# MODULE 3



## UNIT 1 FURTHER DIFFERENTIATION

**CONTENTS**



## 1.0 INTRODUCTION

By now you would have known that the subject differential calculus has a lot to offer to mankind. In order to be able to solve a large number of problems it is important to study the derivative of certain class of functions. You already know that if a function $f$ is a one to one function then $f$ has an inverse $f^{-1}$. The question now is suppose $f$ is a differentiable and one to one function. Will the inverse function $f^{-1}$ be differentiable? And under what conditions will $(f^{-1})$ exist. This is one question among others that you will be able to answer in this unit. In addition problems of relating to motion of a body a along a curve can only be fully described if the derivative known. In this unit higher derivations of function will be discussed so that you and others may be able to solve completely the problem of motion rte: a body along a curve. Optimization of scarce resources can easily be solved with the knowledge of higher derivatives of function most of the functions that have been treated so far are expressed explicitly in terms of one independent variable x. There are certain functions that might not be expressed explicitly, such function fall into the class of functions known as implicit function. They are so called in the sense that dependant and independent variables are expressed implicitly. Finding derivatives of such functions will be discussed; it will save you the time of trying to express the dependent variable in terms of the independent variable before differentiating y.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to

- differentiate the inverse of a function
- evaluate higher derivatives of any given function
- differentiate an implicit function.

## 3.0 MAIN CONTENT

## 3.0 Differentiation of Inverse Functions

You could recall that the inverse of the function $y = x^3$ is given as $y = x^{1/3}$ (see Unit 2)

If $y = x^{1/3}$

$\frac{dy}{dx} - \frac{1}{3} x^{1/3 -1} \quad x^{-2/3}$ and $x \neq 0$

In the above example the function $y = x^3$ is a one to one function and also a differentiable function. Also the function $y = x^{1/3}$ is a one to one and also differentiable at a specified domain provided $x \neq 0$.

You recall that in unit 2 you studied that the composite function of $f$ (x) and its inverse $f^{-1}(x)$ in any order yields the identity function.

i.e. $f(f^{-1}(x)) = f(f^{-1}(X)) = x$

Using the function $f(x) = x^3$ you have that $f^{-1}(x) = x^{1/3}$

Then $f(f^{-1}(x)) = (x^{1/3})^3 = x$

And $f(f^{-1}(x)) = (x^3)^{1/3} = x$

Using the above illustration, you can now differentiate the composite function given as

$f(f^{-1}(x)) = x$

i.e. $\frac{d}{dx}[f(f^{-1}(x))] = \frac{dx}{dx}$

Using the chain rule studied in Unit 6.

Let $f^{-1}(x) = g(x)$

Then $f(g(x)) = x$.

But $\frac{d}{dx}[f(g(x))] = f'^1(g(x)) = \frac{dx}{dx} = 1$

$f^1(g(x)).\ g^1(x) = 1.$

$\Rightarrow \quad g^1(x) = \frac{1}{f^1(g(x))}$

This gives the derivative of inverse of a function

i.e. $(f^{-1}(x)) = \frac{1}{f^1(g(x))}$

**Example**

Let $f(x) = x^3$ Find $(f^{-1}(x))$

**Solution**

$$(f^{-1}(x))^1 = \frac{1}{f^1(f^1(x))} = \frac{1}{3(f^{-1}(x))^2} = \frac{1}{3(x^{1/3})^2}$$

$$\frac{1}{3x^{2/3}}$$

by direct differentiation of y = x you get:

$$\frac{dy}{dx} = x.$$

**Example**

Let $f(x) = x^3 + 1$ find the derivation of the inverse

**Solution**

$$f(x) = x^3 + 1;\ f^1(x) = 3x^2$$

inverse $f^{-1}(x) = (x - 1)^{1/3}$

$$(f^{-1}(x))^1 = \frac{1}{(f^{-1}(x))^1} = \frac{1}{3(f^{-1}(X))^2} = \frac{1}{3((x - 1)^{1/3})^2}$$

The derivative of inverse of the function $f(x) = x^n$ . , $x > 0$

Given that $f(x) = x^n$ and $f^1(x) = x\ 1/n$

$$f^1(X) = nx^{n-1}$$

therefore $(f^{-1}(x))^1 = \frac{1}{f^1(f^{-1}(x))} = \frac{1}{n(f^{-1}(x))^{n-1}}$

$$\frac{d}{dx}[x\ 1/n] = \frac{1}{n}x^{1/n-1} \quad \frac{1}{n(x1/n)n-1} = \frac{1}{n\,x^{1-1/n}}$$

$$\frac{1}{n}x^{1/n-1}$$

Thus for $x > 0$ and $f^1(x) = x^{1/n}$

$$\boxed{\frac{d}{dx}[x^{1/n}] = \frac{1}{n}x^{1/n-1}} \qquad \text{(A)}$$

and for $x \neq 0$ and n odd

$$\frac{d}{dx}[x^{1/n}] = \frac{1}{n}x^{1/n-1}$$

**Examples**

Find the derivative of the following functions.

(I) $y = x^{½}$

(II) $y = x^{I/7}$

(III) $y = x^{p/E}$

(IV) $y = x^{2/5}$

(V) $y = x^{4/3}$

(VI) $y = x^{-5/3}$

**Solutions**

(I) $y = x^{1/2} = \frac{dy}{dx} = \frac{1}{2} x^{-1/2}$

(II) $y = x^{1/7} = \frac{dy}{dx} = \frac{1}{7} x^{-6/7}$

(III) $y = x^{p/E}$

Let $x^{p/q} = (x^{1/q})^p$

$\frac{d}{dx} [x^{p/q}] \quad \frac{d}{dx} ([x^{1/q}])^p$

Let $u = x^{I/q}$ then $y = u^p$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\frac{dy}{du} = p u^{p-1}$ and $\frac{du}{dx} = \frac{1}{q} x^{1/q -1}$

$\frac{dy}{dx} = p u^{P-1} \frac{1}{q} x^{1/q-1} = p(x^{1/q})^{p-1} \cdot \frac{1}{q} x^{1/q-1}$

$= \frac{p}{q} x(\frac{p}{q} - 1)$

$$\boxed{\frac{d}{dx} [x^{p/q}] = \frac{p}{q} x(\frac{p}{q} - 1)} \quad \text{——— (B)}$$

(IV) $y = x^{2/5}$

$$\frac{dy}{dx} = \frac{2}{5} x^{2/5-1} = \frac{2}{5} x^{-3/5}$$

(V) $y = x^{4/3}$

$$\frac{dy}{dx} = \frac{4}{3} x^{4/3-1} = \frac{4}{3} x^{1/3}$$

(VI) $y = x^{-5/3}$

$$\frac{dy}{dx} = \frac{-5}{3} x^{-5/3-1} = \frac{-5}{3} x^{-8/3}$$

The above equation (B) could be extended to the case f (x) = u where u is to the case *f* (x) = u; where u is a differentiable functions of x.

i.e.
$$\frac{d}{dx}(u(x))^{p/q} = \frac{P}{q}[u(x)]^{p/q-1}\frac{d}{dx}[u(x)]$$

For the above to make sense, then u (x) ≠ 0 when q is odd and u (x) > 0 when q is even.

**Example**

(1) If $y = [(x^2 - 1)^{1/5}]$ then

$$\frac{dy}{dx} = \frac{1}{5}(x^2-1)^{1/5-1}\, 2x$$

$$\frac{2x}{5}(x^2 - 1)^{-4/5}$$

(1) Evaluate $\frac{d}{dx}[2x^2 - 7]^{1/3}] = \frac{1}{3}(2x^2 - 7)^{-2/3} \cdot 4x$

$$\frac{4x}{3(2x^2-7)^{2/3}}$$

(3) Evaluate

$$\frac{d}{dx}\left[\sqrt{\frac{x^2+1}{x^2-1}}\right] = \frac{d}{x}\left[\left(\frac{x^2+1}{x^2-1}\right)^{½}\right]$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-½} \cdot \frac{d}{x} = \left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-½} \frac{2x(x^2-1) - 2(x^2+1)}{(x^2-1)^2}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-½} \left(\frac{-4x}{(x^2-1)^2}\right)$$

$$= \frac{-2x}{(\sqrt{x^2+1})(x^2-1)^{3/2}}$$

### 3.2 Implicit Differentiation

So far you have been finding the derivatives of functions of the class of functions whose right side of the equality sign is an expression of one variable (i.e. x). Such functions are said to be explicit functions. However, there are functions such as

$$x^2y = 2xy^2 + 6$$

This type of such is expressed implicitly. To obtain an explicit expression of an implicit expression you resolve in transpose (make subject of formula) the equation in the dependant variable or one variable

Example

$2x^2 + 3y = 6$ transposing for y

yields $y = \frac{2x^2 - 6}{-3}$

However there are implicit functions where it will not be possible to solve for y. Example of such functions are

(1) $x^2 + xy^4 + y^3x + x^3 = 2$

(2) $x^2 + xy^2 + 5x^3 + y^2 = 1$

In the above although it is not possible to solve for y, they can be differentiated by the method of known as implicit differentiation. Appropriate applications of the rules for differentiations u, u v, $u^{1/n}$ ,U, etc. which you have studied in unit 6. You should be able to carry out implicit differentiation. The next question that comes to your mind should be "what is implicit differentiation" this question is best answered by finding the derivative of the functions.

$$x^2 + xy^4 + y^3x + x^3 = 2$$

**Solution:** Differentiating both sides of the equation you get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy^4) + \frac{d}{dx}(y^3x) + \frac{dx^3}{dx} = \frac{d(2)}{dx}$$

$$=2x + \frac{dx}{dx}, y^4 + x\frac{d(y)^4}{dx} + y^3\frac{dx}{dx} + x\frac{d(y)^3}{dx} + 3x^2 = 0$$

$$= 2x + I.\ y^4 = 4xy^3\frac{dy}{dx} + y^3 + 3x\,y^2\frac{dy}{dx} + 3x^2 = 0$$

$$= (2x + y^4 + 3x^2 + y^3) + (4xy^3 + 3x\,y^2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2x + y^4 + 3x^2 + y^3)}{4xy^3 + 3xy^2}$$

Implicit differentiation is useful in finding the slope of a tangent to curves.

**Example:** Find $\frac{dy}{dx}$ of the following functions.

1. $x^2 + y^2$ (2) $y^2 = \frac{x^2 - 2}{x - 1}$

3. $x^2 y^2 + y + 2 = 0$ (4) $x^3 - xy + yl = 0$

5. $2xy - y^2 = x - y$ 6. $(x + y)^2 + (x-y)^2 = x^3 + y^3$

**Solutions**

1. $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(4)}{dx}$

$$=2x^2 + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y},\ y \neq 0$$

2. $\frac{d(x^2)}{dx} = \frac{d}{dx}\left(\frac{x^2 - 2}{x - 1}\right)$

$$-2y\frac{dy}{dx} = \frac{2x(x-1)-(x^2-2)}{(x-1)^2} = \frac{x^2 - 2x + 2}{(x-1)^2}$$

$$2y \frac{dy}{dx} = \frac{x^2 - 2x + 2}{(x - 1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 2x + 2}{2y(x - 1)^2}$$

3. $$\frac{d(x^2y^2)}{dx} + \frac{d(y)}{dx} + \frac{dy}{dx} = 0$$

$$x^2 \frac{d(y^2)}{dx} + y^2 \frac{d(x^2)}{dx} + \frac{dy}{dx} = 0$$

$$x^2 2y \frac{dy}{dx} + y^2 2x + \frac{dy}{dx} = 0$$

$$2xy^2 + (2yx + 1) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{- 2yx^2}{2yx-1}$$

4. $$\frac{d(x^3)}{dx} + \frac{d(xy)}{dx} = \frac{d(y^2)}{dx} = 0$$

$$3x^2 - y \frac{d(x)}{dx} - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$3x^2 - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$3x^2 - y + (2y - x) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} \quad \frac{y - 3x^2}{2y-x}$$

5. $$\frac{d(2xy)}{dx} - \frac{d(y^2)}{dx} = \frac{d(x)}{dx} - \frac{dy}{dx}$$

$$2x \frac{dy}{dx} + 2y - 2y \frac{dy}{dx} = 1 - \frac{dy}{dx}.$$

$$2y - 1 = (2y - 2x - 1) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2y - 1}{2y - 2x - 1}$$

6. $\frac{d(x + y)^2}{dx} + \frac{d(x - y)^2}{dx} = x^3 + y^3$

$$\frac{d(x + y)^2}{dx} + \frac{d(x - y)^2}{dx} = \frac{d\,(x^3 + y^3)}{dx}$$

$$2x(x + y)\left(1 + \frac{dy}{dx}\right) + 2(x - y)\left(1 - \frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$\Longrightarrow 2(x + y) = 2(x - y) + [(2(x + y) - (2x - y)]\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$

$$\Longrightarrow 4x + 4y\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$

$$\Longrightarrow (4x+3y^2)\frac{dy}{dx} = 3x^2 + 4x$$

$$\frac{dy}{dx} \quad \frac{3x^2 - 4x}{4y - 3y^2}$$

**Example**

By differentiating the equation

$x^2 y^2 = x^2 + y^2$ implicitly show that

$c = \frac{k(1 - y^2)}{x - 1}$, where k y = x.

