MODULE 2

UNIT 1 RELATIONS

CONTENTS

1.0 INTRODUCTION

From the concept of ordered pairs, product set or Cartesian product we can draw relations based on propositional functions defined on the Cartesian product of two sets

This is what will be developed in this unit

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive relations as ordered pairs between two sets based on open sentences.
- find the Domain, Range and Inverse of a relation
- define, with examples, the different kinds of relations
- state whether or not a relation defined on a set is a function of the set into itself.

3.0 MAIN CONTENT

3.1 Propositional Functions, Open Sentences

A **Propositional function** defined on the Cartesian product A x B of two sets A and B is an expression denoted by

$$P(x,y)$$

Which has the property that P(a,b), where a and b are substituted for the variables x and y respectively in P(x,y), is true or false for any ordered pair (a,b) $\in$A x B. For example, if A is the set of playwright and B is the set of plays, then

$$P(x,y) = \text{“x wrote y”}$$

Is a propositional function on A x B, In particular,

P (Shakespeare, Hamlet) = “Shakespeare wrote Hamlet”
P(Shakespeare, Things Fall Apart) = “Shakespeare wrote Things Fall Apart”

Are true and false respectively.

The expression P(x,y) by itself shall be called an open sentence in two variables or, simply, an open sentence. Other examples of open sentences are as follows:

Example 1.1: “x is less than y”
Example 1.2: “x weighs y kilograms”
Example 1.3: “x divides y”
Example 1.4: “x is wife of y”
Example 1.5: “The square of x plus the square of y is sixteen”, i.e “$x^2 + y^2 = 16$”
Example 1.6: “Triangle x is similar to triangle y”

In all of our examples there are two variable. It is also possible to have open sentences in one variable such as “x is in the United Nations”, or in more than two variables such as “x times y equals z”

3.2 Relations

A ***relation*** R consists of the following

1. a set A
2. a set B
3. an open sentence P(x,y) in which P(a, b) is either true or false for any ordered pair (a,b) belonging to A x B

We then call R a ***relation from A to B*** and denote it by

$$R = (A, B, P(x,y))$$

Furthermore, if (a,b) is true we write

$$a\ R\ b$$

which reads "a is related to B". On the other hand, if P(a,b) is not true we write

$$a \not R\ b$$

which reads "a is not related to b"

Example 2.1: Let $R_1 = (\Re, \Re\ P(x,y))$ where P(x,y) reads "x is less than y". Then R_1 is a relation since P(a,b), i.e "a< b", is either true or false for any ordered pair (a,b) of real numbers. Moreover, since $P(2, \pi)$ is true we can write

$$2\ R_1\ \pi$$

and since $p(5, \sqrt{2})$ is false we can write

$$5 \not R_1 \sqrt{2}$$

Example 2:2 Let R_2 = (A,B P(x,y)) where A is the set of men, B is the set of women, and P(x,y) reads "x is the husband of y". then R_2 is a relation

Example 2:3 Let R_3 = (N, N, P(x,y)) where N is the natural numbers and P(x,y) reads "x divides y". Then R_3 is a relation. Furthermore,

$$3\ R_3\ 12,\ 2 \not R_3\ 7,\ 5\ R_3\ 15,\ 6\ R_4\ 13$$

Example 2:4 Let R_4 = (A, B, P(x,y)) where A is the set of men, B is the set of women and P(x,y) reads "x divides y". Then R4 is not a relation since P(a,b) has no meaning if a is a man and b is a woman.

Example 2:5 Let R_5 = (N, N, P(x,y)) where N is the natural numbers and P(x,y) reads "x is less than y". Then R_5 is a relation.

Notice that R_1 and R_5 are not the same relation even though the same open sentence is used to define each relation

Let R = (A, B, P(x,y)) be a relation. We then say that the open sentence P(x,y) ***defines a relation*** from A to B. Furthermore, if A = B, then we say that P(x,y) defines a relation in A, or that R is a relation in A.

Example 2:6 The open sentence P(x,y), which reads "x is less than y", defines a relation in the rational numbers

Example 2:6 The open sentence "x is the husband of y" defines a relation from the set of men to the set of women.

