MODULE 2

UNIT 1 INVERSE OF MATRICES

CONTENTS

1.0 INTRODUCTION

The study of inverse of matrix is an important tool in solving many real life problems that can be modeled by any given system of linear equation. For example let Ax = b where A is 1 x 1 matrix and x and b are scalar. A typical solution to the above equation is given as $x = b/A = bA^{-1}$. In this unit you will study how to find A^{-1} if A is a matrix of order greater than 1. This will enable you to solve a system of linear equation by matrix method. You should revise units 4 and 5 since most of the concepts studied there, will be used as essential tools for this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify invertible matrices
- compute the inverse of 2 x 2 and 3 x 3 matrices
- compute the inverse of an n x n matrix by using determinant method.

3.0 MAIN CONTENT

3.1 Invertible Matrices

Definition of inverse of a matrix: Let A be a square matrix of order n. A is said to be invertible if there exists a square matrix B of order n such that AB = BA = I_n

Where I_n is the identity matrix of order n and B is called the inverse of j
Remark: Not all square matrices have inverses. For example let A be a matrix given as.

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix}$$

and suppose that it has an inverse of the form

$$B = \begin{pmatrix} x & y \\ Z & w \end{pmatrix}$$

$$\text{Then } \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ Z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{i.e. } \begin{bmatrix} 3x+6z & 3y+6w \\ x+32 & y+3w \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

therefore 3x + 6z = 1 …………… (i)
3y + 6w = 0 ……………(ii)
x + 32 = 0 ………………(iii)
y + 3w = 1 ……………….(iv)

Equation (i) and (iii) represent two parallel lines, which are not coincident (see unit 3). Similarly, equations (ii) and (iv) are also parallel Therefore, the equations (i) to (iv) have no solutions. Hence there values of x, y, z, w such that AB = 12 and as such the matrix A above inverse.

Any square matrix, which has an inverse, is called a non-singular m; non-invertible matrix.

If a matrix has an inverse, that inverse is unique. In other words if m say has matrix B as its inverse then there will not be any other matrix stand in the place of B. i.e. there will not be any other matrix C such that = CA = I_n except of course C =B.

The above truth can easily be established. That is you can show t matrix A is non-singular then it's inverse is unique.
Assume that B and C are two inverse of A. You want to show that B = C

AB = I (B is an inverse of A)
C(AB)=CI=C
(CA) B = C
I B = C (Since C is an inverse of A)
B=C

Hence the required result. You have shown that B=C.

If matrix A is invertible, the unique inverse matrix of A is written as A^{-1}. Certain properties of invertible matrices will be given with their proofs.

Let A, B be invertible matrices and A be a non-zero constant. Then the following properties hold:

(i) A is invertible

(ii) AB is invertible then $(AB)^{-1} = B^{-1} A^{-1}$

(iii) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(iv) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof

(i) $(\lambda A)(\lambda^{-1}A^{-1}) = (\lambda \lambda^{-1})(AA^{-1}) = 1.1 = 1$ and
$(\lambda^{-1}A^{-1})(\lambda A) = (.\lambda^{-1}\lambda)(A^{-1}A) = 1.1 = 1$

From the above it has been shown that A is the inverse of A^{-1} therefore in symbol you write $(A. A).^{-1} = A^{-1} A^{-1}$.

(ii) $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = (AI)A^{-1}$
$= (AI)A^{-1}$
$= AA^{-1}$
$= I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$\Rightarrow \quad (B^{-1}A^{-1})(AB) = B^{-1(I)}B = B^{-1}IB$

$= B^{-1}B$

$= I$

It has been shown that AB is the inverse of $B^{-1}A^{-1}$. In symbol you write $(AB)^{-1} = B^{-1}A^{-1}$ which gives you the required result.

(iii) Since $AA^{-1} = A^{-1}A = I$ this implies that A is the inverse of A^{-1} therefore in symbol you write $(A^{-1})^{-1} = A$. which is the required result.

(iv) Since $AA^{-1} = A^{-I}A = I$ taking the transposes, you get

$(AA^{-1})^T = (A^{-1}A)^T = I^T$

this implies that $(A^{-1})^T A^T = A^T (A^{-1})^T = I$ thus the inverse of A is $(A^{-1})^T$ therefore in symbol you write $(A^T)^{-1} = (A^{-1})^T$.

SELF ASSESSMENT EXERCISE 1

i. Show that the matrix given as a = $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is non-invertible

ii. Give a definition of the inverse of n x n matrix.

3.2 The Inverse of A Square Matrix

3.2.1 The Inverse of a Non-Singular 2 x 2 Matrix

Given a 2 x 2 matrix of the form

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with an inverse

$A^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

You will find the values of x, y, z, w in terms of a, b, c, d such that

$AA^{-1} = 1$

Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $ax + bz = 1$ …………… (i)
$ay + bw = 0$ ……………(ii)
$cx + dz = 0$ ………………(iii)
$cy + dw = 1$ ……………….(iv)

Solving equation (i) and (iii) simultaneously you obtain x and z as

$$x = \frac{d}{ad-bc}, \ z = \frac{-c}{ad-bc}$$

and solving equation (ii) and (iv) we obtain

$$y = \frac{-b}{ad-bc} \quad w = \frac{a}{ad-bc}$$

In the above inverse of a 2 x 2 square matrix, you can remember it by noting that you interchange the entries in the leading diagonal, change the sign of the other two entries and divide the resulting matrix by the determinant of the matrix, d A I = dd - be).

Hence you can rewrite the inverse as $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

You should note that $|A| \neq 0$ for it to be invertible. All invertible matrices must have $|A| \neq 0$.

To test if a matrix is invertible all you have to do is to test if the determinant is non-zero i.e. $|A| \neq 0$.

Example: Determine whether the following matrices are invertible and find the inverse where it exists.

(i) $A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$ (iii) $D = \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution:

(i) |A | =10-4= 6

$\therefore A^{-1} = \frac{1}{6}$ - $\begin{pmatrix} 5 & -1 \\ 4 & 2 \end{pmatrix}$ = $\begin{pmatrix} 5/6 & 1/6 \\ 2/3 & 1/3 \end{pmatrix}$

(ii) |B| = $\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$ = -4 –(-6) = 2

B^{-1} ½ $\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}$ = $\begin{pmatrix} 2 & 1 \\ -3/2 & -1/2 \end{pmatrix}$

(iii) |C| = $\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$ = 4

C^{-1} = ¼ $\begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix}$ = $\begin{pmatrix} 1 & 0 \\ 3/4 & 1/4 \end{pmatrix}$

(iv) |D| = $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ = 4 – 4 = 0 not invertible

3.2.2 The Inverse of A 3 x 3 Square Matrices

In the previous section you obtained a general formula for an inverse of a 2x2 matrix. In the case of 3 x 3, you might not be able to express the inverse a closed form. However, the inverse can be obtained by solving a system of equations involving 9 unknowns.

Example: Compute the inverse of A = $\begin{pmatrix} 2 & 3 & -1 \\ 2 & 4 & 2 \\ -2 & -2 & 3 \end{pmatrix}$

Solution: You will look for nine constants x^1, x^2, x^3, y^1, y^2, y^3, z $^1z^2$ and z^3 such that

$$\begin{pmatrix} 2 & 3 & -1 \\ 2 & 4 & 2 \\ -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} a_1 & a_{12} & a_{13} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a_1+3b_1-c_1 & 2a_2+3b_2-c_2 & 2a_3+3b_3-c_3 \\ 2a_1+4b_1+2c_1 & 2a_2+4b_2+2c_2 & 2a_3+4b_3+2c_3 \\ -2a_1-2b_1+3c_1 & -2a_2-2b_2+3c_2 & -2a_3-2b_3+3c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$2a_1 + 3b_1 - c_1 = 1$
$2a_1 + 3b_2 - c_2 = 0$
$2a_3 + 3b_3 - c_3 = 0$

$2a_1 + 4b_1 + 2c_1 = 0$
$2a_2 + 4b_2 + 2c_2 = 1$
$2a_3 + 4b_3 + 2c_3 = 0$

$-2a_2 - 2b_1 + 3c_1 = 0$
$-2a_2 - 2b_2 + 3c_2 = 0$
$-2a_3 - 2b_3 + 3c_3 = 1$

Solving each system and nothing that the determinant i.e.

$$\begin{vmatrix} 2 & 3 & -1 \\ 2 & 4 & 2 \\ -2 & -2 & 3 \end{vmatrix} = -2 \neq 0$$

the solution of each of the three systems is given as

$a_1 = -8$ $a_2 = 3.5$ $a_3 = -5$
$b_1 = 5$ $b_2 = -2$ $b_3 = 3$
$c_1 = -2$ $c_2 = 1$ $c_3 = -1$

thus the inverse is given as

$$A^{-1} = \begin{pmatrix} -8 & 3.5 & -5 \\ 5 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix}$$

You could also find the inverse of a 3 x 3 matrix by using elementary row operation. This is method of reducing a square matrix A of order n to an identity matrix I_n by a series of elementary row operation, and then the same series of row operation is applied to I_n to yield the inverse of A.

Example: Let A = $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{pmatrix}$ Find A^{-1}

Solution

1. $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right.$

2. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & -3 \\ 1 & -1 & 3 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right.$ R_{21} (-2)

3. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & -3 \\ 0 & 1 & 5 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right.$ R_{31} (2)

4. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & -3 \\ 6 & 0 & 2 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & -2 & 2 \end{pmatrix} \right.$ R_{23} (½)

5. $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \right.$ R_3(½)

$\begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ -4 & 5 & 3 \\ 1 & -1 & 1 \end{pmatrix} \right.$ R_{21}(-3)

$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ -4 & 5 & 3 \\ 1 & -1 & 1 \end{pmatrix} \right.$ R_{12} (-3)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Bigg| \begin{pmatrix} 14 & -16 & 10 \\ -4 & 5 & -3 \\ 1 & -1 & \end{pmatrix} \quad R_{13}\,(1)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Bigg| \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix} \quad R_{1}\,(½)$$

$$\therefore A^{-1} = \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$$

SELF ASSESSMENT EXERCISE 2

Compute the inverse of the following matrices

1. $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ -1 & 4 & 6 \end{pmatrix}$

$$\text{Ans} = A^{-1} = \begin{pmatrix} 7 & -4 & 1 \\ -8 & 5 & -1 \\ 5 & -3 & 1 \end{pmatrix} \;.\; B^{-1} = \frac{1}{30}\begin{pmatrix} 6 & 10 & -10 \\ 15 & -5 & 5 \\ -9 & 5 & 1 \end{pmatrix}$$

3.3 Invertible Matrices and Determinants

You shall now obtain a general formula for the inverse of an (n x n) matrices by the method of determinant. Let A be an n x n matrix given as

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots a_{2n} \\ . & \\ . & \\ . & \\ a_{n1} & a_{n2}\ldots\ldots a_{nn} \end{bmatrix}$$

Definition: The adjoint of A denoted by adj A is the transpose of the matrix of cofactors of the elements aij of A. (see section 3.3 of unit 4 for definition of cofactor), and is given as

$$\text{Adj } A = \begin{pmatrix} A_{11} & A_{21}\ldots A_{1n} \\ A_{21} & A_{22}\ldots A_{2n} \\ \vdots & \vdots \quad \vdots \\ A_{1n} & A_{2n}\ldots A_{nn} \end{pmatrix}$$

Example: Compute the adjoint of the matrix given as $A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix}$

The element of the 1st row are 2, 0 and -1 and their cofactors are

$$\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7, \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 5, \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2$$

the element of 3rd row are -1, 1 and 3, cofactors are given as

$$\begin{vmatrix} 0 & -1 \\ 2 & -1 \end{vmatrix} = 2, -1 \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = -1, \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} = 4$$

therefore the matrix factor is given as $A^{co} = \begin{pmatrix} 7 & -8 & 5 \\ -1 & 5 & -2 \\ 2 & -1 & 4 \end{pmatrix}$

and $A^* = \text{adj}(A) = \begin{pmatrix} 7 & -8 & 5 \\ -1 & 5 & -2 \\ 2 & -1 & 4 \end{pmatrix}^T = \begin{pmatrix} 7 & -1 & 5 \\ -8 & 5 & -1 \\ 5 & -2 & 4 \end{pmatrix}^T$

furthermore, using matrix multiplication.
Find AA* of the above example.

$$AA^* = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 7 & 1 & 2 \\ -8 & 5 & 1 \\ 5 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note $|A| = 9$.

