MODULE 1

UNIT 1 SETS AND SUBSETS

CONTENTS

1.0 INTRODUCTION

The theory of sets lies at the foundation of mathematics. It is a concept that rears its head in almost all fields of mathematics; pure and applied.

This unit aims at introducing basic concepts that would be explained further in subsequent units. There will be definition of terms and lots of examples and exercises to help you as you go along.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify sets from some given statements
- rewrite sets in the different set notation
- identify the different kinds of sets with examples

3.1 SETS

As mentioned in the introduction, a fundamental concept in all branches of mathematics is that of set. Here is a definition.

"***A set is any well-defined list, collection or class of objects".*** The objects in sets, as we shall see from examples, can be anything: But for clarity, we now list ten particular examples of sets:

Example 1.1 The numbers 0,2,4,6,8
Example 1.2 The solutions of the equation $x^2 + 2x+1 = 0$
Example 1.3 The vowels of the alphabet: a, e, i, o, u
Example 1.4 The people living on earth
Example 1.5 The students Tom, Dick and Harry
Example 1.6 The students who are absent from school
Example 1.7 The countries England, France and Denmark
Example 1.8 The capital cities of Nigeria
Example 1.9 The number 1, 3, 7, and 10
Example 1.10 The rivers in Nigeria

Note that the sets in the odd numbered examples are defined, that is, presented, by actually listing its members; and the sets in the even numbered examples are defined by stating properties that is, rules, which decide whether or not a particular object is a member of the set.

3.1.1 Notation

Sets will usually be denoted by capital letters;

A, B, X, Y,……
Lower case letters will usually represent the elements in our sets:
Lets take as an example; if we define a particular set by actually listing its members, for example, let A consist of numbers 1,3,7, and 10, then we write A={1,3,7,10}

That is, the elements are separated by commas and enclosed in brackets {}.

We call this the **tabular form** of a set

Now, try your hand on this

But if we define a particular set by stating properties which its elements must satisfy, for example, let B be the set of all even numbers, then we use a letter, usually x, to represent an arbitrary element and we write:

$B = \{x \mid x \text{ is even}\}$

Which reads "B is the set of numbers x such that x is even". We call this the set **builders form** of a set. Notice that the vertical line " | " is read "such as".

In order to illustrate the use of ht above notations, we rewrite the sets in examples 1.1-1.10. We denote the sets by A_1, A_2, . A_{10} respectively.

Example 2.1 $A_1 = \{0, 2, 4, 6, 8\}$
Example 2.2 $A_2 = \{x \mid x^2 + 2x + 1 = 0\}$
Example 2.3 $A_3 = \{a, e, i, o, u\}$
Example 2.4 $A_4 = \{x \mid x \text{ is a person living on the earth}\}$
Example 2.5 $A_5 = \{\text{Tom, Dick, Harry}\}$
Example 2.6 $A_6 = \{x \mid x \text{ is a student and x is absent from school}\}$
Example 2.7 $A_7 = \{\text{England, France, Denmark}\}$
Example 2.8 $A_8 = \{x \mid x \text{ is a capital city and x is in Nigeria}\}$
Example 2.9 $A_9 = \{1, 3, 7, 10\}$
Example 2.10 $A_{10} = \{ x \mid x \text{ is a river and x is in Nigeria}\}$

It is as easy as that!

If an object x is a member of a set A, i.e., A contains x as one of its elements, then we write:

$$x \in A$$

which can be read "x belongs to A" or 'x is in A". If, on the other hand, an object x is not a member of a set A, i.e. A does not contain x as one of its elements, then we write;

$$x \notin A$$

It is a common custom in mathematics to put a vertical line " |" or "\" through a symbol to indicate the opposite or negative meaning of the symbol.

Example 3:1 Let A = {a, e, i o, u}. Then $a \in A$, $b \notin A$, $f \notin A$.
Example 3.2 Let $B = \{x \mid x \text{ is even}\}$. Then $3 \notin B$, $6 \in B$, $11 \notin B$, $14 \in B$.

3.1.1 Finite & Infinite Sets

Sets can be finite or infinite. Intuitively, a set is finite if it consists of a **specific number** of different elements, i.e. if in counting the different members of the set the counting process can come to an end. Otherwise a set is infinite. Lets look at some examples.

Example 4:1 Let M be the set of the days of the week. The M is finite

Example 4:2 Let N = {0,2,4,6,8……..}. Then N is infinite

Example 4:3 Let P = {x | x is a river on the earth}. Although it may be difficult to count the number of rivers in the world, P is still a finite set.

The first three sets are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two sets are infinite. If we ever try to count the even numbers, we would never come to the end.

3.1.2 Equality of Sets

Set A is **equal** to set B if they both have the same members, i.e if every element which belongs to A also belongs to B and if every element which belongs to B also belongs to A. We denote the equality of sets A and B by:

$$A = B$$

Example 5.1 Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B, that is {1,2,3,4} = {3,1,4,2}, since each of the elements 1,2,3 and 4 of A belongs to B and each of the elements 3,1,4 and 2 of B belongs to A. Note therefore that a set does not change if its elements are rearranged.

Example 5.3 Let $E=\{x \mid x^2-3x = -2\}$, F={2,1} and G ={1,2,2, 1}, Then E= F= G

3.1.3 Null Set

It is convenient to introduce the concept of the empty set, that is, a set which contains no elements. This set is sometimes called the *null set.* We say that such a set is void or empty, and we denote its symbol $\emptyset$.

Example 6.1 Let A be the set of people in the world who are older than 200 years. According to known statistics A is the null set.

Example 6.2 Let $B = \{x \mid x^2 = 4, x \text{ is odd}\}$, Then B is the empty set.

3.2 Subsets

If every element in a set A is also a member of a set B, then A is called *subset* of B.

More specifically, A is a subset of B if $x \in A$ implies $x \in B$. We denote this relationship by writing; $A \subset B$, which can also be read "A is contained in B".

Example 7.1 The set C = {1,3,5} is a subset of D = {5,4,3,2,1}, since each number 1, 3 and 5 belonging to C also belongs to D.

Example 7.2 The set E = {2,4,6} is a subset of F = {6,2,4}, since each number 2,4, and 6 belonging to E also belongs to F. Note, in particular, that E = F. In a similar manner it can be shown that every set is a subset of itself.

Example 7.3 Let G = {x | x is even}, i.e. G = {2,4,6}, and let
F = {x | x is a positive power of 2}, i.e. let
F = {2,4,8,16…..} Then $F \subset G$, i.e. F is contained in G.

With the above definition of a subset, we are able to restate the definition of the equality of two sets.

Two set A and B are equal, i.e A = B, if an only if $A \subset B$ and $B \subset A$.
If A is a subset of B, then we can also write

$$B \supset A$$

which reads "B is a superset of A" or "B contains A". Furthermore, we write:

$$A \not\subset B$$

if A is not a subset of B.

Conclusively, we state:

1. The null set $\varnothing$ is considered to be a subset of every set
2. If A is not a subset of B, that is, if $A \not\subset B$, then there is at least one element in A that is not a member of B.

3.2.1 Proper Subsets

Since every set A is a subset of itself, we call B a proper subset of A if, first, B is a subset of A and secondly, if B is not equal to A. More briefly, B is a proper subset of A if:

$B \subset A$ and $B \neq A$

In some books "B is a subset of A" is denoted by

$$B \subseteq A$$

and B" is a proper subset of A" is denoted by

$$B \subset A$$

We will continue to use the previous notation in which we do not distinguished between a subset and a proper subset.

3.2.2 Comparability

Two sets A and B are said to be **comparable** if:

$$A \subset B \text{ or } B \subset A;$$

That is, if one of the sets is a subset of the other set. Moreover, two sets A and B are said to be **not comparable** if:

$$A \not\subset B \text{ and } B \not\subset A$$

Note that if A is not comparable to B then there is an element in A which is not in B and … also, there is an element in B which is not in A.

Example 8.1: Let A = {a,b} and B = {a,b,c}. The A is comparable to B, since A is a subset of B.

Example 8.2: Let R – {a,b} and S = {b,c,d}. Then R and S are not comparable, since a $\in$R nd a$\notin$ S and c $\notin$ R

In mathematics, many statements can be proven to be true by the use of previous assumptions and definitions. In fact, the essence of mathematics consists of theorems and their proofs. We now proof our first

Theorem 1.1 If A is a subset of B and B is a subset of C then A is a subset of C, that is,

$$A \subset B \text{ and } B \subset C \text{ implies } A \subset B$$

Proof: (Notice that we must show that any element in A is also an element in C). Let x be an element of A, that is, let x $\in$ A. Since A is a subset of B, x also belongs to B, that is, x $\in$ B. But by hypothesis,
B $\subset$ C; hence every element of B, which includes x, is a number of C. We have shown that x $\in$ A implies x $\in$ C. Accordingly, by definition, A $\subset$ C.

3.2.3 Sets of Sets

It sometimes will happen that the object of a set are sets themselves; for example, the set of all subsets of A. In order to avoid saying "set of sets", it is common practice to say "family of sets" or "class of sets". Under the circumstances, and in order to avoid confusion, we sometimes will let script letters

$\mathcal{A}$, $\mathcal{B}$,……….

Denote families, or classes, of sets since capital letters already denote their elements.

Example 9.1 In geometry we usually say "a family of lines" or "a family of curves" since lines and curves are themselves sets of points.

Example 9.2 The set {{2,3}, {2}, {5,6}} is a family of sets. Its members are the sets {2,3}, {2} and {5,6}.

Theoretically, it is possible that a set has some members, which are sets themselves and some members which are not sets, although in any application of the theory of sets this case arises infrequently.

Example 9.3 Let A = {2, {1,3}, 4, {2,5}}. Then A is not a family of sets; here some elements of A are sets and some are not.

3.2.4 Universal Set

In any application of the theory of sets, all the sets under investigation will likely be subsets of a fixed set. We call this set the ***universal set*** or ***universe of discourse***. We denote this set by U.

Example 10.1 In plane geometry, the universal set consists of all the points in the plane.

Example 10.2 In human population studies, the universal set consists of all the people in the world.

3.2.5 Power set

The family of all the subsets of any set S is called the **power set** of S.
We denote the power set of S by: 2^S

Example11.1 Let M = {a,b} Then
$2^M = \{\{a, b\}, \{a\}, \{b\}, \varnothing\}$

Example 11.2 Let T = {4,7,8} then
2^T = {T, {4,7}, {4,8}, {7,8}, {4}, {7}, {8}, ∅}

If a set S is finite, say S has n elements, then the power set of S can be shown to have 2^n elements. This is one reason why the class of subsets of S is called the power set of S and is denoted by 2^S

3.2.6 Disjoint Sets

If sets A and B have no elements in common, i.e if no element of A is in B and no element of B is in A, then we say that A and B are **disjoint**

Example 12.1: Let A = {1,3,7,8} and B = {2,4,7,9}, Then A and B are not disjoint since 7 is in both sets, i.e 7∈ A and 7∈ B

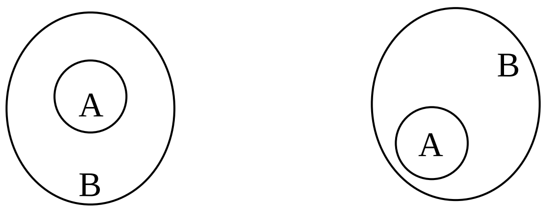
Example 12.2: Let A be the positive numbers and let B be the negative numbers. Then A and B are disjoint since no number is both positive and negative.

Example 12.3: Let E = {x, y, z} and F = {r, s, t}, Then E and F are disjoint.

3.3 Venn-Euler Diagrams

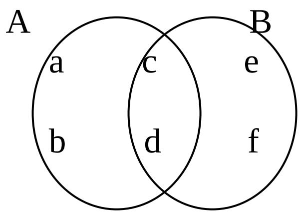
A simple and instructive way of illustrating the relationships between sets is in the use of the so-called Ven-Euler diagrams or, simply, Venn diagrams. Here we represent a set by a simple plane area, usually bounded by a circle.

