MODULE 3

UNIT 1 DIRECTION COSINES

CONTENTS

1.0 INTRODUCTION

One of the consequences of dot products is the direction cosine.

This is a concept involving the use of unit vectors, which you are already familiar with. The coefficient of the components of this unit vectors gives the direction cosines.

From this simple calculation, a lot of other concept will be learnt by you.
In this you will learn about this consequence of dot product and the direction cosine.

2.0 OBJECTIVES

At the end of this unit you will be able to:

- calculate the direction cosines of a given vector

- calculate the angles to which a given vector is inclined to the co- ordinate axis
- find the equation of a plane passing through a point and the perpendicular to the line joining two given vectors.

3.0 MAIN CONTENT

3.1 Direction cosines

If you have a vector reduced to its unit vector, then the magnitude is 1. You can then find the inclination of the vector to the axis easily as it will be 1 cos , 1 cos β, and 1 cos γ represent the vectors inclination to the x, y, and z axes respectively.

To make this easy all you need is to reduce the vector into its unit form, then the coefficient of **i, j** and **k** represent cos, cos β and cos y respectively. You refer to this coefficient as the direction cosine of the vector, since they express the vector's inclination to the axis.

Example 1

Find the direction cosines of the line joining the points (3,2, -4) and (1-2,2)

Solution

Let A (3, 2, -4) and B (l, -1, 2)
Then AB = **b** – **a** = (1 - 3)**i** + (-1 - 2)**j** + (2 - (-4))**k**

= -2**i** - 3**j** + 6**k**

The Unit vector $\mathbf{e}_r = \dfrac{-2\mathbf{i} -3\,\mathbf{j} + 6\mathbf{k}}{\sqrt{(2)^2 + (-3\,)^2 + 6^2}}$

$= \dfrac{2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}}{\sqrt{49}}$

$= \dfrac{-2\mathbf{i} - 3\mathbf{i} + 6\mathbf{k}}{7}$

The direction cosines are $\dfrac{-2}{7}, \dfrac{-3}{7}, \dfrac{6}{7}$

SELF-ASSESSMENT EXERCISE 3

Find the acute angle which the line joining the points (1 -3, 2) and (3, -5, 1) makes with the coordinate axis.

Solution:

a = **i-** 3**j** + 2**k** **b** = 3**i** - 5**j** + **k** **r** = **b** – **a**

= (3 - 1) **i** + (-5 -(-3))**j** + (i - 2)**k**

= 2**i** - 2**j** – **k**

$\therefore e_r = \frac{2\mathbf{i} - 2\mathbf{i} - \mathbf{k}}{\sqrt{4+4+1}}$

$= \frac{2\mathbf{i} - 2\mathbf{j} - \mathbf{k}}{3}$

The direction cosines are $\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}$

and the angles are $\cos^{-1}\frac{2}{3}$, $\cos^{-1}\frac{-2}{3}$, and $\cos^{-1}\frac{-1}{3}$

to x, y, and z axis respectively.
You will have 48.2°, 48.2° and 70.5°

Example 2
Find the angles which the vector **u** = 3**i** - 6**j** + 2**k** makes with the coordinate axes.

Solution
From the previous unit, you should know you are to look for the direction cosines of the vector **u** to start with.
u = 3**i** - 6**j** + 2**k**
$\mathbf{e}_r = \frac{3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}}{\sqrt{9+36+4}}$

$= \frac{3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}}{7}$

The direction cosines are $\frac{3}{7}$, $-\frac{6}{7}$ and $\frac{2}{7}$.

Let α and β and y represent the required angle.

∴ , the inclination to the x-axis is $= \cos^{-1} \frac{3}{7} = 64.1°$

β, the inclination to the y - axis is $R = \cos^{-1} \frac{-6}{7}$

$= 180° - \cos^{-1} \frac{6}{7}$

$= 180° - 31°$

$= 149°$

and y, the inclination to the z -axis is $y = \cos^{-1} \frac{2}{7} = 73.4°$

SELF-ASSESSMENT EXERCISE 1

Find the acute angles which the line joining the points (3, -5, 1) and (1 - 3 + 2) makes with the coordinate axes.

Solution:

You should use relative vector for $\overrightarrow{\mathbf{UV}} = \mathbf{v} - \mathbf{u}$ to get the position vectors of the line joining **u** and **v** where **u** =3**i** - 5**j**+ **k**, and **u** = **i** - 3**j** + 2**k**.

∴ $\overrightarrow{UV} = \mathbf{v} - \mathbf{u} = (1 - 3)\mathbf{i} + (-3 + 5)\mathbf{j} + (2 - 1)\mathbf{k}$ $\mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

∴ The direction cosines of **r** will be the coefficient of the unit + vector **e** in the direction of **r**.

$$\mathbf{e}_v = \frac{-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{4 + 4 + 1}}$$

$$= \frac{-2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

and $= \cos^{-1}\frac{2}{3}$, $\beta = \cos \frac{2}{3}$, $y = \cos^{-1} \frac{1}{3}$

∴ α = 48.2°, β = 48.2°, and y = 70.5°

Note that α should have been $\cos^{-1} 2/3$ which will give 180 - 48.2°, but the question demand for `acute' angle not the obtuse angle, which is only adjacent to the acute angle 48.2°.

SELF-ASSESSMENT EXERCISE 2

For what values of p are the vectors
u = 2p**i** + p**j** - 4**k** and **v** = p**i** - 2**j** + **k** perpendicular?