From above you $AA^* = |A| \Rightarrow AA^* = \frac{1}{|A|}$

It implies that $\frac{A^*}{|A|}$ is the inverse of A

$\therefore A^{-1} \frac{A^*}{|A|}$ Hence $A^{-1} = \frac{1}{9}\begin{pmatrix} 7 & -1 & 2 \\ -8 & 5 & -1 \\ 5 & -2 & 4 \end{pmatrix}$

generally given an n x n matrix A

$$A = \begin{pmatrix} A_{11} & A_{21}....A_{1n} \\ A_{21} & A_{22}....A_{2n} \\ \vdots & \vdots \\ A_{1n} & A_{2n}....A_{nn} \end{pmatrix} \vdots$$

Then $A^{-1} = \frac{AA^*}{|A|}$ where A* = adj (A) and A is det (A)

Example: Find the inverse of (i) $A = \begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}$

Solution: Using the adjoint method

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} A_{12} = -1\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}, A_{13} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1$$

$A_{21} = -1\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = -8, A_{22}\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5, A_{23} = -1\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = -1$

$A_{31} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5, A_{32} = -1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -3, A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$

$A^* = \begin{pmatrix} 7 & -4 & 1 \\ -8 & 5 & -1 \\ 5 & -3 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$

$|A| = 1$

$\therefore A^{-1} = A^* = \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$

Example: Compute A^{-1} if $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 \\ 2 & 1 & 3 & -1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$

$\text{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$

$A_{11} = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & 2 & 2 \end{vmatrix} = -20, A_{12} = -1\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = +1\ A_{13} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix}$

$A_{14} = -1\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = -3, A_{21} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{vmatrix} = 0, A_{22} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = +8$

$$A_{23} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = -4, A_{24} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = +4, A_{31=} \begin{vmatrix} 0 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

$$A_{32} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -2, A_{33} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -2, A_{34=} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & +1 \\ 1 & 2 & 2 \end{vmatrix} = +6$$

$$A_{41} = \begin{vmatrix} 0 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 0, A_{42} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix} = -5, A_{43=} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0$$

$$A_{44} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -5$$

$$A^* = \begin{pmatrix} -20 & 1 & 12 & -3 \\ 0 & 8 & -4 & -4 \\ 0 & -2 & -4 & 6 \\ 0 & -5 & 0 & -5 \end{pmatrix}^T = \begin{pmatrix} -20 & 0 & 0 & 0 \\ 1 & 8 & -2 & -5 \\ 12 & -4 & -4 & 0 \\ -3 & -4 & 6 & -5 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{-1}{20} \begin{pmatrix} -20 & 0 & 0 & 0 \\ 1 & 8 & -2 & -5 \\ 12 & -4 & -4 & 0 \\ -3 & -4 & 6 & -5 \end{pmatrix}$$

4.0 CONCLUSION

In this unit, you have studied how to compute the inverse of a matrix using the determinant method. You have extended the determinant method adjoint method. In other words, you have studied how to find the inverse general n x n square matrix by the method of adjoint. You have also show to find the inverse by the method of elementary row operation these methods will form the necessary tool to solve a system of n equation in n unknown simultaneously, which will be the subject of study in the next unit.

5.0 SUMMARY

In this unit, you have studied how to

- Compute the inverse of a square matrix by the method of elementary row operations
- Compute the inverse of a square matrix by the determinant r
- How to compute the inverse of a square matrix by the adjoint method
- How to identify invertible matrices i.e. $|A| \neq 0$.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the inverse of the following 2 x 2 matrices

(i) $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 3 & -8 \\ 7 & 8 \end{pmatrix}$

For questions (2) and (3)

2. Find the inverse of the following matrices by method of elementary row operations.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 1 \end{pmatrix}$$

3. $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

Find the inverse of the matrices in question 4 and 5 by the adjoint method.

4. $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}$

5. $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$

7.0 REFERENCES/FURTHER READING

The following are recommended textbooks. You could borrow or purchase them.

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UNIT 2 ROW ECHELON FORM AND SYSTEMS OF EQUATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, you shall apply theory and properties of matrices you I studied in the previous units. The main focus here is to solve system linear equation using matrix.

Interestingly all systems of linear equation can be transcribed into a m form, after which appropriate elementary row or column operation is us obtain the solution of the system of linear equation. Basically, all system be transcribed into matrix form as follows Ax = b where A is n x m n and X is m x 1 matrix and b is n x 1 matrix.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a row reduced echelon form of a matrix
- to transcribe system of linear equations into matrix form
- to solve systems of linear equations by echelon form or Gauss elimination.

3.1 Basic Definition

In this unit you will study how to use elementary row operation used in section 3.2 of previous section to solve systems of linear equations. Hi the following definitions will be needed.

Definition: A matrix is said to be in row echelon form if

1. The first non-zero element in each row is 1.

2. If row k does not consist entirely of zeros, the number of leading zero entries in row k + 1 is greater than the number of leading zero entries in row k.

3. If there are rows whose entries are all zero, they are below the row having non-zero entries.

You shall now consider some examples of matrices that are in row echelon form.

1. $A = \begin{pmatrix} 1 & 3 & 2 \\ & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ 2. $B = \begin{pmatrix} 1 & 5 & 4 \\ 6 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 6 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

A counter example of matrices that are not in row echelon form is given below as;

$A^{-1} = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{pmatrix}$ $B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $C^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Matrix A did not satisfy condition (i) above. Matrix B' did not satisfy condition (iii) above. While matrix C^1 did not satisfy condition (ii).

Definition: Elementary Row Operation include

(i) Interchanging of two rows

(ii) Multiplying a row by a non-zero real number

(iii) Adding a multiple of one row to another row.

Definition: Two systems of linear equations involving the same variables are said to be equivalent if they have the same solution set.

What the above definition implies is that if you interchange the order in which two equation of a system are written it will not have effect on the solution set. The re-ordered system will be equivalent to the original system.

Example:

x+2y=2 4x+y=6
3x-y=4 and 3x-y=4
4x+y=6 x+2y=2

have the same solution set. Also if one of the equations in the above* is multiplied through by a non-zero constant, this will not have any the solution set, however the new system will be equivalent to the

Example:

x+y+z= 1 and 3x+3y+3z=3
-3x-4y+2z=8 -3x-4y+2z=8
are equivalent.

Lastly if a multiple of one equation is added to another equation. I system will be equivalent to the original system.

Example:

x + 2y + 3z = 4
x - 2y- z = 1
And 3x-2y+z=6
where x + 2y + 3z = 4 are equivalent.
(2x -4y -2z) + (x + 2y +3z) = 4+2.1 = 3x -2y + 2 = 6

The 3 operations described above for systems of linear equal equivalent to the 3 operations described in the definition above.

Definition: An n x n system is said to be in a triangular form if there equation. The coefficients of the first i-' variables are all zero coefficient of x; is non-zero (i-1,..., n).

Example: The system

4x+2y+z=8 (i)
3y – z =-1 (ii)
2z = 4 (iii)

is in a triangular form.

Because of the triangular form of the system you can easily see that z = 2, substituting in equal (ii) you obtain

$3y-2 = 1 \Rightarrow y = 1.$

Substituting values of z and y into equation (1) you obtain 4x + 2 8 $\Rightarrow x = 1.$

Thus the solution set is given as (1,1,2). The 3 operations described above can reduce any n x n system of linear equation into a triangular form.

3.2 Transcribing a System into a Matrix Form

Example: Let 3×3 system of linear equation be given as
x + 2y + z = -2
3x - y -3z = 3
2x + 3y + 2 = 4

You can easily associate a 3 x 3 matrix whose entries are the coefficients of the (x, y, z).

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$$

The above matrix is called the coefficient matrix of the system of linear equation. If you attach to the coefficient matrix an additional column whose entries are the numbers on the right hand side of the system, you obtain a new matrix.

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 2 & 3 & 1 & 4 \end{array}\right)$$

The above matrix B is called the augmented matrix of the system.

Definition: The process of using the 3 elementary row operations to transform a system of linear equations into one whose augmented matrix is in row echelon form is called Guassian elimination method.

Remark: In unit 6 section 3.2.2 the inverse of 3 x 3 square matrix was obtained by the method of Guassian elimination.. (You could refer to it).

Definition: A matrix is said to e in reduced row echelon form if:

1. The matrix is in row echelon for

2. The first non-zero entry in each row is the only non-zero entry in its column.

Example: The following are matrices in reduced echelon form.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.3 Solving Systems By Row Reduced Echelon Form

Example: Solve the system by echelon form or Guass Jordan elimination method.

$x + 2y - 3z = 3$

$2x - y - z = 11$

$3x + 2y + z = -5$

Solution :

Transcribe the system in matrix form that is;

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

The augmented matrix is given as

$$\left(\begin{array}{ccc:c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$$

Subtract 2 times the 1st row from the second row and also sub the 1st row from the 3rd row to give you a new matrix of this form.

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{pmatrix}$$

Divide 2nd row by -5 to get a new matrix of the form

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -4 & 10 & -14 \end{pmatrix}$$

Add 4 times 2nd row to 3rd row to get a new matrix of this form

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 6 & -18 \end{pmatrix}$$

Observe that through the elementary row operation you have reduced the coefficient matrix to a triangular matrix.

Now divide 3rd by 6 you get a new matrix of the form

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

Adding 2nd row and 3rd row you get a new matrix of the form

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

Add 1st row to 3-times 3rd rows you get 1

$$\left(\begin{array}{ccc:c} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Add 1st row to -2 times 2nd row you get

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

The above matrix is a reduced row echelon form. Finally, rewrite the newest equation in the form of the original one you get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$$

$\Rightarrow$ x = 2, y = -4 and z = -3.

$\therefore$ The solution set is given as (2, -4, -3).
The above process of solving the system of linear equation is also known as Guass Jordan reduction method..

