MODULE 2

UNIT 1 ALGEBRA OF LIMITS

CONTENTS

1.0 INTRODUCTION

You would have been familiar with the word continuous ordinarily to say that a process is continuous is to say that the process goes on without changes or interruptions. In this section the word continuous has almost the same meaning. That is a function is continuous in the sense that you plot the graph continuously without lifting your pencil from the graph paper. In calculus it is demanded that functions must be continuous at points or interval of investigations that is why you must study this unit with some care.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a continuous function at the part $x = x_o$
- recall properties of continuous function
- state theorems on continuous function.
- state the 3 conditions for continuity of a function at a given part.
- identify parts of continuity and discontinuity of a function.

3.0 MAIN CONTENT

3.1 Definitions of a Continuous Function

Consider The Graph shown in Fig. (21 a) and Fig (21b

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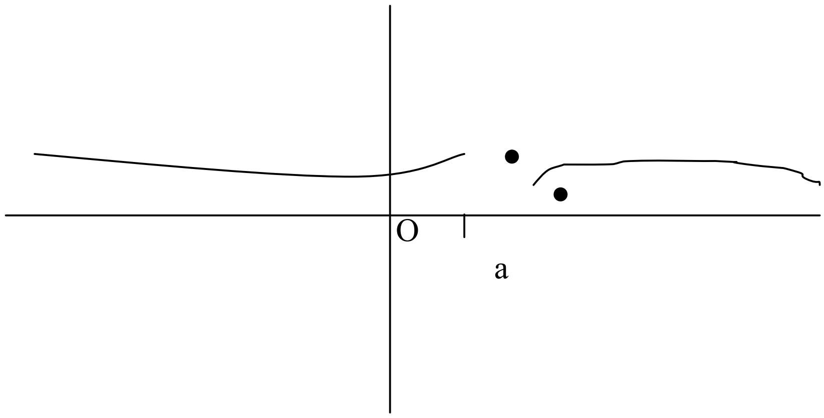


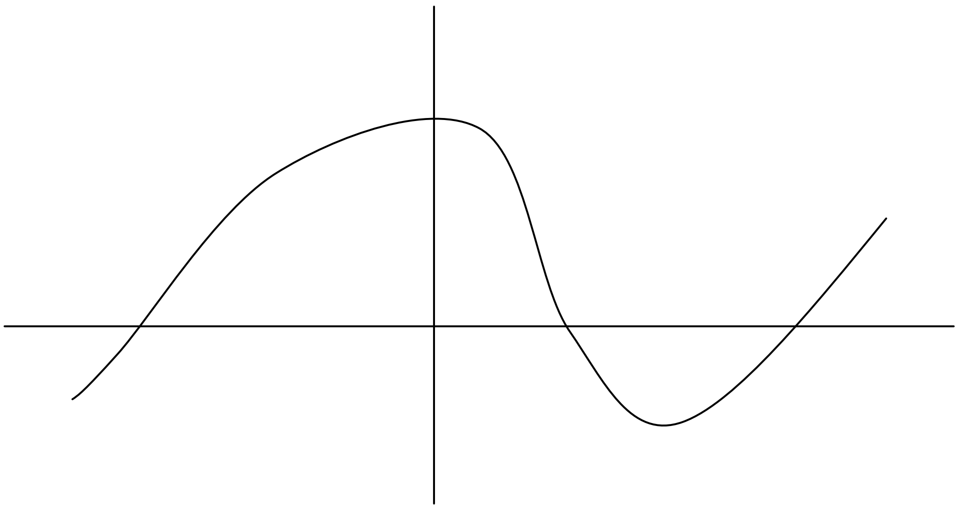


Fig. 21 b.

In Fig 21a the function changes abruptly at the part a: whereas the graph in Fig.21b is continuous.

Definition: A function $f\,(\mathrm{x})$ is said to be continuous at the point x = xo if and only if:

$$\lim_{x \to x_0} f(x) = f(x_o)$$

The above definition can be broken down into 3 main conditions a function must satisfy for it to be continuous at the point $x = x_o$

Definition: A function $f(x)$ is said to be continuous at the point $x = x_o$ if the following 3 conditions are satisfied.

1. $f(x)$ must be defined at $x = x_o$
2. $\lim_{x \to x_0} f(x) = L$ must exist
3. $f(x_o) - L$

Examples:

a. Is the function $f(x) = x^2 - 4$ continuous at the point x

Solution:

Checking for the 3 conditions.

1. Let $x = 2, f(x) = 2^2 - 4 = 0$
2. $\lim_{x \to 2} x^2 - 4 = 0$
3. $\lim_{x \to 2} x^2 - 4 = f(2) = 0$

b. Is the function $f(x) = \frac{1}{x - 1}$ continuous at point x = 1

Solution:

i. let x = 1 $f(x)$ $\frac{1}{1-1} = \frac{1}{0}$

Since division by zero is not possible then f (x) is not defined at the point x = 1. Needless to check for the remaining conditions you can conclude by saying that

$f(x) = \frac{1}{x - 1}$ is not continuous at the point x =1

Definition: A function $f(x)$ which is not continuous at the point $x = x_o$ is said to be discontinuous at that point.

Example:

Determine the points of discontinuities of the function

$$f(x) = \frac{x}{x^2 - 9}$$

Solution:

The same manner a function is said to be continuous (on the left) at $x = x_o$ if. $\lim_{x \to x_0} f(x) = f(x_o)$

Just as the case of limit, a function is said to be continuous at $x = x_o$ if it is both continuous from left and from right at x_o

1. Function that are Continuous on R

Every polynomial function is continuous on R.

i.e.; $f(x) = a_n x^n + a_n, x^{n-1} + ...+ a_{n-1}, x + a_n - 1$

In unit I section 3.3 it was shown that
i. e. ; $\lim f(x) = f(x_o)$ for all polynomial

Thus $f(x)$ is continuous.

2. The Trigonometric Functions

For examples $f(x) = \sin x$ and $f(x) = \cos x$ are continuous on R

$|\sin x| < |x|$ and $|\cos x| < 1$ for all $x \in R$ see the graph of sin x and cos x at Fig. 22a and 22b

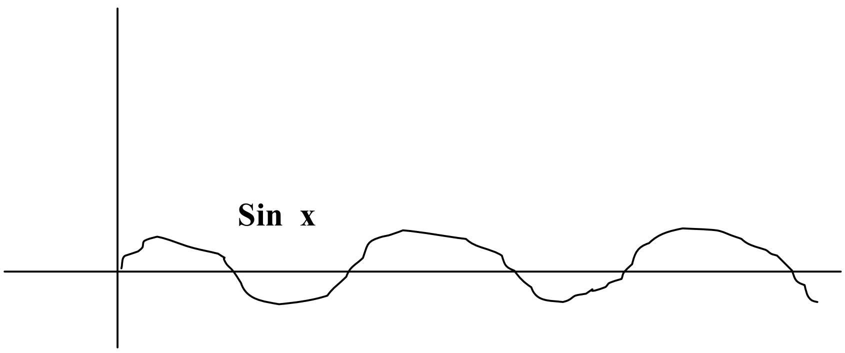


Fig. 22a

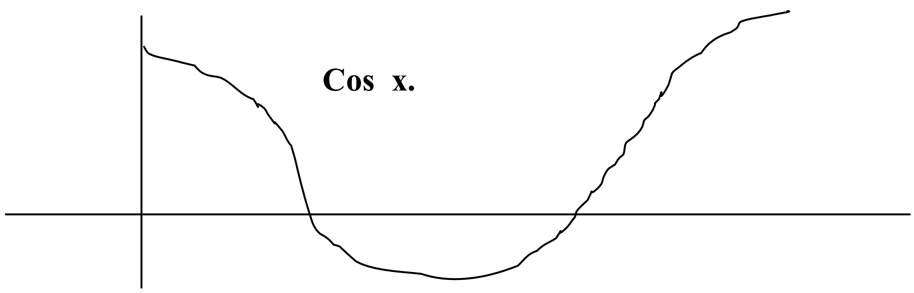


Fig. 22b.

3. Removable Discontinuity

It has earlier been defined that a function that is not continuous at a point $x = x_o$ is discontinuous at that point. However there are basically two types of discontinuities.

If for any function $f(x)$, the $\lim f(x) \neq f(x_o)$ initially, but $x \longrightarrow x_o$ by redefining the function $f(x)$ is done in such a way that $f(x_o) = \lim f(x)\ x \longrightarrow x_o$

then the point x_o is said to be a point of removable discontinuity of $f(x)$.

Example:

Show that $f(x) = \dfrac{x - 4}{x - 2}$

Has a removable discontinuity at point x = 2.

Solution:

Since $f(x)$ is not defined at x = 2. But by appropriate factorization the function.

$$f(x) = \frac{(x - 2)(x + 2) = x + 2}{x - 2}$$

then $\lim\limits_{x \to 2} x=2 = f(2) = 4$

Hence at x = 2 is a point of removable discontinuity.

A very simple way to solve this is to define the domain of the function. You can easily see that the domain D is given as

$$D = \{ x; x \in R, x \neq -3 \text{ or } 3)$$

Therefore the points of discontinuity are -3 and 3.

Finally, another definition of limit will now be given using the familiar $\in, \delta^+$ symbol.

Definition: A function $f(x)$ is said to be continuous at the point $x = x_o$ if for $\in > 0$ there is $\delta > 0$ such that

If $|x - x_o| < \delta$ then $|f(x) - f(x_o)| < \in$

Remark:

The above definition is an extension of the definition of a limit. In the above if we replace $f(x_o)$ with L and remove the restriction that $f(x_o)$ must be defined you get back definition of a limit.

Example:

Show that the function $f(x) = x^2$ is continuous at the point x = 2.

Solution;

Let $\in > 0$ if you can find a $\delta > 0$

Such that

If $| x - 2 | < \delta$ then $| f(x) - f(x_o) | < \in$

Note that $f(2) = 0$ and

$$| x^2 - 4 | = |(x - 2(x + 2)| = | x + 2| \; | x - 2|$$

By keeping x close to 2 we make the factor x - 2 as small as we please and the second factor x + 2 gets close to 4.

If the domain D = [-2,] say then we can be sure that the factor (x + 2)4.

If $x \in [-2, 2]$ then

$X + 2 \;\; 10$ and $| f(x) - f(2) | < \in$

When $| x - 2 | < \delta \mu$

Provided that $\delta \leq \in/10$

If $\in > 0$ then

$| f(x) - f(2) | \;\; 10 \; | (x - 2 | < \in$ if $| x - 2 | < \in /10 = \delta$

In fact the function in the above example is continuous for all points in the interval 1 = [- 2, 2]. When such happens the function $f(x)$ is said to be uniformly continuous on the interval 1.

3.3 Properties of Continuous Functions

1. Uniform Continuity

A function f(x) is said to be continuous in an interval I. If for ∈ > 0 there is a δ > 0 (depending on alone) such that:

If $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \varepsilon$

2. Continuity of Function From Left and Right of a Point

A function f(x) is said to be continuous (on the right) at a point x = x_o

If $\lim\limits_{x \to 0} f(x) = f(x_o)$

With f(x) defined at x = 2. You can say that f(x) is now continuous at the point x =2.

Type II: Non-Removable Discontinuity

If for a given function f(x) the right hand and left hand limits as $x \to x_o$ exist but are unequal i.e.;

$\lim\limits_{x \to x_0} f(x) \neq \lim\limits_{x \to x_0} f(x)$

or if either the $\lim\limits_{x \to x_0} f(x)$ or $\lim\limits_{x \to x_0} f(x)$ does not exist then the function f(x) is said to have a non-removable discontinuity

Example

The function f(x) = sin (1/x) is continuous except for x = 0. The function has non-removable discontinuity at x = 0. Both right and left hand limits do not exist.