Solution

You need to show that the dot product **u.v** = 0 will imply the solved P.
u.v = (2p x p) + (p x -2) + (-4 x 1) = 0
= $2p^2 - 2p - 4 = 0$. A quadratic divide through by 2 equation in p $p^2 - p - 2 = 0$
Factorize to get
(p - 2)(p + 1) = 0.
∴ p = 2 or -1

3.1.3. Orthogonal Unit Vector

Example 3.

Show that **u** =2**i** - 2**j** + **k**/3, **v** =**i** + 2**j** + 2**k**/3
and **w** = 2**i** + **j** - 2**k**/3 are mutually orthogonal unit vectors.

Solution

$$\mathbf{u.v} = \left(\frac{2}{3} \times \frac{1}{3}\right) + \left(-\frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{1}{3} \times \frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\mathbf{v.w} = \left(\frac{1}{3} \times \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3}\right) + \left(-\frac{2}{3} \times \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$$\mathbf{u.w} = \left(\frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{-2}{3} \times \frac{1}{3}\right) + \left(\frac{I}{3} \times \frac{-2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

This proves that they are perpendicular or to use the word in the question, \`orthogonal'.

That they are unit vectors is easily proved by your writing each vector a m**u**, m**v**, m**w** where m is a scalar in this case 1/3 and that the magnitude of **u, v, w** are each $\sqrt{\underline{9}}$, $\sqrt{1}$, 1.9
From the definition of unit vector as a vector whose magnitude is one, you have your proof.

∴ **u, v, w** are mutually orthogonal unit vectors.

3.4 To prove that the diagonals of a rhombus are perpendicular

Example 4
Prove that the diagonals of a rhombus are perpendicular.

Solution:

Draw a rhombus ABCD

as in figure V27

$\vec{DB} = \vec{OA} + \vec{AB} = \mathbf{u} + \mathbf{v}$

$\vec{DC} + \vec{CA} = \vec{DA}$

$\therefore \vec{CA} = \vec{DA} - \vec{DC} = \mathbf{u} - \mathbf{v}$

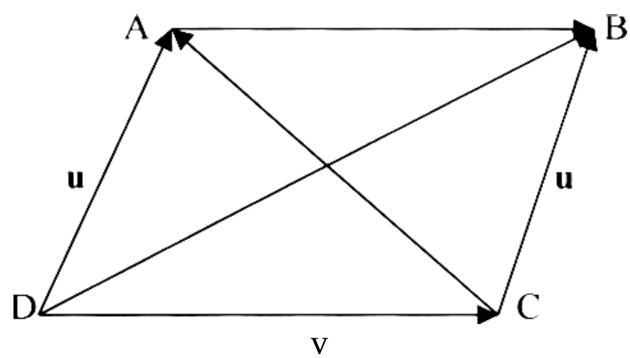


Fig. V27

$\vec{DB} . \vec{CA} = (\mathbf{u} + \mathbf{v}). (\mathbf{u} - \mathbf{v})$ (Difference of two squares)
$= u^2 - v^2 = u^2 - u^2 = 0$. Since **u** = **v** being a rhombus, all sides are equal. You can now conclude that the diagonals DB and CA are perpendicular, because their dot product is zero.

4.0 CONCLUSION

In the study of mathematics, the language of science, every effort is made to reduce any fear associated with calculations. This unit is one of such efforts e.g. The existence of direction cosines makes it easy for you to calculate a lot of quantities such as:

- The inclination of a vector to the axis.
- To prove some simple geometrical problems.

5.0 SUMMARY

In this unit you have come across the following:

- The dot product of two perpendicular vectors is zero.
- The Direction cosine of a vector is the coefficient of the unit vector in the direction of the vectors
- The direction cosines, gives cosine of the indirection of the vector to the x, y, z axes.
- You can use dot product to prove that the diagonals of a rhombus are perpendicular.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that the angle between the vectors
   u = 3**i** + 2**j** - 6**k**, and **v** = 4**i** - 3**j** + **k** is a right angle.
2. For what values of z are the vectors **u** = 2**i** - 2**j** + z**k**
   and **v** = 4**i** + 2**j** - 4z**k** perpendicular.
2. Find the angles which the vector **u** = 4**i** - 3**j** + **k** makes with the co-ordinate axis.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 2 APPLICATIONS OF SCALAR OR DOT PRODUCTS

CONTENTS

1.0 INTRODUCTION

In the last Unit you were introduced to Scalar or Dot product.

In this Unit, an attempt will be made to pick as many examples as possible where the Dot product is applied in calculations including plane Geometry. These calculation have been made easier by the use of dot product even though they are in three dimensions.

You will appreciate these examples, if you are going on to study physics- related subject or higher Geometry.

2.0 OBJECTIVE

At the end of this Unit, you should be able to:

- calculate work done by a given force on a particle through a given displacement using dot product
- determine when the work done on a particle is zero, and maximum
- calculate accurately the Direction Cosines of a vector, hence, find the inclination of the vector to the axis

- calculate the perpendicular to a given plane
- calculate the distance from the origin to a given plane.

3.0 MAIN CONTENT

3.1.1 Work expressed as a scalar Product

Work expressed as a Scalar product work is defined as the force applied multiplied by the distance moved in order words, if there is no distance or displacement of the object, then work done is zero, this also occurs when the F is perpendicular to the displacement .

In terms of Scalar Product, let a force F be applied on an object at an angle θ to the direction of its displacement d. Then

$W = (F \cos \theta)\, d = F.d$

Work is maximum when the force F is parallel to the displacement.