Example: Solve by echelon form the following system of equation
x - 4y-2z = 21
2x + y +2z =3
3x + 2y -z = 2

Solution: Transcribe into matrix form you get

$$\begin{pmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ 2 \end{pmatrix}$$

Next write the above as an augmented matrix you get

$$\left(\begin{array}{ccc:c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & 2 \end{array}\right)$$

Subtract 2 x 1 S' row from 2nd row after which you subtract from 3rd row and you get a new matrix of this form.

$$\begin{pmatrix} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{pmatrix}$$

Next subtract $\frac{14}{9}$ times the 2nd row from the 3rd row. i.e.

$$\frac{-14}{9}\ (0\ 9\ \ 6\ \text{-}39) + (0\ \ 14\ \ 5\ \ \text{-}65)$$

$$= (0\ \ \frac{-14}{3}\ \text{-}28,\ \frac{14 \text{ x } 39}{9} + (0\ \ 14\ \ 5\ \ \text{-}65))$$

$$= (\ 0\ 0\ \ \ -\frac{13}{4}, -\frac{13}{4})$$

The new matrix is given as

$$\left(\begin{array}{ccc:c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -13/3 & -13/3 \end{array}\right)$$

Multiply -3/13 by 3rd row you get new matrix to be

$$\left(\begin{array}{ccc:c} 1 & 4 & 2 & 21 \\ 0 & 1 & 6/9 & -39/9 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Add -6/9 times 3rd row to 2nd row you get new matrix as

$$\left(\begin{array}{ccc:c} 1 & -4 & -2 & 21 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Add - 4times 2nd row to 1st row you get

$$\begin{pmatrix} 1 & -4 & -2 & 21 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Add 2 times 3rd row to 1st row you get a new matrix *given* as

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Write the *above* in the original form you get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ 2 \end{pmatrix}$$

Therefore x = 3, y = -5 and z = 1
Solution set is (3, -5, 1).

SELF ASSESSMENT EXERCISES 1

Transcribe the following system of linear equation into augmented matrix form

1) $3x_1+2x_2 + 3x_3 +x_4 = 5$
$4x_2 + 8x_3 + 2x_4 = -1$
$2x_1 +3x_2 + 3x_4 = 2$
$2x_3 - x_4 = -2$

2) $4x_1 + x_3 = -6$
$2x_1+3x_2+3x_3=1$
$4x_2-x_3=2$

Ans.

1. $\begin{pmatrix} 3 & 2 & 3 & 1 & 5 \\ 0 & 4 & 8 & 2 & -1 \\ 0 & 0 & 2 & -1 & -2 \end{pmatrix}$ 2. $\begin{pmatrix} 4 & 0 & 1 & -6 \\ 2 & 3 & 3 & 1 \\ 0 & 4 & -1 & 2 \end{pmatrix}$

3) Determine which of the following matrices are in row echelon form

(i) $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 & 4 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{pmatrix}$

Ans.

(i) Yes (ii) No (iii) No (iv) No

Solve the following by row echelon form

4) $x + 2y + z = 4$
$3x - 4y - 2z = 2$
$5x + 3y + 5z = -1$

Ans. (2, 3, -4)

5) $x - 2y = 5$
$3x + y = 1$

Ans. (1, -2)

6) $2x + y + 3z = 1$
$4x + 3y + 5z = 1$
$6x + 5y + 5z = -3$

Ans. (-3, 1, 2)

7) $5x - 11y = 3$
$4x - 9y = 2$

Ans. (5, 2)

8) $2x - y + 3z = 1$
$4x + 3y + 5z = 1$
$6x + 5y + 5z = 2$

Ans. (3/4, -1/4, -1/4)

9) $3x - y + 3z = l$
$5x + 4y + z = 2$
$-4x + y + 32 = 2$

Ans. (-2/3, -2!/3, 3'/3)

10) $2x + y + z - w = 2$
$x - 2y + 2z - w = 1$
$2x + 2y - 2z + 3w = -2$
$2x + y + z - w = 7$

Ans. (5, -21/2, -81/2, -8)

4.0 CONCLUSION

You have studied that an n x n linear system can be reduced to a triangular form and by appropriate elementary row operation a unique solution can be obtained by performing back substitution on the triangular system. However such a triangular system can further be reduced to a row echelon form having an augmented matrix of the form (ij B) where I_n is the n x n identity matrix and B is the solution matrix of n x 1 order. The Guass-Jordian elimination method described in this unit is just one of the methods of using properties of a matrix to solve a n x n linear system. In this next unit you will be introduced to two other methods.

5.0 SUMMARY

In this unit you have studied how to;

- Determine whether a matrix is in a reduced row echelon form.
- Transcribe an n x n system of linear equations into augmented matrix form,
- To reduce an n x n linear system to a triangular system by application of elementary row operation.
- To reduce an n x n system of linear equations to row echelon form
- To solve an n x n system of linear equations by the echelon form or Guass- Jordan elimination method.

6.0 TUTOR-MARKED ASSIGNMENT

1. Determine which of the following is in row echelon form.

(i) $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 3 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Solve the following by row echelon form

2. x + 2y = 1
   3x + 1 = 2

3. x + y = 1
   -2x + 3y = 3

4. 6x – 2y = 5
   4x – 3y = 1

5. 2x + y – 2 = 1
   3x + 3y – 5z = 1
   6x + 5y – 2 = -3

7.0 REFERENCES/FURTHER READING

The following are recommended textbooks. You could borrow or purchase them.

Lang, S. (2004). Linear Algebra: *undergraduate Text in Mathematics*. Springer Publishers.

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UNIT 3 DETERMINANT AND SYSTEMS OF EQUATIONS

CONTENTS

1.0 INTRODUCTION

This unit continues where unit 7 stopped in respect of using properties of matrix in solving n x n system of linear equation. The intention in this course is to present various methods of solving n x n linear systems using properties of matrices. So that with little effort on your side and with known software packages suggested at the end of this course you will be able to solve any type of n x n system of linear equations. In this unit, you shall be introduced to two additional methods of solving n x n systems of linear equations. (Refer to introduction of unit 7).

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- solve n x n system of linear equation by matrix inversion
- solve n x n systems of linear equation by adjoint method
- solve n x n system of linear equation by Cramer's rule.

3.0 MAIN CONTENT

3.1 Solving n x n System by Matrix Inversion

In this unit you shall be able to use theory of matrices to solve n x n system of linear equations. However it is of interest to know whether an n x n system has a unique solution. In the previous units you studied that an n x n matrix whose determinant is zero is not invertible. From this you can conclude that an n x n system of linear equations in this forms A x = b has a unique

Solution if the matrix A is non-singular. Consider the following linear system given as

$$A = \begin{matrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{matrix}$$

If the system (A) above has a solution then it is said to be consistent otherwise it is said to be inconsistent. A solution of a system of linear equations where all the X; are zero is called a zero solution or trivial solution, whereas a solution whereby not all the X, are zero is called nonzero or non-trivial solution. Using the matrix the above system (A) can be transcribed into this form AX = b where

$$A = \begin{pmatrix} a_{11} & a_{12}, \ldots . a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ \vdots & \\ a_{n1} & a_{n2} \ldots\ldots a_{nn} \end{pmatrix} \text{ the coefficient matrix}$$

$$X = \begin{pmatrix} z_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where X is the variable matrix and b is the constant term matrix

The above system will have a unique solution if IAI $\neq$ 0. Recall that you have studied equivalent systems in the previous unit. Also recall that a system is said to be homogeneous if A X = 0 that is all the elements of the matrix of constant term are zero, otherwise it is non-homogeneous i.e. AX=b $\neq$ 0.

Generally, a solution of an n x n non-homogeneous system of linear equations AX = b is obtained by adding the general solution of the homogeneous system AX = 0 to a particular solution of a non-homogeneous system AX = b.

The following list of useful properties of the inverse will be given (See unit 6).

(i) The inverse of a non- singular matrix is unique i.e. there is only one matrix B for which AB = BA = I.

(ii) If there are two n x n matrices A and B such that AB = I then A and B are non-singular, A'[1] = B, B''[1] = A, and AB = I.

In view of the above if A is a non-singular matrix i.e.| A| ^0 and AX = b then the solution given as X = A-lb is a unique solution of the system AX =b.

You shall easily see that once we can determine the inverse of A the solution of the AX = b is easy to calculate.

Example: Solve the following system of equation.

1. 2x + 3y -l =3
2x + 4y + 2z = -1
-2x -2y +3z = 1

Solution: Transcribe using matrix solution, you obtain

$$\begin{pmatrix} 2 & 3 & -1 \\ 2 & 4 & 2 \\ -2 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$X = A^{-1} b$

$$A^{-1} = \begin{pmatrix} -8 & 3.5 & -5 \\ 5 & -2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 & 3.5 & -5 & 3 \\ 5 & -2 & 3 & -1 \\ -2 & 1 & -1 & 4 \end{pmatrix} = \begin{pmatrix} -47\frac{1}{2} \\ 29 \\ -11 \end{pmatrix}$$

Example: Solve the following system of equation

3x + y + z = 1
2x + z = 1
2x + 2y +4z = 1

Solution: In matrix notation, you get

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad Ax = b$

where

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -2 & -2 & 1 & 1 \\ -6 & 10 & -1 & 1 \\ 4 & -4 & -2 & 1 \end{pmatrix} = -\frac{1}{8}(-3, 3, -2) = (\frac{3}{8}, -\frac{3}{8}, \frac{1}{4})$$

SELF ASSESSMENT EXERCISE 1

Solve the following systems by direct matrix inversion.

1. 2x + y + 2 = 6
   x + 2y + z = -5
   2x + 2y + 2z = -2

Ans. = $^1/_3$(25, -8, -24)

2. 2x - 4y-2z =2
   3x -3y + 2z =-5
   x- y+ z =-6

Ans. (26, 19, -13)

3. x + z = 3
   3x + 3y+ 4z=-1
   2x + 2y +3z =-2

4. Solve AX = b of question 1 to 3 for the following choices of b respectively.

(i) $b = (-2, -1, 2)^T$
(ii) $b = (1-2, -4)^T$
(iii) $b = (6, 5, -1)^T$

3.2 Solving (n x n) Systems By Determinant Method

In this section you will use the method of computing the inverse of a nonsingular matrix A by adjoint method to solve the system AX = b.

You will also learn a method for solving AX = b using the Cramer's rule. Both methods use the determinant of the matrix A. You will need the concepts you studied in section 3.3 of unit 6. It is advisable that you read section 3.3 of unit 6 before you study this section. The concept you will use here is the same with the one used in the previous section. That is in order to solve AX = b you look for a nonsingular matrix A^{-1} such that $AA^{-1} = I$ - and $X = A^{-1} b$. The various process or method of computing A^{-1} is what you have studied in unit 6. For example you can use the adjoint method to find A^{-1}.

i.e. $A^{'1} = \dfrac{\text{adj}(A)}{|A|} = \dfrac{A^*}{|A|}$

Therefore $X = A^{-1}b = \left(\dfrac{A^*}{|A|}\right)b$

Example: Solve the following system of linear equations.

2x + 3y + z = 9
x + 2y + 3z =6
3x + y + 2z =8

Solution: In matrix notation, you have

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix} \equiv Ax = b.$$

Thus $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

You are now set to find A^{-1} by the adjoint method.

$$|A| = 2\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} - 3\begin{vmatrix}1 & 3\\3 & 2\end{vmatrix} + 1\begin{vmatrix}1 & 2\\3 & 1\end{vmatrix}$$

$= 2(4-3) - 3(2-9) + 1(1-6)$

$= 18$

next you find the cofactor of all the entries in matrix A. i.e.