**Solution**

Given that $x^2 y^2 = x^2 + y^2$

Then $d\,\frac{(x^2 y^2)}{x^2 - 1}$, where k y = x.

**Solution:** Given that $x^2 y^2 = x^2 + y^2$

Then $\frac{d(x^2 y^2)}{x^2 - 1}$, $\frac{d}{d}(x^2 + y^2)$

$$2xy^2 + 2yx^2\frac{dy}{dx} = 2x + 2y\frac{dy}{dx} \qquad \_\_\_\_\_\_\_\_\_ \quad (1)$$

Collecting like terms you equation (1) becomes

$$(2x + 2y)\, dy = 2x - 2xy^2 \qquad \_\_\_\_\_\_\_\_\_\_ \quad (II)$$

Dividing equation II by $(2yx^2 - 2y)$

$$\frac{dy}{dx} = \frac{x - xy^2}{yx^2 - y} = \frac{x(1 - y^2)}{y(x^2 - 1)}$$

since $k = x/y$ then.

**Example**

By differentiating the equation

$x^2 + y – y^2$ implicitly show that

$$\frac{dy}{dx} = \frac{2y - x}{2x + y}$$

**Solution:** Differentiating with respect with y you have:

$$\frac{d(x^2)}{dy} \frac{d(xy)}{dy} - \frac{d(y^2)}{dy} = 0$$

$$2x \frac{dx}{dy} + x + y \frac{d}{d} - 2y = 0$$

$$\frac{dx}{dy}(2x + y) = 2y - x$$

$$\frac{dx}{dy} = \frac{2y - x}{2x + y}$$

## 3.3 Higher Order Differentiation

You will start this section with the study of second derivative of a function where it exist and then extend it to higher order.

Let $y = f(x)$ be a differentiable function of x. Then it has a derivative given as

$$\frac{dy}{dx} = f^1(x)$$

$f^1(x)$ is a function let:

$f^1(x) = g(x)$

Then $g^1(x) = \frac{d^2y}{dx^2}$ which is the second derivative of function $y = f(x)$.

Which is written as $\frac{d^2y}{dx^2}$ or $f^{11}$ (x)

**Example**

Let $y = 4x^3$ Find $\frac{d^2y}{dx^2}$

**Solution**

$\frac{dx}{dy} = \frac{d}{d}(4x^3) = 12x^2$

$\frac{d^2y}{dx^2} = \frac{d}{d}(12x^2) = 24x$

**Example**

$y = x^3 - 2x^2 + x.$ find $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = \frac{d}{d}(x^3 - 2x^2 + x) = 3x^2 - 4x + 1$

$\frac{d^2y}{dx^2} = (3x^2 - 4x + 1)$

$= 6x - 4$

Since you now know what a second derivation of higher order. The idea here is that so long as you have differentiability, you can continue to differentiate $y = f(x)$ from $\frac{dy}{dx}$

$f^{1}(x)$ and $\frac{dy}{dx} = f^{1}(x)$ to form $\frac{dy}{dx^2} = f^{11}(x)$ and $\frac{dy}{dx^2} = f^{11}(x)$ to form $\frac{dy}{dx^3} = f^{111}(x)$

And so on until you get to an $n^{th}$ order

i.e. $\frac{d(y)}{dx} = \frac{dy}{dx}$, $\frac{d}{d} \frac{dy}{dx} = \frac{d^2y}{dx^2}$

$\frac{d}{d} \quad \frac{d^2y}{dx^2} \quad \frac{d^3y}{dx^3} \quad \frac{d}{d} \quad \frac{d^3y}{dx^3} = \frac{d^4y}{dx^4}$

$\frac{d}{d}\left(\frac{(d^{n-1} g)}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$

**Example**

Let $y = x^5 + x^4 + x^3 + 1$

Then $\frac{dy}{dx} = 5x^4 + 4x^3 + 3x^2$

$\frac{d^2y}{dx^2} = 20x^3 + 12x^2 + 6x$

$$\frac{d^3y}{dx^3} = 60x^3 + 24x^2 + 6x$$

$$\frac{d^4y}{dx^4} = 120x + 24$$

$$\frac{d^{555}y}{dx^5} = 120$$

$$\frac{d^6y}{dx^6} = 0$$

$$\frac{d^{>}1y}{dx^1} = \frac{d^8y}{dx^8} = \dots = \frac{d\,''y}{dx^n} = 0$$

In the above example all derivatives of order higher than 5 are identically zero. You can see that derivative of a polynomial function is again a polynomial function. This implies that polynomial functions have derivatives of all order so also, is all rational functions.

**Example**

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-1}{x^2}, \qquad \frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{-6}{x^4} \qquad \frac{d^4y}{dx^4} = \frac{24}{x^5}$$

**Example**

Find $\frac{d^4y}{dx^4}$ If $y = \frac{2x}{x - 1}$

**Solution**

$$y = \frac{2x}{x - 1}$$

$$\frac{dy}{dx} = \frac{-2x}{(x - 1)^4}, \quad \frac{d^2y}{dx^2} = \frac{4x}{(x \_ 1)^3}$$

$$\frac{d^3y}{dx^3} = \frac{-12x}{(x - 1)^4}, \quad \frac{d^4y}{dx^4} = \frac{48x}{(x - 1)^5}$$

**Example**

By differentiating implicitly find $d^2y$ if $x^2 = 1 + y^2$, leave your answer in terms of x and y only.

**Solution**

Given that $x^2 - 1 = y^2$

Then $2x = 2y \frac{dy}{dx}$

$X = y \frac{dy}{dx}$

$1 = \frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}$

$$\Rightarrow \quad 1 = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad 1 - \left(\frac{dx}{dy}\right)^2 = y\frac{d^2y}{dx^2}, \quad \text{where } x/y = dy/dx.$$

$$\frac{d^2y}{dx^2} = \frac{1 - (x/y)^2}{y} = \frac{y^2 - x^2}{y^3}$$

**Example**

By differentiating implicitly find: (1) dy/dx. (2) $d^2y/dx^2$ in the following equations.

(1) $x^2 - y^2 = 4x$

(2) $x^3 - y^3 = 27$

(3) $x^2 + y + y^2 = 1$

(4) $x^2 y^2 = 16$

**Solutions**

(1) $x^2 - y^2 = 4x$

$\frac{d(x^2)}{dx} - \frac{d(y^2)}{dx} = \frac{d(4x)}{dx}$

$2x - 2y \frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{2x - 4}{2y} = \frac{x - 3}{y.}$

Given that $y \frac{dy}{dx} = x - 2$

$$\frac{d}{d}\left(y\ \frac{dy}{dx}\right) = 1 .$$

$$\frac{dy}{dx}\ \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 1$$

$$\frac{d^2y}{dx^2} = \frac{1 - \left(\frac{dy}{dx}\right)}{y} = \frac{1 - \left(\frac{x-2}{y}\right)^2}{y} = \frac{-(y^2 + x^2 - 4x + 4)}{y^3}$$

(2) $x^3 - y^3 = 27$

$$\frac{d(x^3)}{dx} - \frac{(dy^n)}{dx} = 0$$

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$\text{====>}\ \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{d}{dx}(x)$$

$$y^2\ \frac{d^2y}{dx^2} + 2y\frac{d}{dx}.\ \frac{d}{dx} = 2x$$

$$y^2\ \frac{d^2y}{dx^2} = 2(x - y\left(\frac{dy}{dx}\right)^2)$$

$$\frac{d^2y}{dx^2} = \frac{2\,(x - y\left(\frac{x}{y}\right)^4)}{y2} = \frac{2x\ \ 1\left(-\frac{x^3}{y^3}\right)}{y^2}$$

$$= \frac{2x(y^3 - x^3)}{y^5}$$

(3) $x^2\,y + y^2 = 1$

$$\frac{d}{dx}(x^2y) - \frac{d(y^2)}{dx} = 0$$

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$-2xy = (x^2 + 2y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}$$

Differentiating implicitly the equation

$$\frac{d}{dx}(2x\,y)\ \frac{d}{dx}(x^2 \frac{dy}{dx}) + \frac{d}{dx}(2y \frac{dy}{dx}) = 0$$

you get:

$$2y + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 0$$

collecting like terms you get:

$$2y + 4x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x^2 + 2y)\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = 2y + 4x \quad \frac{\left(\frac{-2xy}{x^2+2y}\right) + 2\left(\frac{-2xy}{x^2+2y}\right)^2}{x^2+2y}$$

(4) $x^2 y^2 = 16$

$$\frac{d}{dx}(x^2 y^2) = 16$$

$$2x\,y^2 + x^2\,2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xy^2}{2yx^2} = \frac{-y}{x}$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}(-y) = \frac{d}{dx}\left(x\ \frac{dy}{dx}\right)\ \frac{d}{dx}(-y)$$

$$\frac{dy}{dx} = -2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-2\,dy/dx}{x} = \frac{-2\,(-y/x)}{x} = \frac{2y}{x^2}$$

## 4.0 CONCLUSION

In this unit you have applied rules of differentiation to find derivatives of inverse of a function which is turn lead to differentiation of function such as $y = x^n$ where nEQ. You have studied implicit differentiation will be useful when finding the normal or tangent of curve at a given point. Higher order derivatives of functions, which you studied in this unit, is a very useful tool for studying applications of differentiation. The various solved examples in this unit is given to enable you acquire the necessary tools for further differentiation.

## 5.0 SUMMARY

In this unit, you have studied how to

- Fine the derivatives of inverse of a given function
  i.e.
  $$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f^{1}(f^{-1}(x))}$$
- Differentiate a given equation implicitly
- Fine higher order derivatives of functions

  i.e. $\frac{dy}{dx} \quad \frac{d}{dx}\frac{(dy)}{dx} \quad \frac{d}{dx} \quad \frac{d^2y}{dx^2} \quad = \frac{d^3y}{dx^2}, \ldots,$

  $$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \quad \frac{d^n y}{dx^n}$$
- Differentiate functions with fractional powers i.e. $\frac{d}{dx}(x^{p/q}) = P/q\ x^{(p/q)}$

## 6.0 REFERENCES/FURTHER READING

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## 7.0 TUTOR-MARKED ASSIGNMENT

For exercise (1) - (2) find the derivatives of the inverse of the following functions:

(1) $y = x^2 - 1$

(2) $y = 4x^5 - 2$

(3) $y = \frac{2x}{x - 1}$

(4) $y = \frac{1}{x^3 + 1}$

(5) Find the derivatives of the following functions

(I) $y=x^{1/5}$ (II) $y = x^{1/9}$
(III) $y = x^{3/5}$ (1V) $y = x^{-2/3}$

(6) Find the derivative of the following functions

(1) $y = (x-1)^{1/5}$ (II) $y = (2x^2 - x)^{1/3}$
(III) $y = (x^3 - 1)^{2/3}$

(7) Evaluate $\frac{d}{dx}\sqrt{\left(\frac{x2 - 1}{x + 2}\right)}$

(8) Evaluate $\frac{d}{dx}\left(\frac{x + 1}{x^2 - 2}\right)^{2/3}$

(9) If $y \sqrt{l+x^2} = 2$ find $\frac{dy}{dx}$

(10) If $y = \frac{x - 1}{x + 2}$ show that

$(x+2)^2 \quad \underline{dy} = 3$

dx

(11) If $y = \frac{1}{\sqrt{x^2 + 1}}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(12) Find the value of $\frac{dx}{dx}$ and $\frac{d^2y}{dx^2}$
at the point p(2, 3) if $x^2 + xy = y$.