Example 13.1 Suppose A ⊂ B and, say, A ≠ B, then A and B can be described by either diagram:



Example 13.2 Suppose A and B are not comparable. Then A and B can be represented by the diagram on the right if they are disjoint, or the diagram on the let if they are not disjoint.

Example 13.3 Let A = {a, b, c, d} and B= {c, d, e, f}. Then we illustrate these sets a Venn diagram of the form:



3. 4 Axiomatic Development of Set Theory

In an axiomatic development of a branch of mathematics, one begins with:
1. Undefined terms
2. Undefined relations
3. Axioms relating the undefined terms and undefined relations.

Then, one develops theorems based upon the axioms and definitions

Example 14:1 In an axiomatic development of Plane Euclidean geometry

1. "points" and "lines" are undefined terms
2. "points on a line" or, equivalent, "line contain a point" is an undefined relation
3. Two of the axioms are:

Axiom 1 Two different points are on one and only one line
Axiom 2 Two different lines cannot contain more than one point in common.

In an axiomatic development of set theory:

1. "element' and "set" are undefined terms
2. "element belongs to a set" is undefined relation
3. Two of the axioms are

Axiom of Extension: Two sets A and B are equal if and only if every element in A belongs to B and every element in B belongs to A.

Axiom of Specification: Let P(x) be any statement and let A be any set. Then there exists a set:
$B=\{a \mid a \in A, P(a) \text{ is true}\}$

Here, P(x) is a sentence in one variable for which P(a) is true or false for any a ∈ A. for example P(x) could be the sentence "$x^2 = 4$" or "x is a member of the United Nations"

4.0 CONCLUSION

You have been introduced to basic concepts of sets, set notation etc. that will be built upon in other units. If you have not mastered them by now you will notice you have to come back to this unit from time to time.

5.0 SUMMARY

A summary of the basic concept of set theory is as follows:

- A ***set*** is any well-defined list, collection, or class of objects.
- Given a set A with ***elements*** 1,3,5,7 the ***tabular form*** of representing this set is A = {1, 3, 5, 7}
- The ***set-builder form*** of the same set is A = {x| x = 2n + 1,0 ≤n≤3}
- Given the set N = {2,4,6,8,….} then N is said to be ***infinite***, since the counting process of its elements will never come to an end, otherwise it is ***finite***
- Two sets of A and B are said to be ***equal*** if they both have the same elements, written A = B
- The ***null set***, ∅, contains no elements and is a subset of every set
- The set A is a subset of another set B, written A ⊂ B, if every element of A is also an element of B, i.e. for every x∈A then x∈B
- If B ⊂ A and B ≠A, then B is a ***proper subset*** of A
- Two sets A and B are comparable if A⊂ B and B⊂ A
- The ***power set*** 2^S of any set S is the family of all the subsets of S
- Two sets A and B are said to be disjoint if they do not have any element in common, i.e. their intersection is a null set

6.0 TUTOR-MARKED ASSIGNMENT

1. Rewrite the following statement using set notation:

    - x does not belong to A.
    - R is a superset of S
    - d is a member of E
    - F is not a subset of G
    - H does not included D.

2. Which of these sets are equal: {r,t,s}, {s,t,r,s}, {t,s,t,r}, {s,r,s,t}?

3. Which sets are finite?

    - The months of the year
    - {1,2,3,……99, 100}

- The people living on the earth
- {x | x is even}
- {1,2,3……..)

The first three set are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two set are infinite. If we ever try to count the even numbers we would never come to the end.

4. Which word is different from each other, and why: (1) empty, (2) void, (3)zero, (4) null?

5. Let A = {x, y,z}. How many subsets does A contain, and what are they?

7.0 REFERENCES/FURTHER READING

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UNIT 2 BASIC SET OPERATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall see operations performed on sets as in simple arithmetic. This operations simply give sets a language of their own. You will notice in subsequent units that you cannot talk of sets without reference, sort of, to these operations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Compare two sets and/or assign to them another set depending on their comparability.
- Represent these relationships on the Venn diagram.

3.0 MAIN CONTENTS

3.1 Set Operations

In arithmetic, we learn to add, subtract and multiply, that is, we assign to each pair of numbers x and y a number x + y called the sum of x and y, a number x – y called the difference of x and y, and a number xy called the product of x and y. These assignments are called the operations of addition, subtraction and multiplication of numbers. In this unit, we define the operation ***Union, Intersection*** and ***difference*** of sets, that is, we will assign new pairs of sets A and B. In a later unit, we will see that these set operations behave in a manner somewhat similar to the above operations on numbers.

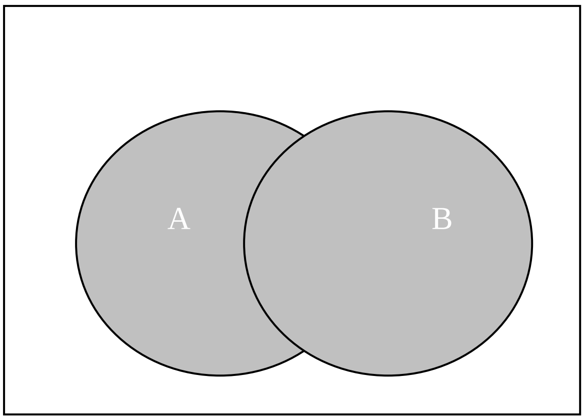
3.1.1 Union

The union of sets A and B is the set of all elements which belong to A or to B or to both. We denote the union of A and B by;

$A \cup B$

Which is usually read "A union B"

Example 1.1: In the Venn diagram in fig 2-1, we have shaded $A \cup B$, i.e. the area of A and the area of B.



$A \cup B$ is shaded

Fig 2.1

Example 1.2: Let S = {a, b. c. d} and T = {f, b, d, g}. Then S T = {a, b, c, d, f, g}.

Example 1.3: Let P be the set of positive real numbers and let Q be the set of negative real numbers. The $P \cup Q$, the union of P and Q, consist of all the real numbers except zero.

The union of A and B may also be defined concisely by

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Remark 2.1: It follows directly from the definition of the union of two sets that $A \cup B$ and $B \cup A$ are the same set, i.e.,
$A \cup B = B \cup A$

Remark 2.2: Both A and B are always subsets of A and B that is,

$A \subset (A \cup B)$ and $B \subset (A \cup B)$

In some books, the union of A and B is denoted by A + B and is called the set-theoretic sum of A and B or, simply, A plus B.

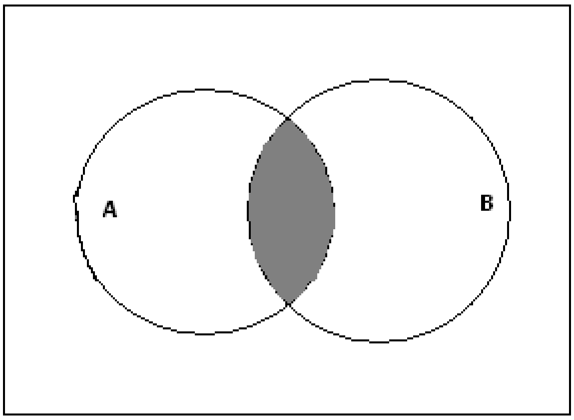
3.1.2 Intersection

The ***Intersection*** of sets A and B is the set of elements which are common to A and B, that is, those elements which belong to A and which belong to B. We denote the intersection of A and B by:

$$A \cap B$$

Which is read "A intersection B".

Example 2.1: In the Venn diagram in fig 2.2, we have shaded $A \cap B$, the area that is common to both A and B.



$A \cap B$ is shaded
Fig 2.2

Example 2.2: Let S = {a, b, c, d} and T = {f, b, d, g}. Then $S \cap T$ = {b, d}

Example 2.3: Let V = 2, 3, 6, ……} i.e. the multiples of 2; and let W = {3, 6, 9,….} i.e. the multiples of 3. Then $V \cap W$ = {6, 12, 18……}

The intersection of A and B may also be defined concisely by

$$A \cap B = \{x \mid x \in A, x \in B\}$$

Here, the comma has the same meaning as "and".

Remark 2.3: It follows directly from the definition of the intersection of two sets that;

$$A \cap B = B \cap A$$

Remark 2.4: Each of the sets A and B contains $A \cap B$ as a subset, i.e.,

$$(A \cap B) \subset A \text{ and } (A \cap B) \subset B$$

Remark 2.5: If sets A and B have no elements in common, i.e. if A and B are disjoint, then the intersection of A and B is the null set, i.e. $A \cap B = \emptyset$.

In some books, especially on probability, the intersection of A and B is denoted by AB and is called the set-theoretic product of A and B or, simply, A times B.

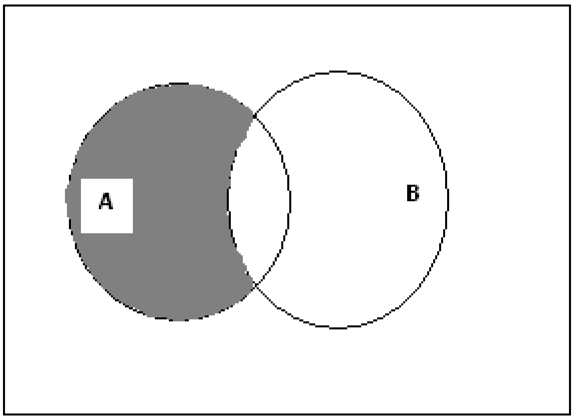
3.1.3 Difference

The difference of sets A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by

$$A - B$$

Which is read "A difference B" or, simply, "A minus B".

Example 3.1: In the Venn diagram in Fig 2.3, we have shaded A – B, the area in A which is not part of B.



A – B is shaded
Fig 2.3

Example 3.2 Let R be the set of real numbers and let Q be the set of rational numbers. Then R – Q consists of the irrational numbers.

The difference of A and B may also be defined concisely by

$$A - B = \{ x \mid x \in A, x \notin B\}$$

Remark 2.6 Set A contains A – B as a subset, i.e.,

$(A - B) \subset A$

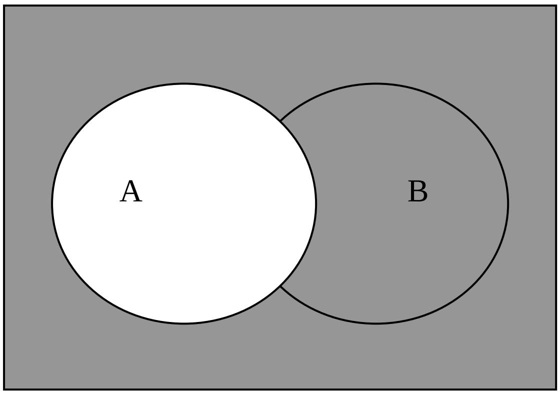
Remark 2.7 The sets (A – B), $A \cap B$ and (B – A) are mutually disjoint, that is, the intersection of any two is the null set.

The difference of A and B is sometimes denoted by A/B or A ~ B

3.1.4 Complement

The complement of a set A is the set of elements that do not belong to A, that is, the difference of the universal set U and A. We denote the complement of A by **A′**

Example 4.1 In the Venn diagram in Fig 2.4, we shaded the complement of A, i.e. the area outside A. Here we assume that the universal set U consists of the area in the rectangle.



A’ is shaded
Fig. 2.4

Example 4.2 Let the Universal set U be the English alphabet and let T = {a, b, c}. Then;

T' = {d, e, f, ….., y, z}

Example 4.3 Let E = {2, 4, 6, …}, that is, the even numbers. Then E′ = {1, 3, 5, …}, the odd numbers. Here we assume that the universal set is the natural numbers, 1, 2, 3,…..

The complement of A may also be defined concisely by;

$A' = \{x \mid x \in U, x \notin A\}$ or, simply,
$A' = \{x \mid x \notin A\}$

We state some facts about sets, which follow directly from the definition of the complement of a set.