$A_{11} = \begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} = (-1)^{1+1}(4-3) = 1$

$A_{12} = \begin{vmatrix}1 & 3\\3 & 2\end{vmatrix} = (1)^{1+2}(2-9) = 7$

$A_{13} = \begin{vmatrix}1 & 2\\3 & 1\end{vmatrix} = (-1)^{1+3}(1-6) = -5$

$A_{21} = \begin{vmatrix}3 & 1\\1 & 2\end{vmatrix} = (-1)^{2+1}(6-1) = -5$

$A_{22} = \begin{vmatrix}2 & 1\\3 & 2\end{vmatrix} = (-1)^{2+2}(4-3) = 1$

$A_{23} = \begin{vmatrix}2 & 1\\3 & 2\end{vmatrix} = (-1)^{2+3}(2-9) = 7$

$A_{31} = \begin{vmatrix}3 & 1\\2 & 3\end{vmatrix} = (-1)^{3+1}(9-2) = 9$

$A_{32} = \begin{vmatrix}2 & 1\\1 & 3\end{vmatrix} = (-1)^{3+2}(6-1) = -5$

$A_{33} = \begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} = (-1)^{3+3}(4-3) = 1$

Let C be the matrix of factor of A given as

$$C = \begin{pmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 9 & -5 & 1 \end{pmatrix}$$

$$A^{-1} = 1 \ \frac{A*}{|A|} = \frac{1}{18} \begin{pmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{pmatrix}$$

Therefore

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ \frac{1}{18} \begin{pmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 9-30+56 \\ 63+6-40 \\ -45+42+8 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 35 \\ 29 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35/10 \\ 9/18 \\ 5/18 \end{pmatrix}$$

SELF ASSESSMENT EXERCISE 2

Solve the following equation

1. $x + 2y + z = 3$
   $3x - y - 3z = -1$
   $2x + 3y + z = 4$

2. $y - z + w = 3$
   $2x - y + z + w = 6$
   $2x + 4y + z - 2w = -2$

Ans: (1) (3,-2, 4) (2) (2.5,-1,-1, 1)

3.3.2 Solving n x n System of Linear Equation by Cramer's Rule

This method is a method derived from the above adjoint method of solving linear system.

Consider an n x n system given as

$$\begin{pmatrix} a_{11}X_1 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}X_1 + \ldots + a_{2n}x_n - b_2 \\ \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = b_n \end{pmatrix}$$

Cramer's Rule: Let A be the coefficient matrix A of the above system (*), which is non-singular and let, b be n x 1 column matrix representing the constant term. Let Ai be the matrix obtained by replacing the ith column of A by b. If x is the unique solution to system (*) (i.e. AX = b) then

$$Xi = \frac{\det (Ai)}{\det (A)} = \frac{|Ai|}{|A|}, i = 1, 2, \ldots n$$

The proof of the above rule will clearly demonstrate the relationship between it and the adjoint method. Proof of the Cramer's Rule:

Given that AX = band I Al $\neq 0$ then

$$X = A^{-1}b$$

But $A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{|A|} \; A * b$

Thus

$$X = \frac{A^* b}{|A|}$$

Example: use Cramer's rule to solve

$4x - 2y = 6$
$-3x + 4y = -2$

Solution: In matrix notation you have

$$\begin{pmatrix} 4 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$\Rightarrow \quad A \quad X \quad = \; b$

$$|A| = \begin{vmatrix} 4 & -2 \\ -3 & 4 \end{vmatrix} = 16 - 6 = 10$$

$$|A_1| = \begin{vmatrix} 6 & -2 \\ -2 & 4 \end{vmatrix} = 24 - 4 = 20$$

$$|A_2| = \begin{vmatrix} 4 & 6 \\ -3 & -2 \end{vmatrix} = -8 + 18 = 10$$

$$\therefore\ x = \frac{20}{10} \qquad y = \frac{10}{10}$$

$$x = 2, \quad y = 1$$

Example: Solve the following systems by Cramer's rule.

1. $x + 2y - 3z = 1$
   $3x - 2y + 2z = 0$
   $2x + 3y - z = 1$

2. $x + y = 1$
   $x + y - z = 3$
   $2x + y + z + 3w = 2$
   $x + 2y + 2y + 2w = 1$

3. $2x + 3y - z = 1$
   $x + 2y - z = 3$
   $-x - y + 3z = -2$

Solution: (1) In matrix notation you have

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 3 & -2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -29$$

$$|A_1| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 2 \\ 1 & 3 & -1 \end{vmatrix} = -6$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 0 & 2 \\ 3 & 1 & -1 \end{vmatrix} = -4$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -2 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 5$$

$$x = \frac{|A_1|}{|A|} = \frac{-6}{-29} = \frac{6}{29}$$

$$y = \frac{|A_2|}{|A|} = \frac{-4}{-29} = \frac{4}{29}$$

$$z = \frac{|A_3|}{|A|} = \frac{5}{-29} = \frac{-5}{29}$$

2. In matrix notation you have

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

hence

$$|A| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 2 \end{vmatrix} = -5$$

$$|A_1| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 2 \end{vmatrix} = 3$$

$$|A_2| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \end{vmatrix} = -8$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \end{vmatrix} = -10$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{vmatrix} = -6$$

$$x = \frac{|A_1|}{|A|} = \frac{3}{-5} \qquad y = \frac{|A_2|}{|A|} = \frac{8}{-5} \qquad z = \frac{|A_3|}{|A|} = \frac{10}{-5}$$

$$w = \frac{|A_4|}{|A|} = \frac{-6}{-5} \Rightarrow (x, y, z, w) \ \frac{-1}{-5} \ (3, -8, 10, -6)$$

SELF ASSESSMENT EXERCISE 3

Use b = $(1, 12, 1, 4)^T$ to solve the above.

Ans. (x, y, z, w) = (6, 5, -1, -5).

3. In matrix notation you have

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3$$

$$|A_1| = \begin{vmatrix} 0 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{vmatrix} = -1 \qquad |A_2| \begin{vmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 3 \end{vmatrix} = 1$$

$$|A_3| = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 1$$

$$x = \frac{|A_1|}{|A|} = \frac{-1}{3} \qquad y = \frac{|A_2|}{|A|} = \frac{1}{3} \qquad z = \frac{|A_3|}{|A|} = \frac{1}{3}$$

$$(x, y, z) = \frac{1}{3}(-1, 1, 1).$$

SELF ASSESSMENT EXERCISE 4

Solve Ax = b in examples (1) and (3) above for the following values of b.

(i) $b = (1\ 1\ 0)^T$ (ii) $b = (2, 4, 8)^T$
(iii) $b = (-3, 3, -6)^T$ (iv) $b = (5, 3, -6)^T$

Ans 1 (i) 1/29 (11,-12,-14) (ii) 1/-29 (-52,-54,-34)
(iii) -1/29 (3,60,-12) (iv) =(l,-4,-4)

3 (i) (-1,1,0) (ii) (-10,8,2)
(iii) (-11, 5,-4) (iv) 1/3(7,-1-4)

4.0 CONCLUSION

In this unit, you have studied how to find the solution to an (n x n) system linear equations. You studied how to find the solution of system Ax =b when |A |and A^{-1} is known. The solution arrived at is unique because of the simple fact that A^{-l} is unique. Once a system Ax = b is gi^1 and AA^{-1} exists. Any method can be used to compute elements of A^{-1} and solution can be computed by direct matrix multiplication i.e. $x = A^{-1}b$. Study the Cramer's rule for solving any n x n system. Although Cramer's rule gives a very convenient method for arriving at the solution of an (n x n) system of linear equation in terms of determinants. To arrive at the final solution we must be ready to compute n + 1 determinants of order n. As you can see evaluating these numbers of determinants, can sometimes be more involving than using the Guass - Jordan reduction or elimination method studied in the previous unit.

5.0 SUMMARY

You have studied in this unit how to;

- Solve an n x n system of linear equation of the type Ax = b using inverse of A i. e x = $A^{-1}b$.
- Solve an n x n system of linear equations Ax = b using adjoint method of computing the inverse of an n x n matrix,
- To use Cramer's rule to find the solution of an n x n system of 1 equation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given that $3x_1 + 2x_2 = -17$
   $4x_1 + 3x_2 = -15$

   where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$

   $b = \begin{pmatrix} -17 \\ -15 \end{pmatrix}$ and by direct inverse method fiond $A^{-1}b$

2. Given the system

$$X_2 + X_3 + X_4 = 0$$
$$3x_1 + x_3 - 4x_4 = 7$$
$$X_1 + X_2 + X_3 + 2X_4 = 6$$
$$2x_1 + 3x_2 + x_3 + 3x_4 = 6$$

   where the inverse A^{-1} of the coefficient matrix is given by

$$A^{-1} = \begin{pmatrix} 11/16 & -1/8 & -1/2 & -1/16 \\ 1/2 & 0 & 1 & -1/2 \\ -13/16` & -1/8 & -1/2 & -7/16 \\ 5/16 & 1/8 & -1/2 & 1/16 \end{pmatrix}$$

Solve for x_1, x_2, x_3

3. Given $2x_1 – 3x_2 + x_3 = -7$
$x_1 + 4x_2 – 2x_3 = 15$
$3x_1 – x_2 + 5x_3 = -14$

let the coefficient matrix in terms of x 1, x2, and x3 be given as

$$B = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ 3 & -1 & 5 \end{pmatrix}$$ where the inverse

$$B^{-1} = \begin{pmatrix} 0.32 & 0.25 & 0.036 \\ -.196 & 0.125 & 0.0893 \\ -.232 & -0.125 & 0.196 \end{pmatrix}$$

4. Let the system of equations:

$x_1 – 3x_2 + 4x_3 = 7$
$3x_1 – 4x_2 + x_3 = 7$
$2x_1 + 7x_2 – 4x_3 = 2$ have coefficient matrix.

$$\begin{pmatrix} 1 & -3 & 4 \\ -3 & -4 & 1 \\ 2 & 7 & -4 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 0.0108 & .193 & .157 \\ .168 & -.145 & -.133 \\ .349 & -0.1566 & .0602 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}$$

5. If Ax = b where

$$A = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 2 & -3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 4 \\ 10 \\ 9 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 7 & -5 & 2 \\ -5 & 4 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$ Find $A^{-1}b$

6. Given that
$3x_1 + 2x_2 + x_3 = 1$
$x_1 - x^2 + 3x_3 = 5$
$2x_1 + 5x_2 - 2x_3 = 0$

$$A^{-1} = \begin{pmatrix} 0.8125 & -.563 & -.4375 \\ -.5 & .5 & .5 \\ -.4375 & .075 & .3125 \end{pmatrix}$$

Find $A^{-1}b$ where $b = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$

7. By adjoint method solve for x_1, x_2, x_3 in the following of linear equations.

$X_1 - 3x_2 - 2x_3 = 8$
$2x_1 + 2x_2 + x_3 = 4$
$3x_1 - 4x_2 + 2x_3 = -3$

8. Apply Cramer's rule to solve

$x_1 - 3x_2 + 2x_3 = 8$
$2x_1 + x2 + x3 = 9$
$3x1 + 2x2 + 3x_3 = 5$

9. Solve by Cramer's rule

$3x_1 + 2x_2 = -17$
$4x_1 + 3x_2 = -15$

10. Let the coefficient matrix in terms of x_1, x_2, and x_3 be given as

$$\begin{pmatrix} 1 & -3 & 4 \\ 3 & 4 & 1 \\ 2 & 7 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solve by Cramer's rule.

11. solve by Cramer's rule the system

$3x_1 + 7x_2 = 11$
$2x_1 + 5x_2 + x_3 = 6$
$2x_2 + 4x_3 = 7$

12. $2x_1 - x_2 + 5x_3 = 6$
$2x_1 - 3x_3 = 4$
$6x_1 - 2x_2 + x_3 = 8$

7.0 REFERENCES/FURTHER READING

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UNIT 4 TRANSFORMATION OF THE PLANE

CONTENTS

1.0 INTRODUCTION

A transformation is a change in the position, size, or shape of a geometric figure (such as a square or triangle). The main transformations are translation, reflection, rotation, and enlargement. Other types of transformations are stretching and shearing. In this unit, you will study 3 types of transformation. You will use the properties of matrices studied in the previous units to perform transformation of a plane into itself.