Example 1

Find the work done in moving an object along a vector $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, if the applied force is $F = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

Solution:

Work done = $\vec{F.r}$
$= (2\mathbf{i} - \mathbf{j} - \mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$
$= (2 \times 3) + (-1 \times 2) + (-1 \times -5)$
$= 6 - 2 + 5$
$= 11 - 2$
$= 9$

SELF-ASSESSMENT EXERCISE 1

Find the work done in moving an object along a straight line from (3, 2, -1) to (2, -l, 4) in a force field given by $F = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Solution:

The resultant displacement is
$\mathbf{d} = \mathbf{r}_2 - \mathbf{r}_1 = (2 - 1 , 4) - (3, 2, -1)$

= **-i** -3**j** + 5**k**
.'. work done W = F.**d**
= (4**i** - 3 **j** + 2**k**). (-**i** - 3**j** + 5**k**)
= (4 x -1) +(-3 x -3) + (2 x 5)
= -4 + 9 + 10
= 19 - 4 =15

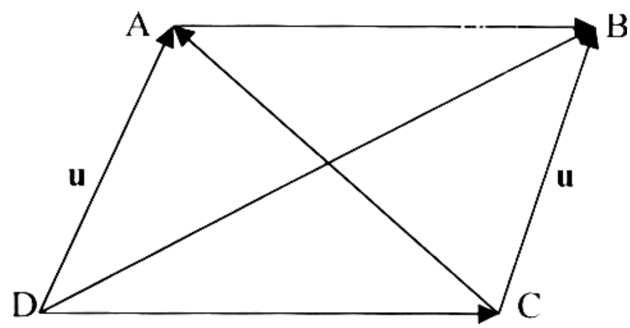
3.2 Perpendicular vectors

Example 2

Prove that the diagonals of a rhombus are perpendicular.

Solution:

Draw a rhombus ABCD
as in figure V28
$\vec{DB} = \vec{OA} + \vec{AB} = \mathbf{u} + \mathbf{v}$
$\vec{DC} + \vec{CA} = \vec{DA}$
.'. $\vec{CA} = \vec{DA} - \vec{DC} = \mathbf{u} - \mathbf{v}$



v

Fig. V28

.'. $\vec{DB}$. $\vec{CA}$ = (u + v). (u - v) (Difference of two squares)
$=u^2 - v^1 = u^2 - u^2 = 0$. Since **u** = **v** being a rhombus, all sides are equal. You can now conclude that the diagonals DB and CA are perpendicular, because their dot product is zero.

3.3 Perpendicular to a plane

Determine a unit vector perpendicular to the plane of **u** and **v**. Where **u** = 2**i** - 6**j** - 3**k** and **v** = 4**i** - 3**j** - **k**.

Solution

You chose a vector **w** = w_1**i** + w_2**j** + w_3**k** to be perpendicular to the plane of **u** and **v**. what you mean here is that w is perpendicular to both **u** and **v** and so their dot product will be zero each.

w.u = $2w_1 - 6w_2 - 3w_3 = 0$

$2w_1 - 6W_2 = 3W_3$ ---- (1)

w.v = $4w_1 + 3w_2 - w_3 = 0$

$4w_1 + 3w_2 = w_3$ ----- (2)

Solve (1) and (2) simultaneously to express w_1 and w_2 in terms of w_3. You will have

$W_1 = \frac{1}{2} w_3$, $w_2 = \frac{-1}{2} w_3$, and

w = w_3 (I**i** - $\frac{1}{2}$ **j** + **k** and so the unit vector in the direction of w,
$\qquad$ 3

ew = $\frac{\mathbf{w}}{w2}$ = w_3 ($\frac{Ii}{}$ -$\frac{1\ j}{3}$ + k)

$\sqrt{w2_3}$ $(\frac{1}{2})^2 + (\frac{-1}{3})^2 + (\frac{1}{})^2$

= $\pm(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k})$

SELF-ASSESSMENT EXERCISE 2

Find a unit vector perpendicular to both u and v where vectors
U = 4**i** – 5 + 3**k**, and **v** = -2**i** + **j** - 2**k**

Solution

Let w = w_1**i** + w_2**j** + w_3**k** be the required unit vector.
Then you can say
w.u =0 **w.v** since **w** is perpendicular to both **u** and **v**.
w.u = $4w_1 - w_2 + 3w_3 = 0$

$\therefore 4w_1 + 3w_3 = w_2$ -------- (1)

w.v = $-2w_1 + w_2 - 2w_3 = 0$

$2w_1 + 2w_3 = w_2$ --------- (2)

Now solve (1) and (2) simultaneous to express w_l and w_3 in terms of w_2. $4w_1$ $+ 3w_3 = 2w_1 + 2w_3$

$4w_1 - 2w_1 = 2w_3 - 3w_3$

$2w_1 = -w_3$ or $w_3 = -2w_1$

$w_1 = \frac{-1w_3}{2}$

Subs. in (1)

$4\left(\frac{-l\ w_3}{2}\right) + 3w_3 = w_2$

$+ w_3 = w_2$

Substituting in (2)

$2w_1 + 2(-2w_1) = w_2$

$2w_1 - 4w_1 = w_2$

$-2w_1 = w_2.$

$\therefore w_1 = \frac{-1w_2}{2}$

$w_3 = w_2$

$\therefore w = w_2(-\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k})$

And $\mathbf{e}w = \frac{w_2(-l\mathbf{i} + \mathbf{i} + \mathbf{k})}{\sqrt{w_2^2(\frac{1}{2})^2 + (1)^2 + (-1)^2}}$

$= \frac{w_2(\mathbf{i} - \mathbf{j} - \mathbf{k})}{w_2\sqrt{\frac{1}{4} + 4/4 + 4/4)}} = \frac{\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{9/4}}$