Remark 2.8 The union of any set A and its complement A′ is the universal set, i.e.,

$$A \cup A' = U$$

Furthermore, set A and its complement A′ are disjoint, i.e.,

$$A \cap A' = \varnothing$$

Remark 2.9 The complement of the universal set U is the null set ∅, and vice versa, that is,

$U' = \varnothing$ and $\varnothing' = U$

Remark 2.10 The complement of the complement of set A is the set A itself. More briefly,

$$(A')' = A$$

Our next remark shows how the difference of two sets can be defined in terms of the complement of a set and the intersection of two sets. More specifically, we have the following basic relationship:

Remark 2.11 The difference of A and B is equal to the intersection of A and the complement of B, that is,

$$A - B = A \cap B'$$

The proof of Remark 2.11 follows directly from definitions:

$A - B = \{x \mid x \in A, x \notin B\} = \{x \mid x \in A, x \notin B'\} = A \cap B'$

3.2 Operations on Comparable Sets

The operations of union, intersection, difference and complement have simple properties when the sets under investigation are comparable. The following theorems can be proved.

Theorem 2.1 Let A be a subset of B. Then the union intersection of A and B is precisely A, that is,

$A \subset B$ implies $A \cap B = A$

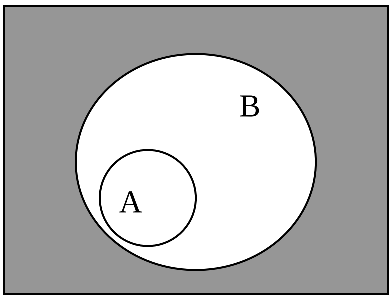
Theorem 2.2 Let A be a subset of B. Then the of A and B is precisely B, that is,

$A \subset B$ implies $A \cup B = B$

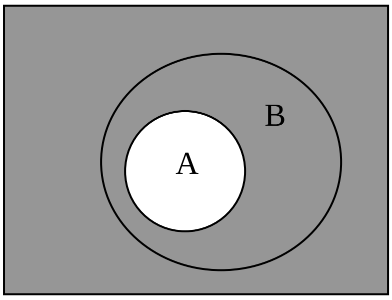
Theorem 2.3 Let A be a subset of B. Then B' is a subset of A', that is,

$A \subset B$ implies $B' \subset A'$

We illustrate Theorem 2.3 by the Venn diagrams in Fig 2-5 and 2-6. Notice how the area of B' is included in the area of A'.



B′ is shaded
Fig 2.5



A′ is shaded
Fig 2.6

Theorem 2.4: Let A be a subset of B. Then the Union of A and (B – A) is precisely B, that is,

$A \subset B$ implies $A \cup (B - A) = B$

4.0 CONCLUSION

You have seen how the basic operations of Union, Intersection, Difference and Complement on sets work like the operations on numbers. These are also the basic symbols associated with set theory.

5.0 SUMMARY

The basic set operations are Union, Intersection, Difference and Complement defined as:

- The ***Union*** of sets A and B, denoted by $\mathbf{A \cup B}$, is the set of all elements, which belong to A or to B or to both.
- The ***intersection*** of sets A and B, denoted by $\mathbf{A \cap B}$, is the set of elements, which are common to A and B. If A and B are disjoint then their intersection is the Null set $\varnothing$.
- The ***difference*** of sets A and B, denoted by **A – B,** is the set of elements which belong to A but which do not belong to B.
- The ***complement*** of a set A, denoted by **A'**, is the set of elements, which do not belong to A, that is, the difference of the universal set U and A.

6.0 TUTOR – MARKED ASSIGNMENT

1. Let X = {Tom, Dick, Harry}, Y = {Tom, Marc, Eric} and Z = {Marc, Eric, Edward}. Find (a) $X \cup Y$, (b) $Y \cup Z$ (c) $X \cup Z$
2. Prove: $A \cap \varnothing = \varnothing$.
3. Prove Remark 2.6: $(A - B) \subset A$.
4. Let U = {1,2,3,…,8,9}, A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find (a) A', (b) B', (c) $(A \cap C)'$, (d) $(A \cup B)'$, (e) (A')', (f) (B – C)'
5. Prove: B – A is a subset of A'.

7.0 REFERENCES/FURTHER READING

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UNIT 3 SET OF NUMBERS

CONTENTS

1.0 INTRODUCTION

Although, the theory of sets is very general, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of ***real numbers,*** which we denote by

$$\Re$$

In fact, we assume in this unit, unless otherwise stated, that the set of real numbers $\Re$ is our universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the ***real number system.***

2.0 OBJECTIVES

After studying this unit, you should be able to do the following:

- represent the set of numbers on the real line
- perform the basic set operations on intervals

3.0 MAIN CONTENTS

3.1 Real Numbers, $\Re$

One of the most important properties of the real numbers is that points on a straight line that can represent them. As in Fig 3.1, we choose a point, called the origin, to represent 0 and another point, usually to the right, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, that is, each point will represent a unique real number and each real number will be represented by a unique point. We refer to this line as the ***real line.*** Accordingly, we can use the words point and number interchangeably.

Those numbers to the right of 0, i.e. on the same side as 1, are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers.* The number 0 itself is neither positive nor negative.

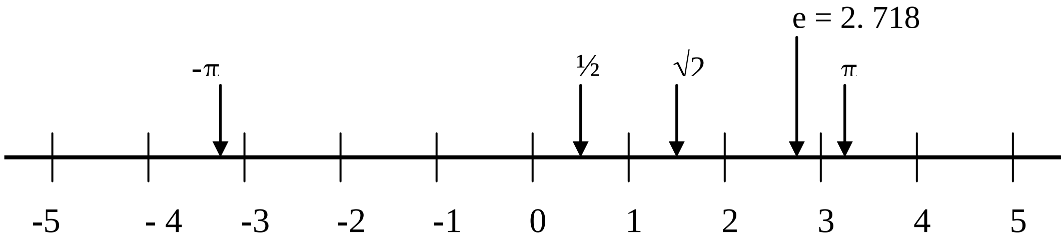


Fig 3.1

3.1.2 Integers, Z

The integers are those real numbers

…, -3, -2, -1, 0, 1, 2, 3,…

We denote the integers by Z; hence we can write
Z = { …, -2, - 1, 0, 1, 2,…}

The integers are also referred to as the “whole” numbers.

One important property of the integers is that they are “closed” under the operations of addition, multiplication and subtraction; that is, the sum, product and difference of two integers is again in integer. Notice that the quotient of two integers, e.g. 3 and 7, need not be an integer; hence the integers are not closed under the operation of division.

3.1.3 Rational Numbers, Q

The *rational numbers* are those real numbers, which can be expressed as the ratio of two integers. We denote the set of rational numbers by Q. Accordingly,

Q = {x | x = p q where p ∉z, q ∈ z}
Notice that each integer is also a rational number since, for example, 5 = 5/1; hence Z is a subset of Q.

The rational numbers are closed not only under the operations of addition, multiplication and subtraction but also under the operation of division (except by 0). In other words, the sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

3.1.4 Natural Numbers, N

The *natural numbers* are the positive integers. We denote the set of natural numbers by N; hence N = {1,2,3…..}

The natural numbers were the first number system developed and were used primarily, at one time, for counting. Notice the following relationship between the above numbers systems:

$$N \subset Z \subset Q \subset \Re$$

The natural numbers are closed only under the operation of addition and multiplication. The difference and quotient of two natural numbers needed not be a natural number.

The *prime numbers* are those natural numbers p, excluding 1, which are only divisible 1 and p itself. We list the first few prime numbers:

2,3,5,7,11,13,17,19…

3.1.5 Irrational Numbers, Q'

The irrational numbers are those real numbers which are not rational, that is, the set of irrational numbers is the complement of the set of rational numbers Q in the real numbers $\Re$; hence Q' denote the irrational numbers. Examples of irrational numbers are $\sqrt{3}$, π, $\sqrt{2}$, etc.

3.1.6 Line Diagram of the Number Systems

Fig 3 -2 below is a line diagram of the various sets of number, which we have investigated. (For completeness, the diagram include the sets of complex numbers, number of the form a + bi where a and b are real. Notice that the set of complex numbers is superset of the set of real numbers.)

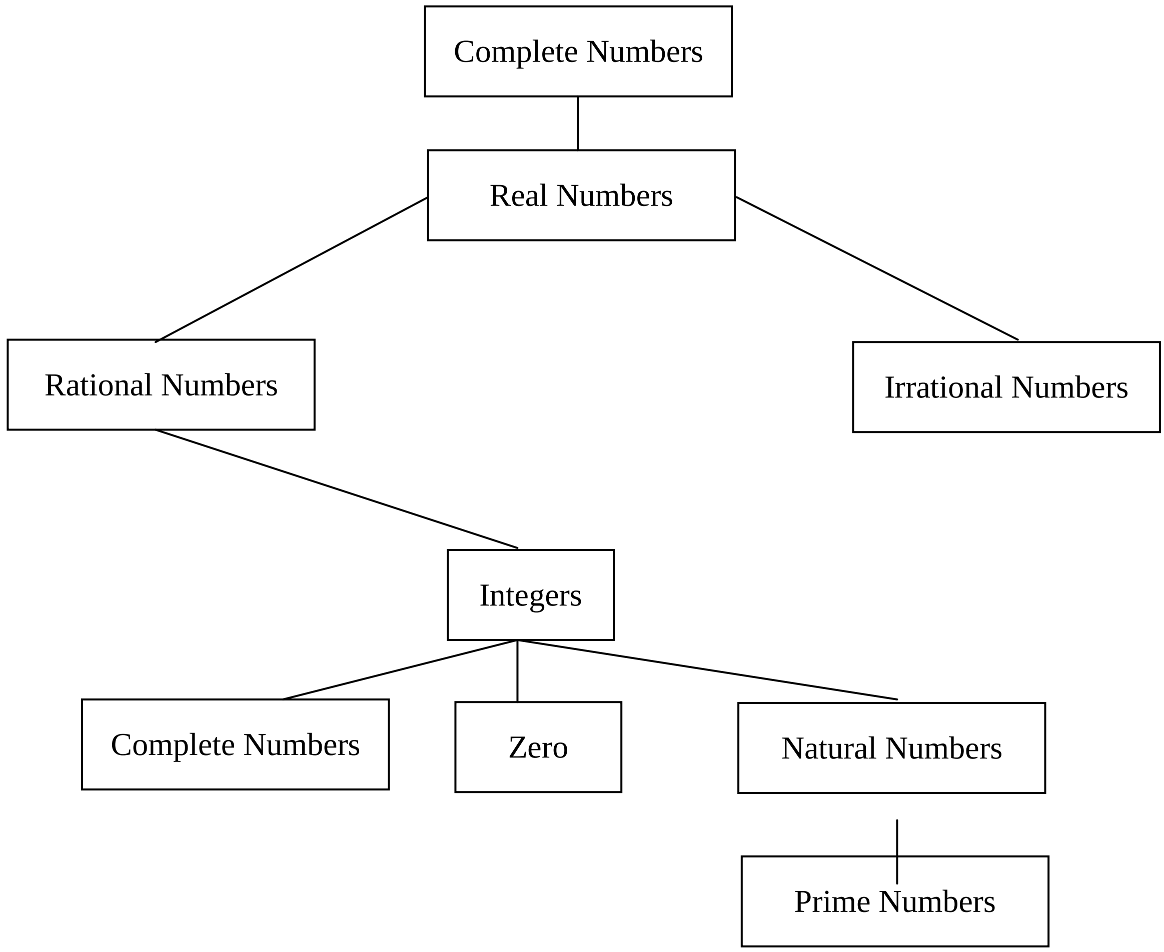


Fig 3.2

3.2 Decimals and Real Numbers

Every real number can be represented by a "non-terminating decimal." The decimal representation of a rational number p/q can be found by "dividing the denominator q into the numerator p." If the indicated division terminates, as for

3/8 = .375

We write 3/8 = .375000

Or 3/8 = .374999…

If we indicated division of q into p does not terminate, then it is known that a block of digits will continually be repeated; for example,

2/11 = .181818…

We now state the basic fact connecting decimals and real numbers. The rational numbers correspond precisely to those decimals in which a block of digits is continually repeated, and the irrational numbers correspond to the other non-terminating decimals.