Example

Determine whether the function f (x) = [x] (the greatest integer function) is continuous at the point x = 3

Solution:

$\lim\limits_{x \to x_0} [x] = 3$and $\lim\limits_{x \to 3} [x] = 2$

Since $\lim\limits_{x \to 3^+} f(x) \neq \lim\limits_{x \to 3^-} f(x)$

Then the function f(x) has a non-removable discontinuity at x = 3.

See Fig 23.

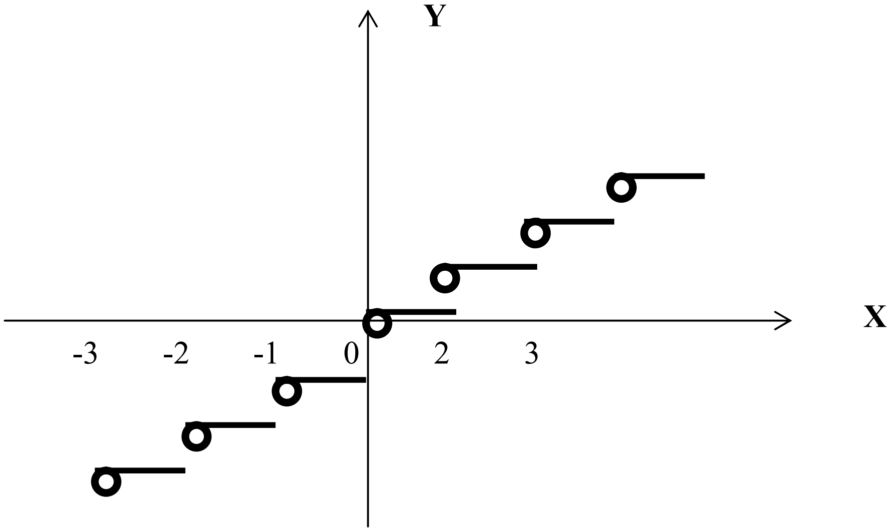


Fig. 23.

The above gives us a picture of a function that is one sided continuous. In fig. 23. the function is continuous from the right and discontinuous from the left.

5. Continuity on [a,b]

If a function *f*(x) is defined on closed interval [a, b) the most continuity we can possibly expect is:

1. Continuity at each point xo of the open interval (a, b).
2. Continuity from the right at a, and
3. Continuity from the left at b.

Therefore any function that satisfies conditions 1 to 3 above is said to be continuous on [a, b]

Functions that are continuous on a closed interval are of special interest to mathematicians, because they possess certain special properties which discontinuous function do not have.

SELF ASSESSMENT EXERCISE 1

Draw the graph of the above function.

1. Define any function that satisfies conditions 1 to 3 above is said to be continuous on [a, b]

2. Fractions that are continuous on a closed interval are of special interest to mathematicians, because they possess certain special properties which discontinuous functions do not have.

Example

The function $f(x) = \sqrt{(1 - x^2)}$ is on the in [-1, 1]

SELF ASSESSMENT EXERCISE 2

Draw the graph of the above function.

3.4 Algebra of Continuous Functions

Recall the theorem, on limits you studied in the last Unit. You will now do the same for continuous function. Using the following theorems it can be shown that continuity is preserved through algebraic operations on functions.

That is:

Theorem 1:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x = x_o$ then the sum $f(x) + g(x)$ is continuous at $x = x_o$

Fortunately enough the proof of the above theorem is not complicated in the sense that all that is required for a function to be continuous at a point $x = x_o$ is that $\lim f(x) = f(x_o)$

From the theorems of limits (see Unit 3).

The $\lim\limits_{x\to x_0} (f(x) + g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$

Therefore the function $f(x) + g(x)$ is continuous since

$$\lim_{x\to x_0} f(x) + g(x) = \lim_{x\to x_0} f(x_o) + g(x_o)$$

Theorem 2:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x_o = x$ then the sum $f(x) + g(x)$ is continuous at $x = x_o$

Proof:

Let $\lim\limits_{x\to x_0} f(x) = f(x_o)$ and

Let $\lim\limits_{x\to x_0} g(x) = g(x_o)$ since

they are continuous at $x = x_o$

Therefore $\lim\limits_{x\to x_0} f(x) \,.\, g(x) = \lim\limits_{x\to x_0} f(x) \,.\, \text{Lim}_{x\to x_0}\, g(x) = f(x) \,.\, g(x)$

hence the function $f(x)\, g(x)$ is continuous at $x = x_o$

Theorem 3:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x = x_o$ then the function $f(x) / g(x)$, $g(x) \neq 0$ is continuous at $x_1 = x_o$

Proof:

The proof is similar to the one above if left as an exercise for you (good proving).

Theorem 4:

If the functions $y=f(x)$ is continuous at the point $x = x_o$ and $z = g(y)$ is continuous at $y = y_o$ where $y_o = f(x_o)$

Then the function $z = g(f(x_o))$ is continuous at point $x = x_o$.

The Proof follows from the fact that

$$\lim_{x \to x_0} f(x) = f(x_o)$$

and $\lim_{y \to y_0} g(y) = g(y_o) = g(f(x_o))$

hence $gf(x)$ is continuous at $x = x_o$

Example:

Use the theorems on continuous function to determine whether the following functions are continuous at the given points.

(i) $f(x) = 6x^2 - 2$ at $x = 2$

(ii) $f(x) = \dfrac{x^2 - 1}{x + 1}$ at $x = 1$

(iii) $f(x) = \begin{cases} x^2 + 9 & x < 3 \\ 6x & x \geq 3 \end{cases}$ at $x = 3$

(iv) $f(x) = \text{Cos}^2 x\,(x^3 + 2x - 1)$ at $x = x_o$

(v) $f(x) = \dfrac{\text{Cos}x}{e^x + \text{Sin } x}$ at $x = x_o$

(vi) $f(x) = \sin(x^2 - 1)$ at $x = x_o$

(vii) $f(x) = \dfrac{\sqrt{x^3 - 1 - x^4}}{x}$ at $x = 2$

(viii) $f(x) = \dfrac{x^2 - 2x - 1}{x - 2}$ at $x = 1$

(viii) Is continuous at $x = x_o$ since cos x is continuous

. . (Cos x) (Cos x) and x3 + 2x + 1 is continuous

.. by theorem 2 $(\text{Cos } x)(\text{Cos}x)(x^3 + 2x + 1)$ is continuous at $x = x_o$

Solution:

(I) $f(x) = 6x^2 - 2$ then

$\lim_{x\to 2} 6x^2 - 2 = 6(2)^2 - 2 = 22$

$f(2) = 6(2)^2 - 2 = 2$

Since $f(x) = f(2)$ it is continuous at $x \rightarrow 2$

(II) $f(x) = \underline{x^2 - 1}$ then $\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} x - 1 = -1 - 1 = -2$

$\lim_{x\to 2} f(x) = f(-1) = -2$

(III) $f(x) = x^2 - 9 \quad x < 3$

$6x \quad x \geq 3$

$\lim_{x\to 3^-} f(x) = 18$ and $\lim_{x\to 3^+} f(x) = 18$

$\lim_{x\to 3^-} f(x) = 18$ and $\lim f(3) = 18$

$\lim_{x\to 3} f(x) = 18$ and $\lim f(3) = 18$

Since $\lim_{x\to 3} f(x) = f(3)$

It is continuous at x = 3

(IV) $f(x) = \text{Cos}^2 x\ (x^3 + 2x + 1)$ at $x = x_o$

Cos x is continuant at x = $x = x_o$

So is (Cos x) (Cos x) by Theorem 2.

x3 + 2x + 1 is continuous since it is a polynorminal therefore

(Cos x) (Cos x) $(x^3 + 2x + 1)$ is product of continuous function which is continuous.

(V) $\frac{\text{Cos } x}{e^x + \text{Sin } x}$ at = $x = x_o$

Cos x, $\in^x$ and sin x are all continuous of $x_o = x$

$\in^x$ + Sin x is continuous by Theorem I

and $\frac{\text{Cos } x}{\in^x + \text{Sin } x}$ is continuous by Theorem II

(VI) $f(x) = \text{Sin}(x^2 - 1)$ at $x = x_o$

since $f(g(x)$ is a continuous function.

If $f(x)$ and $g(x)$ are both continuous

Therefore $\sin(x^2 + 1)$ is continuant at $x = x_o$

(VII) $f(x) = \dfrac{x^3 + 1 - x^4}{x}$ at $x = 2$

$$\lim_{x\to 2} = \frac{x^3+1-x^4}{x} = \frac{\sqrt{8}+1-16}{2} = \frac{3-16}{2} = \frac{-13}{2}$$

$f(2) = \dfrac{-13}{2}$

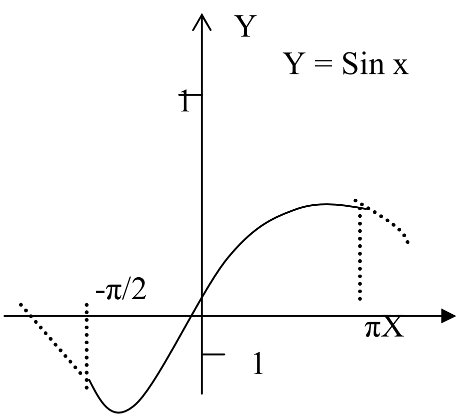
It is continuous since $\lim_{x\to 2} f(x) = f(2)$

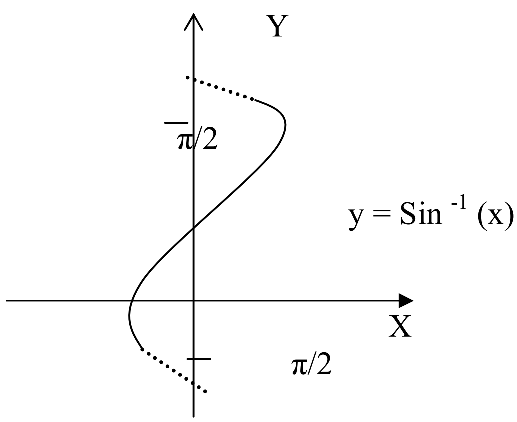
The Theorem of continuity of inverse function will now be stated. This theorem is important in that once a continuous function is defined and it is a one to one function then it becomes easy for you to determine whether the inverse function is continuous. The concept of continuity of functions brings out the hidden beauty in the study of both differential and integral calculus.

Theorem 5: Continuity of Inverse Function

If $f(x)$ is continuous on an interval I and either strictly increasing or strictly decreasing and one to one in the interval I, then there exists an inverse function $f^{-1}(x)$ which is continuous and one to one and either strictly increasing or strictly decreasing.

See Fig (24)a and Fig (24)b





In the interval [-π/2, π/2] the function $f(x) = \text{Sin } x$ is continuous and one to one so is $f(x) = \text{Sin}^{-1} x$ is also a continuous on [-π/2, π/2] and one to one.

4.0 CONCLUSION

In this Unit, you have defined a continuous function. You have used the concept of limit of a function to identify points of continuous and discontinuity of a function in a given interval of points. You have studied theorem on continuous function and used the theorem to examine points of continuities or discontinuities of a function. You are now aware that a function that is continuous at a point $x = x_o$ is defined at that point and that the limit of the function must exist as x approach the point x_o. You are also aware that the a converse is not necessary true i.e. the Limit of a function might exist at a point x_o and not continuous at that point. This logical reasoning will be extended in the next Unit.

5.0 SUMMARY

In this unit, you have studied the following:

(1) the definition of a function.

(2) how to determine points of continuity and discontinuities of a given function.
i.e. $\lim_{x \to x_o} f(x) = f(x_o)$ then x_o is a

point of continuity for the function

(3) how to use the following theorems:
If f and g are continuous functions

Then
(i) $f \pm g$, (ii) fg and f/g are continuous function to determine a function. That is continuous or not.