A transformation can be seen as a mapping or function. In this unit, you shall use the word mapping and transformation interchangeably. You shall use the letter T to denote transformation or mapping. For example, a transformation from 2- dimensional plane R_2 to itself will be denoted by $T: R_2 \rightarrow R_2$. The transformations that will be considered here will all be the linear type. Linear transformation or mapping from one plane to another plane plays an important role in mathematics. This unit will provide you with an introduction to the theory and properties of such mappings. In section 3.2, it is shown that each linear transformation T mapping a 2 dimensional plane R^2 into itself can be represented by a 2 x 2 matrix A. Thus you can conveniently work with the matrix A in place of the mapping T.

Any point in a 2 dimensional plane R^2 is determined by its two coordinates i. e. by an ordered pair of two real numbers (x, y). You denote the position vector of a general point (x, y) in the plane by a 2 x 1 column matrix or vector $V = \begin{pmatrix} x \\ y \end{pmatrix}$

Definition: A transformation (or mapping) T on the plane associates to every point in the plane a unique image point in the plane.

The image of any point under T is represented by T (v).

In R^2 we consider such plane consisting of the rectangular Cartesian coordinates with $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(i and j are known as unit vectors in R^2) with the point $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as the origin

Then a typical point $a = \begin{pmatrix} x \\ y \end{pmatrix}$ in the x - y plane i.e. R^2 is given as

$$a = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = xi$$

$$a = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1 + y$$

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a linear transformation
- define 3 types of transformations namely
  - translation
  - reflection and
  - rotation.
- determine invariant points or lines of a transformation
- construct the matrix of transformation
- combine transformations.

3.0 MAIN CONTENT

3.1 Translation

Definition: A translation is a transformation of the plane to itself in which all points in the plane move by a fixed vector.

Example: if $\begin{pmatrix} x \\ y \end{pmatrix}$ is any point in the plane R^2 and $\begin{pmatrix} a \\ b \end{pmatrix}$ is a fixed vector, then the translation T can be expressed as

$$T: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

or $$T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

Example: The graph of the function

y = f(x) + 2 or y - 2 = f(x)

Can be obtained by a translation of the graph of y = f (x) by 2 units parallel the y - axis.

Example: The graph of the function
y = f (x - 1) can be obtained by a translation of the graph y = f (x) + 1 unit parallel to the x - axis.

Example: In general the graph of the function (y - a) = (x - b) can be obtained by translations of the graph of y = f (x) by a units parallel to the y - axis an by b units parallel to the x - axis.

Example: Describe the graph of (y + 3) = f (x - 2) in terms of the graph
y = f(x).

Solution: The graph (y + 3) = f (x - 2) is the graph of y = f (x) shifted 3 unit; upwards and 2 units to the right.

Example: Given y = f (x) describe the translation needed to obtain the graph of (y- 4) = f (x+3).

Solution: The graph y - 4 = f (x + 3) is the graph of y = f (x) shifted 4 units upwards and 3 units to the left.

Example: Given the graph $x^2 + y^2 = r^2$ describe the translation needed to obtain the graph of $x^2 + y^2 - 36x - 14y + 117 = 0$

Solution $x^2 + y^2 - 36x - 14y + 32 = (x-18)^2 + (y-7)^2 - 18^2 - 7^2 + 3^2 = 0$

$\Rightarrow (x-18)^2 + (y-7)^2 = 18^2 + 7^2 - 117.$

$(x-18)^2 + (y-7)^2 = 256$

$\Rightarrow (x - 78)^2 + (y - 7)^2 = 16^2$

The translation needed is the graph of $x^2 + y^2 = 16^2$ shifted 18 units to the right and 7 units upwards.

Exercises

Given the graph f (x), describe the translation needed to obtain the following graphs;

(i) f(x-2) = y-4

(ii) f(-x + 2) = f(-lx - 2)

(iii) f(x + 5) = y + 8

(iv) f(x-6) = y - 7.

Example: Given the graph $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

Describe the translations needed to obtain the graph
$2x^2 + 4x + 8y^2 - 16y = 30$
$2x^2 + 4x + 8y^2 - 16y = 30$
$2(x^2 + 2x) + 8(y^2 - 2y) = 30$
$2(x + I)^2 + 8(y - I)^2 = 32$

$$\frac{(x + l)^2}{16} + \frac{(y - l)^2}{4} = 1$$

$a = 4 \qquad b = 2$

This is the graph of $\frac{x^2}{16} + \frac{y^2}{4} =$ 1 shifted 1 unit to the left and 1 unit upward.

SELF ASSESSMENT EXERCISE 1

Given the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2}$

Describe the translation needed to obtain the graph of $4x^2 - y^2 + 4y + 8x = 16$

3.2 Linear Transformation

Definition: A transformation T from a plane to itself is said to be linear if the following properties are satisfied.

(i) For any point V and a in the plane
T(u + v) = T(v)+T(u)

(ii) For any number a and a point v in the plane
T (α v) = α T(v).

If T is a linear transformation from the plane to itself it follows from (i) & (ii) above that if a and v are point in the plane and - and 0 are real numbers

$T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

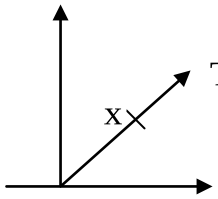
$= \alpha T(u) + \beta T(v)$

Example: Let T be the operator defined by

T(x) = 2x for each x ER^2

Since $T(\alpha x) = 2(\alpha x) = \alpha(2x) = \alpha T(x)$

and T(x + y) = 2 (x + y) = (2x) + (2y) = Tx + Ty. It follows that T is a linear transformation. You can think of T defined above as stretching by a factor of 2. (see fig 9.1)



Example: Consider the transformation T defined by

$T(x) = X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for each $x \in R^2$.

Thus if x = (XL X2) then

$T(x) = (x, 0)^T = (y_1, y_2)^T$

Then

$\alpha_{x1} + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$

and it follows that

$T(\alpha x + \beta y) = (\alpha x_2 + \beta\ y,) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \alpha \left(x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + \beta y_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \alpha T(x) + \beta T(y)$ (*)

This implies that T is a linear transformation on the plane.

The matrix of a linear transformation. It can now be shown that a 2 x 2 matrix gives rise to a linear transformation.

Suppose

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a general 2 x 2 matrix

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is column vector representing a point in the plane, then the transformation T can be defined as

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Thus T takes a point in the plane to another point in the plane. You can easily show that T is linear. Let u and v be any two points in the plane. Let a and P be two real numbers then by he properties of matrix addition and multiplication, you have that

$$\begin{aligned} T(\alpha u + \beta v) &= A(\alpha u + \beta v) \\ &= A(\alpha u) + A(\beta v) \\ &= \alpha A(u) + \beta A(v) \\ &= \alpha T(u) + \beta T(v) \end{aligned}$$

It has been shown that a 2 x 2 matrix is indeed a linear transformation. It should be noted that to define a linear transformation T from the plane to itself, it is sufficient to define it for two-unit vectors i. e.

$$i = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and } j = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is because any point in the plane can be represented as a positive vector of that given point. In other words if v is the position of the point v in the plane then

$$V = x_i + y_j = xe_1 + ye_2 = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The above satisfied the properties of a linear transformation. See equation **Example:** Let T be a transformation on the plane by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ verify if T is linear

Solution:

Since T $\begin{pmatrix} x_1 + & x_2 \\ y_1 & x_2 \end{pmatrix} = T \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$= \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & + & x_2 \\ 0 & & 0 \end{pmatrix}$$

$$= T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Furthermore

$$T \left(\alpha \begin{pmatrix} x \\ y \end{pmatrix} \right) = T \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha x \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} x \\ 0 \end{pmatrix} = \alpha T \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus properties (i) are satisfied hence T is a linear transformation.

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Example: Let T be a transformation defined as T

$$\text{Where } T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the origin is not mapped to the origin. The transformation is not linear. Thus every linear transformation takes the point 0, the origin into the origin.

Example: Let T be defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 4y \\ x + 5y \end{pmatrix}$$

Write out the transformation matrix.

Solution: $T \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} 3x \\ x \end{pmatrix} + T \begin{pmatrix} -4y \\ 5y \end{pmatrix} = xT \begin{pmatrix} 3 \\ 1 \end{pmatrix} + yT \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$\therefore$ the matrix of transformation is given as $\begin{pmatrix} 3 & -4 \\ 1 & 5 \end{pmatrix}$

Example: Given that the matrix of a transformation T is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(i) Determine the image under T of the unit square

Solution: The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

and the image of $\begin{pmatrix}0\\1\end{pmatrix}$ is $\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

the image of $\begin{pmatrix}1\\1\end{pmatrix}$is $\begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$ (see fig. 9.2)

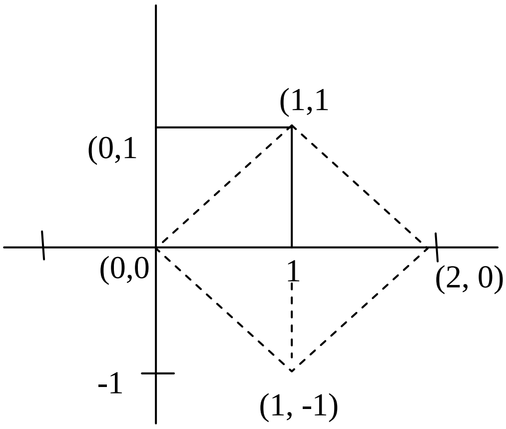


Fig. 9.2

Example: Find the point which is transformed onto the point $\begin{pmatrix}3\\2\end{pmatrix}$ under

$T = \begin{pmatrix}1 & -1\\2 & 2\end{pmatrix}$

Solution: The point $\begin{pmatrix}x\\y\end{pmatrix}$ transformed into $\begin{pmatrix}3\\1\end{pmatrix}$ under T is given by

$$\begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}1 & -1\\2 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$\Rightarrow$ $x - y = 3$

$2x + 2y = 2$

$\Rightarrow$ $y = -l$, $x = 2$

$\Rightarrow$ $\begin{pmatrix}2\\-1\end{pmatrix}$ is the point

3.3 Some Properties of Transformation

3.3.1 Invariant Points and Lines

Suppose T is a linear transformation 1. A point in the plane, which gets mapped to itself under T, is called an invariant point or a fixed point.

That if x is a point in the plane and T(x) = x then x is said to be an invariant point.

Example: The origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ j is an invariant point under any linear

transformation i.e. $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example: The identity matrix 12 is a transformation where all the points in the plane are invariant points under it. Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be any vector in the plane

Then

$$I\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $I \begin{pmatrix} x \\ y \end{pmatrix} = I_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

also $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Thus the identity matrix is the matrix of the identity transformation and is a linear transformation.

Usually, to determine whether a given transformation has an invariant point all you need is to solve the following simultaneous equations.
ax + by + h = x and cx + dy + c = y which is obtained from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

If the above equation does not have a solution you conclude that the transformation under investigation has no variant point. If it has a solution that solution will be a unique solution. If it has infinitely much solution then all points in the plane will be invariant points.

Remark:

(i) The origin is the only invariant point under a zero transformation,

(ii) Under a translation, there is no invariant point except when $\begin{pmatrix} x \\ y \end{pmatrix}$ = 0 and you have the identity transformation.

