MODULE I

UNIT 1 DEFINITION OF A MATRIX AND TYPES OF MATRICES

CONTENTS

1.0 INTRODUCTION

In this unit, you will be introduced to the concept of a matrix. This is simply a convenient way of representing arrays (tables) of numbers. Initially, notations of matrices were merely a shorthand system of solving certain classes of problems in algebra. However, the study of matrices has now extended to almost all branches of mathematics. Matrices are especially useful in the solutions of systems of linear equations, which will be discussed in subsequent units of this course.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a matrix
- list all 6 types of matrices
- identify at least 6 types of matrices.

3.0 MAIN CONTENT

3.1 Definition of a Matrix

A matrix is a rectangular array of numbers that are enclosed within brackets. It can be further defined as a rectangular array of real numbers arranged in horizontal rows and vertical columns and is written as follows:

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The above array of numbers is called an m by n - matrix (written m x n). A matrix has no numerical value. The numbers in the array are called the elements of the matrix. A double subscript is used to denote the location of any given elements. The first subscript gives the row and the second subscript - the column in which the element is located. That is aij is the element in row i and column j. Matrices are denoted by the upper case (Capital letter) boldface roman letters (A, B, etc) and elements, by italicized lower case letters (aij, bij, etc). A matrix with only one row of elements is called a row matrix; similarly, a matrix with only one column of elements is called a column matrix.

Example: Consider the 2x3 matrix

$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \end{bmatrix}$ Its rows are [1, 2, 4] and [3,1, 0]

and its columns are

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

SELF ASSESSMENT EXERCISES 1

Given the following matrices

1. $\begin{bmatrix} 3 & 4 \\ 3 & -1 \\ 2 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} -2 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Write out the rows and columns.
Answers

1. [3, 4] [3, -1] [-2, 3] $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

2. [-2, -1, 0], [3, 1, 4] $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$

3. [2, 1], [0, 2] $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3.2 Types of Matrices

3.2.1 Square Matrices

Definition: Given an m x n matrix. If m=n i.e. when the number of rows = the number of columns = n, then the n x n matrix is called a square matrix of order n.

Example: consider the matrices

1. A = $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ This has order of 2 i.e. a 2 x 2 matrix

2. $\begin{bmatrix} 1 & 3 & 4 \\ 4 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ This has an order of 3 i.e. a 3 x 3 matrix

3.2.2 Zero or Null Matrix

A matrix in which every element is zero is called a zero matrix. Zero matrix or null matrix can have any order. For example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is a zero matrix of order 3, r000

3.2.3 Diagonal Matrix

A diagonal matrix is a square matrix in which every off diagonal element is zero. That is if A = [aij] is a diagonal matrix then aij = 0 for i $\neq$ j.
Example: consider a 3 x 3 square matrix given as

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every other element is equal to zero except the diagonal elements

3.2.4 Triangular Matrices

A square matrix A =(aij) of order n is an upper triangular matrix if all entries below the main diagonal are all zeros. That is, if aij = 0 for i>j where i,j=l,...,n. This is written as follows.

$$A = \begin{bmatrix} a_{11}, & a_{12,} & a_{13}.... & a_{in} \\ 0 & a_{22}, & a_{23}.... & a_{2n} \\ 0 & 0 & a33.... & a3n \\ . & . & . & \\ . & . & . & \\ 0 & 0 & 0....... & a_{nm} \end{bmatrix}$$

Similarly, a square matrix A= (aij) of order n is a lower triangular matrix of its elements aij = 0 for I<j where i,j=l,...,n. In other words, a lower triangular matrix is a square matrix whose entries above the main diagonal are zero, and it is written as follows

$$A = \begin{bmatrix} a_{11}, & 0 & 0.... & 0 \\ a_{21} & a_{22} & 0.... & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1}, & a_{n2} & a_{n3}... & a_{nn} \end{bmatrix}$$

Example: the matrix given as $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

is a upper triangular matrix of order 3.

While $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is a lower triangular matrix of order 4

3.2.4 Unit Matrix or Identity Matrix

A unit matrix or identity matrix is a square matrix of order n having all its diagonal element unity and zero elements elsewhere. It is denoted by the symbol I_n and it is written as follows:

$$I_n = \begin{bmatrix} 1 & 0 & 0......0 \\ 0 & 1 & 0......0 \\ 0 & 0 & 1......0 \\ . & & \\ . & & \\ . & & \\ 0 & 0 & 0......1 \end{bmatrix}$$

The role or importance of the special matrix I_n lies in the fact that it plays *i* role similar to that of the number 1 in a real number system.
Example: The matrix 13 given as

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is the 3 x 3 unit matrix

3.2.5 Equality of Matrices

Two matrices A, B, of the same order m x n are equal, written A=B if any only if the corresponding elements are equal, that is aij = bij for every i,j Where A = (aij) and B= (bij).

Example: The matrices given as

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

$A \neq B$ since A is a matrix of 2x3and B is a matrix of 3x2.

Example: The matrices

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 2^2 & 1 & 5 \end{bmatrix}$$ are equal.

TRANSPOSE OF A MATRIX

Let A be an m x n matrix. Then the transpose of A is a matrix of order n x m obtained from A and denoted by A^T or A^1 . Where A = (aij) is equal to the (aij) of A for all i and j. This is given as follows

$$\begin{bmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ldots\ldots & a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21}\ldots\ldots & a_{m1} \\ a_{12} & a_{22}\ldots\ldots & a_{m2} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{1n} & a_{2n} & a_{mn} \end{bmatrix}$$

3.2.6 Symmetric Matrix

Let A be an n x n square matrix. If the transpose of A is equal to A (i.e., $A^T = A$) then A is called a symmetric matrix. In other words A is a symmetric matrix if $a_{ij} = a_{ij}$ for all i and j.

Example: The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} \text{ is a symmetric matrix}$$

$$\text{i.e. } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

3.2.7 Skew Symmetric

Let A be an n x n square matrix. If $A^T = -A$ then A is said to be a skew symmetric matrix. In other words A is equal to the negative of its transpose. In this case aij = -aji for all i and j.

Example: Given that

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} \quad \text{where } A^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

$$\text{But } -(aij) = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} = A^T$$

i.e. it has been shown that $A^T = -A$. In any skew matrix the elements in the leading diagonal are all equal to zero.

You are quite familiar with the set of complex number. You could recall that for any complex number Z= a + ib the complex conjugate is given as Z = a - ib. You can now discuss a spacing type of matrix where some of the elements are complex numbers.

3.2.8 Hermitian Matrix

A square matrix A = (aij) of order n where $(A^T) = A$ is called a Hermitian Matrix. In any Hermitian matrix the diagonal elements are all real numbers.

Example: The matrix

$$A = \begin{bmatrix} 1 & 1-1 & 2 \\ 1+I & 3 & I \\ 2 & -1 & 0 \end{bmatrix}$$ is a Hermitian matrix. The conjugate of A is given as:

$$\bar{A} = \begin{bmatrix} 1 & 1+I & 2 \\ 1+i & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 1 & 1-I & 2 \\ 1+i & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

3.2.9 Skew Hermitian

A square matrix A = (aij) of order n where the conjugate of the transpose is equal to the negative of the matrix. That is $(A)^T$ = -A. The diagonal elements of a skew Hermittian matrix are either zero or pure imaginary numbers.

Example: The matrix

$$A = \begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & I \\ -2 & i & 0 \end{bmatrix}$$ is a Skew Hermittain.

$$\bar{A} = \begin{bmatrix} -i & 1+i & 2 \\ -1+i & 3i & -i \\ -2 & -i & 0 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} -i & 1+i & 2 \\ -1+i & -3i & -I \\ -2 & -i & 0 \end{bmatrix}$$

4.0 CONCLUSION

In this unit, you have defined a matrix. You have also studied various types of matrices. For example you know that in a square matrix the number of row is equal to its number of columns. You now know how to find the transpose of a matrix. As well as determine whether any two given matrices are equal. In the next unit, you will use the properties of a square matrix to perform some algebraic operations, involving matrices.

5.0 SUMMARY

In this unit you have studied:

- How to define a matrix.
- How to identify various types of matrices e.g. Square matrix, Symmetric matrix, skew symmetric matrix, etc.
- To find the transpose of a given matrix,
- To determine whether two matrices are equal,
- Identify a unit matrix and a null matrix.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Find the order of a matrix A given that

$$A = \begin{bmatrix} 0 & 4 \\ 6 & 1 \\ 1 & 3 \end{bmatrix}$$

2. Given that

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & Z \end{bmatrix}$$

Determine the values of w, x, y, and z for which A = B.

3. If $A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ & 1 & 2 & -1 \\ & 1 & 4 & -5 \end{bmatrix}$

Find (i) A^T (ii) B^T

Given that the following matrices:

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 4 & 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 9 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 0 & 1 & 6 \\ -1 & 0 & 3 \\ -6 & -3 & 0 \end{bmatrix}$ (h) $\begin{bmatrix} 5 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix}$

Determine in each case

4. The square matrixes
5. The skew symmetric mat
6. The symmetric matrix
7. Identity matrix
8. Null matrix
9. Triangular matrix
10 Diagonal matrix.