(13) Find $\frac{dx}{dy}$ if $(x - y) + (x + y)2 = x^2 + y^2$

(14) By differentiating the equation

$2x - y^2 = x^2 - 2y$ show that

$(1- y) \frac{d^2y}{dx^2} = 1 + \left(\frac{x - 1}{1 - y}\right)^2$

(15) What is the value of $\frac{d^2y}{dx^2}$ if $y = x^6$

(16) Find $\frac{d^4y}{dx^4}$ if $y = \frac{x + 1}{x^2}$

(17) Find $\frac{d^3y}{dx^3}$ if $y = \sqrt{\frac{1 + x}{1 - x}}$

(18) Find $\frac{d^4y}{dx^4}$ if $y = 3x^4 + \frac{1}{x}$

(19) show that $\frac{dx}{dy} = \frac{-x}{y}$ if $x^2 y^2 = 16$.

(20) Find $\frac{d^2}{dx^2}\left(\sqrt{\frac{x^2 + 1}{x - 1}}\right)$

# UNIT 2 DIFFERENTIATION OF LOGARITHMIC FUNCTIONS AND EXPONENTIAL FUNCTION

**CONTENTS**



## 1.0 INTRODUCTION

So far you have studied differentiation of functions such as polynomials and rational functions. In this unit you will be studying differentiation of two special functions namely natural log and exponential which have practical applications in real life problems such as computation of compound interest accruing from money deposited or borrowed from financial institutions. Another application where the differentiation of these two special functions could be applied is in the prediction of growth or decay of a radioactive substance. The two functions natural logarithm and exponential functions that will be subject of study in this unit, are related to one another because one is the inverse functions of the others. That is $f^{-1}$(natural logarithm) = exponential function and the $f^{-1}$ (exponential function) is the natural logarithm.

## 2.0 OBJECTIVES

At the end of this unit you should be able to:

- differentiate logarithmic functions
- carry out logarithmic differentiation
- differentiate exponential functions
- find the derivative of the function $a^u$
- find the derivative of the function $\log_a u$.

## 3.0 MAIN CONTENT

### 3.1 Differentiation of the Logarithm Functions

You will review some properties of logarithm functions you are already familiar with.

(1) $x = \log_a y$ if $a^x = y$

(2) the log of a product = the sum of the logs. Keeping the above in mind you should be able to recall the following:

let $f(x) = \log x$

then $f(x, y) = f(x) + f(y)$ …….. (I)

suppose $f(x) = 1$

then $f(1) = f(1 . 1) = f(1) + f(1) = 2 f(1)$

$\Rightarrow$ $f(1) = 2 f(1)$

$\Rightarrow$ $0 = f(1).$ ......... (II)

if this is true for $x > 0$

Then $0 = f(1) = f(x . 1/x) = f(x) + f(1/x)$

$\Rightarrow$ $=$ $f(1/x) = f(x)$ ………….. (III)

Taking $x > 0$ and $y > 0$ then

$f(y/x) = f(y . 1/x) = f(y) + f(1/x) = f(y) – f(x)$

(Using equation II)

$f(y/x) f(y) – f(x)$ .................. (IV)

Still keeping $x > 0$ and using equation I you get

$f(x . x . x.... x) = f(x^n) = f(x) + .....+ f(x)$

$= n f(x)$

$f(x^n) = n f(x).$

Replacing $f(x)$ by log x you get the 3 basic properties.

(1) $\log(x y) = \log x + \log y$

(2) $\log(y/x) = \log x – \log y$

(3) $\log(x^p) = p \log x.$

You will now attempt to derive a formula for the derivative of a logarithm function.

Let log x = $f$ (x) and x > 0 where $f$ is assumed to be a non-constant differentiable function of x which has all the properties of a logarithm stated above let Δx be an increment resulting in a corresponding increment in $f$(x). Then the difference quotient can be formed as

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \underline{\hspace{3cm}} \quad (A)$$

Since $f$ (x) is a logarithmic function.

You can re-write the above equations (A) as

$$f(x+\Delta) - f(x) = f\frac{(x+\Delta x)}{x} = f\left(1+\frac{\Delta x}{x}\right)$$

Hence $$\frac{f(x+\Delta) - f(x)}{\Delta x} = f\frac{(x+\Delta x)}{x}$$

Multiplying equation (B) x/x and noting that $f$ (l) = 0 you get

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{x}\left[\frac{f(1+\Delta x/x) - f(1))}{\Delta x/x}\right]$$

Taking limits of equation (C) as Δx -----> 0 you get

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \lim_{\Delta x\to 0} \frac{1}{x} \quad \lim_{\Delta x\to 0}\left[\frac{f(1+\Delta x/x) - f(1)}{\Delta x/x}\right] = I$$

$$= 1 \quad \lim_{\Delta x\to 0} \left[\frac{f(1+\Delta x/x) - f(1))}{\Delta x/x}\right]$$

$$f^1(x) = 1/x\, f^1(1)$$

$$\Rightarrow \quad \frac{d}{dx}[f(x)] = \frac{d}{dx}[\log x] = \frac{1}{x} \text{ where } f^1(x) = 1$$

(Since $f$ is a non-constant function $f^1(1) \neq 0$).

If u is a differentiable positive function of x (i.e. u(x) > 0 and $u^l$ (x) exist)

$$\text{Then } \frac{d}{dx}(l\,nu) = \frac{d}{du}\ln u \;\frac{du}{dx} = \frac{1}{u} \quad \frac{du}{dx}$$

**Example:** Find $\frac{d}{dx}\log(x^2 + 1)$.

**Solution**

Let $u = x^2 + 1$ here $u > 0 \ \forall \ \varepsilon$ R.

Then $\frac{d}{dx}(x^2 + 1). = \frac{1}{x^2+1} \quad 2x \quad = \frac{2x}{x^2+1}$

**Example:** Find $\frac{d}{dx}\log \frac{1}{x^2 + 1}$

**Solution**

Let $u = \frac{1}{x^2 + 1}$ here $u > 0$ V x $\varepsilon$ R.

Then $\frac{d}{dx} \log\left(\frac{1}{(x^2 + 1)}\right) \frac{d}{dx}(\log u) = \frac{1}{u} \quad \frac{du}{dx}$

$= \quad x^2 +1 \quad \frac{-2x}{(x^2 + 1)^2} \quad = \quad \frac{-2x}{(x^2 + 1)}$

**Example:** Show that $\frac{d}{dx}$ (log /x/ ) = $\frac{1}{x}$

**Solution:** For $x > 0$

$\frac{d}{dx}$ (log /x/ ) = $\frac{d}{dx}(\log x) = \frac{1}{x}$ /x/ = x for $x > 0$

For $x < 0$, / x/ = x .

Therefore $\frac{d}{dx}$(log /x/ ) = $\frac{d}{dx}(\log -x)$

Here let $u = -x \quad \frac{du}{dx} = -1 \quad \frac{d}{dx}(u) = \frac{1}{u}$

$\frac{d}{dx}$ (log /x/ ) = $\frac{1}{u}. (-1) = \frac{1}{x}. (-1) = \frac{1}{x}$

**Example:** Find $\frac{d}{dx} \log \quad /1 - x^2/$

**Solution**

Let $u = 1- x^2$, then $\frac{d}{dx} \log \ /u/$

$$= \frac{1}{1 - x} . - 2x \ = \frac{- 2x}{1 - x^2} \ = \frac{2x}{x^2 - 1}$$

**Example:** Find $\frac{d}{dx} \log \left( \frac{x^4}{x - 1} \right)$, $x \neq 1$.

**Solution**

Let $u = \frac{x\Delta}{x + 1} \quad \frac{du}{dx} = \frac{(3x - 4)x^3}{(x-1)^2},$

Then $\frac{d}{dx} \log = \frac{1}{u} . \frac{du}{dx} = \frac{x - 1}{x^4} . \frac{(3x - 4)x^3}{(x - 1)^2}$

$$= \frac{(3x - 4)}{x(x - 1)}$$

## 3.2 Logarithmic Differentiation

**The Natural Logarithm:** In previous section you have differentiated a general logarithm function. That is the base to which the logarithm is taken was not mentioned. Every logarithm studied so far are mainly of two types $\log_{10} x$ or $\log_e x$. The latter is the one you will study in this section.

Remark: The natural logarithm is that function $f(x) = \log_e x$ that is the logarithm to base e (the number e is taken after Leonard Euler (1707 -1783)(There are logarithm to base other than e or 10.) The interesting thing about the study of differentiation of the natural logarithm is that its definition depends so much on calculus. You will consider the definition after you have studied the second course on calculus i.e. integral calculus. You have to make do with the fact that

$$\text{Ln } x = \log_e x = \text{the natural logarithm.}$$

The above satisfies all the basic properties of a logarithm function reviewed in the previous section.

In practice it has been observed that finding the derivatives of certain functions could be a difficult task.

But with appropriate application of the natural logarithm, derivatives of such functions could easily be found. The method involves taking the natural logarithm Ln of both sides of the given equation before differentiation. This method is called logarithmic differentiation.

**Example**

Suppose y = $f$(x) Find dy/dx

Step 1.

$$\text{Ln } y = \text{In } f(x) \qquad \_\_\_\_\_\_\_ \qquad (I)$$

Step 2. Differentiate both sides with x

$$\frac{d}{dx}(\text{In } y) = \frac{d}{dx}(\text{In } f(x)) \qquad \_\_\_\_\_\_\_\_ \qquad (II)$$

$$\frac{dy}{dx} = y \frac{d}{dx}(\text{In } f(x))$$

Let $y = u^n$ u(x) is a differentiable function of x.

Taking log of both sides you get:

$$\text{Lny} = \text{1n } U^n$$

$$\text{Lny} = \text{nln } U$$

$$\frac{d}{dx}(\text{Iny}) = \frac{d}{dx}(\text{nlnu})$$

$$\frac{1}{y}\frac{dy}{dx} = n\left(\frac{1}{u}\frac{du}{dx}\right)$$

$$\frac{dy}{dx} = n\ y\ \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$= \frac{nu^n}{u}\frac{du}{dx} \quad (\text{since } y = u^n)$$

$$= nu^{n-1}\frac{du}{dx}$$

Which is the same result derived in unit 7.

**Example:** Find dy/dx If $y = x^{x+1}$, $x > 0$

**Solution**

$Y = x^{x+1}$ (taking natural log of both sides)

$\text{Ln } y = \ln (x^{x+1})$

$\text{Lny} = x + 1 \text{ In } x$ (differentiate with x)

$$\frac{1}{y}\frac{dy}{dx} = \text{In } x + \frac{(x+1)}{x}$$

$$\frac{dy}{dx} = \left(\ln x + \frac{(x+1)}{x}\right) y = \left(\ln x + \frac{(x+1)}{x}\right) x^{x+1}$$

$$=(x\ln x + x + 1)x^{x}$$

**Example:** Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)^3 (2x-1)^2}{(x^2 + I)}$

**Solution:**

$$Y = \frac{(x^2+1)^3(2x-1)^2}{(x^2+1)} \quad \text{(taking In of both sides)}$$

$$\text{Lny} = 1\text{n} \left(\frac{(x^2+1)^3 (2x-1)^2}{(x^2+1)}\right)$$

$$\text{Lny} = \text{In } (x^2+1)^3 + \text{In } (2x-1)^2 - \text{In } (x^2+1)$$

$$\frac{d}{dx}(\text{lny}) = \frac{d}{dx}\left(31\text{n } (x^2+1) + 2 \text{ 1n } (2x-1) - \text{In } (x^2+1)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2+1} .2x + \frac{2}{2x-1} .2 - \frac{2x}{x^2+1}.$$

$$\frac{dy}{dx} = \left(\frac{6x}{x^2+} + \frac{4}{2x-1} - \frac{2x}{x^2+1}\right) . \frac{(x+1)^3 (2x-1)^2}{(x^2+1)}$$

$$= 4(3x^2 - x + 1)(x^2+1)(2x-1).$$

**Example:** Find $\frac{dy}{dx}$ if $y\ 3/2 = \frac{(x^2-1)(3x-4)^{1/3}}{(2x-3)5\ (x+1)^2}$

**Solution**

$$y^{3/2} = \frac{(x^2-1)(3x-4)^{1/3}}{(2x-3)^{1/5} (x+1)^2}$$

$$\text{In} \quad y\,3/2 = \text{In} \left( \frac{(x^2 - 1)(3x - 4)^{1/3}}{2x - 3)^{1/5} (x + 1)^2} \right) \quad \text{( taking in of both sides)}$$

$$= 1\text{n}\,(x - 1) + \text{In}\,(3x - 4)^{1/3} - \text{In}\,(2x - 3)^{1/5} - \ln\,(x + 1)^2$$

$$\frac{d}{dx}\left(\frac{3\text{In } y}{2}\right) \frac{d}{dx}\left(\text{In}(x2 - 1) + \frac{1}{3}\ \text{In}(3x - 4) - \frac{1}{5}\ \text{In}(2x - 3) - 2\text{In}(x + 1)\right)$$

$$\frac{3}{2y}\ \frac{dy}{dx} = \frac{2x}{x2 - 1} + \frac{3}{3(3x - 4)} - \frac{2}{5}\left(\frac{1}{2x - 3}\right) - \frac{2}{x + 1}$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2 - 1} + \frac{1}{3x - 4} - \frac{2}{5}\left(\frac{1}{2x - 3}\right) - \frac{2}{x + 1}\right)\frac{2}{3}\left(\frac{(x^2 - 1)(3x - 4)^{1/3}}{(2x - 3)^{1/5} (x + 1)^2}\right)$$

$$\frac{2}{15}\left(\frac{4x^3 + 53x^2 - 1774x + 127}{(x = 1)^{5/3} (2x - 3)^{17/15} [3x - 4]^{7/9} (x - 1)^{1/3}}\right)$$