A line 1 is an invariant line or a fixed line if PEI implies that T(P)E1 Where P is a set of points in the plane.

Example: Determine the invariant point under the transformation T given as

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Solution: You get $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$

i.e. $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + -1 \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\Rightarrow$ $x + 2y - x = -2$
$3x + 2y - y = -4$

$\Rightarrow$ $y = -1$ and $x = -1$

therefore the point is = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3.3.2 Combined Transformation

The following are transformation of the plane to itself.

(i) A translation which moves every point by a constant column vector $\begin{pmatrix} h \\ k \end{pmatrix}$

(ii) A linear transformation with a 2 x 2 matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ I

(iii) A combination of any transformation from (i) or (ii).

Given that T_I and T_2 are both linear transformations of the plane with matrices A_1 and A_2 respectively and A is a constant, then

(1) Their sum $T_1 + T_2$ is a linear transformation with matrix $AI + A_2$

(2) Their difference $T_1 - T_2$ is a linear transformation with matrix $A_1 - A_2$

(3) λT_I, is a linear transformation with matrix λAI.

Example: Given that

$$T_1: \quad \begin{matrix} x \\ y \end{matrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T_2: \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Find (i) T_1+T_2 (ii) T_1-T_2

Solution: T_1+T_2: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$T_1+T_2: \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and T_1-T_2: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

3.3.3 Further Transformations

You will now study other types of transformation.

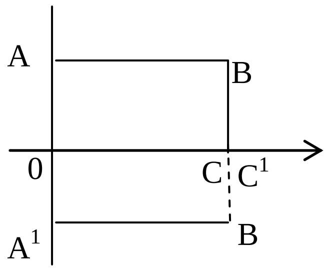
(1.) Reflection:

A reflection in a line through the origin is a transformation of the plane to itself such that the image of a point is at the same distance from the origin (the mirror line) as the given point and the line joining a point and its image perpendicular to the mirror line.

Remark: The mirror line is a line of invariant points for all reflection i. e the mirror line is an invariant line.

Example: Consider a reflection in the x-axis i. e the mirror line is the x - axis where y = 0.

See fig 9.3



here A^1, B^1 and C^1 are the images Of A, B and C respectively

Fig. 9.3

In fig 9.3 OABC is a square with O as

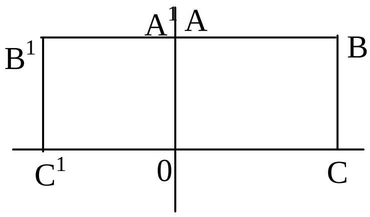
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad B = 1 \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{matrix} A \rightarrow & A^1 & B & B^1 & C & \rightarrow C^1 \end{matrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The matrix of transformation is given as $T_R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example: Consider a reflection in the y - axis (see fig 9.4)



Here A^1, B^1 and C^1 are the images of A B C. Cl respectively where OABC is the unit square.

Fig. 9.4

O → O i.e. $\begin{pmatrix}0\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\0\end{pmatrix}$

A → A i.e. $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

B → B i.e. $\begin{pmatrix}1\\1\end{pmatrix} \rightarrow \begin{pmatrix}-1\\1\end{pmatrix}$

C → C i.e. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-1\\0\end{pmatrix}$

Therefore matrix of transformation is given as $T_R = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

The invariant line is x = 0.

SELF ASSESSMENT EXERCISE 2

Find the matrix of transformation for a reflection in the line

(i) y = x (ii) y = -x

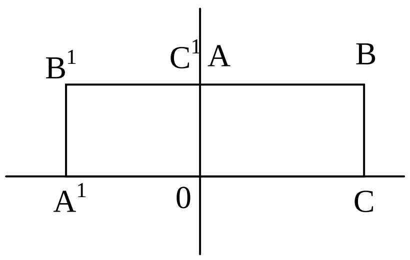
Ans.(i) $T_R = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ (ii) $T_R \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

2. Rotations

A rotation about the origin (i.e. center of rotation) is a transformation of the plane to itself such that the angle from the position vector < point to the position vector of its image (in the anticlockwise direction) is a fixed angle for all points in the plane.

In this transformation the point of invariant is the origin. Therefore you can say that rotation about the origin is linear transformation.

Example: Consider the rotation about the origin through 90°. See fig 9.5



Here A^1, B^1 and C^1 are the images of A, B and C respectively where OABC is a unit square

Fig. 9.5

i.e. $A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow A^1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow B^1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow C^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Therefore the transformation matrix is given as $T_{90} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

SELF ASSESSMENT EXERCISE 3

Find the matrix of transformation for a rotation about the origin through 180°.

Ans.: $T_{180} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

4.0 CONCLUSION

In this unit you have studied transformation of a 2 dimensional plane to itself. You have studied the properties of a near transformation. You have studied 3 types of transformation namely translated reflection and rotation. In each type of transformation studied you have been able to determine the matrix of transformation as well as the invariant point or line.

5.0 SUMMARY

You have studied in this unit how to

- Define a linear transformation
- To determine whether a given transformation is linear or not.
- Find the matrix of transformation
- Determine invariant points or lines of transformation

6.0 TUTOR-MARKED ASSIGNMENT

1. A transformation T mapping $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Determine the matrix of the transformation.

2. Determine the set of invariant points under transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix}$$

3. Prove that points under identity transformation are invariant.

4. The matrix of the transformation T of a plane is $\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$

Determine the images under T of the vertices Q (0, 0), A (4, 0), B (4, 2) And C (0, 2) of a rectangle QABC.

What percentage is the area increased or decreased by the new transformation?

5. The matrix of a transformation T of the plane is $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

Determine the image under T of the unit square.

6. The matrix of a transformation of the plane is $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

Determine the point which is mapped onto the point $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ under T.

7. The matrix of the transformation T of the plane is $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

Find the images under T of

(i) The unit square QABC

(ii) The vertices O (0,0), A (0,6), B (4,0) of a triangle OAB.

8. Given that $T_1: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

and $T_2: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Determine the matrix given by

(i) $3T, + 2T_2$

(ii) T,T_2

9. If T is a linear mapping that map $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\0\end{pmatrix}$ to $\begin{pmatrix}0\\0\end{pmatrix}$

Determine the matrix of the transformation.

10. Determine the matrix of the transformations $T_1 = \begin{pmatrix}2x & - & y\\ y & + & 2y\end{pmatrix}$

and T_2: $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x & + & 2y\\ & & x\end{pmatrix}$

Determine the invariant points under each transformation.

11. Prove that under a linear transformation, the origin is always an invariant point.

12. Prove that the matrix transformation $\begin{pmatrix}4 & -1\\ -3 & 2\end{pmatrix}$

maps all points on the line y = 3x onto themselves.

13. Given that the matrix transformation $\begin{pmatrix}1/2 & 1/253\\ 1/253 & -1/2\end{pmatrix}$

maps all points on the line y = x V 3 onto themselves

14. Construct a matrix that transforms (1,0) onto (3, 2) and (0,1) onto (5,3)

7.0 REFERENCES/FURTHER READING

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UNIT 5 INTRODUCTION TO VECTOR SPACES

CONTENTS

1.0 INTRODUCTION

In this unit you will be introduced to the concept of "Vector spaces" or linear spaces. You are familiar with the concept of a vector in 2 dimensional planes. For example a point x can be regarded as a vector in a 1 dimensional plane, i.e. .x $\in R^{I} \Rightarrow x = \{a: a \in R\}$. This means that the vector x has only one real component. In unit 4, you studied a vector in 2 dimensional planes where it was said that if v is a point in a plane the position vector of that point can be expressed in terms of the unit vectors (ad) respectively. That is v = x $\hat{i}$ + y$\hat{j}$ (see fig 1.1)

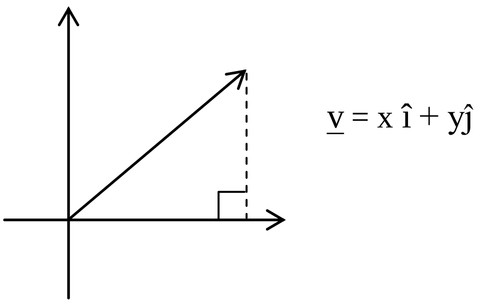

Fig 10.1

The above can be extended to a vector in n dimensional plane. In all your study of mathematics you come across many examples of mathematical objects that can be added to each other and multiplied by real numbers. To start with real numbers can be added together and as well multiplied. However there are other objects that can be added together among such are matrices, real - valued functions, complex numbers, infinite series vectors valued functions and so on. In this unit, you will discuss a general mathematical concept called a vector pace or linear space, which include all these examples and other special ones.

Basically, a vector space involves a set of elements of any kind, which the two basic arithmetic operations of addition and multiplication by numbers can be performed. In this unit the definition of a vector space is given and some basic properties of vector spaces is introduced.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a vector space
- list all the ten axioms of a vector space
- determine whether a given space is a vector space
- state properties of a vector space such as (i) subspace (ii) linear dependence and independence etc.

3.0 MAIN CONTENT

3.1 Vector Space

3.1.1 The Definition of a Vector Space

Let V denote a non-empty set of objects called elements. The set V is called a vector space if it satisfies the following ten axioms, which is listed in 3 groups.

I. Closure Axioms

C_1: Closure under addition.
For every pair of elements x and y e V there corresponds a unique element in V called the sum of x and y denoted by x + y.
i.e. $x + y \in V$.

C_2: Closure under multiplication by real numbers.
For every x e V and every number a there corresponds an element in V called the product of a and x, denoted by a x 1. e a x e V.

II. Axioms for Addition

A_3: Commutative Law: For all x and y in V you have that
$x + y = y + x$.

A_4: Associative Law: for all x, y and z in V you have that
$(x + y) +z = x + (y + z)$.
A_5: Existence of zero element: There is an element in V, denoted by 0, such that x + 0 = x for all xeV.
A_6: Existence of negative (additive inverse). For every xeV, the element (-1) x has the property x + (-1) x = 0.

III Axioms for Multiplication.

M_7: Associated Law: For every x e V and all real numbers a and P you have a (P x) = a P (x)

M_8: Distributive Law for addition in V: For all x and y and y6V and all real α you have
$\alpha(x + y) = \alpha x + ay$
M_9: Distributive Law for addition of Numbers: For all x6V and all real a and P you have
$(\alpha + \beta) = \alpha x + \beta x$.

M_{10}: Existence of identity. For every x e V you have 1.x = x.

The above vector space is sometimes referred to as real vector space since you multiply throughout by real numbers.

For us to determine whether a given set of vector is a vector space, all you need to do is to test if the set of vector satisfies the axioms 1 to 10 enumerated above.
Example 1 shows that the set of real numbers R is a vector space.

Solution: you need to show that R satisfies all the axioms from 1 to 10
C_1 : The sum of any two real numbers is a real number.

C_2: The product of any two real numbers is a real number.

A_3: The set of real numbers is commutative.

A_4: If x, y, z e R then
(x + y) + a = x + (y + z).

A_5: 0 is the zero element of R^1

A_6 For each $x \in V$, there exists an element - $x \in V$ such that x + (-x) = 0. Existence of inverse element.

M_7: $\alpha(x + y) = kx + ky \quad \forall x, y \in V$ and $\alpha \in R$, distributive in terms of scalar multiplication.

M_8 : $= \alpha x + \beta x$ distributivity $\forall x \in V$ and $\alpha, \beta \in R$.

M_9: $(\alpha \beta) x = \alpha (V \beta x) \forall x \in V, \alpha, \beta \in R$.