7.0 REFERENCES/FURTHER READING

Michael O'Nan: Linear Algebra, Harcourt Brace Jovanovich, Inc.

P.D.S.Verma: Engineering Mathematics, Vikas Publishing House Pvt Ltd

UNIT 2 MATRIX ALGEBRA

CONTENTS

1.0 INTRODUCTION

In this unit you will be introduced to how to carry out basic algebraic operation involving matrix. These include addition of matrices, subtraction of matrices and multiplication of matrices. These operations are very important because they occur in real life problems such as solving a system of linear equation. The foundation you acquire in this unit will be useful throughout the duration of your study of the course - Linear Algebra.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the sum of two or more matrices of the same order
- find the product matrix of two or more matrices.

3.0 MAIN CONTENT

3.1 Addition of Matrices

The sum of two or more matrices is obtained by adding corresponding elements. Before you can carry out the above, the two or more matrices must be conformable. In other words two or more matrices are said to be conformable for addition, if they have the same numbers of rows and the same number of columns.

That is, they have the same order or shape. Thus once two or more matrices have different orders, addition will not be possible, since addition involves corresponding entries or elements.

Let A and B be two matrices written as follows:

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ldots\ldots & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{m1} & a_{m2}\ldots\ldots & a_{mm} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12}\ldots\ldots & b_{1n} \\ b_{21} & b_{22}\ldots\ldots & b_{2n} \\ . & & \\ . & & \\ . & & \\ b_{m1} & b_{m2}\ldots\ldots & b_{mm} \end{bmatrix}$$

$$\text{Then} \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12}\ldots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22}\ldots & a_{2n}+b_{2n} \\ . & & \\ . & & \\ . & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2}\ldots & a_{mm}+b_{mm} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -3 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+0 & 3-3 \\ 2-2 & 1+1 & 4+4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 8 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 3 & -2 & 0 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 1 & 8 \end{bmatrix}$$

is not defined since they are not of the same order.

PROPERTIES OF MATRIX ADDITION

(a) A + B = B + A Commutative

(b) A + (B + C) = (A + B) + C Associative

(c) A + 0 = 0+A = A

Matrix addition is commutative as well as associative

3.2 Scalar Multiplication

The product of a scalar k and a matrix A, written kA or Ak is the matrix obtained by multiplying each elements of A by k

$$kA = k\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mm} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{1n} \\ ka_{21} & ka_{22} & ka_{2n} \\ ka_{m1} & ka_{m2} & ka_{mm} \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ find (i) 2A, (ii) – 2A, (iii) 4A, (iv) 4A + 2A

(i) $$2\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2.2 & 2.1 \\ 2.3 & 2.1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix}$$

(ii) $$-2A = -2\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2.2 & -2.1 \\ -2.3 & -2.1 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -6 & -2 \end{bmatrix}$$

(iii) $$4A = 4\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4.2 & 4.1 \\ 4.3 & 4.1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 12 & 4 \end{bmatrix}$$

(iv) $$4A + 2A = \begin{bmatrix} 8 & 4 \\ 12 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 8+4 & 4+2 \\ 12+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 18 & 6 \end{bmatrix}$$

Remark: -A = (-1) A and A-B = A+ (-1) B

Example: Find a, b, c and d if

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 8 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & d \end{bmatrix}$$

Solution: You write each matrix operation as a single matrix i.e.

$$\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} 4+a & 8+a+b \\ -1+c+d & 2d+3 \end{bmatrix}$$

Certain corresponding elements equal to each other to obtain the four linear equations.

3a = 4 +a $\Rightarrow$	2a = 4	$\Rightarrow$	a = 2
3b = 8 + a + b	2b = 8 + a	$\Rightarrow$	b = 5
3c = -1 + c + d	2c = d -1	$\Rightarrow$	c = 1
3d = 2d + 3	d = 3		d = 3

Example: Let

$$A = \begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & -2 \end{bmatrix}$$

Find 3A + 4B – 2C

Solution:

$$3 \begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & -2 \end{bmatrix}$$

$$= \quad + \begin{bmatrix} 4 & -8 & -12 \\ 0 & -4 & 20 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4+0 & -15-18-2 & 3-12+4 \\ 9+0-2 & 0-4+2 & -12+20+2 \end{bmatrix} = \begin{bmatrix} 10 & -25 & -5 \\ 7 & -2 & 10 \end{bmatrix}$$

3.3 Matrix Multiplication

Let A and B be matrices such that the number of columns of A is equal to the number of rows of B. Then the product of A and B, written AB is the matrix with the same number of rows as A and same number of columns as B. the element in ith row and jth column is obtained by multiplying the ith row of A by the jth column of B.

Example: Let A be an n x m matrix and B be an m x n matrix then

$$B = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ldots\ldots & a_{2n} \\ . & & \\ . & & \\ & . & \\ a_{m1} & a_{m2}\ldots\ldots & a_{mm} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12}\ldots\ldots & b_{1n} \\ b_{21} & b_{22}\ldots\ldots & b_{2n} \\ . & & \\ . & & \\ & . & \\ b_{m1} & b_{m2}\ldots\ldots & b_{mm} \end{bmatrix} =$$

$$\begin{bmatrix} c11 & & c1n \\ . & \ddots & \\ . & cij & \\ . & & \\ & & \ddots \\ cn1 & & cnn \end{bmatrix}$$

Where cij = aiibj + $a_{i2}b_{2j}$ +.... + ai_nb_{nj}

If A is an n x m and B is m x n then AB is n x n matrix. But BA is m x m matrix. If the number of columns of A is not equal to the number rows of B, say A is m x n and B is p x m where n # p then the matrix A not defined.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1.2+.-1 & 1.1+2.3 & 1.2+2.4 \\ 3.2+(-1) & 3.1+1.3 & 3.2+1.4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 7 & 8 \\ 5 & 6 & 10 \end{bmatrix}$

Example: Let A be 2 x 3 and B be 3 x 2 matrixes.

What is the size of AB
What is the size of BA

Solution:
AB= (2 x3) (3 x2)=2x2
BA= (3 x2) (2 x3)=3x3

In the above the product AB is defined if the inner numbers in the bracket are equal, and then the product will have the size of the outer numbers in the given order.

Example: Let A = (2, 2) and B = $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

Find (i) AB (ii) BA

Solution:

(i) A=(lx2)and B=(2x3)
AB (1x2)(2x3)=(lx3)

$$\therefore \text{ AB} = [2,\ 2] \begin{bmatrix} 1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} = (2.1+2.2,\ 2.-1+2.2, 0+2.3) = (6,2,6)$$

(ii) BA = (2 x 3)(1 x 2) this is not defined since Example: Let

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 4 & 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 4 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 31 & 53 & 16 \\ 38 & 65 & 24 \end{bmatrix}$$

Example: Let

$$A = \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & -5 \\ 3 & -4 & 1 \end{bmatrix}$$

Find (1) AB (ii) BA

Solution: AB = (3 x 2) (2 x 3) = 3 x 3

(i) $$AB = \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -5 \\ -3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 & -12 \\ -10 & -18 & -6 \\ 0 & 7 & 14 \end{bmatrix}$$

(ii) BA = (2x3) (3 x2) = (2x2)

$$\begin{bmatrix} 1 & -1 & -5 \\ -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 15 & -1 \\ -17 & -11 \end{bmatrix}$$

In (i) above: $[2\ -2] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 + (-2)(-3) = 8$

$[2\ -2] \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -10 -2 = -12$

Thus to obtain the first row of AB you multiply the first row of A by each column of B.

For the second row of AB, you multiply the second row of A by each column of B respectively:

i.e. $[2, 4] \begin{bmatrix} 1 \\ -3 \end{bmatrix} = -10$

$[2, 4] \begin{bmatrix} -1 \\ -4 \end{bmatrix} = 18$

$[2,4] \begin{bmatrix} -5 \\ 1 \end{bmatrix} = -6$

For the third row of AB, you multiply the third row of A by each column of B respectively.

i.e. $[-3, -1] \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$[-3,-1 \begin{bmatrix} -1 \\ -4 \end{bmatrix} = 7$

$[2, 4] \begin{bmatrix} -5 \\ 1 \end{bmatrix} =14$

SELF ASSESSMENT EXERCISES 2

Repeat the above for matrix BA.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & -3 & 1 & 4 \\ -1 & 2 & 1 & 0 \end{bmatrix}$

Given that (cij) = AB

Find (i) c_{12} (ii) c_{23} (iii) C_{14} (iv) AB

Solution: cij = the product of ith row of A and jth column of B.

(i) $c_{12} = [1`,2,3] \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = 2$

(ii) $c_{23} = [1, -1, 2]\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = 6$

(iii) $c_{14} = [1, 2, 3]\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = 8$

(iv) $AB = \begin{bmatrix} -2 & 2 & 10 & 8 \\ -1 & 9 & 6 & 4 \end{bmatrix}$

SELF ASSESSMENT EXERCISES 3

Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$

$D = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

Compute (i) AB (ii) BA (iii) DE (iv) ED

Ans (i) $\begin{bmatrix} 9 & 0 \\ 13 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 20 \\ 1 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 4 & 7 \\ 5 & 6 & 11 \end{bmatrix}$

(iv) $\begin{bmatrix} 14 & 4 & 3 \\ 2 & 2 & 1 \\ 16 & -2 & 1 \end{bmatrix}$

Properties of matrix multiplication. For matrices A, B and c (with same size)

(a) (AB) C = A (BC)
(b) A(B+C)=AB+AC
(c) (B + C) A = BA + CA
(d) k (AB) = (kA) B = A (kB) where k is a scalar.