**Example:** $$y^{\frac{1}{5}} = \frac{x^6}{\sqrt{x + 1}}$$

**Solution**

$$\text{Ln } y^{1/5} = 1\text{n}\left(\frac{x^6}{(x + 1)\ ½}\right)$$

$$\frac{1}{5}\ \text{In } y = 6\ \text{In}x - \frac{1}{2}\ \ln\,(x = 1)$$

$$\frac{1}{5}\ \frac{1}{y}\ \frac{dy}{dx}\ \frac{6}{x} - \frac{1}{2(x+l)}$$

$$\frac{dy}{dx} = 5y\left(\frac{6}{x} - \frac{1}{2(x+l)}\right)$$

$$5.\ \left(\frac{x^6}{(x + 1)^{1/5}}\right)^{1/5} \left(\frac{6}{x} - \frac{1}{2}\right)(x + 1)$$

**Example:** $y = x(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$ find dy/dx

**Solution**

$y = x(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$

$\text{In } y = \text{In } (x(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$ (taking In of both sides.)

$= \ln x + \ln(x-1) + \ln(x^2 + 1) + \text{In }(x - 2) + \text{In }(x^2 - 3)$

$d(\text{In}y) = d[\text{In } x + \text{In }(x - 1) + \text{In }(x^2 + 1) + \text{In }(x - 2) + \text{In }(x^2 - 3)]$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{1}{x-2} + \frac{2x}{x^2-3} . y$$

$$\frac{dy}{dx} = \frac{1}{x} + \left(\frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{1}{x-2} + \frac{2x}{x^2-3}\right)(x(x - 1)(x^2 + 1)(x - 2)(x2 - 3)$$

## 3.3 Differentiation of Exponential Function

You will now be introduced to the function that cannot be changed by any differentiation.

The function $f(x) = e^x$ for all real number x is called the exponential function. At this stage you will review some properties of the exponential function which you are already familiar with.

(I) $\log e^x = x$ for all real number x

(II) $e^x > 0$ for all real number x

(III) $e^{\log x} = x$ for all $x > 0$

(IV) $e^{x+y} = e^x e^y$ for all real x and y

(V) $e^{x-y} = e^x/e^y$ for all real x and y

The derivative of the exponential function is the exponential function. This singular property distinguishes it as the only indestructible function.

i.e. $\frac{d}{dx}(e^x) = e^x$ ___________(I)

To prove the above you start by noting that

$\text{Log } e^x = x$ _________ (2)

Taking the derivative of both sides of equation(2) you get:

$$\frac{d}{d.}(\log e^{x}) = \frac{d}{dx}(x) \qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \quad (3)$$

You can write equation (4) above in a general form. By letting y = e° where u is a real and differentiable function of x.

i.e. $\frac{dy}{dx} = \frac{d}{dx}(e^{u})$ by applying

the chain rule for differentiation you get that

$$\frac{dy}{dx} = e^{u} \frac{du}{dx}$$

Example: find $\frac{dy}{dx}$ if

(I) $y = e^{\sqrt{x}}$ (II) $y = e^{x2}$

(III) $y = e^{(x+1)2}$ (IV) $y = e^{\sqrt{x} +1}$

(V) $y = e^{(x+1)3}$

**Solutions**

(I) $y = e^{\sqrt{x}} = e^{u}$ where $u = x^{2}$

$$\frac{dy}{dx} = e^{u}\frac{du}{dx} = e\sqrt{x} \ . \ \tfrac{1}{2} \qquad x^{-1/2}$$

(II) $y = ex^{2} \qquad e^{u}$, where $u = x^{2}$

$$\frac{dy}{dx} = e^{u}\frac{du}{dx} = e^{x2} \ . \ 2x$$

$$= 2x\, e^{x2}.$$

(III) $y = e^{(x+1)2} \ e^{u}$, where $u = (x+1)^{2}$

$$\frac{du}{dx} = 2(x+1)$$

$$\frac{dy}{dx} e^{u} \frac{dy}{dx} = e^{(x+1)2}, \qquad 2(x+1)$$

$$= 2(x+1)\, e^{(x+1)2}$$

(IV) $y = e^{\sqrt{x+1}}$ $e^u$, wherein $u = (x + 1)^{½}$

$$\frac{du}{dx} \quad \frac{1}{2}(x+12)^{-1/2}$$

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx} = e^{\sqrt{x+1}} \cdot ½ (x+12)^{-1/2}$$

$$= \frac{1e\sqrt{x+1}}{2\sqrt{x+1}}$$

(V) $y = e^{(x+1)3}$ $e^u$, where $u = (x^2 - 1)^2$

$$\frac{du}{dx} = 2(x^2 - 1).2x = 4x(x^2 - 1)$$

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx} = e(x^2 - 1)^2 .. 4x = (x^2 - 1)$$

$$= 4x = (x^2 - 1). e^{(x2 -1)2}$$

**Example:** Find dy/dx if $y = e^x - \ln x$

**Solution**

$$Y = e^{x - \ln x} \Rightarrow \frac{e^x}{e^{\ln x}} = \frac{e^x}{x}$$

$$\frac{dy}{dx} = e^x x^{-1} + (-1)(x)^{-2} ex$$

$$\frac{e^x}{x} - \frac{x}{x^2} = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

**Example:** find dy if $y = e^{\sqrt{x}}$ In $^{\sqrt{x}}$

**Solution:** $Y = uv$, where $v = e^{\sqrt{x}}$ $u = \ln(x)^{½}$

$$\frac{dv}{dx} = \frac{e^{\sqrt{x}}}{2^{\sqrt{x}}}, \frac{du}{dx} = \frac{1}{2x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{\sqrt{x}} \frac{1}{2x} + \frac{e^{\sqrt{x}}}{2}. \quad \text{In}(x)^{1/2}$$

$$= e^{\sqrt{x}}\left(\frac{1}{2x} + \frac{\text{In}x}{4\sqrt{x}}\right)$$

**Example:** if $y = \frac{1}{2}(ex + c^x)$ fine $\frac{dy}{dx}$

**Solution**

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

If $y = x^2e^{-x}$ find $\frac{d^3y}{dx^2}$

**Solution**

Let $y = uv$, where $vv = x^2$, $u = e^{-x}$

$$\frac{dy}{dx} = 2\,x\,e^{-x} - x^2\,e^{-x}$$

$$\frac{d^2y}{dx^2} = 2\,e^{-x} - 4x\,e^{-x} + x^2\,e^{-x}$$

$$\frac{d^2y}{dx^2} = -6e^{-x} + 6x\,e^{-x} - x^2\,e^{-x}$$

## 3.4 Differentiation of the Function $A^u$

You will use the method above to differentiate the function y = au where u is a real differentiable function of x.

if $a > 0$ and

$b = 1n\,a$ —— I

Then $e^b = e^{lna} = a$

Given that u is a differentiable function of x and

$au = e^{lna} = a$

then $In\,a^u = uIna$

To find the derivative of $y = a^u$.

Given that $y = a^u$

Then $\frac{dy}{dx} = \frac{d}{dx}a^u = \frac{d}{dx}(e^{uIna})$

$= e^{uIna} \frac{d}{dx} e. (uIna)$

$= e^{uIna}. Ina \frac{du}{dx}.$

$= a^u \ln a \frac{du}{dx}.$

$\frac{d}{dx}(a^u) = a^{ulna} du$

**Example:** find $\frac{dy}{dx}$ if

(I) $y = 4^{\ln x}$

(II) $y = 2^{-(x2+1)}$

(III) $y = 5^{\sqrt{x}}$

**Solutions**

(1) $y = 4^{\ln x}$

let a = 4, u = Inx

$y=a^u$

$\frac{dy}{dx} = a^u In\ a \frac{du}{dx} = 4^{Inx}. In\ 4 . \frac{1}{x}$

$\frac{4^{In\ x} In4}{x}$

(II) $y = 2^{-(x2+1)}$

$y= a^u$, a=2, $u=-(x^2+ 1)$

$\frac{dy}{dx} = a^u In\ a \frac{du}{dx} = 2^{-(x2+1)}. In\ 2 (-(x^2+1))$

$= 2^{-(x2+1)}. In\ 2. -2x$

$= 2x \ln 2(2^{-(x2+1)}).$

(III) $y=5^{\sqrt{x}}$

$y = a^u$, a = 5, $u = \sqrt{x}$

$\frac{du}{dx} = 5^{\sqrt{x}} . Ln\ 5. \frac{1}{2} x^{-½}$

**Further Examples**

Find $\frac{dy}{dx}$ if

(I) $y = e^x \ln x^3$

(II) $y^2 = e^{-x}$

(III) $y^{2/3} = \left(\frac{x + 1}{(x - 1)}\right)^{1/5}$ $x > 1$

(IV) $y = x^{1/x}$

(V) $y = 1n\,(1nx^2)$

(VI) $x = \ln y$.

**Solution**

(I) $y = e^x \ln x^3$

$= e^x\, 31nx.$

Let $u = e^x$, $v = 31nx.$

$\frac{dy}{du} = u\frac{dv}{dx} + v\frac{du}{dx} = \frac{3}{x}e^x + e^x\, 31nx$

$= 3e^x\left(\frac{1}{x} + \ln x\right).$

(II) $y^2 = e^{-x}$

$21ny = 1ne^{-x} = -x$

$\frac{d}{dx}(2\ln y) = -1$

$\frac{2}{y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{y}{2} = \frac{\sqrt{e\text{-}x}}{2}$

(III) $y^{2/3} = \left(\frac{x+1}{x - 1}\right)^{1/5}$ $x > 1$

$$\ln (y)^{2/3} = \ln\left(\frac{x+1}{x-1}\right)^{1/5} \qquad x > 1$$

$$\frac{2}{3}\frac{dy}{dx} = \frac{1}{5} \ln u \qquad \text{where } u = \left(\frac{x+1}{x-1}\right)$$

$$\frac{2}{3y}\frac{dy}{dx} = \frac{1}{5} \quad .. \qquad \frac{(x-1)-(x+1)}{(x-1)^2}$$

$$\frac{2}{3y}\frac{dy}{dx} = \frac{1}{5}\left(\frac{x+1}{x-1}\right) \cdot \frac{-2}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{3}{10}\left(\frac{-2}{x^2-1}\right)\left(\frac{x+1}{x-1}\right)$$

(IV) $y = x^{1/x}$

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \frac{-1}{x^2} \ln x$$

$$\frac{dy}{dx} = y\; \frac{1}{x^2} - \frac{1}{x^2} \ln x \quad = \quad x^{1/x} \;.\; \frac{1}{x}(1 - \ln x)$$

$$= x^{1/x-1}\,(1 - \ln x)$$

$$X^{1-x/x}\,(1 - \ln x)$$

(V) $y = \ln (\ln x^2)$

$y = \ln\ u, \qquad \text{where } u = \ln x^2$

$$\frac{du}{dx} \quad \frac{1}{x^2} \quad . 2x = \frac{2}{x}$$

$$= \quad \frac{2}{x\ln x^2}$$

(VI) $x = \ln y.$

$x = \ln y.$

$1 = \frac{1}{y}\frac{dy}{dx}$ ====> $y = \frac{dy}{dx}$

but $y = e^x$ ==> $\frac{dy}{dx}$ ex ($e^x = e^{\ln g}$ g = y.)