(4) that all polynomials, cos x and sin x are continuous in R. In the unit that follows this Unit, you will see that all the results on this unit will be used.

6.0 REFERENCES/FURTHER READING

Odili, G. (Ed) (1997): Calculus with Coordinate Geometry and Trigonometry, Anachuma Educational Books, Nigeria.

Osisiogu U.A (1998) An introduction to Real Analysis with Special Topic on Functions of Several Variables and Method of Lagranges Multipliers, Bestsoft Educational Books Nigeria Flanders H, Korfhage R.R, Price J.J (1970) Calculus academic press New York and London. Osisioga U.A (Ed)(2001) fundamentals of Mathematical analysis, best soft Educational Books, Nigeria.

Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition”, John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TOTAL-MARK ASSIGNMENTS

(1) Give a precise definition of the limit of a function $f(x)$ at the point $x = x_o$

(2) State 3 condition a function must satisfy for it to be continuous at the point $x = x_o$

(3) State two properties possessed by a function $f(x)$ which is continuous in a closed interval [a, b].

(4) using the “∈, δ” symbols explain what is meant by saying that a function $f(x)$ is discontinuous at a point x_o

(5) give examples of two types of point of discontinuities of functions. Hence for what values is each of the following functions discontinuous.

$$f(x) = \frac{x}{x+1} \qquad f(x) = \frac{2x+1}{x^2 - 3x + 2}$$

(6) Show that the function $f(x) = \text{Sin } x$ is continuous for $x = x_o$

(7) Determine whether the following functions are continuous at the given points.

(i) $f(x) = \dfrac{x}{x^2 - 2}$ $\qquad x = 1$

(ii) $f(x) = \dfrac{1 - \text{Sin } x}{2 - \text{Cos } x}$ $\qquad x = 0$

(iii) $f(x) = \dfrac{1}{x^3 - 1}$ $\qquad x = 1$

UNIT 2 DIFFERENTIATION

CONTENT

1.0 INTRODUCTION

In this unit you will learn how to differentiate a function or find the derivative of a function at a given point. This will be done by looking at the slope of a line, which you will extend to general case of slope of a curve. Then you will apply the concept studied in unit 2 to study the limiting process of a function along the given line.

After which you use the concept of a slope of a curve and a target at a given point on the curve to solve two type of problems among others namely:

1. given a function $f(x)$, determine those value of x (in the domain of $f(x)$) at which the function is differentiable

2. given a function $f(x)$ and a point $x = x_o$ at which the function is differentiable find the derived function you will finally extend this to differentiation of a polynomials functions. A section on solved problems has been included to sharpen your skills in differentiation.

This unit is a formal bridge between concept studied so far in units 1 to 4 and those you will be studying in units 6 to 10. Therefore carefully read and understand all definitions and solved examples given in this unit – wishing you a successful completion of this unit.

Below are the objectives of this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the slope of a point on a curve
- define the derivative of a function at a given point $x = x_o$
- evaluate the derivative of a function using the limiting process (i.e. Δ - process or from first principle)

- derive standard formula for differentiation of polynomials
- find the derivative of polynomials functions using the Δ - process or a standard formula.

3.0 MAIN CONTENT

3.1 Slope of a Curve

You will start the study this unit by reviewing the following:

i. the coordinate system
ii slope of a line

1. The Coordinate System

This is the system that contains

i) a horizontal line in a plane extending indefinitely to the left and to the right and which is known as x axis or axis of abscussas

ii) a vertical line in the same plane extending indefinitely up and down this is known as the y- axis or aixs of ordinates. A unit is the chosen for both axis. See fig. 25.

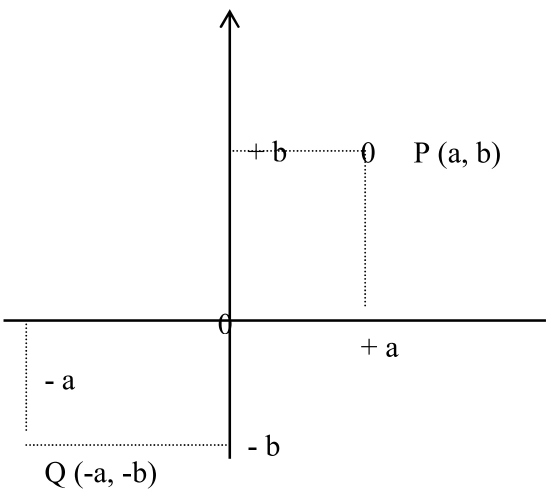


Fig. 25. Showing the coordinate system.

2. The Slope of a Line

You are also familiar with the concept of increment.

For example. If a body starts at a point $Q_1(x_1, y_1)$ and goes to a new position $Q_1(x_2, y_2)$ you say that its coordinates have changed by an increment Δx (i.e. delta x) and Δy(i.e delta y).

Let a body move from point p(2, 4) to Q(4, 6) as shown in Fig (25). Find Δx and Δy.

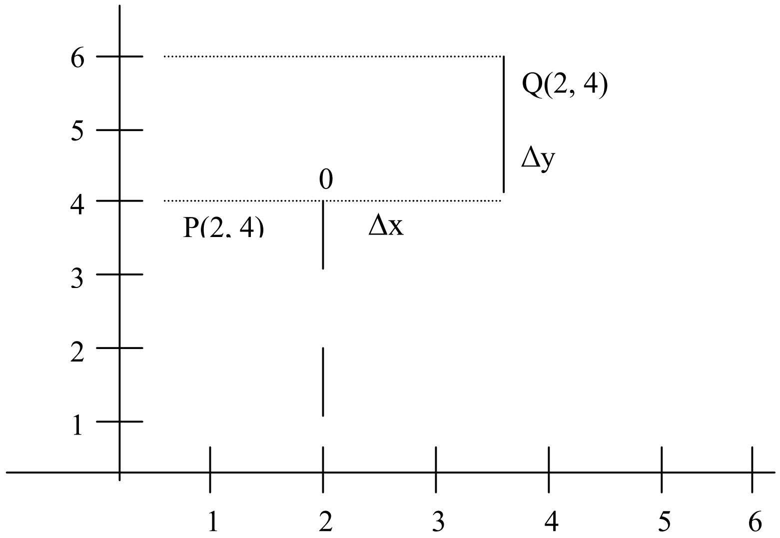


Fig 25.

$\Delta \quad x = 4 - 2 = 2$
$\Delta \quad y = 6 - 4 = 2.$

Generally if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two given points then

$\Delta x = x^2 - x_1$ if P Q
$\Delta x = x_1 - x_2$ if Q P
$\Delta y = y_2 - y_1$ if P Q
$\Delta y = y_1 - y_2$ if Q P

Using the above you can now turn your attention to finding the slope of a line the idea here is that lines in any coordinate plane rise or fall at a constant rate as we move along them from left to right unless, of course they are horizontal or vertical.

You can define the slope or gradient of line as the rate of rise or fall as you move from left to right along the given line.

Example

Describe the slope of the line L in fig 26.

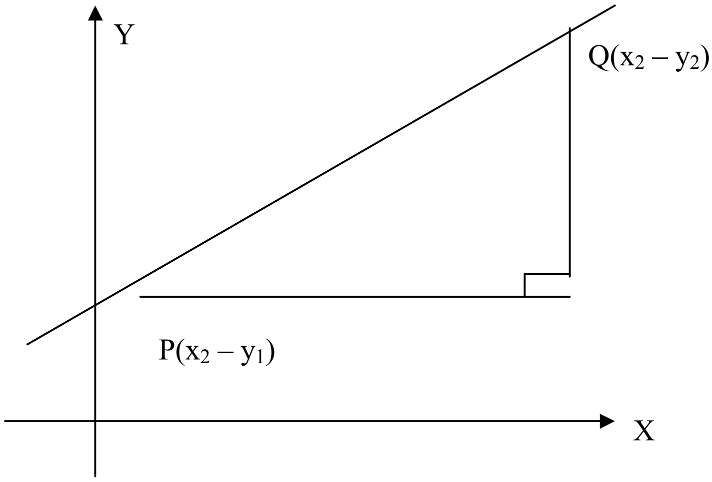


Fig. 26.

As you move from P to Q along line L, the increment $\Delta y = y_2$ is called the rise from P to Q. The increment $\Delta x = x_2 - x_1$ is called the run from P to Q. Since the line L is not vertical line then $\Delta x \neq 0$. The slope of the line L can now be defined as

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_2}{x_2 - x_1} = m$$

Remark

$$m = \frac{y_2 - y_1}{x_2 - x_1}, = \frac{y_1 - y_2}{x_1 - x_2}$$

Example

Let P (4, -2) and Q (-3 , 2) then the slope of the line joining P and Q is given as:

$$m = \frac{2-(-2)}{-3-4} = \frac{4}{-7} = \frac{-2-2}{4-(-3)} = \frac{-4}{7}$$

Slope of a Curve

You will now extend method of finding the slope of line to finding the slope or gradient of curve. To do this you start by finding the slope of a secant line through P and Q by two points on the curve C. See Fig. 27.

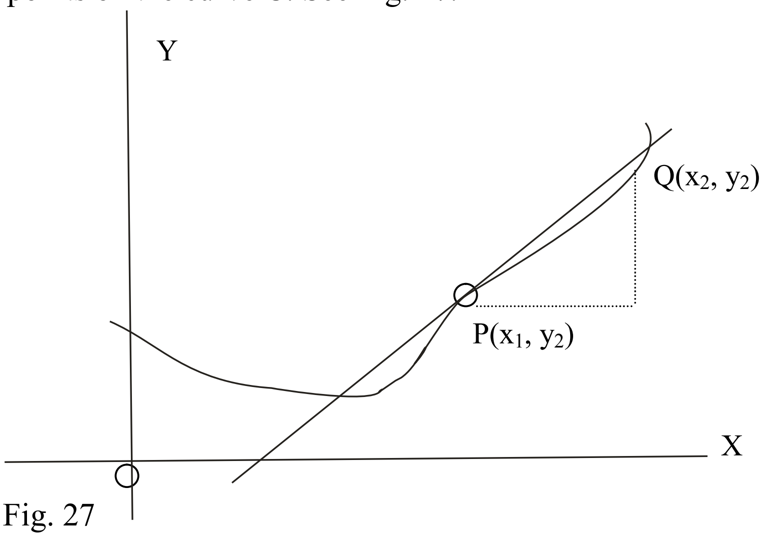


Fig. 27

In fig 27. the slope of the secant line PQ is given as

Slope of PQ $\frac{y_2 - y_{11}}{x_2 - x_{11}} = \frac{\Delta y}{\Delta x}$

$\Delta x, \ \Delta x \neq 0$

The goal is to find the slope of the curve at point P to achieve this goal you hold P fixed and move Q along the curve towards P as you do so, the slope of the secant line PQ will vary. As Q moves closer and closer to P along the curve the slope of the secant line varies by slope of the secant line varies until it approaches a constant limiting value. This limiting value is what is called the slope of the curve at point P. See Fig 28.

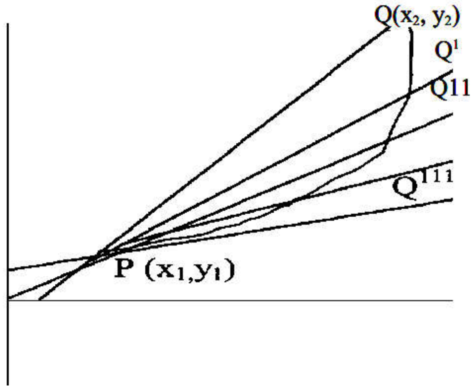


Fig 28.