The transpose operation on matrices satisfies the following properties

(a) $(kA)^T = kA^T$ where k is a scalar
(b) $(A^T)^T=A$

(c) $(A+B)^T=A^T+B^T$
(d) $(AB)^T = B^T A^T$
(e) $(ABC)^T = A^TB^TC^T$
(f) $(A+B+C)^T=A^T+B^T+C^T$

Examples: Given that

$$A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$$

Verify that $(A + B)^T = A^T + B^T$

Solution:

$$A+B = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} \quad B^T = \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

Example: Given that

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Show that $(BA)^T = A^TB^T$

Solution:

$$BA = \begin{bmatrix} 14 & 4 & 3 \\ 2 & 2 & 1 \\ 6 & -2 & 1 \end{bmatrix} \quad (BA)^T = \begin{bmatrix} 14 & 2 & 6 \\ 4 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 & 6 \\ 4 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

Example: Let $C = \begin{bmatrix} 14 & 2 & 6 \\ 4 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

Using the matrices A and B given in the above examples show that
(B+C) A = BA + CA

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 14 & 2 & 6 \\ 4 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$(B+C)\,A = \begin{bmatrix} 15 & 4 & 9 \\ 13 & 2 & -1 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 50 & -2 & 9 \\ 4 & -2 & -1 \\ 14 & -3 & 2 \end{bmatrix}$$

$$\therefore\ BA = \begin{bmatrix} 14 & 4 & 3 \\ 2 & 2 & 1 \\ 6 & -2 & 1 \end{bmatrix} \quad CA = \begin{bmatrix} 36 & -6 & 6 \\ 2 & -4 & -2 \\ 6 & -2 & 1 \end{bmatrix}$$

$$\therefore\ BA + CA = \begin{bmatrix} 14 & 4 & 3 \\ 2 & 2 & 1 \\ 6 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 36 & -6 & 6 \\ 2 & -4 & -2 \\ 8 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & -2 & 9 \\ 4 & -2 & -1 \\ 14 & -3 & 2 \end{bmatrix}$$

Example: Let A = $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$

Compute (i) A^2 (ii) A^3 (iii) f (A) where f (x) = $2x^3+x^2+2$

Solution:

(i) $A^2 = AA = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

(ii) $A^3 = AAA = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 39 \\ 52 & -13 \end{bmatrix}$

(iii) $F(A) = 2A^3+A^2 +21 = 2 \begin{bmatrix} 13 & 0 \\ 10 & 13 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 26 & 78 \\ 104 & -26 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 41 & 78 \\ 104 & -11 \end{bmatrix}$$

4.0 CONCLUSION

In this unit you have studied the addition of two or more matrices. You also studied scalar multiplication over matrix addition. You have seen that matrix addition is possible only between matrices of the same size or order or shape. In other words, you cannot add matrices of different size or shape together. The same applies to matrix multiplication.

You have studied that matrix multiplication is possible between two matrices where the number of rows of the first matrix is equal to the number of columns of the second matrix. There are certain algebraic properties that are exhibited by matrices under addition and multiplication.

Some of these properties could be extended to their transpose respectively. These properties studied in this unit will form part of the foundation for solving systems of linear equations in other units.

5.0 SUMMARY

In this unit, you have studied how to:

- Add two or more matrices of the same size,
- Find the difference between two matrices of the same size,
- Find the product of a scalar k and a matrix A i.e. kA.
- Find the product of two matrices
- (v) To verify the algebraic properties of matrix addition and multiplication such as
    - $A + B = B + A$
    - $(AB)C = A(BC)$
    - $A(B + C) = AB + AC$ etc.

6.0 TUTOR-MARKED ASSIGNMENT

Given that

$A = \begin{bmatrix} 4 & -3 \\ 1 & 3 \end{bmatrix}$ and B $\begin{bmatrix} 1 & -3 \\ -8 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

1.. Find 3A - 4B

2. Verify that $(A+B+C)^T = A^T+B^T+C^T$

3. Show that $(AB)^T = B^TA^T$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$

4. Show that (AB)C=A(BC) if $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 6 \\ 2 & -3 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$

5. Given that $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$

Compute 3A + 2B, for No. 6-10

Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 5 & -3 \end{bmatrix}$

6. Compute A^2 and A

7. If $f(x) = x^3 - x^2 + 2x - 4$ find f (A)

8. Compute B^2

9. Show that $(AB)^2 = A^2B^2$

10. Show that 2(AB) = (2A) B = A (2B)

7.0 REFERENCES/FURTHER READING

Michael O'Nan: Linear Algebra , Harcourt Brace Jovanovich,Inc.

P.D.S.Verma: Engineering Mathematics,Vikas Publishing House Pvt Ltd

UNIT 3 LINEAR EQUATIONS

CONTENTS

1.0 INTRODUCTION

The theory of linear equation is an important tool in solving many practical problems in most aspects of life. It is one of the mathematical tools that has found its application in natural sciences, applied sciences, social sciences and management sciences. Therefore, solutions to problems modeled by linear equations can easily be solved by any computer.

One of the ways is to put these systems in a form of a matrix and then use special techniques to solve them. However in this unit you will only concentrate on the analytical and probably graphical methods of solving linear equations. The extension of these using matrices will be studied in other units of this course.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- solve linear equations involving two unknowns
- identify 3 types of solutions of general linear equations
- find solution to 3 linear equations in 3 unknowns.

3.0 MAIN CONTENT

3.1 Linear Equation in Two Unknowns

A linear equation in two unknown can be written in this form

$Ax + by = c$ where x and y are the unknown variables and a, b, and c are numbers.

A typical solution to the above consist of a pair of numbers, say a = (r_1, r_2) which satisfies the equation

$ar_1 + br_2 = c$

The above can be solved by assigning any value to x and then solved for y or x you assign arbitrary values to y and solve for x.

Example: Given that $4x + 2y = 6$

If you let x = 0 in the equation you get 2y = 6 or y = 3. Thus you have that the pair (0, 3) is a solution. If you substitute x = 1 in the equation you obtain that 2y = 2 or y = 1 hence (1, 1) is another solution. You can have a list of solutions.

x	y
-3	9
-2	7
0	3
1	1
2	-1
3	-3

The above table gives seven possible values for x and the corresponding values for y i.e. seven solution of the equation. Using the above table of values you can plot the graph of the equation 4x + 2y = 6 this is given in fig. 3.1