3.3 Solution Sets and Graphs of Relations

Let R = (A, B, P(x,y)) be a relation. The ***Solution set R**** of the relation R consists of the elements (a,b) in A x B for which P(a,b) is true. In other words

$$R^* = \{(a,b) \mid a \in A, b \in B, P(a,b) \text{ is true}\}$$

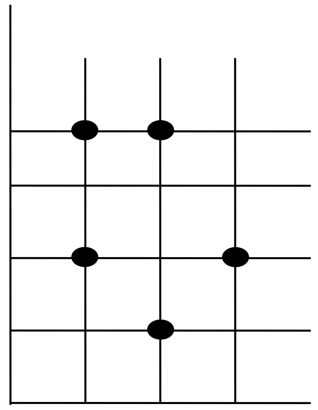
Notice that R*, the solution set of a relation R from A B, is a subset of A x A. Hence R* can be displayed, i.e. plotted or sketched, on the coordinate diagram of A x A

The graph of a relation R from A to B consists of those points on the coordinate diagram of A x A which belongs to the solution set of R.

Example 3:1 Let R = (A, B, P(x,,, y)) where A = {2,3,4}, and B = {3, 4, 5}, and P(x,y) reads "x divides y". Then the solution set of R is:

$$R^* = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$$

The solution set of R is displayed on the coordinate diagram of A x B as shown in Fig.6.2 below



Example 3:2 Let R be the relation in the real numbers defined by

$$y<x + 1$$

The shaded area in the coordinate diagram of ℜ x ℜ shown in Fig. 6.2 above consists of the points which belong to ℜ, the solution set of R, that is, the graph of R.

Notice that ℜ consists of the points below the line y = x + 1. The line y = x +1 is dashed in order to show that the points on the line do not belong to ℜ.

3.4 Relations as Sets of Ordered Pairs

Let R* be any subset of A x B. We can define a relation R = (A, B P(x,y)) where P(x,y) read

"The ordered pair (x,y) belongs to R*"

The solution set of this relation R is the original set R*. Thus to every relation R = (A, B P(x,y)) there corresponds a unique solution set R* which is a subset of A x B, and to every subset R* of A x B there corresponds a relation R = (A, B P(x,y)) for which R* is "R = (A, B, P(x,y)) and subsets R* of A x B, we rederine a relation by the

Definition 6.1: A relation R from A to B is a subset of A x B

Although Definition 6.1 of a relation may seem artificial it has the advantage that we do not use in this definition of a relation the undefined concepts "open sentence" and "variable"

Example 4.1: Let A = {1, 2,3} and B = {a, b}. then

R = {(a, a), (1, b), (3,a)}

Is a relation from A to B. Furthermore

1 R a, 2 $\not R$ b, 3 R a, 3 $\not R$ b

Example 4.2: Let W = {a, b, c}. Then

R = {(a, b), (a,c), (c,c), (c,b)}

is a relation in W. Moreover.

a $\not R$ a, b $\not R$ a, c R c, a R b

Example 4.3: Let R = {x, y) | x $\in \Re$, y$\in \Re$, y<x^2}.

Then R is a set of ordered pairs of real numbers, i.e. a subset of $\Re$ x $\Re$. Hence R is a relation in the real numbers which could also be defined by
R= ($\Re$, $\Re$, P(x,y))
Where P(x,y) reads "y is less than x^2"

Remark 6.1 Let set A have m elements and set B have n elements. Then there are 2^{mn} different relations from A to B, since A x B, which has mn elements, has 2^{mn} different subsets.

4.0 CONCLUSION

In this unit, emphasis has been place on the derivation of relations as ordered pairs between two sets based on open sentences; finding the Domain, Range and Inverse of a relation; defining, with examples, the different kinds of relations and stating whether or not a relation defined on a set is a function of the set into itself.