$= (\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}) \times 2/3$

$= 1/3\mathbf{i} + 2/3\mathbf{j} + 2/3\mathbf{k}$

or $\pm \frac{(\mathbf{i} - 2\mathbf{i} - 2\mathbf{k})}{3}$

3.4 Equation of a plane

Example 3

Find an equation for the plane perpendicular to the vector $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k},$ and passing through the terminal point of the vector $\mathbf{v} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

Solution

let r be the position vector of P (x, y, z) on the plane, and Q the terminal point of v in figure V29 using relative vectors, PQ = **v - r** and is perpendicular to **u**,

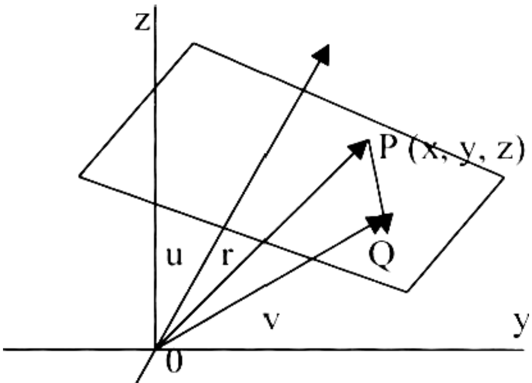


fig V29

then you say
(**v - r**). **u** = 0.
v - u = **r**. **u** = 0 (Distributive law)
v.u = **r. u** is the required equation of the plane, in vector form since it gives the condition required.

You can then write this in the rectangular form as

(x**i** + y**j** + z**k**) (2**i** + 3**j** + 6**k**)
= (**i** + 5**j** + 3**k**).(2**i** + 3**j** + 6**k**)
2x + 3y + 6z =(1 x 2)+(5 x3) + (3 x 6) = 35.
That is the equation for the plane is 2x + 3y + 6z =35

SELF-ASSESSMENT EXERCISE 3

Given that **u** = 3**i** + **j** + 2**k** and **v** = **i** - 2**j** - 4**k** are the position vectors of points p and Q respectively.

(a) Find an equation for the plane passing through Q and perpendicular to line PQ.

Solutions

You must find PQ first. Using relative vectors, without diagram, PQ = v – u

v – **u** = (1 - 3)**i**+ (-2- 1)**j** + (-4 - 2)**k** = 2**i** - 3**j** - 6**k**
Then you let **r** = x**i** + y**j** +z**k** be an arbitrary point on the plane whose equation is required,
(**r** - **u**).(**u** - **v**)=0.
r.(**u** - **v**).= (**u** - **v**) **v** (Distributive law)
or (x**i** + y**j** + z**k**).(-2**i** - 3**j** - 6**k**) = (-2**i** - 3**j** - 6**k**).(**i**- 2**j** - 4**k**)

i.e. -2x - 3y - 6z = (-2 x 1) + (-3 x -2) + (-6 x -4) = 2 + 6 + 24 =30-2
=28.
Or 2x + 3y + 6z = -28

3.5 Distance from the origin to a given plane

Example 4

Using data in Example 3. Find the distance from the origin to the plane.

Solution

The distance from the origin to the plane represents the projection of **v** on **u**, which is **v**. $\mathbf{e}_u$

$$\mathbf{e}_u = \frac{\mathbf{u}}{u} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{4 + 9 + 36.}} = \frac{2\mathbf{i}}{7} + \frac{3\mathbf{j}}{7} + \frac{6\,\mathbf{k}}{7}.$$

The projection of v on u = v $\mathbf{e}_u$

$$= (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}).\left(\frac{2\mathbf{i}}{7} + \frac{3\mathbf{j}}{7} + \frac{6\mathbf{k}}{7}\right)$$

$$=\left(1 \times \frac{2}{7}\right) + \left(5 \times \frac{3}{7}\right) + \left(3 \times \frac{6}{7}\right)$$

$$= \frac{2 + 15 + 18}{7}$$

$$= \frac{35}{7}$$

= 5

4.0 CONCLUSION

You should appreciate the application of direction cosines and dot product in different areas of mathematics. This unit has attempted to bring just a few. You should study them so you'll get familiar with the concept of dot product.

5.0 SUMMARY

- Work done, W = (Fcosθ)d
  = **F**.d = d where,
  F is the force applied, d, the displacement, and θ, the angle inclined to the direction of the displacement.

- The unit vector perpendicular to two vector u and v on a plane is e,,, where **w.u** = **w.v** = 0, gives the simultaneous equation to be solved, to give w and so e,,

- The distance from the origin to the plane represents the projection of **v** on **u** which **v**.e_u.

6.0 TUTOR - MARKED ASSIGNMENT

l. find
   (a) An equation of a plane perpendicular to a given vector u and distant p from the origin.
   (b) Express the equation of (a) in rectangular coordinates.
2. Find a unit vector parallel to the x-y plane and perpendicular to the vector 4**i** - 3**j** + **k**
3 (a) When will the work done on a particle by a force F be (i) (ii) maximum.
  (b) vector **u** = 3**i** -2**j** +**k** and **v** = -2**i** +2**j** + 4**k**. Find the of **u**, and **v** and the angle between them.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 3 THE VECTOR OR CROSS PRODUCT

CONTENTS

1.0 INTRODUCTION

In this unit you will be learning about the second product of vectors the vector or cross product, The vector product has its name from the fact that the product given another vector perpendicular Plane.

You will learn two approaches to the calculation, and so have a choice of which you find easier.