In fig. 27 you will notice that as the secant line PQ moves along the curve towards the point P the slope of the secant line approaches or tends to the slope of the target line at the point P. Interesting the increment Δx tends 0 as Q --->P. Thus it will right to say that the limiting position to the tangent line TP. Therefore the slope of the secant line has a limiting value approximately equal to the slope of the tangent line.

Example

Find the slope of the curve

$Y = x^2$ at the point (x_1, y_l)

Solution

Since the point $P(x_l, y_l)$ lies on the curve then its co-ordinate must satisfy the equation.

$y= x^2$ i.e. $y_l = x_i^2$

Let $Q(x_2, x_2)$ be a second point on the curve $y = x^2$.

If $\Delta x = x_2 - x_1$ =:> $x = \Delta x_2 + x_1$

And $\Delta y = y_2 - y_l$ => $y_2 = \Delta y + y_l$

Since the point Q is on the curve limits coordinates must satisfy the equation

$y_2 = x^2$ that is $y_2 = x_2^2$

Hence $y + \Delta y = (x + \Delta x)^2$

$= x_1^2 + 2x_1 \Delta x + (\Delta x)^2$
$= x\ 1^2 + 2x_1 \Delta x + (\Delta x)^2 - y_l$
$\therefore \quad \Delta y = x_1^2 + 2x_1 \Delta x + (\Delta x)^2 - x_1^2$
$= 2x^1 \Delta x + (\Delta x)^2 \ldots\ldots (A)$

To find the slope of the line PQ you divide both sides of equation (A) by $\Delta x \neq 0$ and equation A becomes:

$\frac{\Delta y}{\Delta x} = Qx + \neq x =$ slope secant PQ

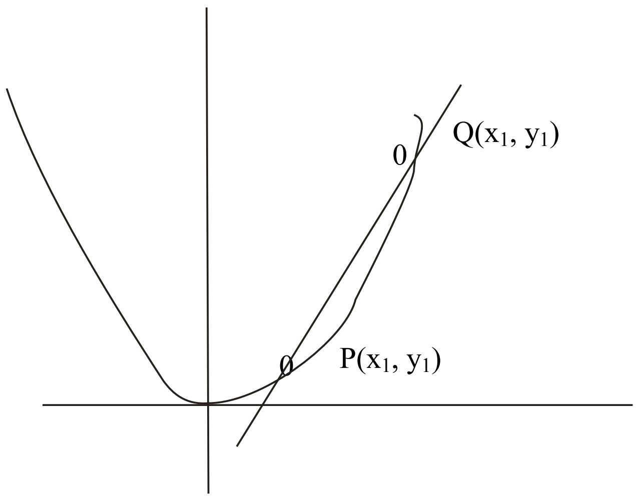


Fig. 29.

As Q gets closer to P along the curve Δx approaches zero and the slope of PQ gets closer to $2x_1$

i. e. $\frac{\Delta y}{\Delta x} \quad 2x, \quad$ as $\quad \Delta x \rightarrow 0$

By definition this means that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x,$$

The value of this limit is the slope of the tangent to the curve or slope of the curve at the point (x_l, y_l). Since the point (x_l, y_1) is chosen arbitrarily (i.e. can be any point on the curve $y = x^2$) you could remove the subscript 1 from x_l and replace x_l by x. Then the slope of the tangent will be given as

$$m = 2x.$$

This is the value of the slope of the curve at any point P(x, y) on the curve.

SELF ASSESSMENT EXERCISE 1

a. Plot the points and find the slope of the line joining them:
(i) (1, 2) and (-3, - ½) (ii) (- 2, 3) (1, -2).

b. Find a formula for the slope of the line y = mx + b at any point on the line.

c. Use the example 2 above to find the slope of the curve $y = x^2$ at the following points (i) (-2, 2) (ii) (3, -2).

3.2 Definition of Derivative of a Function

In this section you will use the concept of the slope of a curve at a given point to defined the derivative of function at a given point

Let P (x_1, y_1) be a point on the curve where;

$x_1 = x_1 + \Delta x$

And $y_2 = \Delta y + y,$

In Fig 29 below $\Delta y = f(x + \Delta x) - f(x)$

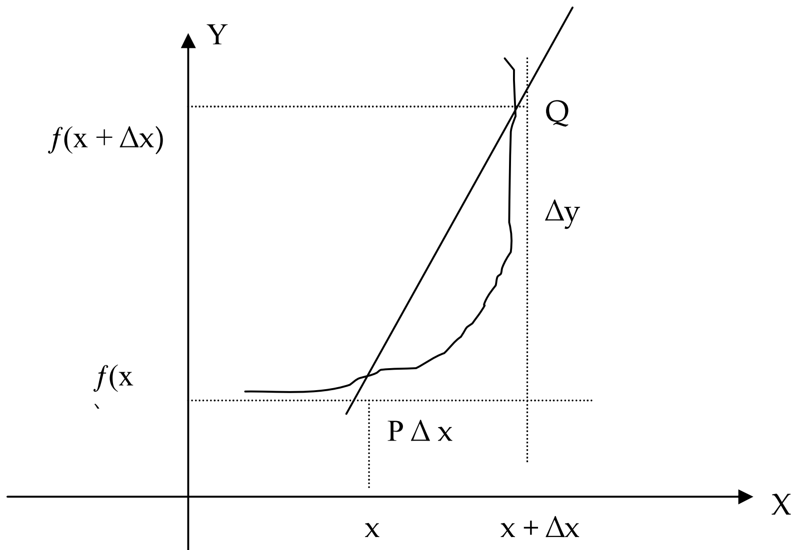


Fig 29.

Then the slope the PQ (i.e. secant) is:

$$M_S = \frac{\Delta y}{\Delta xx} = \frac{f(x + \Delta) - f(x)}{\Delta x} \quad \ldots\ldots \Delta$$

If the slope of the secant m_s approaches a constant value when Δx gets smaller and smaller, then this constant value is the limit of m_s Δx tends to zero. This limit is defined to be the slope of the tangent (m_1) to the curve at point p (x_1 y). The mathematics to describe the above could be symbolized as follows.

$$M_1 = \lim_{Q \to P} m_s = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If the slope of the secant ms approaches a constant value when Δx gets smaller and smaller, then this constant value is the limit of m_s as Δ x tends to zero. This limit is defined to be the slope of the tangent (m_t) to the curve at point P(x, y) . The mathematics to describe the above could be symbolized as follows.

$$M_t = \lim_{Q \to P} m_s = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit above if it exists is related to f(x) at the point x by writing it as f (x) (read f prime of x). which is define as

$$f(x) = \lim_{\Delta x \to 30} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The above limit may exist only for some or all values of x in the domain of definition of the function f(x). That is the limit may fail to exist for other values of x belonging to the domain whenever the limit exist for any point x, belonging to the domain, then the function f(x) is said to be differentiable at the point x.

Remark

All the curves used for or mentioned so far are assumed to be smooth (i.e. continuous) In the previous unit it was mentioned (and it is by definition) that a function that is continuous at a function that is a limit at that point. Since continuity implies existence of lim f(x). It will be shown later that a function that is differentiable at the point $x = x_o$, $x \to x_o$
is continuous at the point $x = x_o$.

Example

In the previous section 3.1 it was shown that the slope of the curve $y = x^2$ at a given point x is 2x. This implies that the function $y = x^2$ possess a derivative whose value at a point x is given to you that the function $f^1(x) = 2x$.

Example

Let us find $f^1(x)$ for the function given as $f(x) = 3x + 2$.

Solution:

This will be curried out in the steps/stages as follows:

Step 1:
For the given function $f(x) = 3x + 2$ you start by defining $f(x + \Delta x)$. This is given as $f(x + \Delta x) = 3(x + \Delta x) + 2$ (i.e. by direct substituting $x + \Delta x$ for x).
$\therefore \quad f(x+ \Delta x) = 3x + \Delta x + 2.$

And $f(x) = 3x+2$

Step 2:
Subtract $f(x)$ from $f(x + \Delta x)$ i.e.

$$f(x+ \Delta x) - f(x) -3x + 3\Delta x + 2 - (3x - 2) = 3\Delta x$$

Step 3:
Divide result of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{3\Delta x}{\Delta x} = 3$$

Step 4;
Evaluate the limit as Δx ----> 0

$$f^1(x) = \lim \frac{f(x + \Delta x) - f(x)}{\Delta x} - s$$

The above process from step 1 to step 4 is referred to as "differentiating from first principle"

Example

Find the $f^1(x)$ if $f(x) = x^2 + 3x$

Solution:

Proceed as before.

Step 1: Write out $f(x + \Delta x)$ and $f(x)$

$$f(x + \Delta x) = (x + \Delta x)^2 + 3(x + \Delta x)$$
$$= x^2 + 2x\,\Delta x + (\Delta x) + 3x + \Delta x$$
$$f(x) = x + 3x$$

Step 2: Subtract f (x) from f(x + Δx) i.e.

$$f(x + \Delta x) - f(x) = x^2 + 2x\Delta x + (\Delta x)^2 + 3x + \Delta x$$

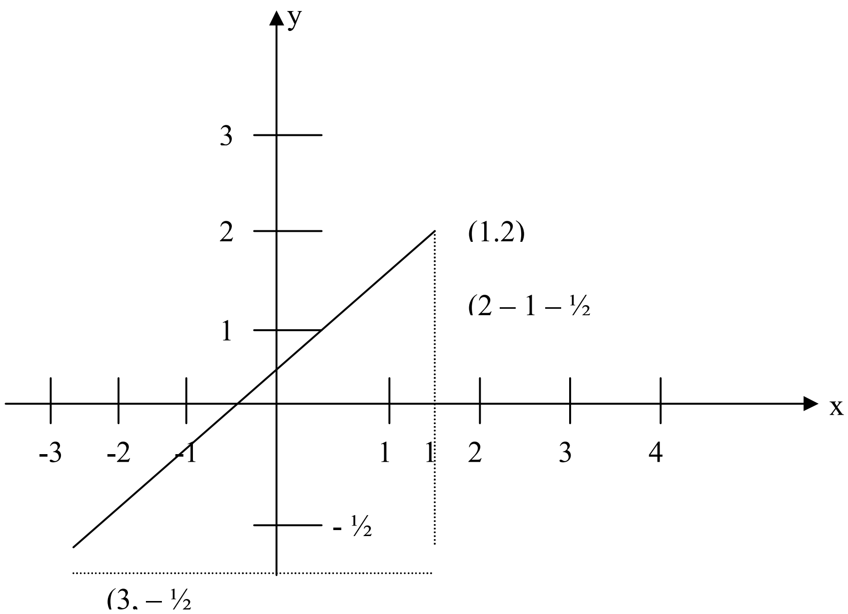
$$-x^2 - 3x$$
$$= 2x\,\Delta x + (\Delta x)^2 + 3\,\Delta x$$

Step 3: Divide the result in step 2 i. e

$$\frac{fx + \Delta x) - f(x)}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2 + 30x}{\Delta x}$$
$$= 2x + 3 + \Delta x$$

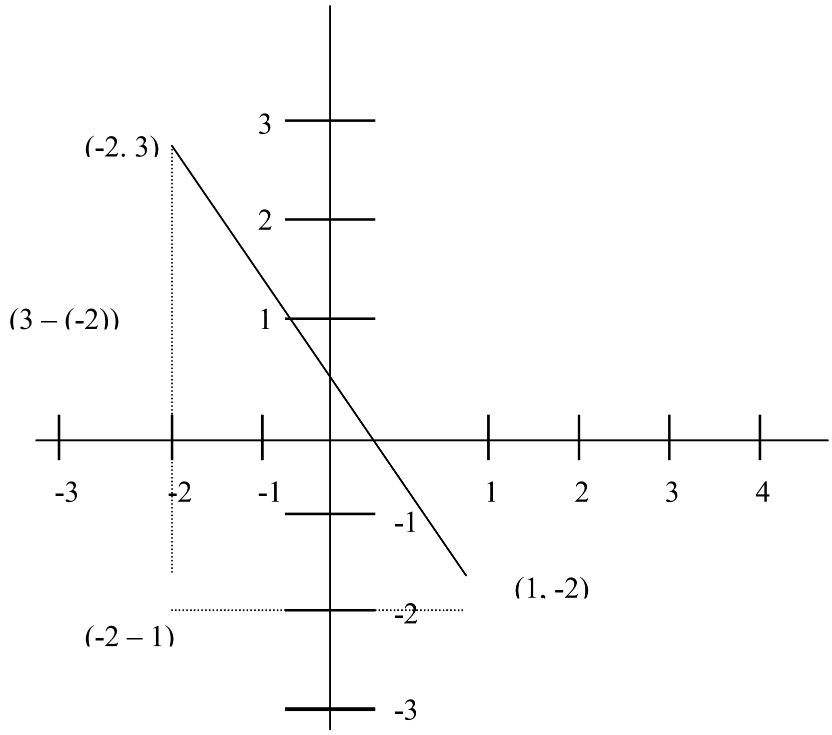
Step 4: Evaluate the limit of result of step 3 as Δx ---->0

$$f^1(x) = \lim_{\Delta x \to 0} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} 2x + 3 + \Delta x = 2x + 3$$



(1 – (-7))

(i) slope= $\frac{21/2}{4}$



(ii) slope = 2/-3

(b) Y = mx + b.
let P(x_1 , y_1) be a point on the line i.e.