## 4.0 CONCLUSION

In this unit you have studied how to differentiate logarithmic and exponential functions. You have studied additional methods of finding the derivative of functions, by application of logarithmic differentiation. Differentiations of certain function that are rigorous have been made easy by the method of logarithmic differentiation. The differentiation of exponential function which is very useful in solving problems of growth or decay and computing compound interest on invested money has been studied by you in this unit. you will use the knowledge gained in this unit to solve problems involving differentiation of trigonometric and hyperbolic functions in the next unit. Make sure you do all your assignments. Endeavour to go through all the solved examples.

## 5.0 SUMMARY

In this unit you have studied how to

(1) differentiate the function $f$ (x) =1nu

i.e. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

(II) differentiate the function $f$ (x) =10$g_a$ u

(III) differentiate the function $f$ (x) = $e^u$

i.e $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

(IV) to final derivative of complicated functions by applying logarithmic differentiation (i.e taking natural logarithmic of both sides of the equation before differentiating).

(V) to find the derivative of the function $f$ (x) = $a^u$

i.e. $\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$

## 6.0 REFERENCES/FURTHER READING

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## 7.0 TUTOR-MARKED ASSIGNMENT

1. Find $\frac{d}{dx} \log, \sqrt{\frac{x+1}{x^2}}$

2. Find $\frac{d}{dx} \log\ / x^4 - 1 /$

3. Differentiate $y = \log \frac{x^2}{x^3 + 1}$

4. Differentiate $y = x^{2x}$

5. If $y = \frac{(x - 1)(x^2 - 1)^{1/5}}{x^2 + 1}$ find $\frac{dy}{dx}$

6. find $\frac{dy}{dx}$ if $\sqrt{y} \left(\frac{x^2 + x}{x^2 - 1}\right)^{1/7}$

7. If $y = \frac{(x + 1)(x - 2)(x^2 + 1)(x^2 - 1)}{x^3}$ find $\frac{dy}{dx}$

8. If $7\sqrt{y} = e^x$ find $\frac{dy}{dx}$

9. find dy/dx if $y = e^{\sqrt{x2+1}}$

10. Given that $y = x e^x$ find $d^3y/dx^3$

# UNIT 3 DIFFERENTIATION OF TRIGONOMETRIC 41 FUNCTIONS

**CONTENTS**

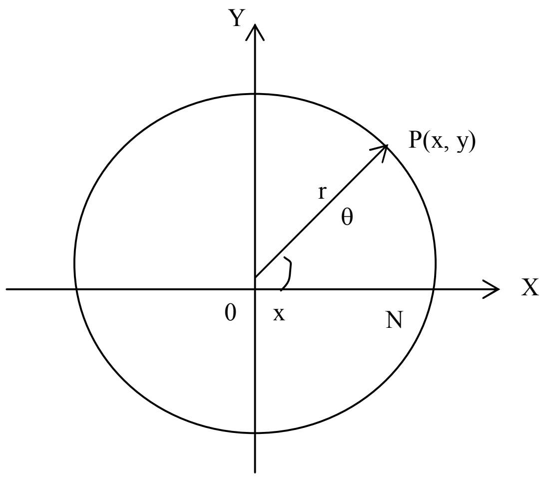


## 1.0 INTRODUCTION

So far you have studied how to differentiate various types of functions such as polynomial, rational, fractional, exponential and logarithm functions. You have applied rules of differentiation to differentiate the sums, products, quotients and roots of these functions. In this unit you will be differentiating the class of functions which are periodic. Such periodic function are best studied using trigonometric ratios such as sine and cosines. You are already familiar with trigonometric ratios of cosines and sines in your SSCE/GCE mathematics. Their properties are briefly studied here as (see Fig 9.1)

(i) $\sin \theta = y$ (ii) $\cos \theta = x/r$ (iii) $\tan \theta =$ (iv) $\text{cosec } \theta = r/y$

(v) $\sec \theta = r/x$ and (vi) $\cot = x/y$




The trigonometric ratios given above are structured by placing an angle of measure θ in standard position at the center of a circle of radius r and finding the ratios of the sides of the triangle O PN.

## 2.0 OBJECTIVES

After studying this unit you should be able to correctly:

i) derive the derivation of the function y = sin x from first principle.
ii) derive the derivatives of trigonometric function such as cos x, tan x, cosec x and sec x.
iii) differentiate combination of various types of trigonometric functions.

## 3.0 MAIN CONTENT

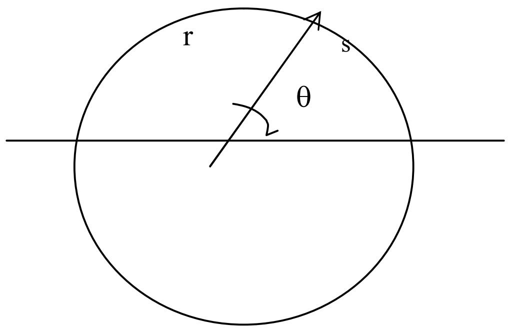
### 3.1 Differentiation of Sines

A good starting point for the differentiation of the trigonometric ratio of sine is imbedded in the concept of evaluating the limit.

$$\lim_{\theta \to \theta} \frac{\sin \theta}{\theta}$$ where θ is measured

in radian ( a radian measure is uniteless)

Fig 9.2




θ (in radian) = s/r

Prove that $\lim_{\theta \to \theta} \frac{\sin \theta}{\theta}$ θ is 1

From the above a direct calculation will not be possible because division by zero is not possible. Therefore, you have to go through a formal proof of the above since you will need to find the derivative of the function $f$(x) = sin u.

**Proof**

Let θ > O and also measured in radian

Let θ be a small angle at the center of the circle (see Fig 9.3) or radian radius r = 1

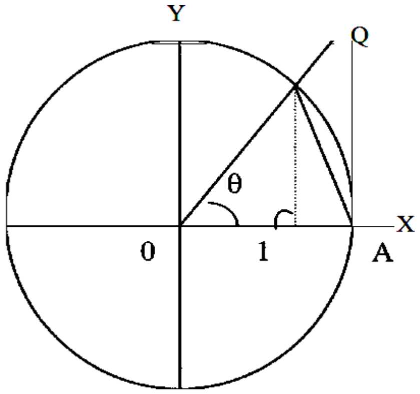



Fig 9.3.

In fig 9.3 OP and OA are side of the angle 0. OA is the targent to the cicle at point A and meets side OP at Q.

Note that

Area of AOPA = ½ base x height
= ½ (OA) (h) = ½ (1) (OP sineθ)
= ½ (1) (1)sinθ = ½ sinθ

Area of sector OPA = $\frac{1}{2} r^2 \theta = \frac{1}{2} (1)^2 \theta$.

Area of ΔOQA = ½ (OA) (QA) = ½ (1) (tan θ)
= ½ tan θ .

Fig. 9.3

Area of ΔOPA < Area of sector OPA < Area of ΔOQA

$$\Rightarrow \quad \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta \qquad \_\_\_\_\_\_ \quad 1$$

Since θ> 0 and small than sin ) > 0
dividing the inequalities in (12) by ½ sin θ you get.

$$1 \leq \theta \quad < \quad 1 \qquad \_\_\_\_\_\_\_\_\_ \quad \text{II}$$

taking the reciprocal (II) you get _________ III

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

taking limits in (III) as θ -->0

$$\lim\ 1 > \lim \frac{\sin\theta}{\theta} > \lim\cos\theta$$

$\theta \to 0 \quad \theta \to 0 \qquad \theta \to 0$

$$1 > \lim_{\theta \to 0} \frac{\sin \theta}{\theta} > 1$$

the above hold for $\theta < 0$. Since cos θ is an even function (see unit 2) i.e.

cos (-θ) - cos θ and sin θ is add i.e.
$\sin(-\theta) = -\sin\theta \Rightarrow \frac{\sin(-\theta)}{-\vartheta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$

Using the above fact you can now derive a formula for $\frac{d}{dx}(\sin u)$

Let Δ u as usual be an increment in u with a corresponding increment

Δy is y. if y = sin U
then $y + \Delta y = \sin(u + \Delta u)$ __________ I

subtracting y from y + Δy you get

$$\begin{aligned}\Delta y &= \sin(u + \Delta u) - y \\ &= \sin(u + \Delta u) - \sin U \qquad \text{\_\_\_\_\_\_\_\_\_\_\_ II}\end{aligned}$$

applying the factor formula i.e. $\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$

to the right side of equation II you get

$$\Delta y = 2 \cos \frac{((u + \Delta u).- u)}{2} \sin \frac{((u + \Delta u) - u)}{2}$$

$$= 2 \cos \left(u + \frac{\Delta u}{2}\right) \sin \frac{\Delta u}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_ III}$$

dividing equation III through by Δu you have

$$\frac{\Delta y}{\Delta u} = 2 \cos u + \frac{\Delta u}{2} \left(\frac{\sin \Delta u/2}{\Delta u}\right)$$

$$= \cos \left(u + \frac{\Delta u}{2}\right) \frac{\sin \Delta u/2}{\Delta u} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_IV}$$

setting $\theta = \Delta u/2$ equation IV becomes

$$\frac{\Delta y}{\Delta u} = \cos (u + \theta) \frac{\sin \vartheta}{\theta}$$

taking limits in equation (V) as $\Delta u \rightarrow 0$

$$\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \rightarrow 0} \left[ \cos (u + \theta) \frac{\sin \theta}{\theta} \right] \qquad \_\_\_\_\_\_\_\_\_\_ \quad \text{VI}$$

$$\lim_{\theta \rightarrow 0} \left[ \cos (u + \theta) \frac{\sin \theta}{\theta} \right]$$

(since $\theta = \frac{\theta u}{2}$ , so as $\Delta u \rightarrow 0$, $\theta \rightarrow 0$)

Equation VI becomes

$$\frac{dy}{du} = \cos U. 1 = \cos U.$$

since U is a differentiable function of x by the chain rule you get

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

i.e. if y = sin u, $\frac{-dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$$\frac{dy}{du} = \cos u,$$

$$\boxed{\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}}$$

The above process is known as differentiation of sin u from first principle or limiting process.

Example find $\frac{dy}{dx}$ if

(i) $y = \sin 5x$

(ii) $y = \sin x^2$

(iii) $y = \sin \sqrt{x}$

(iv) $y = \sin (\text{In } x)$

Solution:

(i) $y = \sin 5x$

Let sin U where U = 5x

Then $\frac{dy}{dx} = \cos U \quad \frac{dy}{dx} = \cos 5x \cdot 5$

$= 5 \cos 5x$

(ii) $y = \cos e^x + \sin x^2$

let $y = \cos u + \sin v$, where $u = e^x$, $v = x^2$.

$\therefore \frac{dy}{dx} = -\sin u \frac{du}{dx} + \cos v \frac{dv}{dx}$

$= -\sin e^x \cdot e^x + \cos x^2 \cdot 2x$
$= 2x \cos x^2 - e^x \sin e^x$.

(iii) $y = \frac{\sin x}{\cos x}$

let $y = \frac{u}{v}$, $u = \sin x$, $v = \cos x$

$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

(iii) $y = \frac{\cos x}{\sin x}$

let $y = \frac{u}{v}$, where $u = \cos x$, $v = \sin x$

$\frac{dy}{} = v \frac{du}{} - u \frac{dv}{}$

$$\frac{dx}{} \quad \frac{\;dx \qquad dx\;}{v^2} = \frac{\sin x(-\sin xx) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

(iv) $y = (\sin x)^{-1}$

let $y = u^{-1}$ and $u = \sin x$

$$\frac{dy}{dx} = \frac{-1}{u^2} \cdot \frac{du}{dx}, \quad \frac{du}{dx} = \cos x.$$

(v) $y = (\cos x)^{-1}$.