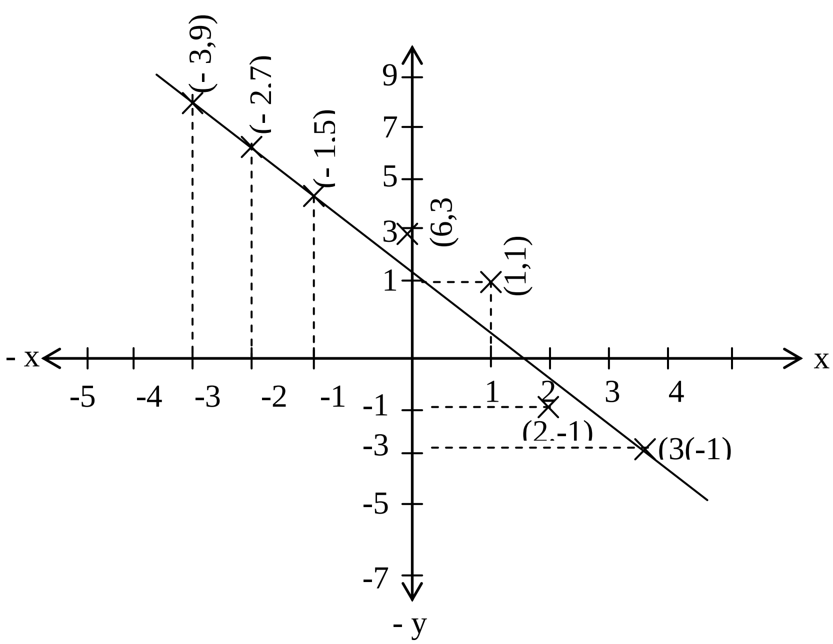


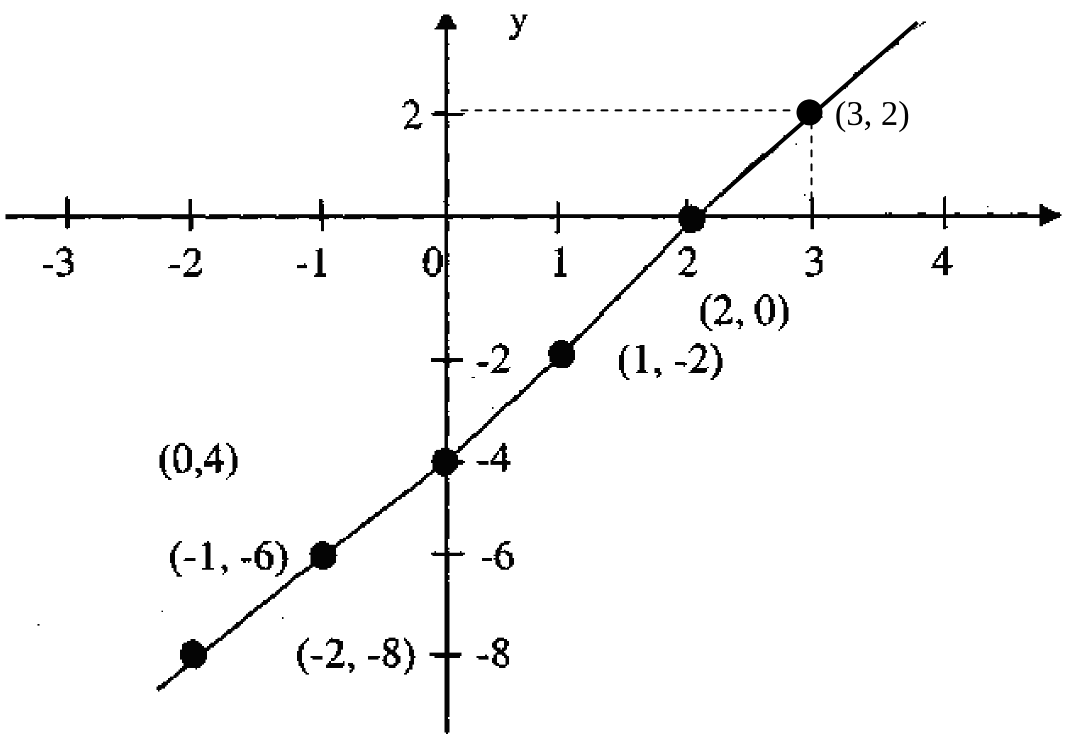
Fig. 3.1

The above graph is a straight line hence the name linear equation. All the solutions to the equation lie on some point on the line.

SELF ASSESSMENT EXERCISES 1

(i) Determine 6 distinct solutions of 6x - 3y =12 and plot its graph. Answer

Y	Y
-2	-8
-1	-6
0	-4
1	-2
2	0
3	2



3.2 Two Linear Equations in Two Unknowns

Given a system of two linear equations in two unknowns x and *y* written as

$$ax + by = c$$
$$dx + ey = f$$

A typical solution to the above is a pair of numbers, which satisfies both equations. Such solution is called a simultaneous solutions of the given equation or a solution of the system of equations. There are 3 possible solutions to any given simultaneous equations. These 3 possible solutions can be described geometrically.

Case I: If the system of equation have exactly one solution, then the lines corresponding to the linear equations intersect in one point as displayed in fig 3.2.

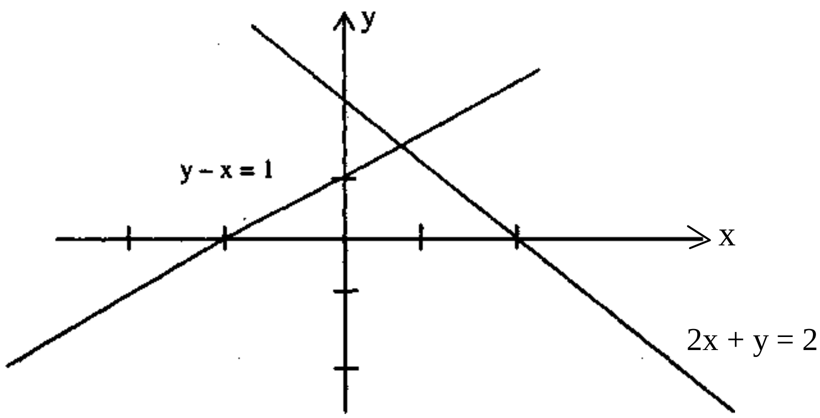
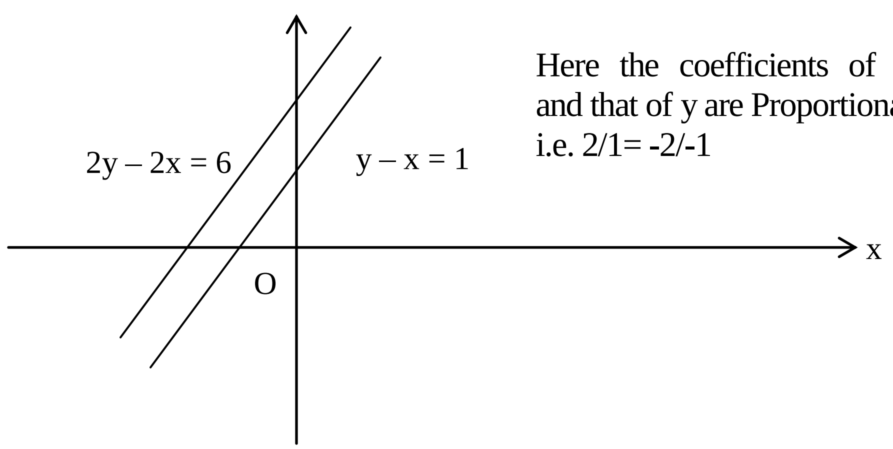


Fig. 3.2

Case II: If the system has no solutions. Hence the lines corresponding to the linear equations are parallel as displayed in fig 3.3



Here the coefficients of x and that of y are Proportional i.e. 2/1= -2/-1

Fig. 3.3

Case III: If the system has an infinite number of solutions. Here the lines corresponding to the linear equations coincide as displayed in fig 3.4.

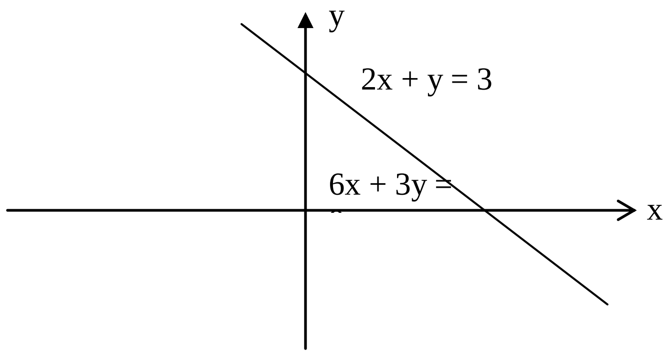


Fig. 3.4

Here the coefficients of x and that of y and the constant term are proportional
i.e. 2/6 = 1/3 =3/9

Examples: Solve the following systems of linear equations

(i) 3x - 2y = 6 (ii) 2x + 4y = 1 (iii) 3x - 2y = 3
x+2y=2 x+ 2y=2 6x-4y=6

Solution: (i) 3x - 2y = 6
x +2y = 2
4 x = 8
x = 2

$\therefore$ 2y = 2 - x or y = (2-x)/2
When x = 2, y= *(2-2)12* = 0.
Thus the pair (2, 0) is a solution to the system.

(ii) 2x + 4y = 1 x+2y=2

The coefficients of x are in the same proportion with coefficients of y.
i.e. 2/1 $\equiv$ 4/2 $\equiv$ 1/a

but in different proportion with the constant term. Therefore the lines are ^ parallel and the system has no solution.

(iii) 3x - 2y = 3
6x-4y=6
Here the coefficients of x and y as well as the constant term are in the same proportion i.e.
3/6 $\equiv$ -2/-4 $\equiv$ 3/6

Hence the lines are coincident and the system has an infinite number of solutions, which correspond to the solutions of either equation.

SELF ASSESSMENT EXERCISES 2

Solve the following system of equations

(i) $3x+6y=10$ (ii) $2x-3y=-7$
$6x+12y=1$ $4x-6y=14$

3.3 General Systems of Linear Equations

You will now, extend your previous knowledge gained so far to a general linear equations. Consider a linear equation with arbitrary number of: unknowns say, x_l, x_2,. .. An which is written as follows

$$a_1x_1+a_2x_2+a_3x_3+......+a_nx_n=6$$

Where a_l, a_2, a_3, ...a_n & b are real numbers. An n - tuple of real numbers $\beta = (r_1, r_2 \ldots r_n)$ is a solution to the general linear equations if the statement

$a_1r_1+a_2x_2+a_3x_3 \ldots .+a_nr_n=b$ is true.

A linear equation is said to be degenerate if the coefficients of the unknow are all zero.

A general system of m linear equations in n unknown x_l, x_2, ...,x_n is of the form

$a_{11}x_1+a_{12}x_2+...+a_{in}x_n=b_1$
$a_{21}x_1+a_{22}x_2+...+a_{2n}x_n=b_2$
$a_{m1}x_1+a_{m2}x_2+\ldots .+a_{mn}x_n=b_m$

Where the aij, bi are real numbers. Any n-tuple of numbers ($3=(r_l, r_2,...r_n)$ which satisfies all the equation is called a solution of the system. There are methods of solving such systems of equation, one of such method is known as the elimination method.