M_{10}: There exist a number called one (1) eR such that 1 .x = x. 1 = x V x e V Existence of identity element.

Example 2:

Prove that R^2 is a vector space.
Here R^2 must satisfy all the ten axioms as outlined above.

C_1 Closure under addition. For every pair of elements x and yeR^2 there corresponds a unique element in R^2 called the sum of x and y denoted by x+ y i.e. x+ y eR^2

C_2 Closure under multiplication. For every x, yeR^2 and every number a there corresponds an element in R^2 called the product of a and x denoted by xeR^2 = α (xi, $x_2) = (\alpha$ xi, $\alpha x_2) \in R^2$

A_3 $x \in R^2 \Rightarrow x = (x_h\ x_2)$
$y \in R^2 \Rightarrow y = (y_1, y_2)$
$\Rightarrow x + y = (x_1, x_2) + y1, y2)$

$= (x_1 + y_1, x_2 + y_2)$
$= (y_1 + x_{1,} y_2 + x_2)$
$= (y_1, y_2) + (x_1 + x_2)$
$y + x$
Thus $\underline{x} + \underline{y} = \underline{y} + x$

A_4 $x+y+z$ $= \{(xi, x_2) + (y_1, y_2)\} +$
$= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$
$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$
$= (x_1, x_1,) + (y_1, z_1, y_2 + z_2)$
$= (x_1, x_2) +)y_1, z_1, y_2 + z_2)$
$= (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)]$
$= x+y+z$

Thus $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$

A_5 Obviously $0 = (0,0)$ e R^2
A_6 Trivially, every element of R^2 has its inverse in R^2

M_7 a (x + y) $= \alpha [(x_h x_2) +(y$
$= \alpha (xi + yi, x_2 + y_2) = (\alpha xi + \alpha yi, \alpha x_2 + \alpha y_2)$
$= (\alpha xi, \alpha x_2) + (\alpha yi, \alpha y_2)$
$= \alpha (xi, x_2) + \alpha (y_h y_2)$
$= \alpha x + \alpha y$

M_8 $(\alpha + \beta)x$ $= (\alpha + \beta (x_1, x_2)$
$= [(\alpha + \beta)x_1, (\alpha + \beta)x_2]$
$= (\alpha x_1, + \beta x_2, \alpha x_2 + \beta x_2)$
$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$
$= \alpha (x_1, x_2) + \beta x_1, x_2)$

$= \alpha \underline{x} + \beta \underline{x}$
Thus $\alpha (x + y)$ $= \alpha x + \beta x$

M_9 $(\alpha \beta)x$ $= (\alpha \beta)(x_1, x_2)$
$= \alpha \beta x_1, + \alpha \beta x2)$
$= \alpha (\beta x_1, + \alpha \beta x2)$
$= \alpha (\beta \underline{x})$
$= \alpha (\alpha x)$

M_{10} There exist $1 \in R^2$ such that
$1.x$ $=l.(xi, x_2).$
$= (1.x_1, l.x_2)$
Thus, $l.x = \underline{x}.l = x \ \forall x \in R^2$

Since x is arbitrary
Hence IR^2 is a vector space.

Example 3: Prove that R^3 is also a vector space.

Solution: The solution here is the same as the case of IR^2 above.

Ci: V x, y e R^3. There exist x + y e R3 closure under addition.

C_2: Closure under multiplication
$\underline{x} \in R^3, \ \alpha \in R^3$
$\alpha x = \alpha \ (xi, x_2, x_3) = (\alpha x_1 \ \alpha x_2, \alpha x_3) \in |R^3$.

A_3 = x + y
$= (xi, x_{2)} x_3) + (y_1, y_2, y_3)$
$= (xi + yi, x_2 + y_2, x_3 + y_3)$
$= (y_1 + x_1 \ y_2 + x_{2)} \ y_3 + x_3)$
$= (y_1 . y_2, y_3) + (xi, x_2, x_3)$
$= \underline{y} \quad + \quad \underline{x}$
$= \underline{y} \quad + \quad \underline{x}$

Thus $\underline{x} + \underline{y}$

A_4 (x + y) + z
$[(x_h \ x_2, x_3) + (yl, y_2, y_3)] + (z_1, z_2, z_3)$
$= (xi + yi, x_2 + y_2, X3 + ys) + (z_1, z_2, z_3)$
$= (*i + yi + z_1, x_2 + y_2 + z_2, x_3 + x_3 + y_3 + z_3)$
$= (xi, x_2, x_3) + (yi + z_1, y_2 + z_2, y_3 + z_3)$
$= (xi, x_2, x_3) + [(yi, y_2, y_3) + z_1, z_2, z_3)]$
$\underline{x} + (\underline{y} + \underline{z})$

Thus $\underline{x} + (\underline{y} + \underline{z}) \ (\underline{x} + \underline{y}) + \underline{z}$

A_5 Trivially 0 = (0, 0, 0) e IR^3
Such that $0 + x = (0, 0, 0) + (x_h \ x_2, x_3)$
$= (xi + 0, x_2 + 0, x_3 + 0)$
$= (x_1, x_2, x_3) = x$
$= (x_1, x_2, x_3) = \underline{x}$

Thus $0 + \underline{x} = \underline{x} + 0 = \underline{x}$

The rest of the axioms are left for you as exercises to test.

You have looked into some examples of vector spaces. Before advancing into some complicated examples of vector spaces, let us now look into some examples of set vectors that are not vector spaces.

If you want to show that a statement made by someone is true, you are only justified to conclude that it is true only when all the statement have been tested one by one and they are all true.

However, if one of the statements happens not to be true, without further analysis, you have the right to conclude that the statements are not true. In practical terms, if a lawyer during cross examination in the court proves beyond all reasonable doubt that one of the statements made by a complainant is not true, then he stands the chance to plead the court to dismiss the case since one statement is not true, he can no longer accept any statement from the complainant.

In a nutshell, if you are asked to show that a set of vector is a vector space, then if you can show that all the axioms are satisfied, you are done, conversely, if you are asked to show that set f vector is not a vector pace, then you can show that at least one of the axioms is not satisfied, then you are done.

Example 4: Show that Z(the set of all the integers) is not a vector space. Solution:
You can pick an example to confirm that statement
Obviously 3£ *z*, *Vi* 6R^1 but
3. $^l/2$ = 3/2 = 1.5 2 z. Thus *%* is not a vector space.

NOTE: The first axiom is satisfied i.e. V x , y $\in$ Z , x + y = y + x $\in$ Z
The satisfaction of that axiom does not make Z a vector space since you can find one of the axioms, which a set of integers fail to satisfy.

SELF ASSESSMENT EXERCISE 1

Show that the following are vector spaces (Hint see the example below).

1. A set of 2 x 2 diagonal matrices

2. IR" where n is finite dimensional (i.e. n $< \infty$)

3. Let $\not\subset$ be a set of complex numbers defined on $\not\subset$ by (a + bi) + (c + di) = (a + c) + (b + d) I and the scalar multiplication defined by
α (a + bi) = α a + α bi
(Where *i* = $\sqrt{}$-1)

4. Let V be the set of all order pairs of real numbers with addition defined by (x_u x_2) + (yi, y_2) = (xi + y,, x_2 + y_2) and scalar multiplication defined by α (xj, x_2) = (α xi, α x_2).

Example: Show that a set of 2 x 2 matrices is a vector space.

Solution:

(i) A3 Addition of 2 x 2 matrices gives 2 x 2 matrices and they are commutative.

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Thus A + B = B +A.

A4 Let A, B and C be element of 2 x 2 matrices. Then (A + B) + C

Then A + (B + C)

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \left\{ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right\}$$

A + (B + C)
Thus (A + B) + C = A + (B + C)

A_5 Trivially,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in 2 \text{ x } 2 \text{ matrix}$$

A_6 $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$

$$\begin{pmatrix} a_{11}-a_{11} & a_{12}-a_{12} \\ a_{21}-a_{21} & a_{22}-a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

M_7 $$\alpha \left\{ \begin{pmatrix} a_{11}+a_{12} \\ a_{21}+a_{21} \end{pmatrix} + \begin{pmatrix} b_{11}+b_{12} \\ b_{21}+b_{22} \end{pmatrix} \right\}$$

$$= \alpha \begin{pmatrix} a_{11}+b_{11} & a_{11}+b_{12} \\ a_{12}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(a_{11}+b_{11}) & \alpha(a_{12}+b_{12}) \\ \alpha(a_{21}+b_{21}) & \alpha(a_{22}+b_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11}+\alpha b_{11}) & \alpha a_{12}+\alpha b_{12}) \\ \alpha a_{21}+\alpha b_{21}) & \alpha a_{22}+\alpha b_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11}+\alpha a_{12} \\ \alpha a_{21}+\alpha a_{22} \end{pmatrix} + \begin{pmatrix} \alpha b_{11}+\alpha b_{12} \\ \alpha b_{21}+\alpha b_{22} \end{pmatrix}$$

$$= \alpha A + \alpha B$$

Thus $\alpha (A+B) = \alpha A + \alpha B$

M_9 $(\alpha + \beta) A = (\alpha + \beta)$

$$\begin{pmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} \end{pmatrix}$$

$$\begin{pmatrix} \alpha a_{11}+\beta a_{11} & \alpha a_{12}+\beta a_{12}) \\ \alpha a_{21}+\beta a_{22} & \alpha a_{21}+\beta a_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11}+\alpha a_{12} \\ \alpha a_{21}+\alpha a_{22} \end{pmatrix} + \begin{pmatrix} \beta a_{11}+\beta a_{12} \\ \beta a_{21}+\beta a_{22} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a_{11}+a_{12} \\ a_{21}+a_{21} \end{pmatrix} + \beta \begin{pmatrix} a_{11}+a_{12} \\ a_{11}+a_{21} \end{pmatrix}$$

$$= \alpha A + \beta A$$

Thus $(\alpha + \beta)A = \alpha A + \beta A$

M_7 $(\alpha + \beta)A = (\alpha\ \beta) \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{21} \end{pmatrix}$

$$= \begin{pmatrix} \alpha\beta a_{11} + \alpha\beta a_{12} \\ \alpha\beta a_{21} + \alpha\beta a_{22} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} \beta a_{11} + \beta a_{12} \\ \beta a_{21} + \beta a_{22} \end{pmatrix}$$

$$= \alpha \left(\beta \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right)$$

$$= \alpha(\beta A)$$

Thus $(\alpha\beta)\ A = \alpha(\beta A)$

M_{10} ∍ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\in$ 2 x 2 matrix

∍ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$= \begin{pmatrix} a_{11} + 0 & a_{12} + 0 \\ 0 + a_{21} & 0 + a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Thus I.A = A.I.
Since A is an arbitrary matrix, it therefore stands the chance to represent all matrices of 2 x 2. Hence a set of 2 x 2 matrices is a vector space.

5. This one is the same as R^2, which has been proven to be a vector space.

You may now turn to the set of vectors that are not vector spaces.

Example

1. N (the set of all the natural numbers) is not a vector space. Since,
$10 \in N : 10 + (-10) = \underline{0}$

2. Z "Z. (the set of all the integers) is not a vector space since if you take α = -1/3 and x = 17, then α x = -1/3 . 17 = -17/3 $\notin$ Z .

3. A set of non-singular 2 x 2 matrices is not a vector space in that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in 2 \text{ x } 2 \text{ matrices (non singular)}$$

and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in 2 \text{ x } 2$ matrices (non singular)

but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

Is a member of non singular matrix.