Y= mx + b.

Q(x_2, y_2) be another point. If

$\Delta X = x_2 - x_1$ and $\Delta y = y_2 - y_1$

==> $x_2 = \Delta x + x_1$ and $y_2 = \Delta y + y,$

but $y_2 = mx_2 + b \Rightarrow \Delta y + y, = (\Delta x + x_1)\, m + b.$

==> $\Delta y + y_1 = mx_1 + m\Delta\, x + b.$

$\Delta y = mx_1 + m\Delta\, x + b - y_1$

$= mx_1 + m\Delta x + b - (mx_1 + b)$ since ($y_1 = mx_1 + b$.)
$= m\Delta x.$

(to get the slope of the line you divide by $\Delta x \neq 0$) this gives.)

$\dfrac{\Delta y}{\Delta x} = m$ = slope of lin y = mx + b.

(c.) from solved example slope of the curve y = x is gin as:

m = 2x therefore slope at

(i) point (-2, 2) is 2. -2 = -4
(ii) point (3, -2 is 2.3 = 6

Example

If $f(x) = x^3 + x$ find f^1 (x0)

Solution

Step 1: write out f (x + Δx) and f(x) i.e.

$f(x + \Delta x) = (x + \Delta x)^3 = (x + \Delta x)$

$= x^3 + 3x^2 \Delta x + 3x (\Delta x)^2 + (\Delta x)$

$f(x) = x^3 + x$

Step 2: subtract f(x) from f (x + Δx) i.e.

$f(x + \Delta x) - f(x) = x^3 + 3x^2 \Delta x + 3x (\Delta x)^2 + (\Delta x)^3 + x + \Delta x - x^3 - x$

$f(x) = x^3 + \Delta x + 3x (\Delta x)^2 + (\Delta x)^3 + \Delta x$

Step 3: Divide result of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^2 + \Delta x}{\Delta xx}$$

$= 3x^3 + 3x\Delta x + (\Delta x)^2 + 1$

Step 4: Evaluate the limit of result in step 3 as Δx 0

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)\, \Delta x}{\Delta x}$$

Based on the discussion so far a formal definition of the derivative of function f(x) at a point x can now be given

Definition: A function f(x) is said to be differentiable (i.e. to have a derivation) at the point x If the limit:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = f^1(x) \text{ exists}$$

Definition: A function $f(x)$ is said to be differentiable at the point x if for $\varepsilon > 0$ there is $\delta > 0$

Such that:

If $0 < /\Delta x/ < \delta$ then $\left| \frac{f(x + \Delta x) - f(x)}{\Delta x -} - f^1(x) \right| < \varepsilon$.

Remark: The limit of the quotient $\frac{f(x + \Delta) - f(x)}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ have various notation.

Such as:

$f^1(x)$ (read f prime of x)

y^1 (read y prime)

(read y clot)

$\frac{dy}{dx}$ (read dee y dee x)

In this course the notation $f^1(x)$ and dy/dx will frequently be used.

3.3 Differentiation of Polynomial Function

In unit 2 you have studied function of this type

$f(x) = a_n x^n + a_{n-1}, \quad x^{n-1} + ... + a_1 x + a_0$

as a polynomial function. You will now investigate the derivation of such function.

1. Differentiation of a constant function

Let $y = k$

Then $f(x) = k$

$f(x + \Delta) = k$

$f(x + \Delta) - f(x) = k - k$

$\frac{f(x + \Delta) - f(x)}{\Delta x} = 0$

$\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = 0$

2. Find the derivative of the function $f(x) = x^n$ at any point x.

Solution

Let $f(x) = x^n$

Step 1: $f(x + \Delta x) = (x = \Delta x)^n$

By binomial expansion:

$$(x + \Delta x)^n = x^n + \binom{n}{1} mx^{n-1} \Delta x + \binom{n}{1} x^{n-2} (\Delta x)^2 + ...+$$

$$\binom{n}{1} (\Delta x)^n$$

(If you are not very familiar with binomial expansion you can read it up in any course in algebra suggested in the unit).

Step 2 Subtract $f(x)$ from $f(x = \Delta x)$

$$f(x + \Delta x) - f(x) = x + nx^{n-I} \Delta x + (n - 1)^n x^{n-1} (\Delta x)^2$$
$$+. . .. +(\Delta x)^n - x^n$$

$$= nx^{n-I} \Delta x + (n-I)nx^{n-1} (\Delta x)^2 +...+ (\Delta x)^n$$

Step 3: Divide result of step 2 by Δx

$$\frac{f(x + \Delta) - f(x)}{\Delta x} = (n - 1)nx^{n-1} + (n - 1)nx^{n-1} \Delta x \ldots + (\Delta x)^{n-1}$$

Step 4: Evaluate limit of result of step 3 as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} nx^{n-1} + (n-1)nx^{n-1} \Delta x +...+ (\Delta x)^{n-1}$$

$$= nx^{n-1}$$

Example

$f(x) = k$ implies that $n = 1$ in that case $f(x) = x$ and $f^1 (x) = 1$

Example

$\frac{dy}{dx}$ if (1) $y = x^8$ (ii) $y = x^5$

Solution

(i) $\frac{dy}{dx} = 8x^{8-1} = 8x^7$

(ii) $\frac{dy}{dx} = 5x^{5-1} = 5^4$

3. Differentiation of the function y = ku where a is a function of x

Solution

Let $f(x) = u$

Then $f(x + \Delta x) = u + \Delta u$

Since $y = ku$

Then $y + \Delta y = k(u + \Delta u)$

$\therefore$ $4y = k(u + \Delta u) - y$

$= k\Delta u$

divide by Δx you obtain

$$\frac{\Delta y}{\Delta x} = k\frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta X} = \lim_{\Delta x \to 0} k\frac{\Delta u}{\Delta x} \quad K \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \frac{du}{dx} \quad \frac{dy}{dx}$$

$$\frac{dy}{dx} = k\frac{du}{dx}$$

Example: Find $\frac{dy}{dx}$ if $y = 5x^4$

Solution

$$\frac{dy}{dx} = \frac{d(5x^4)}{dx} = 5\frac{d(x^u)}{dx} = 5.4x^3 = 20x^3$$

Generally If $y = kx^n$

Then $\frac{dy}{dx} = knx^{n-1}$

3.4 Solved Problems

1. Given the function (i) y = 1/x (ü) y = √x.

Find the derivative by the limiting process i.e. from a suitable difference quotient $\frac{\Delta y}{\Delta x}$ and evaluate the limit as Δx tends to zero.

Solution 1

Step 1. write out $f(x + \Delta x)$ and $f(x)$ i.e.

$$f(x + \Delta x) = \frac{i}{x + \Delta x}$$

$$f(x) = 1/x$$

Step 2. Subtract $f(x)$ from $f(x+ \Delta x)$

$$\text{i.e. } f(x + \Delta x) - f(x) = \frac{1}{x + \Delta x} - \frac{1}{x}$$

$$= \frac{x - (x + \Delta x)}{X (x + \Delta x)}$$

$$= \frac{-\Delta x}{x(x + \Delta x)}$$

Step 3: divide results of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-\Delta x}{x + \Delta x} \Big/ \Delta x = \frac{-1}{x(x + \Delta x)}$$

Step 4: Evaluate limit of result of step 3 as Δx--->0

$$\text{i.e. } \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \frac{1}{x^2}$$

Solution 2

Step 1: write out $f(x+ \Delta x)$ and $f(x)$

$$f(x + \Delta x) = \sqrt{x + \Delta x}$$

$$f(x) = \sqrt{x}$$

Step 2: Subtract $f(x)$ from $f(x+ \Delta x)$

$$\text{i.e. } f(x+ \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$$

Step 3 : Divide the result of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta xx}$$

let $\Delta x = h$

then $$\frac{f(x + h)- f(x)}{h} = \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$$

$$= \frac{(\sqrt{x + h} - \sqrt{x}\ \ (x + h + \sqrt{x})}{(h\ \sqrt{x + h} + \sqrt{x})}$$

$$= \frac{x + h - x}{h\ \sqrt{x + h} + \sqrt{x}} = \frac{h}{h(\sqrt{x + h} = \sqrt{x}}$$

$$= \frac{1}{h\left(\sqrt{x+h} = \sqrt{x}\right)}$$

Step 4: Evaluate the limits of result of step 3 as h ->0

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h(\sqrt{x + h}= \sqrt{x})} - \frac{1}{2\sqrt{x}}$$

2. show that a function $f(x)$ that is differentiable at the point $x = x_o$ is continuous at that point.

Solution

Since the function $f(x)$ is differentiable at $x = x_o$ then $f^1(x_o)$ exists.

i.e. $f^1(x_o) = \lim_{h \to 0} \ f(x_o + h - f(x_o)$ (note $h = \Delta x$

but $$f(x_o+ h) - f(x_o) = \frac{f(x_o + h) - f(x_o)}{h} .h \qquad \text{- I}$$

$\therefore$ $$\lim_{h \to 0} f(x_o + h) - f(x_o) = \lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} .h$$

$$= \lim_{h \to 0} \left(\frac{f(x_o + h) - f(x_o)}{h}\right) .h \quad \lim_{h \to 0} h$$

$= 0$ (since $\lim_{h \to 0} h = 0$) -II

$$\lim_{h \to 0} \left[f(x_o + h) - f(x_o) \right] = \lim_{h \to 0} f(x_o + h) - \lim_{h \to 0} f(x_o)$$

$$= \lim_{h \to 0} f(x_o + h) - f(x_o) \qquad \text{-III}$$

==> then from equations II and III you get $\lim_{h \to 0} f(x_o + h) - f(x_o) = 0$

$$\Longrightarrow \quad \lim_{h \to 0} f(x_o + h) - f(x_o) \qquad \text{-IV}$$

Note that $x - x_0 = h$ then $h \to 0 \Longrightarrow x \to 0$

$$\therefore \quad \lim_{h \to 0} f(x_o + h) = \lim_{x \to 0} f(x_o + h) \ \lim_{x \to 0} f(x_o) = 0$$

Therefore equation IV becomes

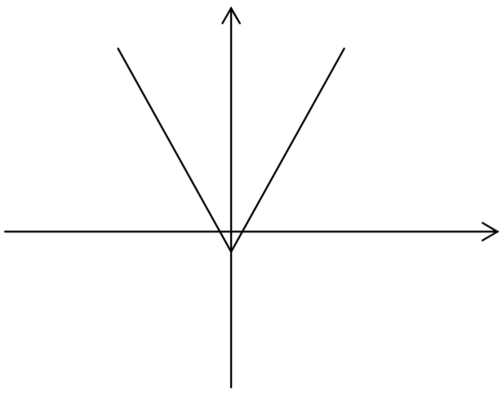
$$\lim f(x) = f(x_0)$$

Which shows that $f(x)$ is continuous at the point $x = x >$ The converse is not true. There exist functions that are continuous at given points but not differentiable at those points.