Example: Solve the linear equations

$3x+2y-z=19$------(i)
$4x-y+2z=4$—— —(ii)
$2x+4y-5z=32$——(iii)

Solution: $3x+2y-z=19$ ---------(i)
Eq(i) x 4 -Eq(2) x 3 $1\ ly-10z=64$

Eq(2) - Eq(3) x 2 $-9y+12z=-60$
$3x+2y-z=19$

$$11y - 10z = 64$$

Eq(4)x9-Eq(5)x 11 $\quad 42z = -84$

$$\therefore z = -2$$

Substituting the value of z in equation (4) you obtain lly+20=64

$11y + 20 = 64$
$11y = 44$
$y = 4$

Substituting the values of y and z in equation
(i) You get
$3x + 8 + 2 = 19$
$3x = 9$
$x = 3$
$\therefore x = 3, y = 4\ z = -2$

SELF ASSESSMENT EXERCISES 3

Solve the following linear equation

(1) $4x+y-3z= 1$
$6x-y-4z=7$
$10x+2y-6z=5$

Ans: $x = 1.5\ y = -2\ z = 1$

(2) $3x+2y-3z=2$
$3x-y - 4z= 14$
$5x+2y-6z = 10$

Ans: x = 6, y = -4, z = 2

(3) $x + y + 3z = 6$
$2x + 2y -3z = 9$
$x + 2y - 6z = - 4$

Ans: x = 12, y = -7, z = 1/3

(4) $3x + 2y - 3z = 2$
$x - 3y - 4z = 14$
$5x + 2y - 6z = 10$

Ans: x = 34, y = - 20, z = 20

(5) $2x + 3y - z = 25$
$2x - 6y - 8z = 14$
$5x + 2y - 6z = 10$

Ans: x = -2, y = l, z = -3

FURTHER COMMENTS ON GENERAL SYSTEMS OF LINEAR EQUATIONS

Let n-tuple of numbers R = (r_l, r_2,...,r_n) be a solution to the systems of m equations in n unknown x_l, x_2, ...x_n written as

$a_{11}x_1$ +......................+$d_{in}x_n$ =b_1
. . .
. . .
. . .
$a_{m1}x_1$ + ……………+ $a_{mn}x_n$ = b_m

The above system of linear equation is Sajd to be degenerate if the coefficients aij of all the unknowns. x_1, x_2 ..., x_n are all zero. There are two cases that can occur when aij = 0

Case I: If one of the constants $b_1 \neq 0$ i.e. The system has an equation of the form: $0xi + 0x_2 + \ldots + 0x_n = b_i$ with $b_i \neq 0$, then the system has no solution.

Case II: If every constant bi = 0 i.e. every equation in the system is of this form $0x_1, + 0x_2 + \ldots + 0xn = 0$, then the system has every n-tuple of real numbers as a solution..

Example: The system

$0x_1 + 0x_2 + 0x_3 = 1$
$0x_1 + 0x_2 + 0x_3 = 0$
$0x_1 + 0x_2 + 0x_3 = 0$

has no solution since one of the constants on the right hand side is not zero.

The system

$0x_1 + 0x_2 + 0x_3 = 0$
$0x_1 + 0x_2 + 0x_3 = 0$
$0x_1 + 0x_2 + 0x_3 = 0$

has every 3 - tuple (3 = (r_l, r_2.... r_n) as a solution since all the constants on the right hand side are zero.

HOMOGENEOUS SYSTEMS OF EQUATION

A system of linear equation is called homogeneous if it can be written in this form.

$$
\begin{array}{l}
a_{11}x + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\
a_{21}x + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\
. \qquad\qquad . \qquad\qquad . \\
. \qquad\qquad . \qquad\qquad . \\
. \qquad\qquad . \qquad\qquad . \\
a_{m1}x + a_{m2}x_2 + \dots + a_{nm}x_n = 0
\end{array}
$$

The above homogeneous equation has the trivial solution given as (0,..., 0). It also has a non-trivial solution whenever n> m. the method of finding solutions to such system of equation shall be the subject of study in the preceding units.

4.0 CONCLUSION

In this unit you have studied a general system of equations. Starting from a simple linear equation of several unknown to 2 simultaneous linear equations in 2 unknowns. You have use simple algebraic manipulation to solve for the unknowns in a 2 by 2 systems of equation and 3 by 3 systems of equation You have studied how to determine whether a system has (i) a solution (ii) no solution (iii) infinite number of solution. You have also studied how t(determine whether s system of linear equation is degenerate or non-degenerate system. You have studied a homogeneous system.

5.0 SUMMARY

In this unit you have studied how to:

- Determine the 3 types of solutions of a general system of line;
- equation (a) Unique solution (b) no solution (c) Infinite Solutions
- Classify system of linear equations into (a) degenerate type (b) homogeneous type,
- Solve system of linear equation by elimination method.

6.0 TUTOR-MARKED ASSIGNMENT

Determine whether the following systems have (i) solution (unique solution) (ii) Infinite solutions (iii) no solutions.

1.. $x - 2y = 3$
$-3x + 6y = 9$

2. $2x + 2y = 2$
   $x + y = 1$

3. $x + 2y = 3$
   $x + y = l$

Solve the following systems of linear equations:

4. $5x-3y-2z = 31$
   $2x+6y+3z=4$
   $4x+2y-z=30$

5. $x-3y+z=-l$
   $2x+y-4z=-l$
   $6x-7y+8z=7$

6. Determine whether the systems of linear equations are homogeneous.
   $3x+y-5=0$
   $2x+3y-8=0$
   $x-2y+3=0$

7. Solve the system of equations
   $2x - 5y + 2z = 7$ ------ (1)
   $x + 2y - 4z = 3$ --------(2)
   $3x - 4y - 6z = 5$ ------- (3)

8. Determine the values of x_1, x_2, & x_3
   $2xi - 2x_2 - x_3 = 3$ ------- (1)
   $4x, + 5x_2 -2x_3 = -3$ ----- (2)
   $3xi + 4x_2 -3x_3 = -7$ ------- (3)

9. Consider the homogeneous system.

   $x +2y +3z = 0$------(1)
   $-x +3y+2z =0$-------- (2)
   $2x+ y-z =0$------- (3)

7.0 REFERENCES/FURTHER READING

Michael O'Nan: Linear Álgebra, HARCOURT BRACE JOVANOVICH INC..New York

UNIT 4 DETERMINANTS

CONTENTS

1.0 INTRODUCTION

The need to find an efficient way to solve a large system of linear equation has made the study of determinants imperative. Determinants are restricted to only square materials. In other words, determinants are only evaluated for matrices with matrices whose number rows are equal to the number of column. This has made square matrices to be very important in the study of linear algebra. In this unit, determinants of square matrices of order 2 will be discussed. Evaluation of determinant of a general order n will be introduced.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- evaluate the determinant of a 2 x 2 matrix
- evaluate the determinant of 3 x 3 matrix
- evaluate the determinant of any n x n matrix.

3.0 MAIN CONTENT

3.1 Determinant of a 2 X 2 Matrix

The determinant of a matrix A is written as |A| = det (A) and it is a single specific number. For example the determinant of a 1 x 1 matrix| (a) is the number itself,

$de = |a| = a.$

Consider a linear equation in one unknown x

$ax = b$

In the previous unit a can stand as a 1 x 1 matrix and x as a one component vector. If $|a| \neq 0$ i.e. $a \neq 0$ then the equation ax = b will have a unique solution, given as

$$x = b/a$$

However in a case |a| = 0 or a = 0 then the equation has no solution if b # 0 and will have every number as a solution if b = 0. The matrix a is called the coefficient matrix for the equation ax = b. The determinant of a coefficient matrix plays a vital role in solving any system of equation. This will be discussed later in this unit and the next one.

For now it can be said that if $\det(A) \neq 0$ i.e. determinant of a coefficient matrix is not equal to zero then the equation will have a unique solution. You shall now consider the determinant of 2 x 2 matrix. Let the matrix A be given as

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ then the}$$

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: find

$$\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = 2.5 - 1.4 = 10\text{-}4 = 6$$

$$\begin{vmatrix} 4 & 5 \\ 2 & 4 \end{vmatrix} = 4.4 - 2.5 = -16 - 10 = -26$$

Example: let $A = \begin{bmatrix} 2 & 3 \\ 14 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 \\ 1 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ -1 & -4 \end{bmatrix}$

Find (i) | A| (ii) |B| (iii) |C| (iv) |AB| (v) |ABC|

Solution:

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix}$ = 2(-4) – (3.4) = - 8-12 = - 20,

(ii) $\begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix}$ = (6.4) – (1.5) = 24 -5 = 19,

(iii) $\begin{vmatrix} -2 & -1 \\ -1 & 4 \end{vmatrix}$ = (-4.2) – (-1.1) = -8- (-1) = -8 + 1 = -7

(iv) AB = $\begin{bmatrix} 2 & 3 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 28 \\ 12 & -4 \end{bmatrix}$

|AB| = $\begin{vmatrix} 11 & 28 \\ 12 & -4 \end{vmatrix}$ = - 380

(v) ABC = $\begin{bmatrix} 2 & 3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -6 & -101 \\ 28 & 28 \end{bmatrix}$

|ABC| = 2660

In evaluating the above determinant pf a 2 x 2 matrix, you can use the orientation described in the diagram below.

$$|AB| = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- +

Here the arrow slanting downward indicates the product "ad" in the positive sense. And the arrow slanting upward indicates the product "cb" in the negative sense. Thus, the determinant is addition of these products i.e.