Let $y = u^{-1}$ and $u = \cos x$

$$\frac{dy}{dx} = \frac{-1}{u^2} \quad . \quad \frac{du}{dx} = \frac{-1}{\cos^2 x} - \sin x$$

$$\frac{\sin x}{\cos^2 x}.$$

(vi) $y = \cos(\sin^2 x)$

let $y = \cos(u)$

let $u = v^2$ where $v = \sin x$

AQ

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dv} = 2v, \quad \frac{dv}{dx} = \cos x.$$

$$\frac{dy}{dx} = -\sin(\sin^2 x). \; 2\sin x \cdot \cos x.$$

$$= 2\sin x \cos x \sin(\sin^2 x)$$

**Differentiation of tan u.**

Since $\tan u = \dfrac{\sin u}{\cos u}$

Let $y = \tan u = \dfrac{\sin u}{\cos u}$

Using example (III) above we get:

$$\frac{dy}{dx} - \frac{d}{du}(\tan u) = \frac{\cos u(c \cos u)-(-\sin u)(\sin u)}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u} = \sec^2 u.$$

$$\boxed{\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}}$$ ~

Exercise: Derive the formula for the derivative of cot u, where u is a differentiable function of x. (see example (IV) above).

$$\boxed{\frac{d}{d}(\cot u) = \text{cosec}^2 u \frac{du}{dx}}$$

**Differentiation of sec u**

Let $y = \sec u = \frac{1}{\cos u,} = (\cos u)^{-1}$

$$\frac{d}{dx}(\sec u) = -1 (\cos u)^{-2} . (- \sin u) \frac{du}{dx}$$

$$= \frac{\sin u}{\cos^2 u} \frac{du}{dx}$$

$$\frac{\sin u}{\cos u} . \frac{1}{\cos u} \frac{du}{dx}$$

$$= \tan u \sec u \frac{du}{dx}$$

$$\boxed{\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}}$$

**SELF ASSESSEMENT EXERCISE 1**

Derive the formula for the derivative of cosec u, where u is a differentiable function of x (see example (IV) above)

$$\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}$$

## 3.1 Differentiation Of Other Trigonometric Functions

Example: Find $\frac{dy}{dx}$ if

(i) $y = \cot \sqrt{x}$ (ii) $y = \sqrt{x} \tan(\sqrt{1- x})$

(iii) $y = \sec^2 2x$ (iv) $y = \tan x \sec x.$

(v) $y = \tan (x^2 + \sec x)$ (vi) $x \cos 2y = y \sin x$

(vii) $y = \frac{2x}{\cos 3x}$ (viii) $y = \frac{x + \cos^2 x}{\sin x}$

(vx) $y = \cot^2 x \tan x$ (x) $y = e^x \cos x^2 y = \sin y.$

**Solution**

(i) $y = \cot\sqrt{x}$

$y=\cot u , u = x^{½}$

$\frac{dy}{dx} = -\text{cosec}^2 u \frac{du}{dx}, \quad \frac{du}{dx} = \frac{1}{2} x^{-½}$

$\frac{dy}{dx} = -\text{cosec}^2 u (x) \frac{1}{2\sqrt{x}} \quad = \frac{- \text{cosec}2(\sqrt{x})}{2\sqrt{x}}$

(ii) $y = \sqrt{x}- \tan(\sqrt{1 - x})$

let $y = uv, \quad u = \sqrt{x}, \quad v = \tan z, \quad z = (I - x)^{½}$

$\frac{dy}{dx} = \frac{du}{dx}.v + u \frac{dv}{dx}$

$\frac{\tan z.}{2\sqrt{x}} + \sqrt{x} . \left(\frac{dv}{dz} . \frac{dz}{dx}\right)$

$= \frac{\tan (1 – x) ½}{2\sqrt{x}} + \sqrt{x} \left(\sec2 (I – x) ½ . \frac{1}{2} (1 – x) – ½\right)$

$$= \frac{\tan(1-x)^{½}}{2\sqrt{x}} + \frac{\sqrt{x}}{4}\left(\frac{\sec^2\left(\frac{1}{\sqrt{1-x}}\right)}{\sqrt{(1-4)}}\right)$$

(iii) $y = \sec^2 2x$

$y = \sec^2 (u)$ , $u = 2x$

$y = (\sec u)^2$

$\frac{dy}{dx} = 2\sec u. \frac{d}{d} (\sec u)$

$2 \sec u. - (\tan u \sec u) . 2$

$\frac{dy}{dx} = 4 \sec^2 (2x) \tan 2x.$

(iv) $y = \tan x \sec x.$

let $y = uv$ $u = \tan x$ $v = \sec x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, $\frac{du}{dx} = \sec^2 x \frac{dv}{dx}$ $\tan x \sec x$

$\frac{dy}{dx} = \tan x(\tan x \sec x) = \sec 2\ xtnx.$

(v) $y = \tan (x^2 + sce\ x)$

$y = \tan u$, $u = x^2 + \sin x.$

$\frac{dy}{dx} \sec^2 u \frac{du}{dx}$, $\frac{du}{dx} = 2x = \cos x$

$\frac{dy}{dx} = \sec^2 (x^2 + \sin x)(2x + \cos x)$

(vi) $x \cos 2y = y \sin x$

using implicit differentiation you

if $u = \cos 2y.$

$\frac{du}{dx} = 2\frac{dy}{dx}(- \sin 2y),$

.'. $- 2x \sin 2y \frac{dy}{dx} + \cos y = \sin x \frac{dy}{dx} + y \cos x.$

$\therefore \quad \cos y - y\cos x = (\sin x + 2x\sin 2y)\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{\cos y - y\cos x}{\sin x + 2x\sin 2y}.$

(vii) $y = \frac{2x}{\cos 3x}$

let $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 2x, \quad v = \cos 3x, \quad \frac{du}{dx} = 2, \quad \frac{dv}{dx} = -3\sin 3x.$

$$\frac{dy}{dx} = \frac{2\cos 3x - 2x.(-3\sin 3x)}{\cos^2 3x.}$$

$$= \frac{2\cos 3x + 6x\sin 3x}{\cos^2 3x.}$$

(viii) $y = \frac{x + \cos^2 x}{\sin x}$

$y = \frac{u}{v,} = x + \cos^2 x, \; v = \sin x.$

$\frac{du}{dx} = 1 - 2\cos x.\sin x \quad \frac{dv}{dx} = \cos x.$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dsx}}{v^2.}$$

$$= \frac{\sin x(1 - 2\cos x\sin x) - (x + \cos^2 x)\cos x}{\sin^2 x}$$

(ix) $y = \cot^2 x \tan x$

let $y = uv, \quad u = \cot^2 x, \quad v = \tan x$

$\frac{du}{dx} = 2\cot x(-\text{cosec}^2 x)$

$\frac{dv}{dx} = \sec^2 x.$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \cot^2 x.\sec^2 x + \tan x.(-2 \cot x \operatorname{cosec}^2 x)$

$= \quad \cot x(\cot x\sec^2 x - \tan x \operatorname{cosec}^2 x)$

$= \quad \frac{-(1 - \operatorname{cosec}^2 x + \operatorname{cosec}^2 x . \cos^2 x)}{(\cos^2 x - 1).}$

(x) $y = e^x \cos x^2 y = \sin y.$

let $u = e^x, \quad v = \cos z, \quad z = x^2 y$

$\frac{du}{dx} = ex , \quad \frac{dv}{dz} = -\sin z, \quad \frac{dz}{dx} = 2xy + x^2 \frac{dy}{dx}$

$\frac{duv}{dx} = u\frac{dv}{dx} + \frac{du}{dx}$

$\cos y\, dy + e^x (=(\sin x^2 y) = (2xy + x^2 dy) + \cos x^2 y.e^x$

*collecting like terms*

$\cos y \frac{dy}{dx} + e^x \sin x^2 yx^2 dy = -e^x \sin x^2 y2xy + \cos x^2 ye^x$

$\Rightarrow \frac{dy}{dx}(\cos y + x^2e^x \sin x^2 y) = e^x (\cos x^2 y - 2xy \sin x^2 y)$

$\frac{dy}{dx} = \frac{ex (\cos x^2 y - 2xy \sin x^2 y}{\cos y + x^2e^x \sin x^2y.}$

## 4.0 CONCLUSION

In this unit you have studied now to derive the derivative of $f(x) = \sin x$ from first principle i.e. using the limiting process. You have extended it to finding basic formula for the derivative of cos x, tan x, cosec x, and sec x., You have used rules for differentiation studied unit 8 to find the derivatives of functions involving trigonometric functions.

## 5.0 SUMMARY

In this unit you have studied how to;

(i) Derive the formula $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ from first principle.

(ii) Use $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ to derive the formula $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$.

(iii) Differentiate functions involving various combination of trigonometric functions. Such as $\cos(\sec^2 x) \times \sin^2 3x^2$ etc.

(iv) How to differentiate functions involving inverse hyperbolic functions such as arc sin h u, arc cos h u and arc tan h u.

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## 7.0 TUTOR-MARKED ASSIGNMENT

(1) find $\lim_{x \to 1} \frac{(x - 3)}{\sin(x - 3)}$

(2) $\lim_{x \to 3} \frac{\sin(x^2 - 1)}{\sin(x^2 - 1)}$

(3) find the $\frac{dy}{dx}$ if

(i) $\cos y = \sin x^2$
(ii) $y = \cos(\ln x)$
(iii) $y = \tan(x^2 - 1)$

(4) If $2\sin y = \cos(\tan x)$ find $\frac{dy}{dx}$

(5) Find dx if $y = \sqrt{x}\cos(\sqrt{x})$

(6) Find $\frac{dy}{dx}$ if $y^2 = \tan x \operatorname{cosec} x$.

(7) Given that $y = e^x \cos(e^x)$

(8) Derive the formula $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

(9) Derive the formula $\frac{d}{dx}(\cos ecu) = -\operatorname{cosec} u \cot u \frac{du}{dx}$

(10) Derive the formula $\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u \frac{du}{dx}$

# UNIT 4 DIFFERENTIATION INVERSE TRIGONOMETRIC FUNCTIONS AND HYPERBOLIC FUNCTIONS

**CONTENTS**



## 1.0 INTRODUCTION

You have already studied how to differentiate trigonometric functions of sins, cosines, tangent, secant and cosecant. In this unit you will study how to differentiate their respective inverses. The derivatives of inverse trigonometric functions are very useful in evaluating integral, of a certain trigonometric functions. Therefore your understanding of this unit will help you tremendously in the course on integral calculus i.e. calculusis.

In this unit you shall also differentiate a special class of function that is derived as a combination of exponential $e^x$ and $e^{-x}$ which you are already familiar with in previous units. These combination produce functions that are called hyperbolic functions. They are engineering problems.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- differentiate the inverse trigonometric functions such as arc(sin u), arc (cos u) arc (tan u), arc (sec u) and arc (cosec u)
- find the derivative of the inverse hyperbolic function of arc (sin hu) and arc (cos hu) etc.

## 3.0 MAIN CONTENT

### 3.1 Differentiation of Inverse Sine and Cosines Functions

In this unit you will use the knowledge you acquired when you studied unit 2 and unit 9 to study the inverse of a trigonometric function. This section is important because the concept you will study here will be useful in the second course of calculus. Recall that the inverse of a function $f(x)$ is that function. $f^{-1}(x)$ for which its composite with $f(x)$ yields the identical function:

$$\text{i.e.} \quad f\,(f^{-1}\,(x)) = f^{1}\,(f\,(x)) = x.$$

You could begin the study of differentiation of trigonometric functions by examining the inverse of the sine function. Consider the equation

$$x = \sin y$$

In this equation you can show that infinitely many values of y corresponds to each x in the interval [-1,1 ] i.e. only one of these values y lies in the interval

$$\left( \frac{-\pi}{2}, \frac{\pi}{2} \right).$$

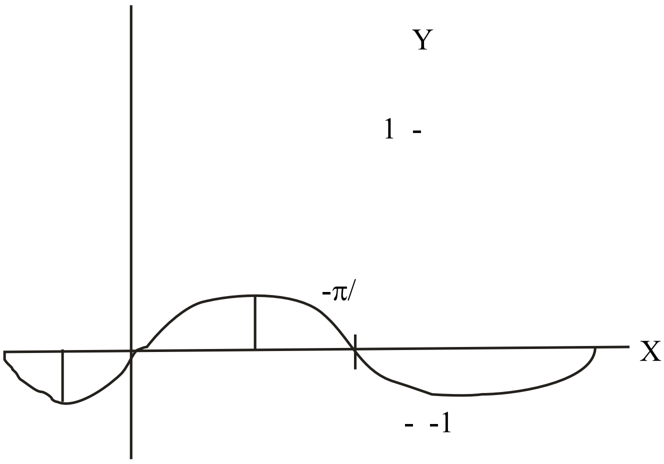
For example if x = ½ then you might wish to know the values of all angles y such that sin y = ½ . These two angles y = 30° and y = 150° will come readily to your mind. Multiples of these two angles will give the sine value to be ½ .

i.e. sin 30° = ½ , sin 150° = ½ . sin k y = ½

for k = 1, 2, ..., and y = π/6, note that

$$150^0 = 5(30°).$$

Consider the graph of y = sin x as shown in fig. 9.3.