3.3 Inequalities

The concept of "order" is introduce in the real number system by the

Definition: The real number a is less than the real number b, written $a < b$

If $b – a$ is a positive number.

The following properties of the relation $a < b$ can be proven. Let a, b and c be real numbers; then:

P_1: Either $a < b$, $a = b$ or $b < a$.
P_2: If $a < b$ and $b < c$, then $a < c$.
P_3: If $a < b$, then $a + c < b + c$
P_4: If $a < b$ and c is positive, then $ac < bc$
P_5: If $a < b$ and c is negative, then $bc < ac$.

Geometrically, if $a < b$ then the point a on the real line lies to the left of the point b.

We also denote $a < b$ by $b > a$
Which reads "b is *greater then* a". Furthermore, we write

$$a \leq b \text{ or } b \geq a$$

if $a < b$ or $a = b$, that is, if a is not greater than b.

Example 1.1 $2 < 5$; $-6 \leq -3$ and $4 \leq 4$; $5 > -8$

Example 1.2 The notation $x < 5$ means that x is a real number which is less than 5; hence x lies to the left of 5 on the real line

The notation $2 < x < 7$; means $2 < x$ and also $x < 7$; hence x will lie between 2 and 7 on the real line.

Remark 3.1 Notice that the concept of order, i.e. the relation $a < b$, is defined in terms of the concept of positive numbers. The fundamental property of the positive numbers which is used to prove properties of the relation $a < b$ is that the positive numbers are closed under the operations of addition and multiplication. Moreover, this fact is intimately connected with the fact that the natural numbers are also closed under the operations of addition and multiplication.

Remark 3.2 The following statements are true when a, b, c are any real numbers:

1. $a \leq a$
2. if $a \leq b$ and $b \leq a$ then $a = b$.
3. if $a \leq b$ and $b \leq c$ then $a \leq c$.

3.4 Absolute Value

The absolute value of a real number x, denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

that is, if x is positive or zero then $|x|$ equals x, and if x is negative then $|x|$ equals –x. Consequently, the absolute value of any number is always non-negative, i.e. $|x| \geq 0$ for every $x \in \Re$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e. the point 0. Moreover, the distance between any two points, i.e. real numbers, a and b is $|a - b| = |b - a|$.

Example 2.1 $|-2| = 2$, $|7| = 7$. $|-\pi| = \pi$

Example 2.2 The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5, i.e. x must lies between -5 and 5 on the real line. In other words,
$|x| < 5$ and $-5 < x < 5$ have identical meaning. Similarly,
$|x| \leq 5$ and $-5 \leq x \leq 5$ have identical meaning.

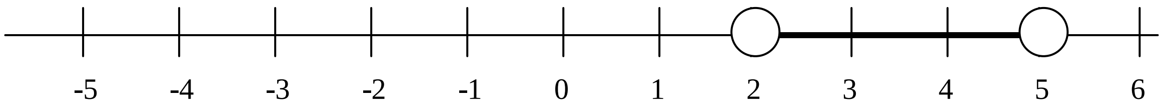
3.5 Intervals

Consider the following set of numbers;

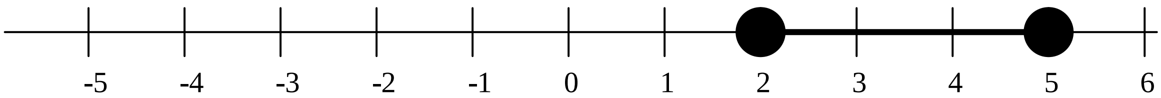
$A_1 = \{x | 2 < x < 5\}$
$A_2 = \{x | 2 \leq x \leq 5\}$
$A_3 = \{x | 2 < x \leq 5\}$
$A_4 = \{x | 2 \leq x < 5\}$

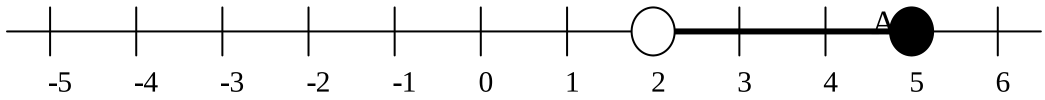
Notice, that the four sets contain only the points that lie between 2 and 5 with the possible exceptions of 2 and/or 5. We call these sets intervals, the numbers 2 and 5 being the endpoints of each interval. Moreover, A_1 is an ***open interval*** as it does not contain either end point: A_2 is a ***closed interval*** as it contains bother endpoints; A_3 and A_4 are ***open-closed*** and ***closed-open*** respectively.

We display, i.e. graph, these sets on the real line as follows.

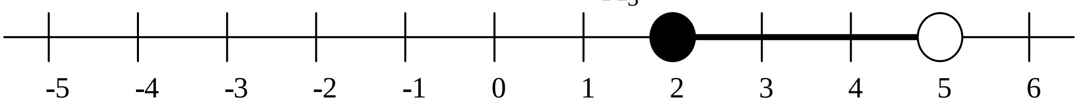


A_1





A_3



A_4

Notice that in each diagram we circle the endpoints 2 and 5 and thicken (or shade) the line segment between the points. If an interval includes an endpoint, then this is denoted by shading the circle about the endpoint.

Since intervals appear very often in mathematics, a shorter notation is frequently used to designated intervals, Specifically, the above intervals are sometimes denoted by;

$$A_1 = (2, 5)$$
$$A_2 = [2, 5]$$
$$A_3 = (2, 5)$$
$$A_4 = [2, 5)$$

Notice that a parenthesis is used to designate an open endpoint, i.e. an endpoint that is not in the interval, and a bracket is used to designate a closed endpoint.

3.5.1 Properties of Intervals

Let $\mathfrak{I}$ be the family of all intervals on the real line. We include in $\mathfrak{I}$ the null set $\varnothing$ and single points a = [a, a]. Then the intervals have the following properties:

1. The intersection of two intervals is an interval, that is,

$$A \in \Im, B \in \Im \text{ implies } A \cap B \in \Im$$

2. The union of two non-disjoint intervals is an interval, that is,

$$A \in \Im, B \in \Im, A \cap B \neq \emptyset \text{ implies } A \cup B \in \Im$$

3. The difference of two non-comparable intervals is an interval, that is,

$$A \in \Im, B \in \Im, A \not\subset B, B \not\subset A \text{ implies } A - B \in \Im$$

Example 3.1: Let A = {2, 4), B = (3, 8). Then
$A \cap B = (3, 4)$, $A \cup B = [2, 8)$
$A – B = [2, 3]$, $B – A = [4, 8)$

3.5.2 Infinite Intervals

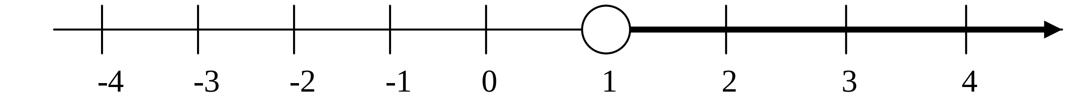
Sets of the form

$A = \{x \mid x > 1\}$
$B = \{x \mid x \geq 2$
$C = \{x \mid x < 3\}$
$D = \{x \mid x \leq 4\}$
$E = \{x \mid x \in \Re\}$

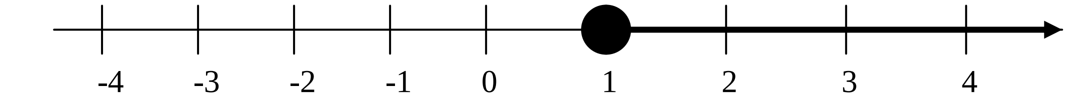
Are called infinite intervals and are also denoted by

$A = (1, \infty)$, $B = [2, \infty)$, $C = (-\infty, 3)$, $D = (-\infty, 4]$, $E = (-\infty, \infty)$

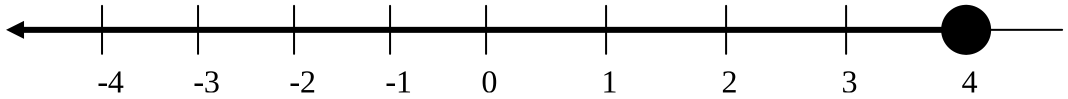
We plot these intervals on the real line as follows:



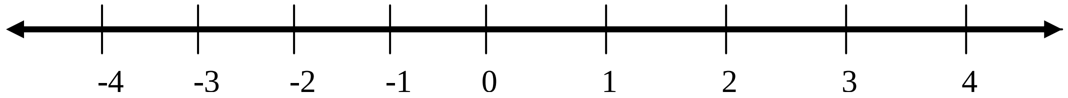
A is shaded



C is shaded

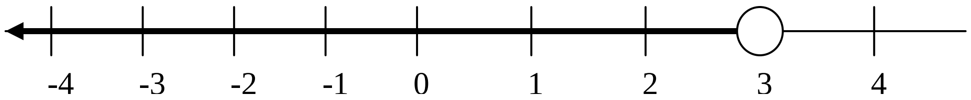


D is shaded



E is shaded

B is shaded



3.6 Bounded and Unbounded Sets

Let A be a set of numbers, then A is called ***bounded*** set if A is the subset of a finite interval. An equivalent definition of boundedness is

Definition 3.1 Set A is ***bounded*** if there exists a positive number M such that

$$|x| \le M$$

for all $x \in A$. A set is called ***unbounded*** if it is not bounded

Notice then, that A is a subset of the finite interval [- M, M].

Example 4.1 Let A = {1, ½ , 1/3,…..}. Then A is bounded since A is certainly a subset of the closed interval [0, 1].

Example 4.2 Let A = {2, 4, 6,…..}. Then A is an unbounded set.

Example 4.3 Let A = {7, 350, -473, 2322, 42}. Then A is bounded

Remark 3.3 If a set A is finite then, it is necessarily bounded.
If a set is infinite then it can be either bounded as in example 4.1 or unbounded as in example 4.2

4.0 CONCLUSION

The set of real numbers is of utmost importance in analysis. All (except the set of complex numbers) other sets of numbers are subsets of the set of real numbers as can be seen from the line diagram of the number system.

5.0 SUMMARY

In this unit, you have been introduced to the sets of numbers. The set of real numbers, $\Re$, contains the set of integers, Z, Rational numbers, Q, Natural numbers, N, and Irrational numbers, Q'.

Intervals on the real line are open, closed, open-closed or closed-open depending on the nature of the endpoints.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove: If a < b and B < c, then a < c
2. Under what conditions will the union of two disjoint interval be an interval?
3. If two sets R and S are bounded, what can be said about the union and intersection of these sets?

7.0 REFERENCES/FURTHER READING

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UNITS 4 FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, you will be introduced to the concept of functions, mappings and transformations. You will also be given instructive and typical examples of functions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify functions from statements or diagrams
- state whether a function is one-one or onto
- find composition function of two or more functions

3.0 MAIN CONTENTS

3.1 Definition

Suppose that to each element in a set A there is assigned by some manner or other, a unique element of a set $\Re$. We call such assignment of ***function.*** If we let f denote these assignments, we write;

$$f : A \longrightarrow B$$

which reads "f is a function of A onto B". The set A is called the ***domain*** of the function f, and B is called the ***co-domain*** of f. Further, if $a \in A$ the element in B which is assigned to a is called the ***image*** of a and is denoted by;

$$f(a)$$

which reads "f of a".

We list a number of instructive examples of functions.