2.0 OBJECTIVE

At the end of this unit you should be able to calculate correctly the cross product of two or three vectors using the expansion or determinant method

3.0 MAIN CONTENT

3.1 The Cross Product Definition

The cross or vector product of **u** and **v** is a vector **w** = **u** x **v**. The magnitude of **u** x **v** is defined as the product of the magnitude of u and v and the sine of the angle θ between them.

The direction of the vector **w** = **u** x **v** is perpendicular to the plane of **u** and **v** = 2**i** – **j** + 5**k** and such that u. v and w form a right – handed system. in symbols, u x v= uv sin θ_e θ < 0, where e is a unit vector indicating the direction of u x v, if u = v or if u is parallel to v, then sin θ = 0 and you will define u x v =0

3.2 Algebraic laws on cross product

1 u x v = -v x u commutative law fails u x v ≠ v x u
2 ux (v x w) = u x v + u x w distributive law
3 m(u x v) = (mu) x v = u x mv) = (u x v)m where m is scalar.
4 i x i= i x j =k x k = 0 = I x I x sine 0 = I x I x O = O.i x j = k, j x k = j k x j = i
5 If u = x_1i + x_2j + x_3k and v = y_1i + y_2j + y_3k

Then

$$\mathbf{u \times v} = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

i.e. the determinant of the matrix formed

6. The magnitude of **u**. **v** is the same as the area of a parallelogram with side u and v.
7. If **u** x **v** = 0 and **u** and **v** are not null vectors. Then u and v are parallel.

Remember your right-handed rule i - j - k - i - j but if reversed, you get minus.

Example 1

Show that | u x v ^{2}| + | u.v |^2 = | u |^2 | v |^2

Solution

| u x v | ^{2} + | u.v | ^{2}

| uv sin θ_e|^2 + |uv cos θ|^2

= u^2 v^2 $\sin^2\theta$ + u^2v^2 $\cos^2\theta$
= u^2v^2 (sin $^2\theta$ + $\cos^2\theta$)
= u^2 v^2 x 1 (sin $^2\theta$ + $\cos^2\theta$ = 1)
= u^2v^2

3.3 Cross product of the unit vectors

Example 2

Evaluate (a) (2**j**) x (3**k**) (b) (3**i**) x (2**k**)
(c) 2**j** x **i** – 3**k**

Solution

a. (2j) x (3k) = 6j x k = 6i

b. (3i) x (-2k) = - 6i x k = 6j c. 2j x i - 3k = 2k - 3k = -k.

3.4 The use of Determinant

Example 3

If **u** = 2**i** - 3**j** - **k** and **v** = **i** + 4**j** - 2**k**
Find (a) (**u** x **v**) (b) (**v** x **u**)
(c) (**u** + **v**) x (**u** - **v**)

Solution

(a) **u** x **v** = $\begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & +4 & -2 \end{vmatrix}$

Determinant of the 3 x 3 matrix form from i. j. k and the coefficients of u and v.

$$= \mathbf{i} \begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} \ \mathbf{j} \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

= **i**[6 – (-4)] –**j** [-4 – (-1)] + **k**[8 – (-3)]

= 10**i** – 3(-**j**) + 11**k**

= 10**i** + 3**j** + 11**k**

3.6 Expansion method

note that **i** x **i** = **j** x **j** = **k** x **k** = 0

i x j = kj x k = i,k x i = j and i x k = -j,k x j = -i,j x i = k

(2i - 3j - k) x (i + 4j -2k)

= 2i x (i x 4j - 2k) - 3j x (i+ 4j - 2k) – k x (i + 4j - 2k)

= 2i x i + 8i x j – 4i x k -3j x i -12j x j + 6i x k. –k x i -4k x j + 2k x k.

= 0 x 8k - 4(j) - 3(-k) – 0 + 6i – j - 4(-1) + 0

= 10i + 3j + Ilk.

Remark.

Which is not too bad once you remember the rule. Here is a hint you could use:-
Let your mind be fixed on the right order of the alphabets i, j, k.

Same letters give zero. Two different letters that sound right, give the missing letter. e.g. i x j = k, j x k = i.

But when the two letters are in the wrong order, the result is still the missing letter, but negative e.g. j x i = - k, i x k = - j, k x j = -i.
Of course the determinant removes all these problems from the calculation.

Now you have an example to complete, so go ahead and do it right this time.

$$\text{(b) } v \times u = \begin{vmatrix} I & j & k \\ 1 & 4 & 2 \\ 2 & -3 & -1 \end{vmatrix} = i \begin{vmatrix} 4 & -2 \\ -3 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix}$$

= i (-4 (+6)) –j (-1 + 4)) + k (-3 – 8)

= (-4 – 6)I – j(-1 + 4) + k (-11)

= 10i – 3j – 11k

Which proves (u x v) = -(v x u)

yv = (2 + 10)i + (-3 + 4)j + (-1 -2)k

= i – 7j + 3k

u – v = (2 – 1)i + (-3 – 4)j + (i – 7j + k)

$$\begin{vmatrix} i & j & k \\ 3 & 1 & 7 \\ 1 & -7 & +1 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - j\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} -21 & -1 \\ 1 & -7 \end{vmatrix}$$

= i (I – (21) – j (3 – (-)) + k (-21 – 1)

= - 20i – 6j – 22k

SELF-ASSESSMENT EXERCISE I

If **u** = 3**i** – **j** + 2**k** **v** = 2**i** + **j** - **k**
And w = **i** - 2**j** + 2**k**.
Find (i) **u** x **v** (ii) **u** x **w**