+ad - cb = ad - cb

Example: Compute the determinant of each matrix

(i) $\begin{bmatrix} x-y & y \\ y & x+y \end{bmatrix}$ (ii) $\begin{bmatrix} x-y & x \\ x & x+y \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 \\ -4 & -4 \end{bmatrix}$ (iv) $\begin{bmatrix} -a & 1 \\ 1 & b \end{bmatrix}$

Solution:

(i) $\begin{bmatrix} x-y & y \\ -y & x+y \end{bmatrix}$ = (x – y) (x + y) – (y) (-y) = $x^2 - y^2 + y^2 = x^2$

(ii) $\begin{bmatrix} x-y & x \\ x & x+y \end{bmatrix}$ = = (x – y) (x + y) – (x) (-x) = $x^2 - y^2 - x^2 = y^2$

(iii) $\begin{bmatrix} -1 & 2 \\ -4 & 4 \end{bmatrix}$ = (-1) (4) –(-2) (-4) = -4 -8 = -12

(iv) $\begin{bmatrix} -a & 1 \\ 1 & b \end{bmatrix}$ = (-a) (b) – (1) (1) = -ab -1

Example: Determine those values of p for which

$\begin{vmatrix} P & P \\ 2 & 4p \end{vmatrix}$ = 4p2 -2p = 0

$\Rightarrow$ $2p^2 - p = 0$
$\Rightarrow$
p(2p-1) = 0 $\Rightarrow$ p = 0 and 2p -1 = 0 $\Rightarrow$ p = 1/2
if p = 0 or p = ½ the determinants is zero

SELF ASSESSMENT EXERCISE 1

1. Compute the determinants of the following matrices.

(i) A = $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ (ii) B = $\begin{bmatrix} -1 & -1 \\ -4 & -2 \end{bmatrix}$ (iii) C = $\begin{bmatrix} -2 & 1 \\ -3 & 4 \end{bmatrix}$

iv D = $\begin{bmatrix} -4 & -3 \\ 4 & 5 \end{bmatrix}$ (v) E = $\begin{bmatrix} 2 & -10 \\ -10 & 35 \end{bmatrix}$

2. using the above matrix evaluate
(i) |A-B| (ii) |AB| (iii) $|A^2\text{-}B^2|$
Ans:

3.2 Determinants Of 3 x 3 Matrix

Let a 3 x 3 matrix be given as

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then the determinant of A is defined as follows

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1 c_2 – a_1b_3c_2 – a_2b_1c_3 – a_3b_2c_1$
$= a_1(b_2c_3 - b_3c_2) + b_1 (a_3c_2 - a_2c_3) + C_1(a_2b_3 - a_3b_2)$

The above could also be written as

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_2 & c_2 \end{vmatrix} \quad -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \quad + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Remark: each of the 2 x 2 matrix above can be obtained by deleing, in the original 3x3 matrix, the row and column containing its coefficient; i.e.

$$A_1 \begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix} - b_1 \begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix} + c_1 \begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}$$

Example: Compute the determinant

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & -7 & 2 \\ 4 & -8 & 1 \end{bmatrix}$$

$$\det (A) = \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 5 & -7 & 2 \\ 4 & -8 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -7 & 2 \\ -8 & 1 \end{vmatrix} -0 \begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix} +3 \begin{vmatrix} 5 & 7 \\ 4 & -8 \end{vmatrix}$$

$= 1(-7 - (-8.2)) - 0 + 3(5.(-8) - (4. -7))$
$= 1(-7 + 16) + 3(-40 + 28) = 9 - 36 = -27$

Example: Compute the above determinant using elements of the 3rd column as the coefficients the determinants of the 2 x 2 matrix.

Solution:

$$3 \begin{vmatrix} 5 & -7 \\ 4 & -8 \end{vmatrix} -2 \begin{vmatrix} 1 & 0 \\ 4 & -8 \end{vmatrix} +1 \begin{vmatrix} 1 & 0 \\ 5 & -7 \end{vmatrix}$$

$= 3(-40+28)-2(-8-0)+l(-7-0)$
$= -36+16-7=-27$

The above example has shown that the determinant of a 3 x 3 matrix can be expressed as a linear combination of three determinants of order two with coefficient from any row or from any column and not just from the first row.

Example: Compute the determinant of the matrix using (i) 2nd column (ii) 2nd row (iii) 3rd row as coefficients of the determinants of the 2 x 2 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

(i) $2\begin{bmatrix} 4 & 3 \\ 5 & -1 \end{bmatrix} +2\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} +5\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$

$= -[2(-4) +2(-1) +5((3) -42)] = -(-8-2-45) = 55$

(ii) $4\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} +2\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} +3\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$

$- [4(-2 - 15) + 2(-1) +3(5)]$
$= - 68 -2 + 15 = + 55$

(iii) $0\begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix} -5\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}$

3.3 Determinant of n x n Matrix (Optional)

You shall now study the determinant of a general n x n matrix. Before doing that, consider the following definitions.

Definition let A = (aij) be an n x n matrix. The determinant of A denoted[A] is defined
$| A| = \sum (-1)\ 3\ (k)\ a_1k_1,\ a_2k_2\ ...a_nk_n$

Where the sum is taken over all $k = (k_1,..., k_n)$ and ∂ (k) $\begin{cases} 0 \text{ if k is even} \\ 1 \text{ if k is odd} \end{cases}$

Example: The determinant of a 1 x 1 matrix is all by the above definition.

Example: Let A be a 2 x 2 matrix given as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then

$$|A| = (-1)^0\ a_{11}\ a_{22} + (-1)^1\ a_{12}\ a_{21} = a_{11}\ a_{22} - a_{12}a_{21}$$

Example: Let A be a 3x3 matrix given as A =

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$|A| = a_{11}a_{22}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

If you continue using this formation the determinant of a 4 x 4 matrix would involve the sum of 24 products of four elements. Therefore there is need to use a simpler formula. This will be done by introducing the concept or notations of minors and cofactors.

Definition: Let A = (aij) be an n x n matrix. Let Mij be the (n-1) x (n-1) matrix obtained from A by deleting (in the original matrix) the row and column containing aij. The determinant of Mij is called the minor of aij. The cofactor Aij is hereby defined as
$Aij = (-1)^{1+j}$ det (Mij).

Recall in section 3.2 you studied how to compute the determinant of a 3 x 3 by using the same method described in the above definition

Given that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $|A| = a_{11}(a_{22}a_{33} – a_{32}a_{23}) – a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

the above can be written as

$| A | = a_{11} \det(M_{11}) - aj_2 \det(M_{12}) + a_{13} \det(M_{13})$

OR

$|A| = a_{11} A_{11} + a_{12}A_{12} + a_{13}A_{13}$ ……… (*)

Equation (*) is called the cofactor expansion of det (A) using the first row of A. Other cofactor expansions are given as follows

$| A | = -a_2i(ai_2a_{33} - ai_3a_{32}) + a_{22} (a_na_{33} - a_{)3}\, a_{3]})$
$- a_{23} (a_ua_{32} - ai_2a_30$
$= a_2]A_2i + a_{22}A_{22} + a_{23}A_{23}$

also

$| A | = a_{3)}A_{3|} + a_{32}A_{32} + a_{33}A_{33}$

You can also get the above cofactor expansion by using the columns instead of the rows of A.

i.e. $|A| = anAn + a_2iA_2i + a_3iA_3i$
$= ai_2A_{12} + a_{22}A_{22} + a_{32}A_{32}$
$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

Example: Compute the determinant of a 4 x 4 matrix given as A=

$$A = \begin{vmatrix} 0 & 1 & 3 & 0 \\ 0 & 2 & 10 & 0 \\ 0 & 2 & 0 & 6 \\ 4 & 0 & 1 & 3 \end{vmatrix}$$

Solution: The cofactor expansion of the above determinant will involve four 3 x 3 determinants. Much labour can be reduced by expanding along the row or column that contains the most zeros. Therefore you will expand along the 1st column since it contains 3 zeros. Hence

|A| =0A11+0A21+0A31+4A14

$$= -4 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 10 & 0 \\ 2 & 0 & 6 \end{vmatrix}$$

Using the same criteria you have

$$-4 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 10 & 0 \end{vmatrix} = -4 \left[6 \begin{vmatrix} 1 & 3 \\ 2 & 10 \end{vmatrix} \right] = -4.6\,(10 - 6) = -96$$

Example: Compute the determinant of a 5 x 5 matrix given as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & 2 & 2 & 6 & 8 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution: 1st row has highest number of zero as elements. Therefore you will expand along the 1St row i.e. the cofactor expansion of the determinant will involve five 4 x 4 matrix.

$$|A| = 1 \begin{vmatrix} 3 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix} + 0\,A_{12} + 0\,A_{13} + 0\,A_{14}$$

$$= 1[-2 \begin{vmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 6 \end{vmatrix} = -2 \left[6 \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} \right]$$

$$= -2.6(3-(-1) = -2.6.4 = 48$$

From the above examples you have seen that using cofactor to e determinant is generally more efficient than using the definition. Especially if the matrix in question is a sparse matrix i.e. matrix with many zero entries.