If you interchange the letters(variables) x and y in the original equation y = sin x you will clearly see the what is being discussed so far in this section .

That is x = sin y, x ε [ -l, 1 ] and $\left[ -\pi/_{2}, \ \pi/_{2} \right]$

In the interval [- π/2, - π/2] the function f (x) = sin x is a one to one function (see Fig 9.3 no horizontal line cuts the graphs only once).

Therefore within the interval [- π/2, - π/2] the inverse exist and it called the inverse sine function and it is written as y = arc sin x (or sin-' (x)) (see Fig 9.4 and 9.5.)

Remark: You will use the arc sin x frequently to represent the inverse sine function. The notation $\sin^{-1}(x)$ could be used if you are sure you will not confuse it with the function

$$\frac{1}{\sin x}.$$

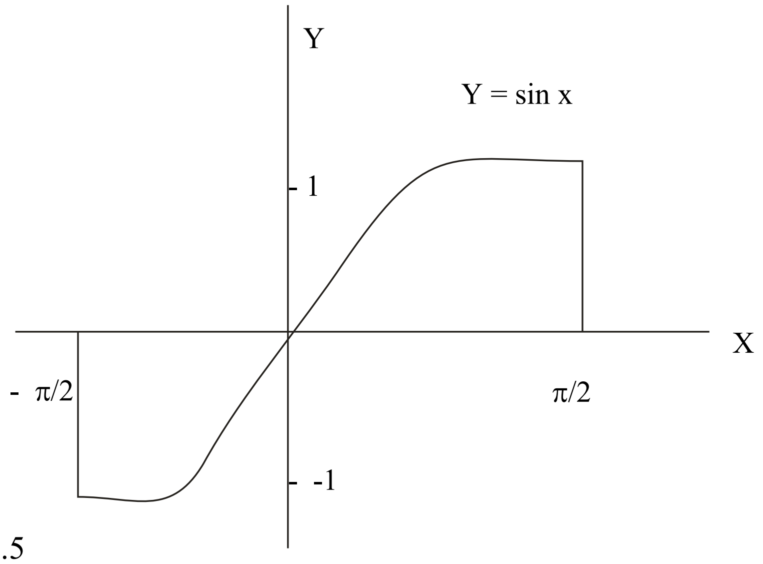



Fig: 9.5

In Fig 9.4 the function y = sin x is a continuous so also is y = arc (sin x).

The function $\frac{dy}{dx} = \frac{d}{d2}(\sin x) = \cos x$

is defined in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. and there is no x ε $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

such that cos x = 0. So also the derivative of the inverse sine function does not take any value zero in the open interval ( -1, 1 ) i.e.

$$d(\text{arc} \sin x) \neq 0 \ \forall \ x \ \varepsilon \ (-1, 1)$$

With the above information you can now proceed to derive a formula for the derivative of arc sin x.

Let $f(x) = \sin x$

Then $f^{-1}(x) = \text{arc} \sin x$

Note that $f\,(f^{-1}(x)) = x$.

Therefore $\frac{d}{dx} - (f\,(f^{-1}(x)) = \frac{d}{dx}(x) = 1$

$\Rightarrow \quad \frac{d}{dx}\sin(\arcsin x) = 1.$

$\Rightarrow \quad \cos(\arcsin x)\,\frac{d}{dx}(a \arcsin x) = 1.$

Hence $\quad \frac{d}{dx}(\arcsin x) = \frac{1}{\cos(a \arcsin x)}$

$= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}}$

note that : $\cos^2 x + \sin^2 x = 1 \Rightarrow \quad \cos x = \sqrt{1 - \sin^2 x})$

thus $\quad \frac{d}{dx}(\text{arc} \sin x) = \frac{1}{\sqrt{1 - x^2}}$

note also that;

$\sin(\text{arc} \sin x) = x$ then $(\sin(\text{arc} \sin x)^2 = x^2)$

you could also derive the above formula by applying implicit differentiation Given that:

$y = \text{arc} \sin x$

$\Rightarrow \quad \sin y = x$

*then* $\Rightarrow \quad \frac{d}{d}(\sin y) = \frac{dx}{dx}$

$\cos y \,\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}.$

putting $\quad y = \text{arc} \sin x$ and $\sin y = x$ you get :

$\frac{d}{dx}(\text{arc} \sin x) = \frac{1}{\sqrt{1 - x^2}}$

let y = arc sin u where u is a differentiable function of x.

Then;

$$\frac{d}{dx}(\text{arc sin } u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx.}$$

**Differentiation of Inverse Cosine**

Given that y = arc cos u

Let cos y = u

Then $\frac{d}{dx}(\cos y) = \frac{dy}{dx}$

$-\sin y \frac{dy}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sin y} \frac{dy}{dx}$

$\sin y = \sqrt{1 - \cos^2 y}$. where cos y = u

Therefore

$$\frac{d}{dx}(\text{arc cos } u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx.}$$

**Differentiation of Inverse Tangent**

Given that y = arc tan u

Let tan y = u

then $\frac{d}{dx}(\tan y) = \frac{du}{dx}$

$\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{\text{Sec}^2 y} \frac{du}{dx.}$

(note that if $\tan^2 y = \sec^2 y$ and tan y == u) therefore $\frac{dy}{dx} = \frac{1}{1 + \tan^7 y} \frac{du}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$

hence

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

**Differentiation of arc sec u.**

Given that y = arc sec u

Let sec y = u

$\frac{d}{dx}(\sec y) = \frac{du}{dx}$

$\sec y^t \text{an } y \frac{dy}{dx} = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \frac{du}{dx}$

*(note that* $\tan y = \pm \sqrt{\sec^2 y - 1}$ *and* $\sec y = u$*)*

therefore $\frac{dy}{dx} \quad \frac{1}{u \pm \sqrt{u^2 - 1}} \quad \frac{du}{dx.}$

hence

$$\frac{d}{dx}(\text{arc tan } u) = \left(\frac{1}{/u/\sqrt{u^2 - 1}}\right) \frac{du}{dx.}$$

**DIFFERENTIATION OF y = arc cot u**

Give y = arc cot u

Let cot y = u

$\frac{d}{dx}(\cot y) = \frac{du}{dx}$

$- \text{cossec}^2 y \frac{dy}{dx} = \frac{dx}{dx}$

$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y} \frac{du}{dx}$

(but $\operatorname{cosec}^2 y = 1 + \cot^2 y$, $\cot y = u$)

then

$$\frac{d}{dx}(\text{arc cot } u) = \frac{1}{1 + u^2}\frac{du}{dx.}$$

**DIFFERENTIATION OF y = arc cosec u.**

Given that y = arc cosec u.

Then $\operatorname{cosec} y = u$

$$-\cot y \operatorname{cosec} y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\cot y \operatorname{cosec} y}\frac{du}{dx.}$$

but $\cot y = \pm\sqrt{\operatorname{cosec}^2 y - 1}$ $\operatorname{cosec} y = u$

then $$\frac{dy}{dx} = \frac{-1}{u\sqrt{u^2 - 1}}\frac{du}{dx.}$$

Therefore

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{/u/\sqrt{u^2 - 1}}\frac{du}{dx.}$$

**Examples**

Find dy/dx if;

1. $y = \text{arc sin } x^2$
2. $y = \text{arc cos } 2x^3$
3. $y = \text{arc tan } (x + 1)^2$
4. $y = \text{arc cot}\left(\frac{x+1}{x-1}\right)$
5. $y = x^2 (\text{arc sec } 2x)$.

**Solutions**

(1) $y = \text{arc sin } x^2$

let $y = \text{arc sin } u,\ u = x^2$

$$\frac{dy}{dx} = \frac{1}{1 - u^2}\frac{dy}{dx} = \frac{1}{1 - (x)^2}2x$$

$$= \frac{2x}{\sqrt{1 - x^4}}$$

(2) $y = \text{arc cos } 2x^3$

let $y = \text{arc cos } u, u = 2x^3$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} = \frac{-1}{\sqrt{1-(2 \times 3)^2}} .6x^2$$

$$=- \frac{6x^2}{1-4x^6}$$

(3) $y = \text{arc tan } (x+1)^2$

let $y = \text{arc tan } u, u = (x+1)^2$

$$\frac{dy}{dx} = \frac{1}{1+u^2}, \frac{du}{dx} = \frac{1}{1+(x+1)^4} .2(x+1)$$

$$= \frac{2(x+1)}{1+(x+1)^4}$$

(4) $y = \text{arc cot } \left(\frac{x+1}{x-1}\right)$

let $y = \text{arc cot } u, u = \left(\sim \frac{x+1}{x-1}\right)$

$$\frac{dy}{dx} = \frac{-1}{1+u^2}\frac{du}{dx} = \frac{-1}{\left(\frac{x+1}{x-1}\right)^2} \cdot \frac{-2}{(x-1)^2}$$

$$= \frac{1}{x^2+1}$$

5. $y = (x^2 \text{ arc sec } 2x)$

let $y = u\,v$ and $u = x^2$, $v = \text{arc sec } 2x$

let $v = \text{arc sec } z, z = 2x.$

$$\frac{dv}{dx} = \frac{1}{|2|\sqrt{2^2-1}} . \frac{d2}{dx} = \frac{1}{2x\sqrt{4x^2-1}} .2x$$

but

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \frac{x^2}{\sqrt{4x2-1}} + (arc\ \sec\ 2x).2x.$$

$$\Longrightarrow \frac{dy}{dx} - \frac{x^2}{4x^2-1} + 2x(\text{arc sec } 2x)$$

## 3.3 Differentiation of Hyperbolic Functions

You are already familiar with the differentiation of exponential function $e^x$ and $e^{-x}$. These combinations occur in two basic forms ½ ($e^x$ + e-') and ½ ($e^x - e^{-x}$). They occur so frequently that they have to be given a special attention. The types of function described above are known as hyperbolic functions (see unit 2 sec 3.2 for more details).

**Definition:** The hyperbolic sine and cosine are functions written as

Sin h x = ½ ($e^x + e^{-x}$) and

Cos h x = ½ ($e^x - e^{-x}$)

(Recall that the word hyperbolic is formed from the word hyperbola see unit 2 sec. 3.2).

Given sin h x and cos h x defined above you can easily form other hyperbolic function of tangent cotangent, secant and cosecant by noting that

Sin h x = ½ ($e^x + e^{-x}$) and

Cos h x = ½ ($e^x$ – é$^{-x}$)

Then

1. $\tan h\, x = \frac{\sin\ h\, x}{\cos h\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

2. $\cot h\, x = \frac{\cos h\, x}{\sin h\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

3. $\text{cosec}\, h\, x = \frac{1}{\sin h\, x} = \frac{2}{e^x + e^{-x}}$

4. $\sec hx = \frac{1}{\cos h\, x} = \frac{2}{e^x + e^{-x}}$

You will briefly review some of the identities associated with hyperbolic functions they follow the same pattern with those derived for trigonometric functions.

Note that the equation of a unit hyperbola is given as

$$x^2 - y^2 = 1$$

if you put x =cos h θ and sin h θ = y

then $x^2 = \cosh^2 \theta$ and $y^2 = \sin h^2\, \theta$

$\Rightarrow \cos h^2 - \sin h^2\, \theta = 1$ _________ (1)

Then by substituting appropriately you get the following identities

$1-\tan h^2\theta = \text{cosec } h^2\theta$ __________ (2)

$\cot h^2\theta - 1 = \text{cosec } h^2\theta$ __________(3)

The identities will be useful in finding the derivative of inverse hyperbolic functions.

**DIFFERENTIATION OF Sin h u.**

Let y = sin h u., where u is a differentiable function of x.

then:

$$\frac{dy}{dx} = \frac{d}{dx}(\sin h\, u = \frac{d}{dx}\left(\frac{(e^u - e^{-u}}{2}\right)$$

$$\frac{\frac{d}{dx}e^u - \frac{d}{dx}e^{-u}}{2}$$

$$\frac{e^u\frac{d}{dx} + e^{-u}\frac{d}{dx}e^{-u}}{2}$$

$$= \frac{1}{2}\left(e^u + e^{-u}\right)\frac{du}{dx}$$

$$= \cos h \frac{du}{dx.}$$

$$\frac{d}{dx}(\sin h\, u) = \cos h\, u \frac{du}{dx}$$

**DIFFERENTIATION OF cos h u**

Let y = cos h u

Then:

Therefore

$$\frac{d}{dx}(\cos h\, u) = \sin\ h\, u \frac{du}{dx}$$

**DIFFERENTIATION OF tan h u.**

Let y = tan h u.