Solution

u x v = (3i - j + 2k) x (2i + j - k)
= 3i x 2i + 3i x j + 3i x - k – i x 2i – j x i – j x -k + 2k x 2i + 2k x j + 2k - k
= 0 + 3k - 3(-j) -2 (-k) x O x i + 4(j) + 2(-i) x O
=3k + 3j + 2k + i + 4 - 2i
= (1 - 2)i + (3 + 4) + (3 + 2)k
= -i + 7j + 5k

(ii) u x w = (3i – j+2k) x (i – 2j + 2k)

$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= -i\begin{vmatrix} & 2 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \text{ x } k\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}$$

= i(-2 - (-)) - j(6 – 2 + k(-6 - (-1))

= i (-2 + 4) - j(4) + k(-6 + 1)

= 2i - 4j - 5k

3.7 Cross Product of 3 Vectors

Examples

Using the data in exercise 1.
u = 3i – j + 2k v = 2i + j – k and w =i - 2j + 2k , find (u x v) x w first find u x v =i+ 7j + 5k
in Exercise 1. ... u x (v x w)
= (-i + 7j + 5k) x (i - 2k)

$$= \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & -2 \end{vmatrix}$$

$$= i \begin{vmatrix} 7 & 5 \\ -2 & -2 \end{vmatrix} - j \begin{vmatrix} -1 & 5 \\ 1 & -2 \end{vmatrix} + k \begin{vmatrix} -1 & 7 \\ 1 & -2 \end{vmatrix}$$

= i (14 – (-10)) – j (-2 – 5) + k (2 – 7)

= 24i + 7j – 5k.

SELF-ASSESSMENT EXERCISE 2

Find u x (v x w), with the same u = 3i - j + 2k, v = 2i + j – k and w = i - 2j + 2k.

Solution:

$$v \times w = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} - j \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

= i(2 - (2) - j (4- (- 1)) + k (4 - 1)

= 0i - 5 j -5k
= u x (v x w) = (3i – j + 2k) x (0i - 5j - 5k)

$$= \begin{matrix} i & j & K \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{matrix}$$

$$= i\begin{vmatrix} -2 & 2 \\ -5 & -5 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 0 & -5 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 0 & -5 \end{vmatrix}$$

= i(5 – 1 – 10) – j (-15 – 0) + k (-15)

= 15i + 15j – 15k

Which proves that cross product is not associative. u x (v x w) + (u x v) x w.

4.0 CONCLUSION

So now you have learnt another product of vectors this time, the result of the product is another vector as opposed to scalar in dot product, you can remember the two by their name. In the next unit you will take more application of vector product.

5.0 SUMMARY

In this unit you have learnt that

- The product of two vectors called the vector or cross product is a perpendicular vectors their plane. in symbols. u v = uv sin 0_e 0< 0< t where e is the unit vector perpendicular to the plane of u and v , -u x v = 0
  If a = v or parallel.

- Algebraic laws The commutative law does not hold u x v = - v x u Distributive law holds over addition u x (v x w) = u x v + u + w scalar multiples.
  m(u x v) = (mu) x v = u x (mv) = (u x v)m. Where m is scalar
  i x j = i x j = k x k = 0 = i + 1 sin 0 = i x i x 0 = 0, i x j = k, i x k = i, k x i =j .

- If **u** = xi + yj + zk and v = 2i – j + 5k
  = xi x uj + zk

  Then u x v = $\begin{matrix} i\,j & k & \\ X_1 & y_1 & z_1 \\ X_2 & y_2 & z_2 \end{matrix}$

- The magnitude of (**u** x **v**) is the same as the area of a parallelogram with sides' **u** and **v**.

- If **u** x **v** =0 and u and v are not null vectors then **u** and **v** are parallel

6.0 TUTOR - MARKED ASSIGNMENTS

1. If **v** =2**i** + **j** - **k**, and **u** = **i** -2**j** + 2**k**
   Find (a) **v** x **u** (b) **u** x **v**

2. Evaluate each of the following. (a) 2**j** x (3**i** - 4**k**)
   (b) (**i**- 2**j**) x **k**
   (c) (2**i** - 4**k**) x (**i** x 2**j**)

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 4 APPLICATIONS OF VECTOR PRODUCTS IN AREAS

CONTENTS

1.0 INTRODUCTION

In this unit, you are presented with different application of Vector or cross product and you will discover why you refer to mathematics as the language of Science.

You should, take time to study and practice the exercises and examples given.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- calculate accurately, the area of triangle, parallelogram, square and rhombus using given data.
- recognize and use the definitions of quadrilaterals with their difference or common properties

3.0 MAIN CONTENT

3.1 Definitions of types of quadrilaterals

At this point, you should, revise the definitions of the quadrilaterals and their properties.

- A quadrilateral is a 4 -sided polygon (quad.)

- A rectangle is a quadrilateral with each angle as right angle (90°)
- A square is a rectangle with all sides equal.
- A rhombus is a quad with all sides equal in length.
- A parallelogram is quad with two pairs of parallel sides.
- A trapezium is a quad. With one Pair of parallel sides
- A kite is a quad, with a diagonal as line of symmetry

From these definitions, you can conclude that rectangles, squares, and rhombuses are parallelograms.

But the diagonals of rectangle do not intersect at right angles, while those of rhombus and square bisect each other at right angle.

Also, the diagonals of the square are equal but those of rhombus are not equal.

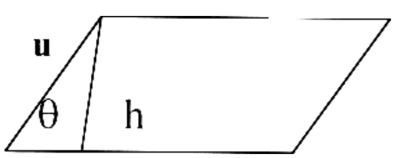
3.2 Area as a Vector

Usually you think of Area as a Scalar quantity. However, in many applications in physics (e.g. in fluid mechanics or in electrostatics) you also want to know the orientation of the area.