SELF ASSESSMENT EXERCISE 2

Compute the determinants of the following matrices using 1 cofactor method.

(1) $A = \begin{bmatrix} 9 & 20 & 3 \\ 4 & 8 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ (2) $B = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 0 & 2 & 1 \\ 3 & 0 & 6 & 0 \\ 4 & 0 & -1 & 0 \end{bmatrix}$

(3) $C = \begin{bmatrix} 2 & 1 & 2 & 3 & 1 \\ 0 & 3 & 1 & 4 & 0 \\ 0 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 1 & 2 & -6 & -2 \end{bmatrix}$

Ans. (1) – 36 (2) 81 (3) -10

4.0 CONCLUSION

In this unit, you have studied how to compute the determinant of a 2 x 2 matrix. You also use the knowledge to compute determinant of a 3 x 3 matrix. The cofactor method of evaluating the determinant of an n x n matrix was introduced. You studied with examples that the cofactor method of evaluating the determinant of an n x n matrix is far simpler than using or applying the definition given at the beginning of the section 3.3.

The evaluation of determinant of an n x n matrix will be very useful in solving systems of n linear equations in n unknowns.

Also this cofactor method studied in this unit will be very useful when investigating or verifying some properties of determinants in the next unit.

5.0 SUMMARY

You have studied in this unit how to:

- Compute the determinant of a 2 x 2 matrix i.e. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- Compute the determinant of a 3 x 3 matrix
- Compute the determinant of an n x n matrix.

6.0 TUTOR-MARKED ASSIGNMENT

1. Compute the determinant of the following 2 x 2 matrices

(i) $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$

(ii) $\begin{bmatrix} -2 & 4 \\ -1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -10 & 12 \\ 13 & 14 \end{bmatrix}$

(iv) $\begin{bmatrix} 20 & 2 \\ 20 & 4 \end{bmatrix}$

2. Determine those values of p for which $\begin{vmatrix} -p & 4 \\ -2 & 2p \end{vmatrix} = 0$

3. Compute the determinant of

(i) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 1 & 3 \\ 4 & -1 & 3 \end{bmatrix}$

4. Compute the determinant of

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

7.0 REFERENCES/FURTHER READING

Seymour Lipschutz (1974). Schaum Outline series: Theory and Problems of Linear Algebra. McGraw Hill Int Book Company NY

S. A. Ilori, O Akinyele (1986). Elementary Abstract and linear Algebra. Ibadan University Press.

UNIT 5 PROPERTIES OF DETERMINANTS

CONTENTS

1.0 INTRODUCTION

In this unit you shall study more on determinants of matrices by investigating some basic properties of determinant. Changes in some of the elements of the row or columns might affect the sign of the determinant. There are certain theorems that explain this changes that can occur along the expansion of the cofactors of the determinants. Such theorem will be studied here without giving their proofs.

The concept of the adjoint of a matrix will be introduced in this unit. The Cramer rule that will be discussed in the next unit will be based on the topics that will be studied in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify and verify 4 properties of a determinant
- state the basic theorems without proof involving
  - (i) determinant of a transpose
  - (ii) triangular matrix
- compute the adjoint of a matrix
- manipulate and compute determinants resulting from row operations.

3.0 MAIN CONTENT

3.1 Basic Properties

Basic properties of a determinant of a matrix are hereby given as follows:

1. Let A be an n x n matrix. If the elements in a row (column) arc zero, the | A | = 0.

Example: Given $A = \begin{bmatrix} 5 & -7 & 7 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$$A = 5\begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} + 7\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 7\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}$$

=5.0+7.0+7.0 = 0

2. The determinant of the transpose of a matrix is equal to determinant of the matrix

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$= a_1(b_2c_3 - b_3c_2) – b_1 (a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$
$= a_1b_2c_3 – a_1b_3 c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1 a_2b_3 – c_1a_3b_2$
$= - a_1b_2 c_3 + a_2c_1b_3 + c_1a_2b_3 – a_1b_3c_2 – b_1a_2c_3 – c_1a_3b_2$

$$A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$= a_1(b_2c_3 – c_3b_3) – a_2 (b_1c_3 – c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1b_2c_3 – a_1c_2 b_3 - a_2b_1c_3 + a_2c_1b_3 + a_3 b_1c_2 – a_3c_1b_2$

$= - a_1b_2 c_3 + a_2c_1b_3 + a_3b_1c_2 – a_2b_1c_3 – a_1c_2b_3 – a_3c_1b_2$

Example: Let A = $\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$

|A| = 3 – (-4.2) = 11

$A^T = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}$

$|A^T| = \begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}$ = 3-(-8) = 11

A, then

$|A^T = |A|$

Example:

Let A = $\begin{bmatrix} 2 & -1 & 3 \\ 3 & -1 & 0 \\ 4 & 5 & 0 \end{bmatrix}$

Then $A^T = \begin{matrix} 2 & 3 & 4 \\ -1 & -1 & 5 \\ 3 & 0 & 0 \end{matrix}$

|A| = 3 $\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$ =3 (3 x 5-(-l $_x$ 4)) = 57

A^T = 3 $\begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$ = 3(15-(-1 x4)) = 57

Example: Show that) $|B^T|$ = |B| if

$$B = \begin{bmatrix} -1 & 2 & 1 & 2 \\ 3 & -1 & 1 & 1 \\ 4 & -2 & 3 & 4 \\ 5 & 3 & 2 & 1 \end{bmatrix}$$

$$B = -1 \begin{vmatrix} -1 & 1 & 1 \\ -2 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 & 1 \\ 4 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -1 & 1 \\ 4 & -2 & 4 \\ 5 & 3 & 1 \end{vmatrix} -2 \begin{vmatrix} 3 & -1 & 1 \\ 4 & -2 & 3 \\ 5 & 3 & 2 \end{vmatrix} = 18$$

$$B^T = \begin{vmatrix} -1 & 3 & 4 & 5 \\ 2 & -1 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 1 \end{vmatrix}$$

$$= -1- \begin{vmatrix} 1 & -2 & 3 \\ 1 & 3 & 2 \\ 1 & 4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & -2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 1 \end{vmatrix} = 4 \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$- 5 \begin{vmatrix} 2 & 1 & -2 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 18$$

The above property shows that rows and columns of a determine interchangeable.

3. If every element of a row (column) of a matrix A is multiplied nonzero constant then, the determinant of A is multiplied by that r constant

Example:

$$\text{Let } |A| = \begin{vmatrix} a1 & -a2 & a3 \\ b1 & b2 & b3 \\ c1 & c2 & c3 \end{vmatrix}$$

$= a_2b_2c_3 – a_1c_2b_3 – a_2b_1c_3$
$a_2c_1b_3 + a_3b_1c_2 – a_3c_1b_2$

$$\text{and } |\lambda A| = \begin{vmatrix} \lambda a1 & \lambda a2 & \lambda a3 \\ b1 & b2 & b3 \\ c1 & c2 & c3 \end{vmatrix}$$

$= \lambda (a_1b_2c_3 – a_1c_2b_3)$
$+ \lambda (a_2c_1b_3 – a_2b_1c_3)$
$+ \lambda (a_3b_1c_2 – a_3c_1b_2)$

$= \lambda (a_1b_2c_3 – a_1c_2b_3 + a_2c_1b_3 – a_2b_1c_3 + a_3b_1c_2 – a_3c_1b_2)$
$= \lambda |A|$

Example: Let A = $\begin{bmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Let $\lambda = 2$. Multiply 2nd row of matrix A by 2 you get

$$A^1 = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Where A^1 is the matrix obtained after multiplying the 2nd row of A by 2

To show that $(A^1) = |2A| = |\ 2\ A|$

$$|A = 2 \begin{vmatrix} -1 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} + & 1 \\ 1 & 2 \end{vmatrix} 1 \begin{vmatrix} +2 & 1 \\ 1 & 1 \end{vmatrix} - 1$$

= 2)-2-1) + (2 -1) + 2(1-(-1))
= -6 + 1 + 4 = -1

$$|A^1| = 2 \begin{vmatrix} -2 & +2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$

= 2 (-4-2) + (4-2) + 2(2+2)
= 2 (-6 +1 + 4) = 2 (-1) = -2

4. Determinant of triangular matrices

$$\text{Let } A = a_{11} \begin{pmatrix} a_{11} & 0 & 0 \ldots\ldots & 0 \\ a_{21} & a_{22} & 0 \ldots\ldots & 0 \\ \vdots & & a_{33} & \vdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{pmatrix}$$

Be a triangular matrix i.e. a matrix having zero below or above the leading diagonal, then the determinant of A is given as

$|A| = a_{11}a_{22} \ldots a_{nn}$

Here the determinant is the product of all the elements in the leading diagonal. You can easily see this if you use the first row to expand the determinant and induction on n.