Example

Is the absolute value function, which you are very familiar with (see unit 1)

This function $f(x) = /x/$ is continuous at $x = 0$ but not continuous at $x = 0$ see fig 5.6.



The limit $f^1(0)$ does not exist

i.e. $$f^1(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = -1$$

$$f^1(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$$

Since $\lim_{h \to 0^+} f(x) \neq \lim_{h \to 0^-} f(x)$

then $f^1(x)$ at x = 0 does not exist

4.0 CONCLUSION

You have studied how to find the slope a line you have extended this to finding the slope of a curve by evaluating the limit of the slope tangent line at a given point on the curve. You have related the value of this limit with the slope of the curve at any point. You have defined this limit whenever it exist as the derivative of the function at the point x. You have used these definitions to find the derivative of certain functions such as:

$$y = k,\ y = x,\ y = x^2,\ y = 1/x,\ y = \sqrt{x}, \text{ and } y = kx^n$$

by the method of limiting process. You are now posed to use materials studied here to find derivatives of sum, product quotients of functions in the next unit. Make sure you do all your assignment correctly. You will definitely need them, because you will refer to them directly or indirectly in this remaining part of this course.

5.0 SUMMARY

The principle of using limiting processes to derive the derivatives of a function is called the first principle. The principle must be well understood because it is very useful in Advanced Analysis which you will study as you progress in the study of Mathematics.

6.0 REFERENCES/FURTHER READING

Odili, G. (Ed) (1997): Calculus with Coordinate Geometry and Trigonometry, Anachuma Educational Books, Nigeria.

Osisiogu U.A (1998) An introduction to Real Analysis with Special Topic on Functions of Several Variables and Method of Lagranges Multipliers, Bestsoft Educational Books Nigeria Flanders H, Korfhage R.R, Price J.J (1970) Calculus academic press New York and London. Osisioga U.A (Ed)(2001) fundamentals of Mathematical analysis, best soft Educational Books, Nigeria.

Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TUTOR-MARKED ASSIGNMENT

1) Differentiate from th the derivative e first principle the following functions:
   (a) $f(x) = 3x^4 + 56x^3$
   (b) $f(x) = e^{2x} + 5x + 6$
2) If $y = x^n$, show that the derivative of y is nx^{n-1} using the first principle.

UNIT 3 RULES FOR DIFFERENTIATION I

CONTENT

1.0 INTRODUCTION

In this unit you will learn a few simple rules for differentiation of all sorts of functions constructed from the ones you are already familiar with so far you have observed that the study of differential calculus started with the study of behaviour of functions and their limits at different points. In the previous unit you studied that functions that are differentiable are continuous. This implies that rules governing the results on continuous functions in respect of algebra of continuous functions can easily be extended to differentiable functions.

In this unit you will formulate rules based on theorems on limits and theorems on continuous functions you will be expected to apply the rules formulated in this unit to differentiate sum and product of polynomials function. You will then use the same rules throughout the remaining part of this course. Until the following study style will be adopted; first the rules will be stated with example. Then the rules will be justified.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive the following rules for differentiation
    a. Sum rule
    b. Difference rule
    c. Product rule
- differentiate all types of polynomial functions.

3.0 MAIN CONTENT

3.1 Differentiation of Sum of Functions

Sum Rule

The derivative of the sum of finite number of functions is the sum of their individual derivatives.

i.e.

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Solution

The proof for the case of the derivative of the sum of two functions u and v will be given first after which the prove of a finite sum will then follow.

Example 1

Let $u = x^2$, $v = 2/x$ find $d/dx\ (u+v)$

$$\frac{d}{dx}(x^2 + 2/x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2/x) = 2x - \frac{2}{x^2}$$

Now let $y = u + v$ where $u = u(x)$ and $v = v(x)$ both $u(x)$ and $v(x)$ are differentiable

Let Δx be an increment in x which will give a corresponding increments in y, and v given as Δy, Δu and Δv respectively.

Then $\Delta y = (u + \Delta u) + (v + \Delta v)$

$\Delta y = (u + \Delta u) + (v + \Delta v) - (u + v)$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \quad \text{(divide by } \Delta x\text{)}$$

$$\lim \frac{\Delta y}{\Delta x} = \lim \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = \lim\left(\frac{\Delta u}{\Delta x}\right) + \lim \frac{\Delta v}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

$$\therefore \frac{d(u+v)}{dx} = \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}$$ which is the required result.

If $y = u + v + w$

Then $y = z$, where $z = v + w$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{dx} \quad \text{but} \quad \frac{dz}{dx} = \frac{\Delta v}{\Delta x} + \frac{dw}{dx}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{dw}{dx}$$

This can be extended to a finite sum of differentiable functions i.e.

if $y = u_1 + u_2 + \ldots\ldots. + u_n$ then

$$\frac{d(u_l + u_2 + .. + u)}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \ldots\ldots + \frac{du_2}{dx}$$

Example 2

Find $\frac{dy}{dx}$ if $y = 3x^3 + 2x^2 + x + 1$

Solution

First find the derivative of each of the term and then add the result.

$$\frac{dy}{dx} = \frac{d(3x^3)}{dx} + \frac{d(2x^2)}{dx} + \frac{d(x)}{dx} + \frac{d(1)}{dx}$$

$$=3.3x^2 + 2.2x + 1 + 0$$
$$= ax^2 + 4x + 1$$

Example 3

Find the dy/dx if.

(i) $y = x^3 + 1/x$ (ü) $y = 1/3\, x^3 + ½\, x^2 + 2x$

(iii) $y = 5x^7 + 4x^5 + l0x$ (iv) $y = x^4 + 5x^2 + 1$

Solutions

(i) $y = x^3 + 1/x$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^3) + \frac{d}{dy}(1/x) = 3x^2 - \frac{1}{x^2}$$

(ii) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x$

$$\frac{dy}{dx} = \frac{d}{dx}(1/3x^3) + \frac{d}{dx}(½\, x^2) + \frac{d}{dx}(2x)$$

$$= 3.\ 1/3x^2 + 2.\ ½x + 2$$

$$= x^2 + x + 2$$

(iii) $y = 5x^7 + 4x^5 + l0x$

$$\frac{dy}{dx} - \frac{d(4x)}{dx} + \frac{d(5x^2)}{dx} + \frac{dy(1)}{dx}$$

$$= 4x^3 + 2.5x + 0$$

$$= 4x^3 + l0x$$

In unit 5, it was shown that

$\frac{d(ku)}{dx} = k\frac{du}{dx}$ where k is a constant and u is a differentiable function.

3.2 Differentiation of Difference of Functions

The above will be used to establish the difference rule.

Difference Rule: The derivative of the difference of a finite number of functions is the difference of the individual derivatives.

$$\boxed{\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx.}}$$

Example 4

Find dy/dx if $y = x^2 (x - 1)$

Let $u = x^2$ and $v = x - 1$

Then $uv = x^2(x - 1) = x^2 - x$

$$\frac{d(uv)}{dx} = \frac{d (x^3 - x)}{dx} = \frac{d (x^3)}{dx} = \frac{d(x)}{dx}$$

In unit 2 you studied that the product of two functions will result to another function. For some complicated functions it might not be very easy to carry out a straight forward multiplication or find the product before differentiable. Therefore arises the need to find a rule that will side track finding of the product of functions before differentiation. That rule is what you will derive now.

Product Rule

Suppose y = uv is the product of two differentiable functions of x then the derivative is given as:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

Proof: Let y = uv where u and v are differentiable functions of x. Let Δx be an increment in x which will result in a corresponding increments in y, a and v given as Δy, Δu and Δx respectively.

Then $\Delta y + y = (u + \Delta u) (v + \Delta v)$

$= uv + v \Delta u + u \Delta v + \Delta u \Delta v$

Subtract y from Δy + y you get

$\Delta y = u v + v\Delta u = u \Delta v + \Delta u \Delta v - u v$
$= v\Delta u + u\Delta v + \Delta u\Delta v$

Next
divide by $\Delta x \neq 0$

$$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \Delta v\frac{\Delta v}{\Delta x}$$

Evaluate the limit of dy/dx as $\Delta x \to 0$ you get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(v\frac{\Delta u}{\Delta x} + u\frac{\Delta x}{\Delta x} \Delta v \frac{\Delta v}{\Delta x} \right)$$

$$= v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \lim \frac{\Delta v}{\Delta x}$$

The $\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \Delta x = \frac{du}{dx} . 0 = 0$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + 0.\frac{dv}{dx}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

This can now be extended to the case of product of a finite number of functions:

i.e. if $y = u_l u_2 u_3 \ldots u_n$ then

$$\frac{d(u_l u_2 u_3 \ldots u_n)}{dx} = \frac{du_l}{dx}(u_2 \ldots u_n) + u_l \frac{du_2}{dx}(u_3 \ldots u_n)$$

$$+ \ldots + (u_l \ldots u_{n-1})\frac{du_n}{dx}$$

Example 5

Differentiate the following functions

(i) $y = (x^2 - 1)(x + 3)$

(ii) $y = (x^3 – 2x)(x^2 - 1)$

(iii) $y = (x^2 + 1)(2x^2 - 1)(x - 1)$

(iv) $y = (x^2 - 1)(3x - 1)(x^4 - 1)(3x^2 - 1)$

Solutions

(i) $y = (x^2 - 1)(x + 3)$

let $u = (x^2 - 1)$ and $v = (x + 3)$

then $\frac{dy}{dx} = u\frac{du}{dx} + v\frac{dv}{dx} = (x^2 - 1)\frac{d(x+3)}{dx} + (x+3)\frac{d(x^2 - 1)}{dx}$

$= (x^2 - 1)\,1 + (x + 3)(2x)$
$= x^2 - 1 + 2x^2 + 6x$
$= 3x^2 + 6x - 1$

(ii) $y = (x^3 - 2x)(x^2 - 1)$

Let $u = (x^3 - 2x)\ v + (x^2 - 1)$

$\frac{dy}{dx} = u\frac{du}{dx} + v\frac{dv}{dx} = (x^3 - 2x)\frac{d(x^2 - 1)}{dx} + (x^2 - 1)\frac{d(x - 2x)}{dx}$

$= (x - 2x)(2x) + (x^2 - 1)(3x^3 - 2x)$

$= 2x^4 - 4x^2 + 3x^4 - 3x^2 - 2x^2 + 2$

$= 5x^4 - 9x^2 + 2.$

(iii) $y = (x^2 + 1)(2x^2 - 1)(x - 1)$

$y = u\ v\ w$

$\frac{dy}{dx} = u\ v\frac{dw}{dx} + u\ w\frac{dv}{dx} + v\ w\frac{du}{dx}$

let $u = (x^2 + 1)$ $v = 2x^2 - 1$ $w = x - 1$

$\frac{dy}{dx}$ $\frac{d(uvw)}{dx} = \frac{du}{dx}(v\ w) + u\frac{dv}{dx}w + (u\ v)\frac{dw}{dx}$