Suppose you want to calculate the rate at which water in a stream flows through a wire loop of a given area. This rate will obviously be different if we place the loop parallel or perpendicular to the flow, when the loop is parallel, the flow through it is Zero. So you will now see how the vector product can be used, to specify the direction of an area,

3.3 Area of parallelogram

= h /v/
= /u/ sin θ v
= /u x v/



Flg. V30

i.e. the area of a parallelogram with vectors u and v as sides is the modulus of its cross product. Fig.V30

3.4 Area of Parallelogram, given diagonals.

Example 2

The diagonals of a parallelogram are given by a **=** 3**i - **4**j -k**
and b **=** 2**i** + 3**j**-6**k**.

(a) Show that the parallelogram is a rhombus

(b) Find the Area of the parallelogram.

Solution:-

(a) **a. b** = (3x2) + (-4x3) + (-1x-6)
= 6 -12 + 6
=12 - 12
=0
:. The diagonals are perpendicular
:. The figure could be rhombus or square a_2 = 9+16+1 =26.

b_2 = 4|+9+|36 =49 =7.

Since the magnitudes of the diagonals are not equal, it follows that the parallelogram is a rhombus.

(b) The Area of the parallelogram is ½ab sin θ where θ is the angle between the diagonal which in this case is a right angle.

½ab sin θ = ½a x b

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ 3 & -4 & -1 \\ 2 & 3 & -6 \end{vmatrix}$$

=½(24 + 3)i - (-18 + 2)j + (9 + 8) k

= ½ (27i + 16j +17k)

$=\frac{1}{2}\sqrt{27^2 + 16^2 + 17^2}$

$= \frac{1}{2}\sqrt{729 + 256 + 289}$
$= \frac{1}{2}\sqrt{1247}$

= ½ (35.31)

= 17.7 sq.units.

Exercise. l

Find the area of the triangle having sides with position vector
a = 3**i** + **j** -2**k**, **b** =**i** + 3**j** + 4**k**, and **c** = 4**i** – **j** - 6**k**

Solution

$\overrightarrow{\mathbf{AB}}$ = **b** – **a** = (-1 - 3) i + (3 - 1)j + (4 + 2)k
= -4i + 2j + 6k.

$\overrightarrow{\mathbf{BC}}$ = **c - b** = (-1 - 4)**i** + (3 + 2)**j** + (4 + 6)**k**
= 5**i** + 5**j** + lO**k**.

Area of triangle ABC =½AB X BC

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ -4 & 2 & 6 \\ -5 & 5 & 10 \end{vmatrix}$$

= ½ (20 - 30)i - (-40 + 30)j + (-20 + 10)k

= ½ (-l0i + 10j - l0k)
= ½ (100 + 100 + 100)

= ½√300

=½ x 10√3

= 5√3

= 8.66 sq. units.

3.5 Parallel vectors

If **u x v** = 0 and if **u** and **v** are not Zero, show that **u** is parallel to **v**.

Solution

If **u x v** = 0, then you have I **uv** I sinθ, = 0. Then sin θ = 0 and θ is 0° or

180°.

Example 1

Show that $|u \times v|^2 + |u.v|^2 = |u|^2 |v|^2$

Solution

$= |u \times v|^2 + |u.v|^2$

$= |uv \sin \theta_e|^2 + |uv \cos \theta|^2$

$= u^2 v^2 \sin^2\theta + u^2 v^2 \cos^2\theta$

$= u^2 v^2 (\sin^2\theta + \cos^2\theta)$
$= u^2 v^2 \times 1$ $(\sin^2\theta + \cos^2\theta = 1)$

$= u^2 v^2$

4.0 CONCLUSION

The applications of the definition of vector or cross product, in many fields of learning make calculation easier and faster.

It is useful in Area of plane shapes. With vector as sides you should take note of the type of information given, and required in a problem for example if you are given two points, Pand Q, you might need to get the relative vector

q - p to express PQ as a vector. But you might be given the vector representing **PQ** already as just u = x**i** + y **j** + z **k**. Just take time to read questions carefully before answering them.

5.0 SUMMARY

You have learnt in this unit, the following
1) Area of parallelogram is /**u** x **v**/ where **u** and **v** represent vector forming the sides.
2) Area of triangle = ½ /**u** x **v**/
3) Area of parallelogram given diagonals $d_1 d_2$, is $\frac{1}{2} d_1 d_2 \sin Ø = \frac{1}{2} / d_1 \times d_2 /$.

6.0 TUTOR - MARKED ASSIGNMENTS

1. Calculate the cross product of the vectors
   u = 3**i** + **j** + 2**k** and
   v = 2**i** + 3**j** + **k**.

2. Two sides of triangle are formed, by the vectors u = i + 3j - k and v= 2 i + j + Sk. Calculate the area of the triangle.

3. Determine a unit vector perpendicular to the plane of a=2i + 6j + 3k and v =4i+3j+k

7.0 REFERENCES/ FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 5 APPLICATION OF VECTOR PRODUCTS IN PHYSICS

CONTENTS

1.0 INTRODUCTION

In the last unit, you learnt about the application of cross product in areas.

In this unit, you will see the application of the cross product, this time in physics or mechanics, to be specific.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Calculate moments of a force **F** about a point **Q** with position vector **r** given
- Calculate relative velocity **V** given the angular velocity $\dot{\omega}$ and r the relative vector of point P
- Calculate correctly the torque (Γ), given the force **F** on a particle at position **r**.

3.0 MAIN CONTENT

3.1.1 Moment of a force F about a point P

Consider a force **F**= x**i** + y**j** + z**k** applied a point Q with position vector **r** fig. V3 the moment of **F** about the point Q is **rxF**

A force given by **f** = 3**i** +2**j** - 4**k** is applied at the point P(1-1,2). Find the moment of **f** about the point Q (2,-1,3).