Example 1.1 Let f assign to each real number its square, that is, for every real number x let $f(x) = x^2$. The domain and co-domain of f are both the real numbers, so we can write

$$f : \Re \rightarrow \Re$$

The image of -3 is 9; hence we can also write f(-3) = 9 or $f : 3 \rightarrow 9$

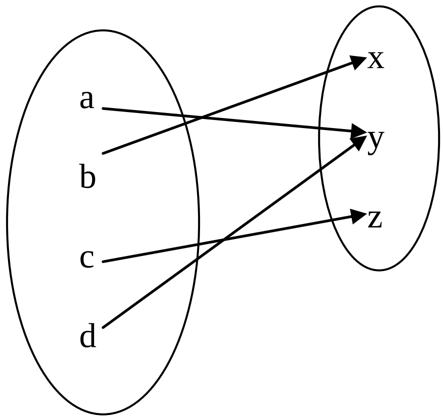
Example 1.2 Let f assign to each country in the world its capital city. Here, the domain of f is the set of countries in the world; The co-domain of f is the list of capital cities in the world. The image of France is Paris, that is, f(France) = Paris

Example 1.3 Let A = {a, b, c, d} and B = {a, b, c}. Define a function f of A into B by the correspondence f(a) = b, f(b) = c, f(c) = c and f(d) = b. By this definition, the image, for example, of b is c.

Example 1.4: Let A = {-1, 1}. Let f assign to each rational number in $\Re$ the number 1, and to each irrational number in $\Re$ the number -1. Then f: $\Re \rightarrow A$, and f can be defined concisely by

$$f(x) = \begin{cases} 1 \text{ if x is rational} \\ -1 \text{ if a is irrational} \end{cases}$$

Example 1.5: Let A = {a, b, c, d} and B = {x, y, z}. Let f: A → B be defined by the diagram:



Notice that the functions in examples 1.1 and 1.4 are defined by specific formulas. But this need not always be the case, as is indicated by the other examples. The rules of correspondence which define functions can be diagrams as in example 1.5, can be geographical as in example 1.2, or, when the domain is finite, the correspondence can be listed for each element in the domain as in example 1.4.

3.2 Mappings, Operators, Transformations

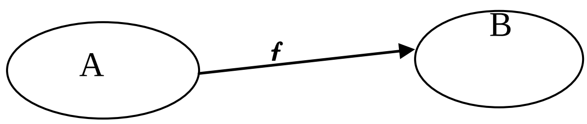
If A and B are sets in general, not necessarily sets of numbers, then a function f of A into B is frequently called a mapping of A into B; and the notation

f: : A ---> B

is then read "f maps A into B". We can also denote a mapping, or function, f of A into B by

$$A \xrightarrow{f} B$$

Or by the diagram



If the domain and co-domain of a function are both the same set, say

f: A ---> A

then f is frequently called an ***operator*** or ***transformation*** on A. As we will see later operators are important special cases of functions.

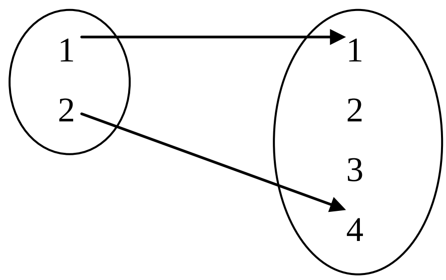
3.3 Equal Functions

If f and g are functions defined on the same domain D and if f(a) = g(a) for every a $\in$ D, then the functions f and g are equal and we write

$$f = g$$

Example 2.1: Let f(x) = x^2 where x is a real number. Let g(x) = x^2 where x is a complex number. Then the function f is not equal to g since they have different domains.

Example 2.2: Let the function f be defined by the diagram



Let a function g be defined by the formula g(x) = x^2 where the domain of g is the set {1, 2}. Then f = g since they both have the same domain and since f and g assign the same image to each element in the domain.

Example 2.3: Let f: $\Re \rightarrow \Re$ and g: $\Re \rightarrow \Re$. Suppose f is defined by f(x) = x^2 and g by g(y) = y^2. Then f and g are equal functions, that is, f = g. Notice that x and y are merely dummy variable in the formulas defining the functions.

3.4 Range of a Function

Let f be the mapping of A into B, that is, let f: A $\rightarrow$ B. Each element in B need not appear as the image of an element in A. We define the range of f to consist precisely of those elements in B which appear and the image of at least one element in A. We denote the range of f: A $\rightarrow$ B y f(A)

$$f(A)$$

Notice that f(A) is a subset of B. i.e f(A)

Example 3.1 Let the function f: $\Re \rightarrow \Re$ be defined by the formula f(x) = x^2. Then the range of f consists of the positive real numbers and zero.

Example 3.2 Let f: A $\rightarrow$ B be the function in Example 1.3.
Then f(A) = {b, c}

3.5 One – One (Injective) Functions

Let f map A into B. Then f is called a ***one-one or Injective function*** if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, f: A $\rightarrow$ B is one-one if f(a) = f(a') implies a = a` or, equivalently, a = a` implies f(a) $\neq$ f(a`)

Example 4.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then f is not a one-one function since f(2) = f(-2) = 4, that is, since the image of two different real numbers, 2 and -2, is the same number, 4.

Example 4.2: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^3$. Then f is a one-one mapping since the cubes of the different real numbers are themselves different.

Example 4.3: The function f which assigns to each country in the world, its capital city is one-one since different countries have different capitals, that is no city is the capital of two different countries.

3.6 Onto (Subjective) Function

Let f be a function of A into B. Then the range f(A) of the function f is a subset of B, that is, f(A) $\subset$ B. If f(A) = B, that is, if every member of B appears as the image of at least one element of A, then we say "f is a function of A onto B", or "f maps A onto B", of "f is an ***onto or Subjective function".***

Example 5.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then f is not an onto function since the negative numbers do not appear in the range of f, that is no negative number is the square of a real number.

Example 5.2: Let f: A $\rightarrow$ B be the function in Example 1.3. Notice that f(A) = {b, c}. Since B = {a, b, c} the range of f does not equal co-domain, i.e. is not onto.

Example 5.3: Let f: A $\rightarrow$ B be the function in example 1.5: Notice that

f(A) = {x, y, z} = B

that is, the range of f is equal to the co-domain B. Thus f maps A onto B, i.e. f is an onto mapping.

3.7 Identity Function

Let A be any set. Let the function f: A → A be defined b y the formula f(x) = x, that is, let f assign to each element in A the element itself. Then f is called the identity function or the identity transformation on A. We denote this function by 1 or by 1_A.

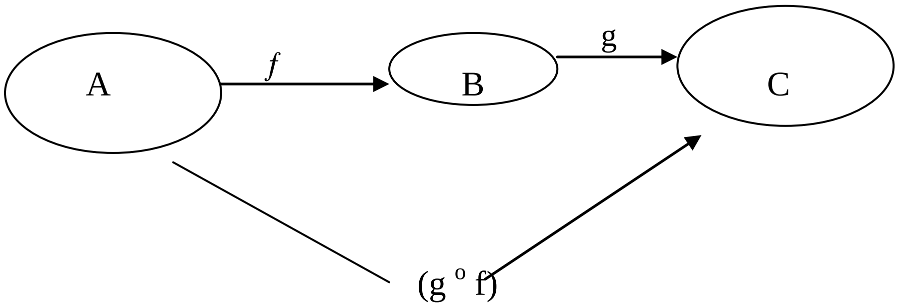
3.8 Constant Functions

A function f of A onto B is called a ***constant function*** if the same element of b∈B is assigned to every element in A. In other words, f: A → B is a constant function if the range of f consists of only one element.

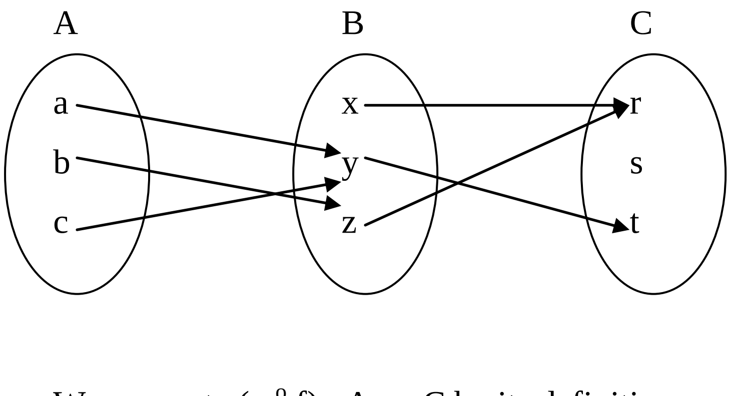
$(g \circ f) : A \rightarrow C$ by

$$(g \circ f)(a) \equiv g(f(a))$$

Here ≡ is used to mean equal by definition. We can now complete our diagram:



Example 7.1: Let f: A → B and g: B → C be defined by the diagrams



We compute $(g \circ f)$: A → C by its definition:

$(g \circ f)(a) \equiv g(f(a)) = g(y) = t$
$(g \circ f)(b) \equiv g(f(a)) = g(z) = r$
$(g \circ f)(c) \equiv g(f(a)) = g(y) = t$

Notice that the function (g o f) is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.

Example 7.2: To each real number let f assign its square, i.e. let $f(x) = x^2$. To each real number let g assign the number plus 3, i.e. let $g(x) = x + 3$. Then

$(g^{o} f)(x) \equiv f(g(x)) = f(x+3) = (x+3)^2 = x^2 + 6x + 9$
$(g^{o} f)(x) \equiv g(f(x)) = g(x^2) = x^2 + 3$

Remark 4.1: Let f: A → B. Then

$$I_B {}^{o} f = f \text{ and } f {}^{o} 1_A = f$$

that is, the product of any function and identity is the function itself.

3.9.1 Associativity of Products of Functions

Let f: A → B, g: B → C and h: c → D. Then, as illustrated in Figure 4-1, we can form the production function $g^{o} f$: A → C, and then the function $h^{o} (g^{o} f)$: A → D.

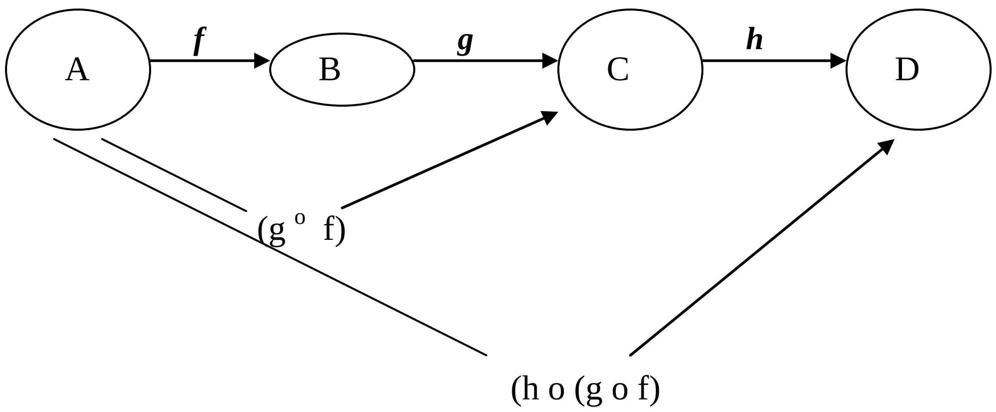


Fig. 4.1

Similarly, as illustrated in Figure 4-2, we can form the product function h^{o} g: B --> D and then the function (h o g) o f: A → D.

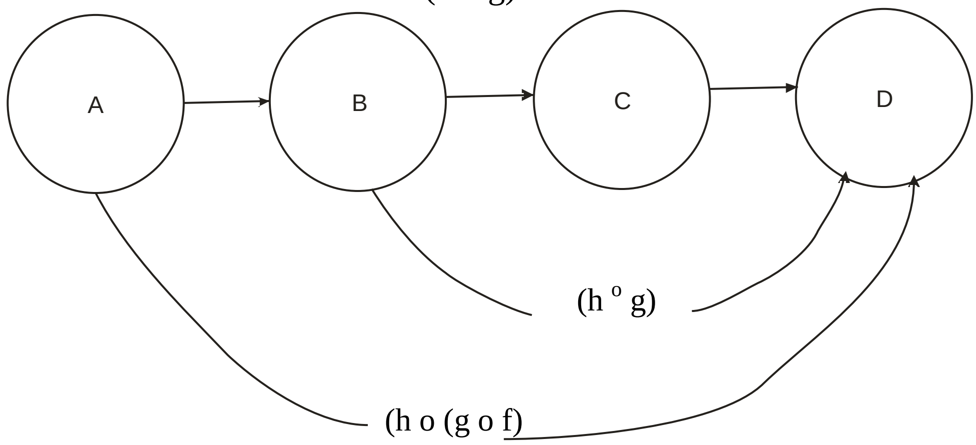


Fig 4.2

Both h^{o} (g^{o} f) and (h^{o} g) ^{o} f are function of A into D. A basic theorem on functions states that these functions are equal. Specifically,

Theorem 4.1: Let f: A → B, B → C and h: C → D. Then
(h o g) o f = h o (g o f)

In view of Theorem 4.1, we can write

$$h^{o} g^{o} f: A \to D$$

without any parenthesis.