$= 2x(2x^2 - 1)(x^2 - 1) + (x + 1)\,4x\,(x - 1) + (x^2 + 1)(2x^2 - 1)\,1$

$= 10x^4 - 8x^3 + 3x^2 - 2x - 1$

(iv) $y = (x^2 - 1)(3x - 1)(x^4 - 1)(3x^2 - 1)$

Let $u = (x^2 - 1)$ $v = (3x - 1)$, $w = (x^4 - 1)$ $z = (3x^2 –$

1) $\frac{dy}{dx} \quad \frac{d(uvwz)}{dx} = \frac{du}{dx} v\,w\,z + u \frac{dv}{dx} w\,z = uv \frac{dw}{dx} ≠ u\,v\,w \frac{dz}{dx}$

$= 2x\ (3x - 1)(x^4 - 1)(3x^2 - 1)$

$+ (x^2 - 1)(3)(x^4 - 1)(3x^2 - 1)$

$+ (x^2 - 1)(3x - 1)(4x^3)(3x^2 - 1)$

$+ (x^2 - 1)(3x - 1)(x^4 - 1)\ 6x$

Example 6

More solved samples find dy/dx if

(i) $y = \left(\frac{1}{x}\right) \left(\frac{1}{x - 1}\right) (x^2 - 1)$

(ii) $y = (2x - 1)\left(\frac{1}{x - 1}\right) (\sqrt{x})$

(iii) $y = \left(\frac{1}{x}\right)(\sqrt{x})(x^2 - 1)$

Solutions

(i) $y = \left(\frac{1}{x}\right) \left(\frac{1}{x - 1}\right)(x^2 - 1)$

let $u = \left(\frac{1}{x}\right)$, $v = \left(\frac{1}{x - 1}\right)$ $w = (x^2 - 1)$

$\therefore \frac{dy}{dx} = \frac{-1}{x^2} \cdot \frac{1}{(x-1)} (x^2 - 1) - \frac{1}{(x(x-1)^2)} \cdot (x^2 - 1) + \frac{2}{(x-1)}$

(ii) $y = (2x - 1)\left(\frac{1}{x - 1}\right) (\sqrt{x})$

$u = (2x - 1)$, $v = \left(\frac{1}{x - 1}\right)$, $w = (\sqrt{x})$

$\therefore \frac{dy}{dx} = 2. \frac{1}{x + 1} \sqrt{x} - \frac{2x -}{(x + 1)} \sqrt{x} + \frac{1}{2} \frac{(2x - 1)}{(x + 1)\sqrt{x}}$

$= \frac{x\sqrt{x}}{(x + 1)} - \frac{2x^{3/2}}{(x + 1)^2} + \frac{1}{(x + 1)^2} \sqrt{x} - \frac{1}{2(x + 1)\sqrt{2}}$

(iii) $y = \left(\frac{1}{x}\right)(\sqrt{x})(x^2 - 1)$

$y = \frac{1}{\sqrt{x}}(x^2 + 1)$

Let $u = \frac{1}{\sqrt{x}} = x^{-½}$ $\quad v = (x^2 + 1)$

$\frac{dy}{dx} = \frac{-1}{2x^{3/2}}(x^2 + 1) + 2(\sqrt{x})$

$= \frac{3}{2}\sqrt{x} \quad \frac{1}{2x^{3/2}}$

4.0 CONCLUSION

In these unit you have acquired necessary techniques or methods of differentiation. These techniques are governed by rules which you have just studied. You will be required to apply these rules when dealing with differentiation of transcendental functions. The rules of differentiation you have studied as follows: sum rule, Difference rule, Product rule, Quotient rule and Composite rule. You have used the rules in the examples given in the unit some of these examples with little modification or changes. Endeavor to solve all your exercises. It builds confidence in you.

5.0 SUMMARY

In this unit you have studied the following rules for differentiation.

1) Sum rule: $\frac{d\,(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2) Difference rule: $\frac{d\,(u - v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3) Product rule: $\frac{d(uv)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

In this unit you have studied the following

(i) the slope of a line i.e. $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

(ii) the slope of a curve at a given point i.e. $\lim_{\Delta x \to 0} \Delta y / \Delta x$

(iii) the definition of derivative of a function f (x) at a point x i.e. $f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

(iv) how to differentiate a polynomial function using the limiting process i.e. evaluating the limit of a suitable quotient such as $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$

(v) that not all continuous functions are differentiable e.g. $f(x) = /x/$ is continuous at x = 0 but not differentiable at x = 0.

6.0 REFERENCES/FURTHER READING

Odili, G. (Ed) (1997): Calculus with Coordinate Geometry and Trigonometry, Anachuma Educational Books, Nigeria.

Osisiogu U.A (1998) An introduction to Real Analysis with Special Topic on Functions of Several Variables and Method of Lagranges Multipliers, Bestsoft Educational Books Nigeria Flanders H, Korfhage R.R, Price J.J (1970) Calculus academic press New York and London. Osisioga U.A (Ed)(2001) fundamentals of Mathematical analysis, best soft Educational Books, Nigeria.

Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TUTOR-MARKED ASSIGNMENT

Differentiate with respect to x

(1) $4x^{10}$

(2) $3x^7$

(3) ¼ $x^{7/2}$

(4) $2x^{-2}$

(5) $4x^4 - 8x^3 + 2x$

(6) $3\sqrt{7}x$

(7) $(x + 1)(x^2 - 1)$

(8) $(x + 1)(x + 2)(x + 3)$

(9) $(x^2 + 1)(4x^2 - 1)$

(10) $(x^3 - 1 (x)$

(11) drive the product rule.

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

(12) show that $\frac{d}{dx}(u+v) = v\frac{du}{dx} + u\frac{dv}{dx}$

(13) $\frac{d}{dx}(x^2 - 1)(2x + 1) =$

(14) $\frac{d}{dx}((3x)^2 - 1)(x - 1) =$

(15) $\frac{d}{dx}(3x^{½} - 1)(x^2 - 1) =$

UNIT 4 RULES FOR DIFFERENTIATION II

CONTENTS

1.0 INTRODUCTION

In the previous unit you have studied rules of differentiation namely

1) Sum rule
2) Difference rule
3) Product rule

In this unit you will also extend the properties of differentiability to include quotient rule and differentiation of a composite function. This unit will follow the same concept used in the previous unit. The introductions in unit 7 will fit perfectly as an introduction to this unit.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- Derive the formula for quotient rule from first principle using the limiting process.
- Derive the Chain rule.
- Differentiates all types of rational functions with denomination of this type $a\,x^n + a_l\,xn^{-1} + ... + \ c/n$

3.0 MAIN CONTENT

3.1 Differentiation of Quotient of Functions

Quotient Rule. The derivation of the quotient of two functions is given as:

i.e.

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$

Example 1

Find dy/dx if $y = \frac{x^2 - 2}{x - 1}$

Solution

Let $u = x^2 - 2, \quad v = x - 1$

$$\frac{d}{dx} = \frac{(x-1)2x - (x^2 - 2)}{(x-1)^2}$$

$$= \frac{2x^2}{(x-1)} - \frac{x^2 - 2}{(x-1)^2}$$

Quotient Rule

Suppose y = u/v $\qquad v \neq 0$

Is the quotient of two differentiable functions of x then the derivative is given as

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$

Proof: Let $y = {}^{u}\!/_{v}$, where u(x) and v(x) are both differentiable at a domain where $v \neq 0$. Let Δx be an increment in x in the given domain. Then Δy, Δu and Δv are corresponding increments in y, u and v respectively.

Then

1. $\Delta y + y = \frac{(u + \Delta u)}{(v + \Delta u)}$ where $\Delta v + v \neq$

   subtracting y from $\Delta y + y$ you get

   $$\Delta y = \frac{u + \Delta u}{v + \Delta u} - \frac{u}{v}$$

   $$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

2. $\frac{v\Delta u - u\Delta v}{v(v + \Delta V)}$

   division by $\Delta x \neq 0$ yields

   $$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$

Taking the limit as Δx tends to zero you obtain

3. $\lim d = \lim_{\Delta x \to 0} \left(\frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta x)} \right)$

$$\frac{\lim_{\Delta x \to 0} v \frac{\Delta u}{\Delta x} - \lim_{\Delta x \to 0} u \frac{\Delta v}{\Delta x}}{\lim_{\Delta x \to 0} v (v + \Delta v)}$$

As $\Delta x \to 0 \lim (v + \Delta v) = \lim v \lim (v + \Delta v)$
$= \lim v (\lim v + \Delta v)$
$= v(v+0)$

Recall that $\lim_{\Delta x \to 0} \Delta v = \lim_{\Delta x \to 0} \Delta v \quad \Delta x = dv \quad .0$

$\lim_{\Delta x \to 0} v(v + \Delta v) = V^2$

Hence from (3) you get:

$$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{V^2}$$

$$\frac{dy}{dx} = \lim \frac{\Delta y}{\Delta x}$$

Example 2

If $y = \frac{x^2 + 2}{x^3}$ Find $\frac{dy}{dx}$

Solution

Let $u=x^2 + 2, v = x^3$

Then $\frac{dy}{dx} = \frac{(x^3) (2x) - (x^2 +2) (3x^2)}{(x^3)^2}$

$$\frac{2}{x^2} - \frac{3(x^2 +2)}{x^4}$$

Example 3

Differentiate the following functions if

(i) $y = \frac{x3 + 1}{x - 1}$

(ii) $y = \frac{x + 1}{\sqrt{x}}$

(üi) $y = \frac{x^2 + 1}{\sqrt{x} + 1}$

(iv) $y = \frac{3x^2 + 2x + 1}{x^2 - 1}$

Solution

(1) $y = \frac{x^3 + 1}{x - 1}$, $y = x^3 + 1$, $v = x - 1$

$$\frac{dy}{dx} \quad \frac{(x - 1)3x^2 - x^3 + 1}{(x-1)^2} = \frac{3x^2}{x-1} - \frac{(x^3 + 1)}{(x-1)^2}$$

(ii) $y = \frac{x + 1}{\sqrt{x}}$ $u = x + 1$, $v = \sqrt{x}$

$$\frac{dy}{dx} = \frac{x - ½ (x + 1) (x^{-1/2})}{x}$$

$$= \frac{1}{\sqrt{x}} - \frac{1}{2}$$

$$= \frac{x - 1}{2x^{3/2}}.$$

(üi) $y = \frac{x^2 + 1}{x + 1}$ $u = 3x^2 + 2x + 1$, $v = x^2 - 1$

$$\frac{dy}{dx} = \frac{(6x + 2)(x^2 - 1) - 2x(3x^{22} + 2x + 1)}{(x^2 - 1)^2}$$

$$\frac{6x + 2}{x^2 - 1} - \frac{2x(3x^2 + 2x + 1}{(x^2 - 1)^2}$$

Let $u = g(x)$ and $y = u^n$ then

$$\frac{dy}{dx} = \frac{du"}{dx} = nu^{n-1}\frac{du}{dx}$$

Recall that in unit 5 you proved that $\frac{dy}{dx} = mx^{n-1}$ when u = x. You will use the same principle to show the above, i.e. Let $y = u^n$ where u is a differentiable function of x and $n \in N$.

Therefore Δy and Δu are corresponding increment in y and u respectively.