Solution

Let **QP** = **r** = **p** – **q** = (1 - 2) i + (-1- (-1))j + (2 - 3) k.

∴ **r** = **-i**+ 0**j** – **k**

The moment of **F** about the point Q is **r x F**

r x F = (- **i** - k.) x (3**i** + 2**j** - 4**k**)

$$= \begin{vmatrix} I & j & k \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

= (0 - 2)i - (4 - (-3)) j + (-2 - 0)k.
= -2**i** - 7**j** - 2**k**.

3.2 Angular velocity

A rigid body rotates about an axis through point O with angular speed ὠ
The linear velocity **v** of a point P of the body with position vector **r** is given by

v = ὠ x **r** where ὠ is the vector with magnitude 1ὠI whose direction is that in which a right handed screw would advance under the given vocation.

The vector ὠ is called the angular velocity.

SELF-ASSESSMENT EXERCISE 2

The angular velocity of rotating rigid body about an axis of rotation is given by

ὠ = 4**i** + **j** – 2**k**. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is 2 **i** - 3 **j**+ **k**.

Solution

v = ὠ x **r** ὠ = 4**i** + **j** - 2**k** **r** = 2**i** - 3**j** + **k**.
∴ v = (4 **i** + **j** - 2 **k**) x (2 **I** - 3 **j** + **k**)

$$\therefore v = \begin{vmatrix} i & j & K \\ 4 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

= (I - 6)i - (4 - (-4) j + (-12 - 2) k

v = -5i - 8j - 14k is the linear Velocity required

3.3 Torque

The torque due to a force **F** which acts on a particle at position **r** is defined by
ᴦ = **r x F**

You may wish to know that torque is a measure of the ability of an applied force to produce a twist, or to rotate a body.

Note that a large force applied, parallel to **r** would produce no twist, it would only pull.

Only **F** sin θ_e, i.e. the component of F perpendicular to **r** produces a `torque. The direction of torque is along the axis of rotation. This is precisely what the equation i = **r x F** is telling you.

For example, since **r x r** is a zero vector, a force along **r** yield zero toque. The direction of i is given by the right-handed rule (Fig. V31.)

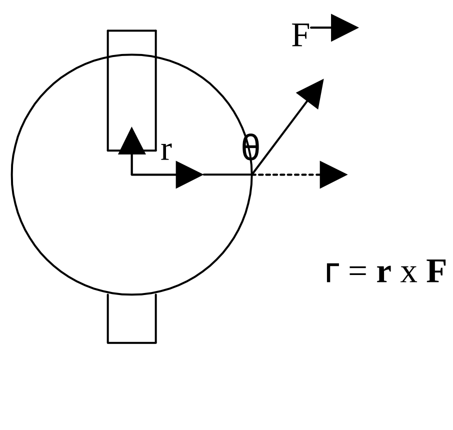


Fig. V31.
Torque **r** due to a force **F**

SELF-ASSESSMENT EXERCISE 3

Consider a force **F**= (-3**i** + **j** + 5) Newton, acting at a point P (7$\mathbf{i}_2$+ 3**j**+ **k**) m, what is the torque in NM about the origin?

Solution

The displacement of P with respect to origin is **r** where **r** = 7 **i** + 3**j**+ **k** m
ᴦ = **r x F** = (7 **i** + 3**j**+ **k**) x (-3**i** + **j** + 5**k**)

$$\therefore \Gamma = \begin{vmatrix} i & j & K \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

= (15 -1)i - (35 - (-3)) j + (7 - (-9)) k
= (14i - 38j +16 k) NM.

Example 6

Find **p** and **q** such that the vectors
w = p**i**+ 3 **j** and **v** =2 **i** + q**j** are each

Parallel to **u** = 5**i** + 6**j**

Solution

If **w** and **u** are to be parallel to u, then
w x u = 0, and **v x u** =0.
w x u = (p$\mathbf{i}_2$ + 3**j**) x (5**i** + 6**j**) = 0.
6P **k** -15 **k**= (6P -15) **k** =0.
6P - 15 = 0. Because k ≠0
P = 15/6
= 2.5

v x u = (2**i** + q **j)** x (5**i**+ 6 **j**)
= 12 **k** - 5q**k**=0.
= 12 **k** - 5 **q** = 0, k ≠ 0. q= 12/5 = 2.4

4.0 CONCLUSION

The applications of the definition of vector or cross product, in many fields of learning make calculation easier and faster.

It is used in calculating moments.

It is used in calculating angular velocity. It is also useful in calculating (ᴦ) torque.

Once again you are advised to take note of the type of information given and required in a given problem.

5.0 SUMMARY

You have learnt in this unit, the following:

- The moments of force F about a point Q with position vector r_2 x F
- The linear velocity **v** =ώ x **r** where ώ is the angular velocity and **r** the position vector of the point through which the body rotates about an axis.
- Torque (ᴦ) is **r** x **F** where torque is the force to rotate a body with a force **F** at position vector r_2.

6.0 TUTOR-MARKED ASSIGNMENT

1. A force given by **F** = 2**i** + 3**j** - 3**k** is applied by at the point P (3, -2, 4). Find the moment of F about the point Q (1, -3, 2)
2. Consider a force F = (-3**i** + **j** + 5**k**) Newton acting at a point P = (5**i** + 2**j** + 3**k**) meter.
   What is the torque(ᴦ)?

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum’s and Problems of Advanced Calculus, McGraw – Hill N. Y.