3.10 Inverse of a Function

Let f be a function of A into B, and let b ∈ B. Then the ***inverse*** of b, denoted by

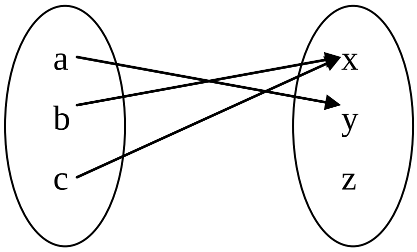
$$f^{-1}(b)$$

Consist of those elements in A which are mapped onto b, that is, those element in A which have m as their image. More briefly, if f: A → B then

$$f^{-1}(b) = \{x\}\ x \in A; f(x) = b\}$$

Notice that f^{-1} (b) is always a subset of A. We read f^{-1} as "f inverse".

Example 8.1: Let the function f: A ---> B be defined by the diagram



Then $f^{-1}(x) = \{b, c\}$, since both b and c have x as their image point.
Also, $f^{-1}(y) = \{a\}$, as only a is mapped into y. The inverse of z, $f^{-1}(z)$, is the null set ∅, since no element of A is mapped into z.

Example 8.2: Let f: ℜ ---> ℜ, the real numbers, be defined by the formula $f(x) = x^2$. Then $f^{-1}(4) = \{2, -2\}$, since 4 is the image of both 2 and -2 and there is no other real number whose square is four. Notice that $f^{-1}(-3) = \emptyset$, since there is no element in ℜ whose square is -3.

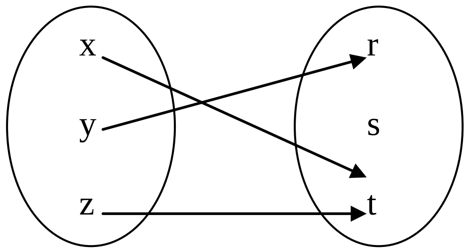
Example 8.3: Let f be a function of the complex numbers into the complex numbers, where f is defined by the formula $f(x) = x^2$. Then $f^{-1}(3) = \{\sqrt{3}i, -\sqrt{3}i\}$, as the square of each of these numbers
is -3.

Notice that the function in Example 8.2 and 8.3 are different although they are defined by the same formula

We now extend the definition of the inverse of a function. Let f: A --> B and let D be a subset of B, that is, $D \subset B$. Then the inverse of D under the mapping f, denoted by $f^{-1}(D)$, consists of those elements in A which are mapped onto some element in D. More briefly,

$$f^{-1}(D) = \{x \mid x \in A, f(x) \in D\}$$

Example 9.1: Let the function f: A --> B be defined by the diagram



Then f^{-1} ({r, s}) = {y}, since only y is mapped into r or s. Also f^{-1} ({r, t}) = {x, y, z} = A, since each element in A as its image r or t.

Example 9.2: Let f: $\Re$ ---> $\Re$ be defined by $f(x) = x^2$, and let
D = [4, 9] = $\{x \mid 4 \le x \le 9\}$

Then

f -1 (D) = $\{x \mid -3 \le x \le -2 \text{ or } 2 \le x \le 3\}$

Example 9.3: Let f: A ---> B be any function. Then f^{-1} (B) = A, since every element in A has its image in B. If f(A) denote the range of the function f, then

$$f^{-1}(f(A)) = A$$

Further, if $b \in B$, then

$$f^{-1}(b) = f^{-1}(\{b\})$$

Here f^{-1} has two meanings, as the inverse of an element of B and as the inverse of a subset of B.

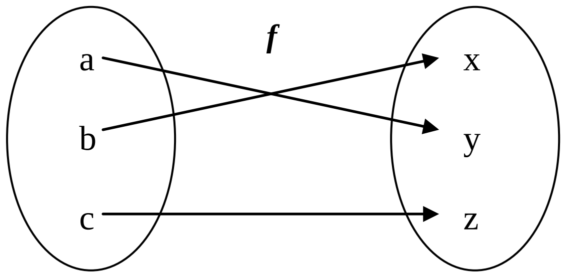
3.11 Inverse Function

Let f be a function of A into B. In general, $f^{-1}(b)$ could consist of more than one element or might even be empty set $\varnothing$. Now if f: $A \rightarrow B$ is a one-one function and an onto function, then for each $b \in B$ the inverse f^{-1} (b) will consist of a single element in A. We therefore have a rule that assigns to each $b \in B$ a unique element $f^{-1}(b)$ in A. Accordingly, f^{-1} is a function of B into A and we can write

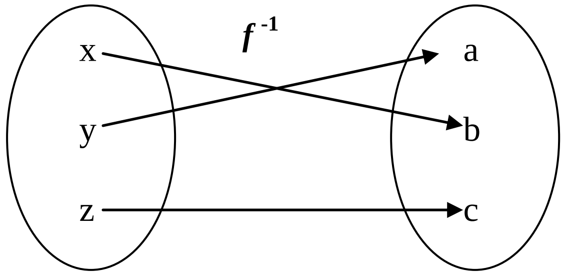
$$f^{-1} : B \rightarrow A$$

In this situation, when f: $A \rightarrow B$ is one-one and onto, we call f^{-1} the inverse function of f.

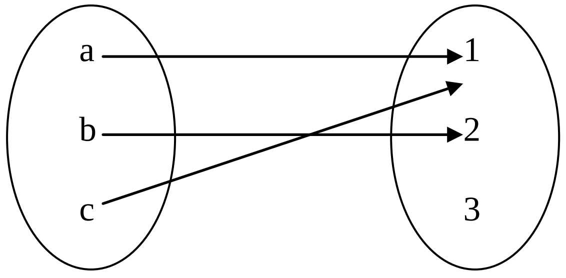
Example 10.1: Let the function f: A → B be defined by the diagram



Notice that f is one-one and onto. Therefore f -1, the inverse function exists We describe f^{-1}: B → A by the diagram



Example 6.1: Let the function f be defined by the diagram:

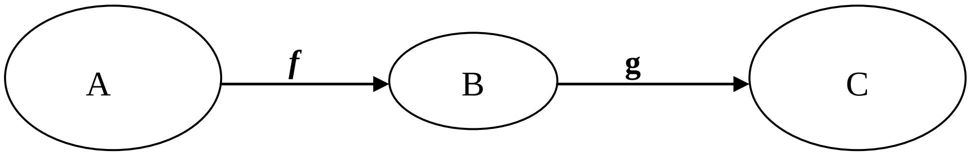


Then f is a constant function since 3 is assigned to every element in A.

Example 6.3: Let f: ℜ → ℜ be defined by the formula f(x) = 5. Then f is a constant function since 5 is assigned to every element.

3.9 Product Function

Let f be a function of A and B and let g be a function of B, the co-domain of f, into C. We illustrate the function below.



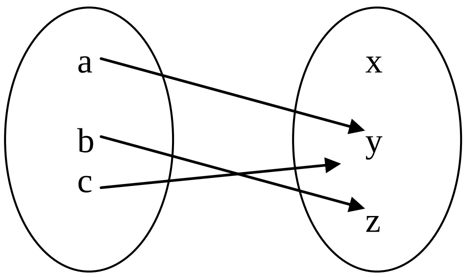
Let a ∈ A; then its image f (x) is in B which is the domain of g. Accordingly, we can find the image of f (a) under the mapping of g, that is, we can find g (f(a)). Thus, we have a rule which assigns to each element a∈A a corresponding element (f(a))∈ C. In other words, we have a function of A into C. This new function is called the ***product function*** or ***composition function*** of f and g and it is denoted by

$$(g \circ f) \text{ or } (gf)$$

More briefly, if f: A → B and g: B → C then we define a function

Notice further, that if we send the arrows in the opposite direction in the first diagram of f we essentially have the diagram of f^{-1}.

Example 10.2: Let the function f: A → B be defined by the diagram

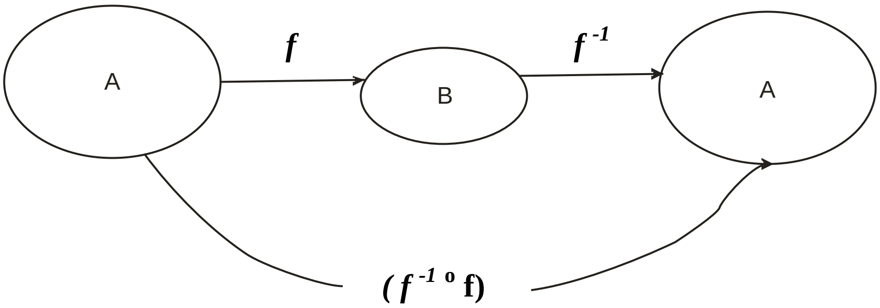


Since f(a) = y and f(c) = y, the function f is not one-one. Therefore, the inverse function f^{-1} does not exist. As $f^{-1}(y) = \{a, c\}$, we cannot assign both a and c to the element y ∈ B.

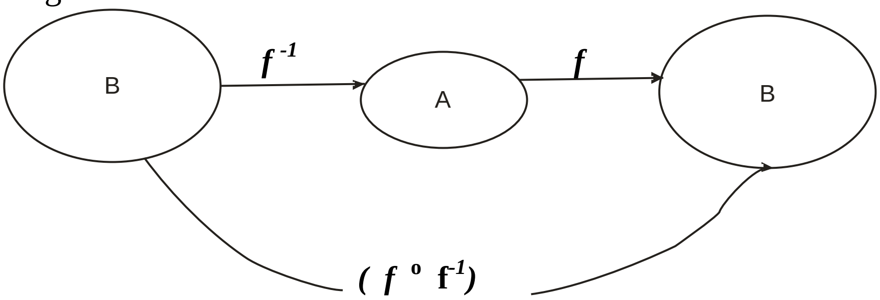
Example 10.3: Let f: ℜ → ℜ, the real numbers, be defined by $f(x) = x^3$. Notice that f is one-one and onto. Hence f^{-1}:
ℜ → ℜ exists. In fact, we have a formula which defines the inverse function, $f^{-1}(x) = \sqrt[3]{x}$.

3.11.1 Theorems on the inverse Function

Let a function f: A → B have an inverse function f^{-1}: B → A. Then we see by the diagram



That we can form the product ($f^{-1} \circ f$) which maps A into A, and we see by the diagram



That we can form the product function ($f \circ f^{-1}$) which maps B into B. We now state the basic theorems on the inverse function:

Theorem 4.2: Let the function f: A → B be one-one and onto; i.e. the inverse function f^{-1}: B → A exists. Then the product function

$$\underline{(f^{-1} \circ f):}\ A \to A$$

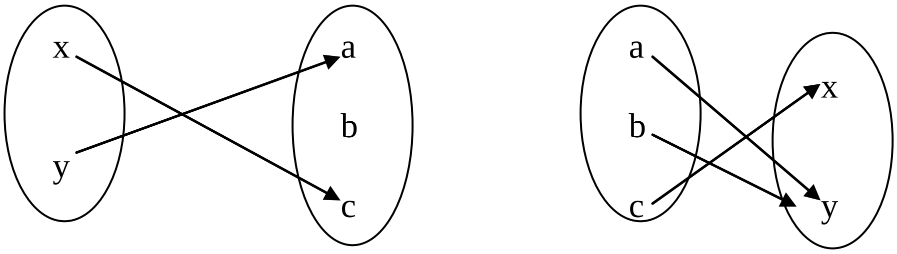
is the identity function on A, and the product function

$$\underline{(f \circ f^{-1}):}\ B \to B$$

is the identity function on B.