Then:

$$\frac{dy}{dx} = \frac{d}{dx}(\tan h) = \frac{\sin h\, u}{\cos h\, u}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos h\, U \frac{d}{dx}(\sin h\, U) - \sin h\, U \frac{d}{dx}(\cos h\, U)}{\cos h^2 u}$$

$$= \frac{1}{\cos h^2 u} = \sec h^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan h\, u) = \sec h\, u \frac{du}{dx}$$

**SELF ASSESSMENT EXERCISE 1**

Using the method above:

(1) show that $\frac{d}{dx}(\cot h\, u) = -\operatorname{cosec} h^2 u \frac{du}{dx}$.

(2) show that $\frac{d}{dx}(\sec h\, u) = -\sec h\, u \tan h\, u \frac{du}{dx}$

(3) show that $\frac{d}{dx}(\operatorname{cosec} h\, u) = -\operatorname{cosec} h\, u \cot h\, u \frac{du}{dx}$

**Examples**

Find $\frac{dy}{dx}$ if

(i) $y = \tan h\, 3x$ (ii) $y = \cos h^2\, 5x$

(iii) $y = \sin h\, 3x^2$ (iv) $y = \sec h^3\, 2x^2$

(v) $\sin h\, x = \tan y$.

**Solution**

(i) $y = \tan h\ 3x$

let $y = \tan h\ u,\ u = 3x$

$\frac{dy}{dx} = \sec h^2 u \frac{du}{dx} = \sec h^2 (3x).\ 3$

$= 3 \sec h^2 3x.$

(ii) $y = \cos h^2 5\ x$

let $y = \cos h^2 5x, \qquad u = 5x$

$\frac{dy}{dx} = 2 \cosh u.\sin h\ u \frac{du}{dx}$

$= \quad 2 \cos h\ 5x \sin h\ 5x.\ 5.$

$= \quad 10 \cos h\ 5x \sin h\ 5x.$

(iii) $y = \sin h\ 3x^2$

let $y = \sin h\ u, \quad u = 3x^2$

$\frac{dy}{dx} = \cos h\ u \frac{du}{dx} = \cos h\ 3x^2 .\ 6x$

$= 6x \cos h\ 3x^2.$

(iv) $y = \sec h^2 2x^2$

let $y = \sec h^3 u, \quad u = 2x^2.$

$\frac{dy}{dx} = 3\sec h\ u\ ( -\sec h\ u \tan h\ ).4x$

$= - 3 \sec h^2 2x^2 .\ \sec h\ 2x^2 \tan h\ 2x^2 .\ 4x$

$= 12x \sec h^3 2x^2 \tan h\ 2x^2$

(v) $\sin h\ x = \tan y.$

$\cos h\ x = \sec h^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{\cos h\ u}{\sec h^2 y}.$

## 3.4 Differentiation of Inverse Hyperbolic Functions

In this section you will adopt the same pattern used in studying the differentiation of inverse trigonometric function to finding the derivative of the derivative of the inverse hyperbolic functions. In this course only the following hyperbolic inverse will be treated.

(I) Inverse hyperbolic sine i.e. y = arc sin h x

(II) Inverse hyperbolic cosine i.e. y = arc cos h x

(III) Inverse hyperbolic tangent i.e. y = arc tan h x.

**DIFFERENTIATION OF arc (sin h u)**

Let y = arc sin h x

⇒ sin hy=x

⇒ $\frac{1}{2}(e^{y} - e^{-y}) = x$.

$e^{y} – e^{-y} = 2x$ / multiplying through by $e^{y}$ we get.

$e^{2y} - 1 = 2x\, e^{y}$

⇒ $e^{2y} - 2xe^{y} = 0$

let $e^{y} = P$

then $P^{2} - 2xp - I = 0$

solving for P you get

P = $\frac{1}{2}(2x \pm \sqrt{4x^{2} + 4})$

$e^{y} = x \pm \sqrt{x^{2} + 1}$

Now find $\frac{dy}{dx}$ by Logarithmic differentiation

i.e take In of both sides

In ey = In (x + $\sqrt{x2 + 1}$) (note $e^{y}$ . 0 hence you drop the minus sign.)

y = ln (x +$x^{2}$ + 1) y = In u

$\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx}$, u (x + $\sqrt{x2 +1}$

$\frac{du}{dx} = 1 + \frac{x}{\sqrt{x^{2} + 1}}$

therefore $\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{x\sqrt{\ +1}}\right)$

$= \frac{1}{\sqrt{x^2} + 1}$

| |
|---|
| $\frac{d\,(\text{arcsin } hu)}{dx} = \frac{1}{\sqrt{x^2 + 1}} \frac{du}{dx}$ |

**DIFFEREENTIATION OF arc cos h u**

Let y = arc cos h x

.'. cos h y = x

$\frac{1}{2}(e^y + e^{-y}) = x$

$e^y + e^{-y} = 2x$/ multiplying through by $e^y$

$e^{2y} + 1 - 2x\ e^y = 0$

let $P = e^y$ you get a quadratic equation of the form.

$P^2 - 2x\,P + 1 = 0.$

Solving for P you get;

$P = \frac{1}{2}(2x \pm \sqrt{4x^2 - 4}\ )$

$\therefore$ $ey = x \pm \sqrt{x^2} - 1$

for $e^y > 0$ then

$e^y = x + \sqrt{x^2} - 1$

To find $\frac{dy}{dx}$ by logarithmic differentiation you take natural logarithm of both sides and get;

$Ln\ e^y = In\ (x + \sqrt{x^2} - 1)$

$\frac{dy}{dx} = \frac{d}{d^2}\ In\ u,$ $u = x + \sqrt{x^2} - 1$

$= \frac{1}{u} \frac{du}{dx},$ $\frac{du}{dx} = 1 + \frac{x}{\sqrt{x^2} - 1}$

$$\frac{1}{x + x^2 - 1} \quad 1 + \left(\frac{x}{\sqrt{x^2 - 1}}\right)$$

$$= \frac{1}{\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\text{arcos hu}) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}$$

**DIFFERENTIATION OF arc tan h u.**

Let $y = \text{arc tan h } x$

$\Rightarrow \quad \tan h\, y = x$

$$\frac{\sin h\, y}{\cos h\, y} = x.$$

$$\Rightarrow \quad \frac{e^y - e^{-y}}{ey + e^{-y}} = x$$

Multiplying through by $e^{-y}$

$$\frac{e^{2y} - 1}{e^{2y} + 1} = x$$

$\Rightarrow \quad e^{2y} - 1 = (e^{2y} + 1)\, x$

collecting like terms

$e^{2y} - x\, e^{2y} = x + 1$

$(1 - x)\, e^{2y} = x + 1$

$$e^{2x} = \frac{x + 1}{1 - x}$$

Differentiating by taking natural logarithm of both sides you get:

$$\text{Ln } e^{2y} = 1\text{n } \frac{x + 1}{1 - x}$$

$$2y = \text{In}\left(\frac{x + 1}{1 - x}\right)$$

$$2\frac{dy}{dx} = \frac{d}{dx}\text{In } u = \frac{1}{u}\frac{du}{dx,} \quad u = \left(\frac{x + 1}{1 - x}\right)$$

$$\frac{dy}{dx} = \frac{1}{u\, .\, dx,}\, .\, \frac{du}{dx} \quad \frac{du}{(x - 1)^2} \quad -2$$

$$2\frac{dy}{dx} = \left(\frac{1 - x}{x + 1}\right) . = \frac{1}{1 - x^2}$$

$$\therefore \quad \boxed{\frac{d}{dx}(\text{arcos hu}) = \frac{1}{1 - u^2}\frac{du}{dx}}$$

Example: Given that cos $h^2$ y - sin $h^2$y = 1

Show that $\frac{d}{dx}(\text{arc h u}) = \pm \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}$

**Solution**

Let y= arc cos h u

Then cos h y = u

$$\frac{d}{dx}(\cos h\, y) = \frac{du}{dx}$$

$$\sin h\, y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sin h\, y} \frac{du}{dx}$$

but $\quad \cos h^2 y - \sin h^2 y = 1$

$\therefore \quad \cos h^2 y - 1 = \sin h^2 y$

$\Rightarrow \quad \sin h\, y = \pm \sqrt{}\cosh^2 y - 1$

but cos h y = u

then sin h y = $\sqrt{}u^2$ - 1

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\pm\sqrt{u - 1}} \frac{du}{dx}$$

**SELF ASSESSEMENT EXERCISE 2**

Use the above exercise to show that

$$\frac{d}{dx}(\text{arc sin h u}) = \frac{1}{1 + u^2} \frac{du}{dx}$$

**Example**

Find $\frac{dy}{dx}$ If

(1) y = arc sin h (4x) (II) y = arc tan h (sin x)

(III) y= arc cos h (In x) (IV) y = arc cos h (cosx)

**Solution**

(1) let y =arc sin h u, u = 4x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$
$$= \frac{1}{\sqrt{1 + 16x^4}} = \frac{4}{1 + 16x^4}$$

(II) let y = arc tan u, u = sin x.

$$\frac{dy}{dx} = \frac{1}{1 - u^2} \frac{du}{dx} = \frac{1}{1 - (\sin x)^2} . \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{1 - \sin^2 x} = \frac{1}{\cos x}$$

(III) let y = arc cos h u, u = (In x)

$$\frac{dy}{dx} \quad \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(\ln x)^2 - 1}} . \frac{1}{x} = \frac{1}{x\sqrt{(\ln x)^2 - 1}}$$

(IV) let y = arc cos h u, u = cos x

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx} \quad \frac{du}{dx} = -\sin x$$

$$\frac{dx}{dx} \quad \frac{-\sin x}{\sqrt{\cos^2 x - 1}}$$

## 4.0 CONCLUSION

In this unit you have studied three types of functions and their respective derivative that is inverse trigonometry, hyperbolic and inverse rules for differentiation to differentiate functions involving inverse trigonometric and hyperbolic functions. You have been exposed to numerous examples involving the differentiation of these function discussed. Some of the examples were repeated in another format for example some of the examples used in unit 8 were used to explain the concept of differentiation of trigonometric and hyperbolic functions. This is a deliberate attempt so that you will master the technique studied in this unit. The differentiation of inverse function of trigonometric and hyperbolic will be very useful when studying the next course on calculus that is integral calculus. Make sure you go through the example thoroughly because you will need them in the second course in calculus.

## 5.0 SUMMARY

In this unit you have studied how to:

(1) Derive the formula for inverse trigonometric function such as

$$\frac{d}{dx}(\text{arc sin } u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}(\text{arc cos } u) = -\frac{I}{\sqrt{u^2-1}}\frac{du}{dx} \text{ etc.}$$

(II) Derive the formula

(a) $\frac{d}{dx}(\sin h\, u) = \cos h\, u \frac{du}{dx}$ etc

(b) $\frac{d}{dx}(\cos h\, u) = \sin h\, u \frac{du}{dx}$

(III) Differentiate functions involving inverse hyperbolic functions such as arc sin h u, arc cos h u, arc tan h u etc.

## 6.0 REFERENCES/FURTHER READING

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## 7.0 TUTOR-MARKED ASSIGNMENT

(1) Find $\frac{dy}{dx}$ if $y = e^x$ arc sin (In x)

(2) Find $\frac{dy}{dx}$ if $y= \frac{\text{arc cot}(\sqrt{1- x})}{\sin h\, x^2}$

(3) Find $\frac{dy}{dx}$ if $\cos h^2\, 2(\sin x)$

(4) Find $\frac{dy}{dx}$ if $\sin h^2\, y = \tan h\, (x)$

(5) Derive the formula $\frac{d}{d}(\tan h\, u) = \sec h^2\, u\, \frac{du}{dx}$

(6) Derive the formula $\frac{d}{d}(\text{arc sin h } u) = \frac{1}{1 + u^2} \frac{du}{dx}$

(7) Find $\frac{dy}{dx}$ if $y = \frac{\sinh^3 (e^{2x})}{\ln(\sin x)}$

(8) Derive the formula $\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1 + u^2} \frac{du}{dx}$

(9) Derive the formula $\frac{d}{dx}(\text{arc sec } u) = \frac{du}{|u|\sqrt{u^2 - 1}}$

(10) Derive the formula d (arc cos u) $= \frac{- du}{1 - u^2}$