Example: Determinant of the identity matrix 1n is equal to 1 for all n.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|I_3| = (1 - 0) = 1$

$$\begin{bmatrix} 1 & 0 & 0\ldots\ldots & 0 \\ 0 & \ddots & 0\ldots\ldots & 0 \\ \cdot & & 1 & \\ \cdot & & & 1 \\ \cdot & & & 1 \\ 0 & & 0\ldots\ldots & 1 \end{bmatrix}$$

$$|1_n| = \begin{bmatrix} 1 & 0 & 0\ldots\ldots & 0 \\ 0 & 1 & 0\ldots\ldots & 0 \\ \cdot & & \ddots & \\ & & 1 \quad \cdot & \cdot \\ \cdot & & & 1 \\ \cdot & & & \ddots \\ 0 & & 0\ldots\ldots & 1 \end{bmatrix} \begin{aligned} &= 1\,|1_{n-1}| \\ &= .\ 1\ldots.|1_2| = 1 \end{aligned}$$

5. The determinant of the product matrix AB is equal to the product the determinants.

Let A and B be two square matrices of the same order.
Then $|AB| = |A|\ |B|$

Example: Given that

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$|A| = -1 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 3$$

$$|B| = -1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$|AB| = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 3 \\ -1 & 6 & 9 \\ -2 & 1 & 1 \end{pmatrix}$$

$$|AB| = 2 \begin{vmatrix} 6 & 9 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 6 \\ -2 & 1 \end{vmatrix} = 3$$

$\Rightarrow$ |AB| = |A| |B| = 3

Example :

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$|A| = 1 \begin{vmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = -1$$

$$|B| = 1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2$$

$$|AB| = \begin{vmatrix} 11 & 7 & -5 & 0 \\ 1 & 1 & -1 & 2 \\ 13 & 10 & -8 & 8 \\ 1 & 2 & -1 & 2 \end{vmatrix} = 11 \begin{vmatrix} 1 & -1 & 2 \\ 10 & -8 & 8 \\ 2 & -1 & 2 \end{vmatrix} - 7 \begin{vmatrix} 1 & 1 & 2 \\ 13 & -8 & 8 \\ 1 & -1 & 2 \end{vmatrix}$$

$$+(5) \begin{vmatrix} 1 & 1 & 2 \\ 13 & 10 & 8 \\ 1 & 2 & 2 \end{vmatrix} = -2$$

$\Rightarrow |AB| = |A|\ |B| = -2 = -1.2$

Now consider

$$|BA| = \begin{vmatrix} 0 & 0 & 1 & -3 \\ 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & -3 \\ 2 & 2 & -1 & -1 \end{vmatrix} = -1 \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 & 4 \\ 1 & 2 & 5 \\ 2 & 2 & -1 \end{vmatrix} = -2$$

$\therefore\ |AB| = |BA| = |B|\ |A| = |A|\ |B|$

3.2 Row Operations

1. Addition to one of a multiple of another row does not change the value of the determinant of a matrix. In other words the determinant remains unchanged when you add to one row a multiple of another row.

Example: Let

$$|A| = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1b_2c_3 - a_1c_2b_3 + a_2b_1c_3 - a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$$

and $|A^1| = \begin{vmatrix} a_1+\lambda b_1 & a_2+\lambda b_2 & a_3+\lambda b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1 + \lambda\ b_1(b_2c_3 - c_2b_3) + a_2 + \lambda\, b_2\ (b_1c_3 - c_1b_3)$
$a_3 + \lambda\, b_3\ (b_1c_2 - c_1b_2)$

$= \ a_1(b_2c_3 - c_2b_3) + a_2\ (b_1c_3 - c_3b_3) + a_3\ \ (b_1c_2 - c_1b_2)$
$+\lambda\ \ b_1(b_2c_3 - c_1b_2) + \ \lambda\ \ b_2(b_1c_3 - c_1b_3)$
$+\lambda\ \ (b_1c_2 - c_1b_2)$

$$= |A| + \lambda \begin{vmatrix} b_{11} & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(Two rows are identical) hence
$= |A| + \lambda\ .0 = |A|$

Example: Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

$$A^1 = \begin{bmatrix} -1 & 3 & 8 \\ -2 & 0 & 4 \\ 1 & 4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} - 4\begin{vmatrix} -1 & 0 \\ 1 & 4 \end{vmatrix} = -6$$

$$|A| = \begin{vmatrix} 0 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix} + 8\begin{vmatrix} -2 & 0 \\ 1 & 4 \end{vmatrix} = -6$$

2. Suppose A_1, A_1^1, A_2, A_2^1 and A_3, A_3^1 are rows of a 1 x 3 matrices respectively then you have

(i) $$\det\begin{bmatrix} A_1 + A_1^{1} \\ A_2 \\ A_3 \end{bmatrix} = \det\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \det\begin{bmatrix} A_1^{1} \\ A_2 \\ A_3 \end{bmatrix}$$

(ii) $$\det\begin{bmatrix} A_1 \\ A_2 + A_2^{1} \\ A_3 \end{bmatrix} = \det\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \det\begin{bmatrix} A_1 \\ A_2^{1} \\ A_3 \end{bmatrix}$$

(iii) $$\det\begin{bmatrix} A_1 \\ A_2 \\ A_3 + A_3^{1} \end{bmatrix} = \det\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \det\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

The above properties are useful in the expansion and factorization determinants.

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 4 \\ -1 & 0 & 1 \end{bmatrix}$

Hence,

Show that $|A| = |A^1| = |B|$

Solution:

$|A| = -1 \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} + 1\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

= - 1 (-3 -2) + 1 (4 – 3)

= 5 + 1 = 6.

$|A^1| = -1 \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} + 1\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

= (12 + 2) + 1 (-4 + 3)

= -14 -1 = -15

$|B| = -3 \begin{bmatrix} -2 & 3 \\ -1 & 0 \end{bmatrix} = -3\ (0+3) = -9$

$\therefore\ |A| + +A^1| = 6 – 15 = -9$

$\therefore$ |A| + +A[1]| = |B|.

$$\text{and } |B| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|A| = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3 (b_1c_2 - c_1b_2)$
$- |A| = a_1(c_2b_3 - b_2c_3) - a_2(c_1b_3 - b_1c_2) + a_3(c_1b_2 - b_1c_2)$

$= |B| = a_1(c_2b_3 - b_2c_3) - a_2(c_1b_3 - b_1c_2) + a_3(c_1b_2 - b_1c_2)$

therefore |B| = =-|A|

Example: Given

$$A = \begin{vmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\text{Let } B = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\therefore \ |A| = 2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 2(1 - 4) = -6$$

$$\text{and } |B| -2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -2(1 -4) = 6$$

SELF ASSESSMENT EXERCISE 1

Use the following matrices

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 3 & 2 & 5 \\ 2 & -1 & 3 \end{pmatrix} B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & 0 & 5 \\ 3 & 2 & 5 \\ 2 & -1 & 3 \end{pmatrix} D = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 0 \\ 3 & 1 & 3 \end{pmatrix}$$

To compute
(i) $|A^T|$ (ii) |B| (iii) |C| (iv) |D|

Hence

(2) Shows that $|A^T| = |A|$

(3) Shows that $|AB| =| B | . \lambda A| = | A| . | B|$

(4) Use matrices B and D to show that if the elements of any row are the sums or difference of two or more terms, the determinant may be written as the sum or difference of two determinants.

(5) Use matrices A, AT, C and CT above to show that adding a multiple of one row (or column) of a matrix to another row (or column) of the matrix leaves the determinant unchanged.

4.0 CONCLUSION

In this unit you have examined various ways determinants of matrices can be computed. You have studied the effects certain row operations such as

1. Interchanging any two of the rows of a matrix,
2. Adding a multiple of one row (or column) of a matrix to another row (or column) of the matrix etc. have on the value of the determinants. You have studied that row operations have the same effect on determinants as the column operations since the det A = det AT. The properties of determinants studied in this unit are very useful in solving systems of linear equations.

5.0 SUMMARY

In this unit you have studied;

- Basic properties of determinants of matrices such as
- det (A) = det (A^T)
    - $|AB| =|A| .|B| =|B| .| A|$

- ○ $|A| = a_{11}a_{22}\ldots a_{1n}$ for all n x n triangular matrices etc.

- Row (column) operations such as
  - ○ The determinant remains unchanged when you add to one row a multiple of another row.
  - ○ Interchanging the i^{th} and j^{th} rows
  - ○ Multiplying the i^{th} row by a nonzero constant
  - ○ Adding the j^{th} row multiplied by a scalar A, to the i^{th} row.

6.0 TUTOR-MARKED ASSIGNMENT

1. If $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

What relation exists between det (A), det (B) and det (BA)?

2. If $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix} B = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 2 & -8 \\ 0 & -2 & 0 \end{pmatrix}$

Find (i) $|A|$ (ii) $|B|$ (iii) find a scalar such that $\lambda |B| = |A|$
In problems 3-7 use properties of determinants to evaluate the determinant.

3. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 6 \\ 2 & -1 & 4 \end{vmatrix}$

4. $\begin{vmatrix} 3 & 6 & 9 \\ 3 & 1 & 6 \\ 2 & -1 & 4 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 3 & 9 \\ 3 & 1 & 6 \\ 2 & -1 & 4 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 3 & 6 & 4 \end{vmatrix}$

7. Compute

$$\begin{vmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{vmatrix}$$

7.0 REFERENCES/FURTHER READING

K.A STROUD: Engineering Mathematics (Fifth Edition) Palgrave.