Theorem 4.3: Let f: A → B and g: B → A. Then g is the inverse function of f, i.e. g = f – 1, if the product function ($g \circ f$): A → A is the identity function on A and ($f \circ g$): B → B is the identity function on B.

Both conditions are necessary in Theorem 4.3 as we shall see from the example below



(a) (b)

Now define a function g: B → A by the diagram (b) above.

We compute (g ^{o} f): A → A,

(g ^{o} f)(x) = g (f(x)) = g(c) = x and

(g ^{o} f)(y) = g (f(y)) = g(a) = y

Therefore the product function (g ^{o} f) is the identity function on A. But g is not the inverse function of f because the product function (f ^{o} G) is not the identity function on B, f not being an onto function.

4.0 CONCLUSION

I believe that by now you fully grasp the idea of functions, mappings and transformations. This knowledge will be built upon in subsequent units.

5.0 SUMMARY

Recall that in this unit we have studied concepts such as mapping and functions. We have also examined the concepts of one-to-one and onto functions. This concept has allowed us to explain equality between two set. We also established in the unit that the inverse of f: A – N usually denoted f^{-1}, exit, if f is a one-to-one and onto function.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let the function f: $R^{\#} \rightarrow R^{\#}$ be defined by
   f (X) = {1 if x is rational]
   {-1 if x is irrational.
   a. Express f in words
   b. Suppose the ordered pairs (x + y, 1) and (3, x – y) are equal.
   Find x and y.
2. Let M = {1, 2, 3, 4, 5} and let the function f: M → $\Re$ be defined by
   $f(x) = x^2 + 2x - 1$
   Find the graph of f.
3. Prove: A x (B ∩ C) = (A x B) ∩ (A x C)
4. Prove A ⊂ B and C ⊂ D implies (A x C) ⊂ (B x D).

7.0 REFERENCES/FURTHER READING

Lipschutz; S. (2004). *Schaum's Outline Series: Theory and Problems of Set Theory and Related Topics,* pp. 1 – 133. McGraw – Hill NY

Onwukwe, C. E. (2003). *Introduction Statistics*. Calabar: Favoured Trust Limited

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UNIT 5 PRODUCT SETS AND GRAPHS OF FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we are going to define a type of set that not only gives a better understanding of Cartesian coordinate but also brings the concept of real-valued functions to the fore.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the ordered pairs, given two sets
- find the ordered pairs corresponding to the points on the Cartesian coordinate diagram
- find the graph of functions
- state whether or not a set of ordered pairs of a given set, say A, is a function of A into itself.

3.0 MAIN CONTENT

3.1 Ordered Pairs

Intuitively, an ***ordered pair*** consists of two elements, say a and b, in which one of them, say a, is designated as the first element and the other as the second element. An ordered pair is denoted by (a, b)

Two ordered pairs (a, b) and (c, d) are equal if and only if
a = c and b = d.

Example 1.1: The ordered pairs (2, 3) and (3, 2) are different
Example 1.2: The points in the Cartesian plane shown in fig 5.1 below represent ordered pairs of real numbers.
Example 1.3: The set {2, 3} is not an ordered pair since the elements 2 and 3 are not distinguished
Example 1.4: Ordered pairs can have the same first and second elements such as (1, 1), (4, 4) and (5, 5).

Although the notation (a, b) is also used to denote an open interval, the correct meaning will be clear from the context.

Remark 5.1: An ordered pair (a, b) can be defined rigorously by

$$(a, b) = \{ \{a\}, \{a, b\} \}$$

From this definition, the fundamental property of ordered pairs can be proven:

$$(a, b) = (c, d) \text{ implies } a = c \text{ and } b = d$$

3.2 Product Set

Let A and B be two sets. The ***product set*** of A and B consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. it is denoted by

$$A \times B.$$

Which reads "A cross B". More precisely

$$A \times B = \{ (a, b) \mid a \in A, b \in B\}$$

Example 2.1: Let A = {1, 2, 3} and B = {a, b}. Then the product set
A x B = {(1, a), (1, b), (2, a), (2, b), (3, a) (3, b)}
Example 2.2: Let W = {s, t}. Then
W x W = {(s, s), (s, t), (t, s), (t, t)}
Example 2.3: The Cartesian plane shown in Fig 5.1 is the product set of the real numbers with itself, i.e. $\Re \times \Re$

The product set A x B is also called the ***Cartesian Product*** of A and B. it is named after the mathematician Descartes who, in the seventeenth century, first investigated the set $\Re$ x $\Re$. It is also for this reason that $\Re$ x $\Re$, as pictured in Fig. 5.1, is called the Cartesian Plane.

Remark 5.2: If set A has n elements and set B has m elements then the product set A x B has *n times m* elements, i.e. *nm* elements. If either A or B is the null set then A x B is also the null set. Lastly, if either A or B is infinite and the other is not empty, then a x B is infinite.

Remark 5.3: The Cartesian product of two sets is not commutative; more specifically,

$$A \times B \neq B \times A$$

Unless A = B or one of the factors is empty.

3.3 Coordinate Diagrams

You are familiar with the Cartesian plane $\Re$ x $\Re$, as shown in Fig 5.1 below. Each point P represents an ordered pair (a, b) of real numbers. A vertical through P meets the horizontal axis at *a* and a horizontal line through P meets the vertical axis at *b* as in Fig. 5.1.

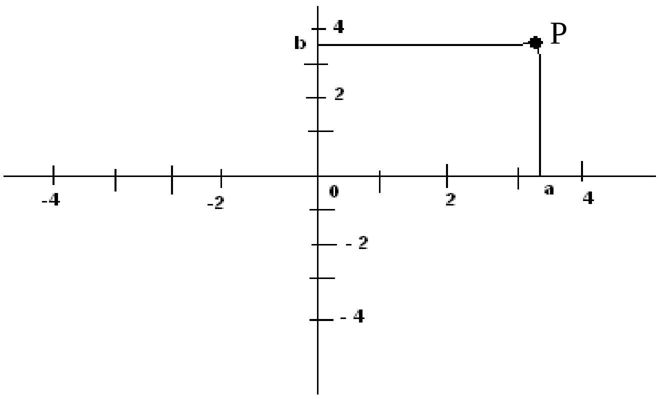


Fig. 5.1

The Cartesian product of any two sets, if they do not contain too many1 elements, can be displayed on a coordinate diagram in a similar manner. For example, if A = {a, b, c, d} and B = {x, y, z}, then the coordinate diagram of A x B is as shown in Fig 5.2 below. Here the elements of A are displayed on the horizontal axis and the elements of B are displayed on the vertical axis. Notice that the vertical lines through the elements of A and the horizontal lines through the elements of B meet 12 points. These points represent A x B in the obvious way. The point P is the ordered pair (c, y).

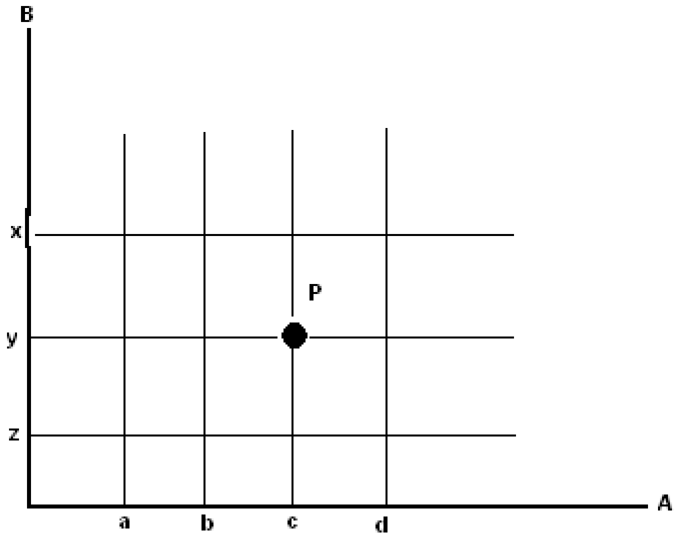


Fig 5.2

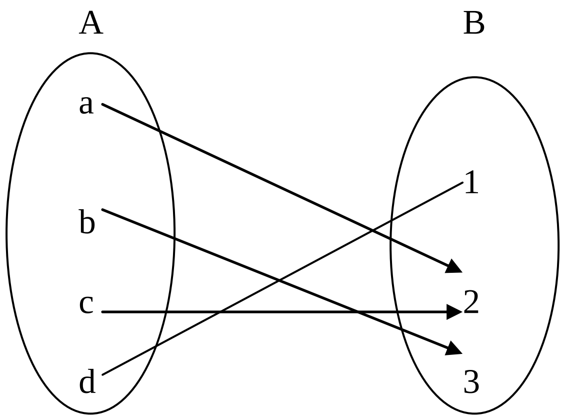
3.4 Graph of a Function

Let f be a function of A into B, that is, let f: A ---> B. The graph f* of the function f consists of all ordered pairs in which $a \in A$ appears as a first element and its image appears as its second element. In other words,

$$f^* = \{(a, b) \mid a \in A, b + f(a)\}s$$

Notice that f*, the graph of f: A --> B, is a subset of A x B.

Example 3.1: Let the function f: A --> B be defined by the diagram



Then f(a) = 2, f(b) = 3, f(c) = 2 and f(d) = 1. Hence the graph of f is

$$F^* = \{(a, 2), (b, 3), (c, 2), (d, 1)\}$$

Example 3.2: Let W = {1,2,3,4}. Let the function f: W --> $\Re$ be defined by

$$f(x) = x + 3$$

Then the graph of f is

$$f^* = \{(1, 4), (2, 5), (3, 6), (4, 7)\}$$

Example 3.3: Let N be the natural numbers 1, 2, 3,………Let the function g: N --> N be defined by

$$g(x) = x^3$$

Then the graph of g is

$$g^* = \{1,1), (2,8), (3, 27), (4, 64),.....\}$$

3.4.1 Properties of the Graph of a function

Let f: A ---> B. We recall two properties of the function f. First, for each element $a\in A$ there is assigned an element in B. Secondly, there is only one element B which is assigned to each $a\in A$. In view of these properties of f, the graph f* of f has the following two properties:

Property 1: For each $a\in A$, there is an ordered pair $(a, b) \in f^*$
Property 2: Each $a\in A$ appears as the first element in only one ordered pair in f*, that is
$(a, b) \in f^*$, $(a, c) \in f^*$ implies $b = c$

In the following examples, let A = {1,2,3,4} and
B = {3,4,5,6}

Example 4.1: The set of ordered pairs {(1,5), (2,3), (4,6)} cannot be the graph of a function of A into B since it violates property 1. Specifically, $3\in A$ and there is no ordered pair in which 3 is a first element.

Example 4.2: The set of ordered pairs

$$\{(1,5), (2,3), (3,6), (4,6), (2,4)\}.$$

cannot be the graph of a function of A into B since it violates Property 2, that is, the element $2\in A$ appears as the first element in two different ordered pairs (2, 3) and (2,4)

3.5 Graphs and Coordinate Diagrams

Let f* be the graph of a function f: A → B. As f* is a subset of A x B, it can be displayed, i.e. graphed, on the coordinate diagram of A x B.