Hence non-singular 2 x 2 matrix is not a vector space.

Problems:

Show that the following are not vector spaces.

1. Let S be a set of all ordered pairs of real numbers defined by $(x_1, X_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and the scalar multiplication is defined by $\alpha(x_1, x_2) = (\alpha\ X_1\ \alpha x_2)$

2. Let S be a set of all ordered pairs of real numbers defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, X_2 + y_2)$ and scalar multiplication by $a(x_1,x_2) = (l, x_2)$

3.2 Subspaces Definition

Definition

A subset of vector space, which is also a vector space, is called a subspace. In other words, a subset of vector space is termed a subspace if the following conditions hold. Let S be a subset of a vector space V then,

(i) if x, y $\in$S then, x + y$\in$ S

(ii) if x$\in$S $\alpha \in 1R^1$ then $\alpha x \in$ S
Therefore S is closed under scalar multiplication and vector addition.

Example 1

Let S = $\{(xi, x_2, x_3)^T : x_t = x_2\}$ Show that S is a subspace of IR^3

Solution:
You only need to show that conditions (i) and (ii) above are satisfied

(i) $x = (x_1, x_1, x_2)^T \in S$ then
$\alpha x = \alpha\ (x_1, x_1,\ x_2)^T = (\alpha x_{1;}\ \alpha x_1,\ \alpha x_2)^T \in S$

(ii) If $(x_1, x_1, x_2)^T$ and $(a, a, b)^T$ are arbitrary elements of S then,
$(x_1, x_1, x_2)^T + (a, a, b)^T = (x, + a, x_1 + a, x_2 + b)^T \in S$
$(x_1 + a, x_1 + a, x_2 + b)^T \in S$
Hence S is a subspace of IR^3

(ii) Show that S the set of all 2 x 2 symmetric matrices is a subspace of 2 x 2 matrices.

$$\text{Let } A \in S \Rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\text{and } B \in S \Rightarrow B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

$$\text{So} \quad A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{12} + b_{12} & a_{22} + b_{22} \end{pmatrix} \in S$$

$$\text{Also } \alpha A = a \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{12} & \alpha a_{22} \end{pmatrix} \in S$$

Hence a set of 2 x 2 symmetric matrices is a subspace of 2 x 2 matrices.

3.3 Rank of a Matrix

Definition: The rank of a matrix is the number of non-zero rows of the matrix after the matrix has been reduced to echelon form. By this definition, you only need to reduce any given matrix to row echelon form and the number of non-zero rows is obtained as the rank.

Example 1: Find the rank of A if

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Now,

$$\begin{pmatrix}1 & -2 & 1\\ 2 & 1 & 0\\ 3 & 0 & 1\end{pmatrix} \to \begin{pmatrix}1 & -2 & 1\\ 0 & 5 & -2\\ 0 & 6 & -2\end{pmatrix}$$

$$\to \begin{pmatrix}1 & -2 & 1\\ 0 & 5 & -2\\ 0 & 0 & 2\end{pmatrix} \quad \text{in the Echelon form.}$$

From the above, the rank of A, r (A) = 3 and this is also called the dim (A) is dim (A) = 3.

Example 2:

$$\text{Let } A = \begin{pmatrix}1 & -2 & 3\\ 2 & -5 & 1\\ 1 & -4 & 7\end{pmatrix} \to \begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 5\\ 0 & 0 & 0\end{pmatrix} \text{in Echelon form}$$

Here the rank of A or dim (A) = 2. Then the non-zero rows form the basis vector of the row space o the reduced echelon.

3.4.1 Linear Dependence

Definition:

Given a set of vector V_1, V_2 ...v_n, V*.
V* is a linear combination of V_1, v_2, va ...v_n. If V*can be expressed as the sum of scalar multiples of V_1, V_2 ...v_n,

i.e. $V^* = \alpha_1 v_1 + \alpha v_2 + ... + \alpha_n v_n \ \forall \ \alpha_1 \in IR^1$ more compactly, we have that

$$V^* = \sum_{i=1}^{n} \alpha i Vi = i$$

At times, we can take V * = 0 then $\sum_{i=1}^{n} \alpha i \ Vi = 0$

If this happens when at least one $\alpha \ i \neq 0$, then we say that the set of vector v_1, v_2 ... v_n are linearly dependent. But conversely of it is only true when all the α i = 0 then the set of vectors are said to be linearly independent.

Example 2

Which of the following collection of vectors are linearly independent in IR^3

(a) $(1,1, 1)^T$ $(1,1, 0)^T$ $(1, 0, 0)^Y$
(b) $(1,0, 1)^T$ $(9,1,0)^T$
(c) $(1.2, 4)^T$, $(2, 1, 3)^T$, $(4, -1, 1)^T$

Solution

(a) $$\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}+\beta\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\\alpha\\\alpha\end{pmatrix}+\begin{pmatrix}\beta\\\beta\\0\end{pmatrix}+\begin{pmatrix}\alpha\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

so $\alpha+\beta+\alpha=0$

$\alpha+\beta=0$

$\alpha=0$

$\Rightarrow$ $\beta=0,\ \alpha=0$

$\alpha=\beta=\alpha=0$

the vectors are linearly independent.

(b) $$\alpha\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\0\\\alpha\end{pmatrix}+\begin{pmatrix}0\\\beta\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\\beta\\\alpha\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad \alpha=0$$

$\Rightarrow\beta=0\Rightarrow\alpha=\beta=0$

Hence the vectors are linearly independent.

(c) $$A=\begin{pmatrix}1&2&4\\2&1&-1\\4&3&1\end{pmatrix}$$

$$|A|=\begin{vmatrix}1&2&4\\2&1&-1\\4&3&1\end{vmatrix}=0$$

Hence the system has in addition to a trivial solution a nontrivial solution, which means that there, exists at least a $\neq$ 0 such that

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence the vectors are linearly dependent.

3.4.2 Spanning Sets: Basis of Vectors

Definition: If an arbitrary vector from a vector space can be written as a linear combination of a set of vectors from the same vector space, then the set of vectors can be said to span the space.

If the spanning set of vectors that spanned the vector space are themselves linearly independent then the set of vectors are said to form a basis of the vectors space.

SELF ASSESSMENT EXERCISE 2

i. Determine the null space of each of the following matrices,

(i) $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & -3 & -1 \\ -2 & 4 & 6 & 3 \end{pmatrix}$

ii. Determine whether or not the following are spanning sets for IR^2

(i)	v_1	=	(2,1)	v_2	=	(3, 2)
(ii)	v_1	=	(2,3)	v_2	=	(4, 6)
(iii)	v_1	=	(1,2)	v_2	=	(-1, 1)
(iv)	v_1	=	(-1,2)	v_2	=	(l, -2) V_3 = (2, -4)

iii. Which of the following are spanning set for P_3. Justify your answer.

(i) v, = l, $v_2 = x^2$ $v_3 = x^2 - 2$
(ii) {x + 2, x2 - l}
(iii) {x + 2, x+ I, x2 - 1}

iv. Which of the following set of vectors are linearly independent.

(i) $(1,1,1)^T$, $(1,1,0)^T$, $(1,0,0)^T$

(ii) $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0, & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 \\ 0, & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

WRONSKIAN OF FUNCTIONS

The Wronskian, W(x) is defined

$$\begin{vmatrix} f_1 & f_2 \ldots\ldots\ldots\ldots f_n \\ f_1 & f_2 \ldots\ldots\ldots f^1_n \\ f^{11}_1 & f^{11}_2 \ldots f^{11}_n \\ \vdots & \vdots \quad \vdots \\ f^n_1 & f^n_2 \ldots\ldots f^n_n \end{vmatrix}$$

Where $f^{(n)}$ refers to $\frac{d^n f}{dx^n}$

If the W(x) ^ 0, then the functions fl, f2 ... fn are linearly independent and linearly dependent if W(x) = 0

Example 1

Determine whether the functions are linearly independent 1, x2, x2 - 2

Solution:

W (1, x^2, $x^2 - 2$) =

$$\begin{vmatrix} 1 & x^2 & x^2 - 2 \\ 0 & 2x & 2x \\ 0 & 2 & 2 \end{vmatrix}$$

= (2x)2 - (2x)2 = 0.

Hence the functions are linearly dependent.

Example 2

Are the functions sine x, cos x, linearly independent?

$$W(\text{Sin } x, \text{Cos } x) = \begin{vmatrix} \text{Sin } x & \text{Cos } x \\ \text{Cos } x & -\text{Sin } x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$
$$= -(\sin^2 x + \cos^2 x)$$
$$= -1$$

Hence the functions sinx and cos x are linearly dependent.

SELF ASSESSMENT EXERCISE 3

Determine whether the following functions/vectors are linearly independent.

(a) e^x, e^x

(b) I, x^2, x^2, x^3

(c) Cos TIX, Sin TCX

(d) $l, e^x + e\text{-}^x, e^x\text{-}e^x$

(e) $x^{3/2}, x^{5/2}$

(f) $\log_e x$, 100

(g) tan x, sec x

The idea of row space and column helps us to understand the system of linear systems. The system Ax = b can be written in the form

$$x_1 \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots\ldots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

From the above, $A\underline{x} = \underline{b}$ will be consistent if and only if $\underline{b}$ can be expressed as a linear combination of the column vectors of A. Thus $A\underline{x} = \underline{b}$ is consistent if and only if b is in the column space of A. If on the other hand $\underline{b}$ is replaced with zero vector, then the above

$$x_1a_1 + x_2a_2 + \ldots + x_n a_n = \underline{0} \quad *$$

It follows from * above that the system Ax = 0 has only the trivial solution, x = 0 if and only if the column vectors of A are linearly independent.

4.0 CONCLUSION

In this unit, you have studied one of the important concepts of modern linear algebra, solution to problems can easily be solved using standard algorithms based on the properties of a vector space. Using matrices, you have investigated the properties of a vector space. You know how to determine whether a set of vectors is linearly independent or not. You know how to find the rank of a matrix and its relation to a set of linearly independent vectors.

5.0 SUMMARY

In this unit, you have studied how to

- Define a vector space
- Determine whether a set is a vector space
- Define a subspace of a vector space
- Define and compute the rank of a matrix.
- Determine whether a set of vectors is linearly independent or dependent,
- Define a basis of vector space.

6.0 TUTOR-MARKED ASSIGNMENT

1. Let T= $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = 2x_1 \right\}$

   Show that T is a subspace of IR^2

2. Let K= { x IR^1 : x$\in$ (0, 1)}
   Show that K is not a subspace of IR^1

3. Let M be an n x n identity matrix. Show that M is not a subspace of the matrix M x n.

4. Let P be an n x n zero matrix. Prove that P is a subspace of m x n matrix

5. Let S = $\left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : \in IR^1 \right\}$ Prove that S is NOT a subspace of IR^2

6. Given that $3x_1 - x_2 = 0$
   $x_1 + x_2 = 0$

   Show that the solution set of the above system is actually; of IR^2. (Hints: do not solve the system).

7. Which of the following is true and why? Give examples t your points.

- Any subset of a vector space is a subspace
- Any subspace is automatically a vector space

8. Show that the solution to the given system of linear equation is a subspace of IR^2. $2x_1 – x_2 = 0$
$2x_1 - x_2 = 0$

9. Let A be an m x n matrix. Let N (A) denote the set of all s the homogeneous system. Ax = 0. Prove that N (A) is a s IR^1.

10. Determine N (A) if A $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

7.0 REFERENCES/FURTHER READING

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