Then $y + \Delta y = (u + \Delta u)^n$
By trinomial equation you get that:

$$y + \Delta y = u^n + nu^{n-1}\Delta u + (\text{terms in u and higher powers of } (\Delta u)) .$$

From the above subtract y from $y + \Delta y$ You get:

$$\frac{dy}{dx} = nu^{n-I}\frac{du}{dx} + 0$$

(iv) $y = (x+1)^4 (x - 1)^3 (x^2 + 1)^2$

Let $u = (x + 1)^4$, $v = (x- 1)^3$, $w = (x^2 + 1)^2$

Then $\frac{du}{dx} = 4(x+1)^3$ $\frac{dv}{dx} = 3(x- 1)^2$ $\frac{dw}{dx} = 4x(x^2 + 1)$

$$\frac{dy}{dx} = 4(x \ \ 1)^3 (x - 1)^3 (x^2 + 1)^2 \ 3(x -1)^4 (x + 1)^2 (x - 1)^2$$
$$+ 4(x + 1)^4 (x - 1)^3 (x^2 + 1)$$

If n is negative instead of $y = u^n$ you have

$$Y = u^{-n} = \frac{1}{u^n}$$

Using the quotient rule you will get that

$$\frac{dy}{dx} = \frac{d\left(\frac{1}{U^n}\right)}{dx} \quad \frac{u^n \frac{d(l)}{dx} - 1\frac{d(u^n)}{dx}}{(u^n)^2}$$

$$= \frac{0 - nu^{n-1}\frac{du}{dx}}{u^{2n}}$$

$\frac{dy}{dx} = (- nu^{n-1} \; u^{-2n}) \frac{du}{dx}$

Given that:

(1) $y = x^4 + \frac{1}{x^4}$ (2) $Y = x^3(x - 1)^{-2}$

(3) $y = 2x(4x^2 - 3)^{-3}$ Find dy/dx in each case.

Solutions

(1) $y = x^4 + \frac{1}{x^4} = x^4+X^{-4}$

$\frac{dy}{dx} = 4x - 4/x^5$

(2) $Y = x^3(x - 1)^{-2}$

let $u = x^3$ $v = (x - 1)^{-2}$

$\frac{dy}{dx} = 3x^2(x-1)^{-2} – 2x^3 (x - 1)^3$

$\frac{3x^2}{(x - 2)^2} - \frac{2x^3}{} \quad (x - 2)^3$

(3) $y = 2x(4x^2 - 3)^{-3}$

let $u = 2x$ $v = (4x^2 - 3)^{-3}$

$\frac{dy}{dx} = \frac{2}{(4x^2 - 3)^3} - \frac{48x^2}{(4x^2 - 3)^4}$

3.2 The Chain Rule for Differentiation

You will now learn how to differentiate composite functions, which you studied in unit 2. i.e. if *f* (x) and g(x) are functions defined in the same domain then *f* og =*f*(g (x))

Remark: Now suppose that u(x) is a differentiable functions of x, then u (x) changes $\frac{du}{dx}$ times as fast as x does. If *f* changes n times as fast as g and g changes n times as fast as w, then *f* changes mn times as fast as w.

Suppose a function y = *f* (g(x)) where *f* and g are both differentiable functions of x.

Then $\frac{dy}{dx} = \frac{d}{dx}$ [f (g(x))]

This implies that by the above remark f (g(x)) changes f^{1}(g(x)) times as fast as g(x) And g(x) changes g^{1}(x) x g^{1}(x) times as fast as x.

f^{1} (g(x) g^{1} (x) is the derivative of f (g(x) with respect to x . which means that

d [f (g(x))] = f^{1} (g(x) g^{1} (x) .

The above is called the chain rule for the differentiation of a composite function (i.e, function of function). In the previous section you have studied that if u is a differentiable

function of x and $y = u^n$ then $\frac{dy}{dx} = nun^{-1} \frac{du}{dx}$ for n εQ

The coefficient of the term $\frac{dy}{dx}$ of the above equation can be written as

$$nu^{n-1} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du}, \quad \frac{du}{dx}$$

To prove the chain rule you assume that y = f (u) is differentiable at point $u = u_0$, then an increment Δu will produce a corresponding increment Δy is y such that:

$$\Delta y = f^{1}(u_0)\, \Delta u + \varepsilon 1\ \Delta u \qquad \_\_\_\_\_\_\_ \qquad (A)$$

If u = g(x) is a differentiable at a point $x = x_0$, then an increment Δx produces a corresponding increment Δy such that

$$\Delta u = g^{1}(x_0)\, \Delta x + \varepsilon_2\, \Delta x \qquad \_\_\_\_\_\_\_\_ \qquad (B)$$

If $\varepsilon_1 \rightarrow 0$ then $\Delta u \rightarrow 0$

And if $\varepsilon_2 \rightarrow 0$ then $\Delta x \rightarrow 40$

Combining equations (A) and (B) you get:

$$\Delta y = (f^{1}(u_0)\ \varepsilon_1)\ \Delta x\ (g^{1}(x_0) + \varepsilon 2) \quad \_\_\_\_\_\_\_\_\_\_ \quad (C)$$

Dividing equation (C) by $\Delta x \neq 0$ you get

$$f^{1}(u_0)\ g^{1}(x_0) + f^{1}(u_0)\ \varepsilon_2 + g^{1}(x_o)\ \varepsilon_l + \varepsilon_l\ \varepsilon_2 \quad \_\_\_\_\_\_\_ \quad (D)$$

Taking limits on both side D you get

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to O} f^{1}(u_0)\ g^{1}(x_0) + f^{1}(u_0)\ \varepsilon_2 + g^{1}(x_o)\ \varepsilon_l + \varepsilon_l\ \varepsilon_2 \quad (E)$$

(E)

But as $\Delta x \to 0 \quad \Rightarrow \quad \varepsilon_2 \to 0 \quad \Rightarrow \Delta u \to 0 \quad \Rightarrow \quad \varepsilon_1 \to 0$
Then Equation (E) becomes

$$\left(\frac{dy}{dx}\right)_{x^0} = f^{1}(u_0)\ g^{1}(x_0)$$

which is $\left(\frac{dy}{du}\right)_{u_o} \left(\frac{du}{dx}\right)_{x_o}$

since x_0 and u_0 are chosen arbitrarily then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The above could be written as

$$(fog)^{1}(x) = f^{1}(g(x))\ g^{1}(x)$$

or $\quad (fog)^{1} = f^{1},(g(x))\ g^{1}(x).$

Solution

Let $u = \frac{1}{x - 1}$

$$y = u^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = 4u^3, \quad \frac{du}{dx} = \frac{-1}{(x - 1)^2}$$

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \; 4\left(\frac{1}{x-1}\right)^3 \; \frac{-1}{(x - 1)2}$

Example 4

Differentiate the following function with respect to x

(1) $y = (1- 3x)^{-1}$ (2) $y = \left(\frac{2x}{x^2 - 1}\right)^3$

(3) $y = \left(\frac{1}{x - 1} - \frac{1}{x + 1}\right)^3$ (4) $y = \left(\frac{x^2 + 2}{x^2 - 1}\right)^4$

(5) $y = (2x^4 + x^2 - 1)^6$ (6) $y = \left(\frac{x^3}{2} - \frac{x^{-2}}{2} - \frac{x^{-1}}{1}\right)$

Solutions

(1) $Y = (1-3x)^{-1}$

$y = u^{-1}$ $u = 1 - 3x$

$\frac{dy}{du} = -u^{-2}$ $\frac{du}{dx} = - 3.$

$\frac{dy}{dx} = - u^{-2}$ $-3 = 3u^{-2} = \frac{3}{(1-3x)^2}$

(2) $y = \left(\frac{2x}{x^2 - 1}\right)^3$

Let $y = u^3$, $u = \frac{2x}{x^2 -1}$, $\frac{dy}{du} = 3u^2$

$$\frac{du}{dx} = \frac{-2(x^2+1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3\left(\frac{2(x^2+1)}{(x^2-1)}\right)^2 \cdot \left(\frac{-2(x2+1)}{(x^2-1)^2}\right)$$

$$\frac{-24x^2(x^2+1)}{(x^2-1)^4}$$

(3) $y = \left(\frac{1}{x-1} - \frac{1}{x-1}\right)$

$u = \left(\frac{1}{x-1} - \frac{1}{x-1}\right) \quad y = u^3$

$\frac{du}{dx} = \frac{-1}{(x-1)^2} + \frac{1}{(x+1)^2} \quad \frac{dy}{du} = 3u^2$

$$dy = 3\left(\frac{1}{x+1} - \frac{1}{(x-1)^2}\right)^2 (x + 1)^2 \left(\frac{1}{} - \frac{1}{}\right) \text{ x } -1$$

$$\frac{-48x}{(x-1)^4(x+1)^4}$$

(4) $y = \left(\frac{x^2+2}{x^2-1}\right)^4$

Let $u = \left(\frac{x^2+2}{x^2-1}\right)^4 \quad y = u^4$

$$\frac{du}{dx} \quad \frac{2x}{(x^2-1)} - \frac{2x(x^2+2)}{(x^2-1)^2} = \frac{-6x}{(x^2-1)^2}$$

$\frac{dy}{} = 4u^3\, du$

$$dy = 4\left(\frac{(x^2+2)}{(x^2-1)^2}\right)^3 = \left(\frac{-6x}{(x^2-1)^2}\right)$$

$$\frac{-24x(x^2+1)^3}{(x^2-1)^5}$$

5) $y = (2x^4 + x^2 - 1)^6$

$u = 2x^4 + x^2 - 1, \quad y = u^6$

$\frac{du}{dx} = 4x^3 + 2x, \quad \frac{dy}{du} = 6u^5$

$\frac{dy}{dx} = 6(2x^4 + x^2 - 1)^5 \quad (4x^3 = 2x)$

(6) $y = \left(\frac{x^3}{3} - \frac{x^2}{2} - \frac{x}{1}\right)^-$

Let $u = \frac{x^3}{3} - \frac{x}{2} - \frac{x}{1}$

$\frac{dy}{du}$ u-2, $\frac{du}{dx} = x2 - x - 1$

$$\frac{dy}{dx} = \frac{-(x^2 - x - 1}{\left(\frac{x^3}{3} - \frac{x^2}{2} - x\right)^2}$$

4.0 CONCLUSION

In this unit you have derived rules for differentiating quotient of functions. You have also derived the important chain rule. As mentioned in the previous unit all these rules are very important because you will use them throughout the remaining part of this course.

5.0 SUMMARY

In this unit you have studied the following rules for differentiation.

(i) Quotient rule

$d\left(u\right)$ [redacted] $= \dfrac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$

(ii) Chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

You have used these rules differentiate polynomials and rational functions.

6.0 REFERENCES/FURTHER READING

Godwin Odili (Ed) (1997): Calculus with Coordinate Geometry and Trigonometry, Anachuma Educational Books, Nigeria.

Osisiogu U.A (1998) An introduction to Real Analysis with Special Topic on Functions of Several Variables and Method of Lagranges Multipliers, Bestsoft Educational Books Nigeria Flanders H, Korfhage R.R, Price J.J (1970) Calculus academic press New York and London. Osisioga U.A (Ed)(2001) fundamentals of Mathematical analysis, best soft Educational Books, Nigeria.

Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

7.0 TUTOR-MARKED ASSIGNMENT

Differentiate the following functions with respect to x.

(1) $4x^3 + 2x^2 + \frac{1}{x}$

(2) $\frac{x + 1}{2x}$

(3) $x (x - 4)^3$

(4) $\frac{x^2}{x - 1}$

(5) $\frac{x + 1}{x\sqrt{x}}$

(6) $\frac{x3 - 2x + 1}{x^2 - 1}$

(7) $\left(x - \frac{2}{x}\right)^5$

(8) $\frac{6x}{(x^2 - 1)^2}$

(9) $\frac{2(x^2 + 1)}{(x^2 - 1)^2}$

(10) $\frac{4}{(x - 1)^5}$

(11) $x^3 (x - 2)^{-1}$

(12) $x^3 - \frac{1}{x^3}$

(13) $(x^2 + x)^3 (3x + 1)^3$

(14) $(x + 1) (2x_2 - 1) (2x - 1)$

(15) $(x - 1)^4 (x + 1)^2$