Example 5.1: Let $f(x) = x^2$ define a function on the interval $-2 \le x \le 4$. Then the graph of f is displayed in Fig 5.3 below in the usual way:

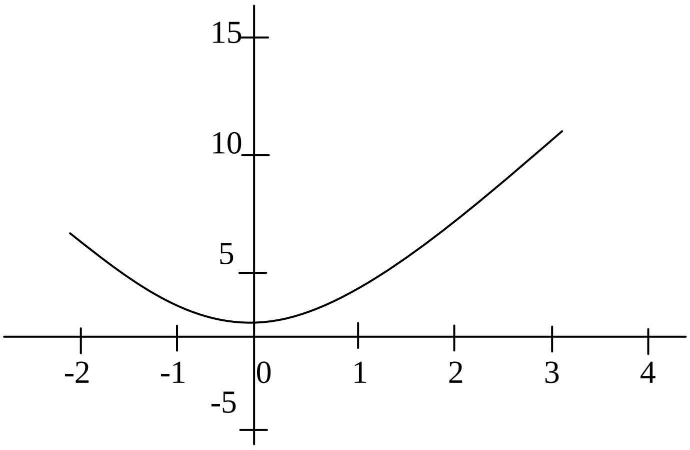
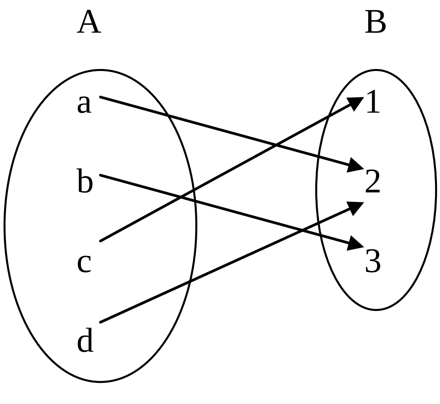


Fig 5.3

Example 5.2: Let a function f: A → B be defined by the diagram shown in Fig 5.4 below

Here f*, the graph of f, consist of the ordered pairs (a, 2), (b, 3), (c, 1) and (d, 2). Then f* is displayed on the coordinate diagram A x B as shown in Fig 5.5 below.



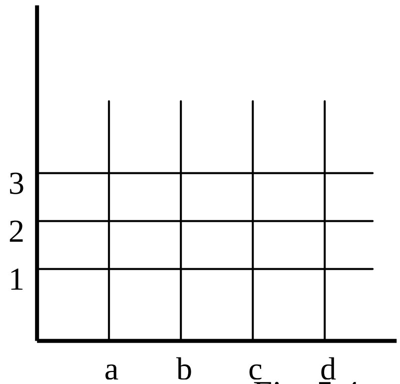


Fig. 5.4

Fig. 5.5

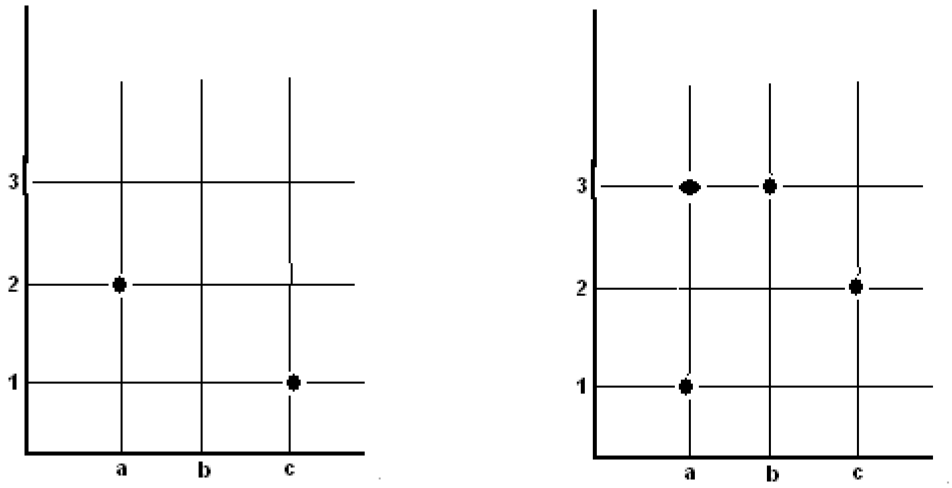
3.5.1 Properties of Graphs of Functions on Coordinate Diagrams

Let f: A → B. Then f*, the graph of f, has the two properties listed previously:

Property 1: For each $a \in A$, there is an ordered pair $(a, b) \in f^*$
Property 2: If $(a, b) \in f^*$ and $(a, c) \in f^*$, then b = c.
Therefore, if f* is displayed on the coordinate diagram of A x B, it has the following two properties:

Property 1: Each vertical line will contain at least one point of f*
Property 2: Each vertical line will contain only one point of f*

Example 6.1: Let a = {a, b, c} and B = {1, 2, 3}. Consider the sets of points in the two coordinate diagrams of A x B below.

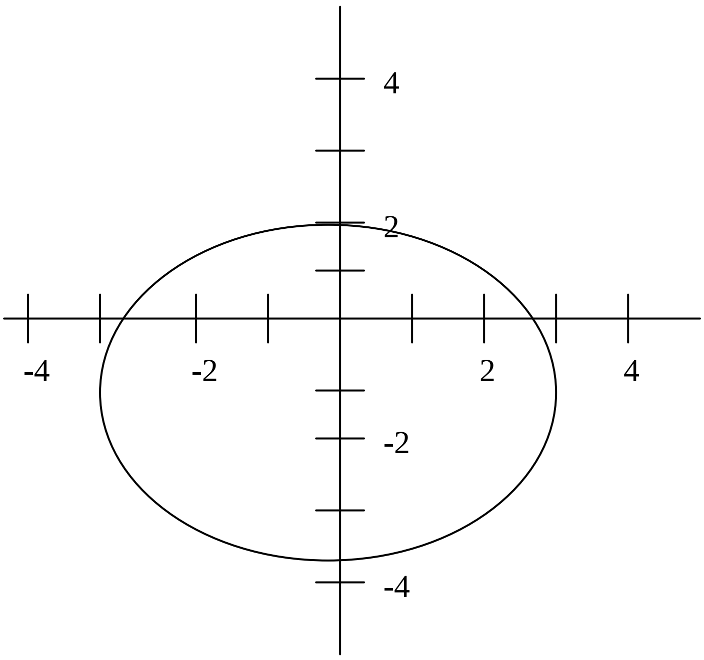


(1) (2)

In (1), the vertical line through *b* does not contain a point of the set; hence the set of points cannot be the graph of a function of A into B.

In (2), the vertical line through *a* contains two points of the set, hence this set of point cannot be the graph of a function of A into B.

Example 6.2: The circle $x^2 + y^2 = 9$, pictured below, cannot be the graph of a function since there are vertical lines which contain more than one point of the circle.



$x^2 + y^2 = 9$ is plotted

3.6 Functions as Sets of Ordered Pairs

Let f* be a subset of A x B, the Cartesian product of sets A and B; and let f* have the two properties discussed previously:

Property 1: For each a ∈ A, there is an ordered pair (a, b) ∈f*.
Property 2: No two different ordered pairs in f* have the same first element.

Thus, we have a rule that assigns to each element a ∈ A, the element b ∈ B that appear in the ordered pair (a, b) ∈ f*. Property 1 guarantees that each element in A will have an image, and Property 2 guarantees that the image is unique. Accordingly, f* is a function of A into B.

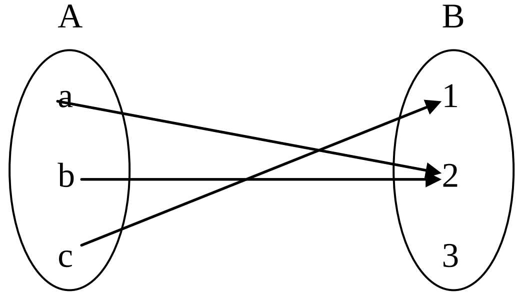
In view of the correspondence between functions f: A → B and subset of A x B with property 1 and property 2 above, we redefine a function by the

Definition 5.1: A function f of A into B is a subset of A x B in which each a ∈ A appears as the first element in one and only one ordered pair belonging to f.

Although, this definition of a function may seem artificial, it has the advantage that it does not use such undefined terms as "assigns", "rules", "correspondence".

Example 7.1: Let A = (a, b, c) and B = (1, 2, 3). Furthermore, let
f = {(a, 2), (c, 1), (b, 2)}

Then f has Property 1 and Property 2. Hence f is a function of A into B, which is also illustrated in the following diagram:



Example 7.2: Let V = {1, 2, 3} and W = {a, e, I, o, u}.Also let

f + {(1, a), (2, e), (3, 1), (2, u)}

Then f is not a function of V into W since two different ordered pairs in f, (2, e) and (2, u), have the same first element. If f is to be a function of V into W, then it cannot assign both e and u to the element $2 \in V$.

Example 7.3: Let S = {1,2,3,4} and T = {1,3,5}. Let

$$f = \{(1,1), (2, 5), (4, 3)\}$$

Then f is not a function of S into T since $3 \in S$ does not appear as the first element in any ordered pair belonging to f.

The geometrical implication of Definition 5.1 is stated in.

Remark 5.4: Let f be the set of points in the coordinate diagram of A x B. if every vertical line contains one and only point of f, then f is a function of A into B.

Remark 5.5: Let the function f: A $\rightarrow$ B be one-one and onto. Then the inverse function f^1 consists of those ordered pairs which when reversed, i.e. permuted, belong to f. More specifically,

$$f^1 = \{(b, a) \mid (a, b) \in f\}$$

3.7 Product Sets in General

The concept of a product set can be extended to more than two sets in a natural way. The Cartesian product of sets A, B, and C, denoted by

$$A \ x B x C$$

Consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$ and $c \in C$. Analogously, the Cartesian product of n sets A_1, A_2,……A_n, denoted by

$$A_1 \text{ x } A_2 \text{ x} \ldots \text{ x } A_n$$

Consists of all ordered n-tuples (a_1, a_2,…,an) where $a_1 \in A$,…., $a_n \in A$. Here an ordered n-tuple has the obvious intuitive meaning, that is,, it consists of n elements, not necessarily distinct, in which one of them is designated as the first element, another as the second element, etc.

Example 8.1: In three-dimensional Euclidean geometry each point represents an ordered triplet, i.e. its *x*-component, its *y*-component and its *z* – component.

Example 8.2: Let A = {a, b}, B = {1, 2, 3} and C = {x, y}. Then

A x B x C = {(a, 1, x), (a, 1, y), (a, 2, x)
(a, 2, y), (a, 3, x), (a, 3, y)
(b, 1, x), (b, 1, y), (b, 2, x)
(b, 2, y), (b, 3, x), (b, 3, y)

4.0 CONCLUSION

In this unit you have studied concepts such as ordered pairs, product sets, co-ordinate diagram, functions as set of ordered pairs.

We have also learnt about how to represent function on a graph. We require the mastery of the above concepts in the understanding of the subsequent units.

5.0 SUMMARY

That an ordered pair is denoted by (a, B), a $\in$ A and b $\in$ B. Two ordered pairs (a, b) and (c, d) are if and only if a = c, and b = d.

That is A and B are two sets such that a $\in$ A and b$\in$B them the product of A and B is denoted by A × B = {(a, b)} a $\in$ A, b $\in$ B}

That the Cartesian place is the product set of real number with itself i.e
IR × IR

That the concept of product can be extended to more than two sets in a natural ways i.e if A, B and C are sets then the product of A, B and C is denoted as
A × B × C = {(a, b, c) a $\in$ A, b $\in$ B, c C}

Generally the Cartesian product of n sets A_1, A_2, ……..A_n is denoted by

A_1 A_2 A_3 …. × A_n = {(a1, a2, a3 ……an)}
a_1 A_1, a_2 A_2 ………, an $\in$ An }

6.0 TUTOR-MARKED ASSIGNMENTS

1. Suppose the ordered pairs (x + y, 1) and (3, x – y) are equal. Find x and y.
2. Let M = {1,2,3,4,5} and let the function f: M $\to \Re$ be defined by

   $$f(x) = x^2 + 2x - 1$$

   Find the graph of f.

3. Prove: A x (B $\cap$ C) = (A x B) $\cap$ (A x C)
4. Prove A $\subset$ B and C $\subset$ D implies (A x C) $\subset$ (B x D).

7.0 REFERENCES/FURTHER READING

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