **k**), (4**i** - 2**j** - 2**k**)
= (2 x 4) + (y x -2) + (1 x -2)
= 8 - 2y - 2
.-.8 - 2y – 2 = 0
2y = 6 y = 3

SELF-ASSESSMENT EXERCISE 4

Show that the vectors **u** = 3**i** - 2**j** + **k**,
v = **i** -3**j**+ 5**k**, and **w**= 2**i** + **j** - 4**k** form a right-angled triangle.

Solution

You will show, first, that the vectors from a triangle.
You will need to apply the triangle law to the sides of the triangle by checking if

u + **u** + **w** = 0 OR

(i) **u** + **v** = **w** or **u** + **w** = **v** or **v** + **w** = **u**
that is, will the sum of two of the vectors give the third vector (Fig. V25)

(ii) The resultant of the three vectors is zero. This will remove the doubt of the three positions vectors being collinear.

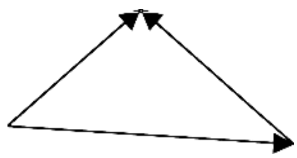


Fig. V25

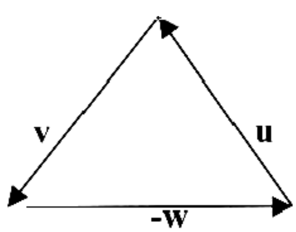


Fig. V26

(1) By trial, you will discover that
v + **w** is = (1 +2)**i** + (-3 + 1)**j** + (5 - 4)**k**
= 3**i** - 2**j** + **k** = **u** given
(iii) **v** + **w** – **u** = 3**i** – 2 + **k** -(3**i** - 2**j** + **k**) = 0

Therefore the vector to form a triangle, for the triangle to be right angled, the dot product of the two vectors containing the right angle must be zero. Again by trial **u.w** = 0. from
u.v = (3**i** - 2**j** + **k**) . (**i** - 3**j** + 5**k**)
= 3 + 6 + 5 = 0

v.w = (3**i** - 2**j** + **k**).(2**i** + **j** - 4**k**)= 6 – 2 – 4 = 6 – 6 = 0 and so the triangle is right angled.

4.0 CONCLUSION

The scalar or dot product and the algebraic laws on it has a lot of properties , some of which you have studied in this unit.

You can calculate the angle between two vectors and the projection of a vector along another vector.

Using dot product you can easily prove the cosine rule.

5.0 SUMMARY

In this Unit you have learnt:

1. The angle between two vectors **u** and **v**, θ is cos θ
   $\cos\theta = \frac{\mathbf{u.v}}{uv}$, $= \cos^{-1} \frac{\mathbf{u.v}}{uv}$

2. The projection of vector **u** along **v** is
   $\frac{\mathbf{u.v.}}{|\mathbf{v}|}$ and of **v** along **u** = $\frac{\mathbf{u.v}}{|\mathbf{u}|}$

3. The equation of a plane passing through a vector **v** and perpendicular to a vector **u** is (**v** - **r**) . **u** = 0 where r is the position vector of an arbitrary P (x, y, z).**v.u** - **r.u** = 0 or **v.u** = **r.u.**

6.0 TUTOR - MARKED ASSIGNMENT

1. Vector **u** = 3 **i** -2**j** + **k** and **v** = -2**i** + 2**j** + **4** k

(i) Find the magnitudes of **u**, and **v.**
(ii) The angle between **u** and **v**.
(iii) Find the projection of the vector **u** +½**v** along **u**.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.