Here is the summary

5.0 SUMMARY

- The expression P(x,y) is called an ***open sentence in two variables*** *x and y.*
- R = (A, B P(a,b)) is called ***relation*** from A to B where (a,b) ∈ A x B
- R* = {(a,b) | a∈A, b∈B, P(a,b) is true} is the ***solution set*** of the relation R.
- A relation R from A to B is a subset of A x B
- R^1 = {(b,a) | (a,b) ∈ R}is the ***inverse relation*** of R
- If every element in a set is related to itself, then the relation is said to be ***reflexive***
- For a relation R in A, if (a,b) ∈ R implies (b,a) ∈ R, then R is a ***symmetric*** relation
- For a relation R in A, if (a,b) ∈ R and (b,a) ∈ R implies a = b, then R is an ***anti-symmetric relation***
- For a relation R in A, if (a,b) ∈ R and (b,c) ∈ R implies (a,c) ∈ R, then R is a ***transitive relation***
- A relation in a set is an ***equivalence relation*** if it is reflexive, transitive and symmetric
- D = {a | a ∈ A, (a,b) ∈ R}, the set of all first elements of the ordered pairs which belong to the relation from A to B, is the ***domain*** of the relation.
- E = {b|b∈B, (a,b) ∈ R}, the set of all second elements of the ordered pairs which belongs to the relation from A to B, is the ***range*** of the relation

6.0 TUTOR-MARKED ASSIGNMENT

1. Consider the relation R = {(1,5), (4,5), (1,4), (4,6), (3,7), 7,6)}
   Find (1) the domain R, (2) the range of R, (3) the inverse R.

2. Le E= {1,2, 3}. Consider the following relation in E.

7.0 REFERENCE/FURTHER READING

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UNIT 2 FURTHER THEORY OF SETS

CONTENTS

1.0 INTRODUCTION

In this unit, we lay a basic foundation of a branch of mathematics (Logic) that studies laws associated with the set operations; intersection, union and complement. You will do well to follow closely the reasoning presented in the text.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- prove identities using the table of laws of the algebra of sets
- find the dual of any identity
- generalize the operations of union and intersections of sets
- find the possible partitions of a set
- state the relationship between Equivalence relations and Partitions.

3.0 MAIN CONTENT

3.1 Algebra of Sets

Set under the operations of union, intersection and complement satisfy various laws, i.e. identities. Below is a table listing laws of sets, most of which have already been noted and proven in unit 2. One branch of mathematics investigate the theory of set by studying those theorems that follow from these laws, i.e. those theorems whose proofs require the use of only these laws and no others. We will refer to the laws in Table 1 and their consequences as the algebra of sets.

LAWS OF THE ALGEBRA OF SETS			
Idempotent Laws			
1a.	$A \cup A = A$	1b.	$A \cap A = A$
Associative Laws			
2a.	$(A \cup B)\cup C = A\cup(B\cup C)$	2b.	$(A\cap B)\cap C = A\cap(B\cap C)$
Commutative Laws			
3a.	$A\cup B = B\cup A$	3b.	$A\cap B = B\cap A$
Distributive Laws			
4a.	$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$	4b.	$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
Identity Laws			
5a.	$A\cup\varnothing = A$	5b.	$A\cap U = \varnothing$
6a.	$A\cup U = U$	6b.	$A\cap\varnothing = \varnothing$
Complement Laws			
7a.	$A\cup A' = U$	7b.	$A\cap A' = \varnothing$
8a.	$(A')' = A$	8b.	$U' = \varnothing$, $\varnothing' = U$
De Morgan's Laws			
9a.	$(A\cup B)' = A'\cap B'$	9b.	$(A\cap B)' = A'\cup B'$

Table 1

Notice that the concept of "element" and the relation "a belongs to A" do not appear anywhere in table 1. Although, these concepts where essential to our original development of the theory of sets, they do not appear in investigating the algebra of sets. The relation "A is a subset of B" is defined in our algebra of sets by.

$$A \subset B \text{ means } A \cap B = A$$

As examples, we now prove two theorems in our algebra, of sets, that is, we prove two theorems, which follow directly from the laws in Table 1. Other theorems and proofs are given in the problem section.

Example 1.1 Prove: $(A \cup B) \cap (A \cup B') = A$

	Statement	*Reason*
1.	$(A\cup B) \cap (A\cup B') = A\cup(B\cap B')$	1.Distributive Law
2.	$B \cap B' = \varnothing$	2. Substitution
3.	$\therefore (A\cup B)\cap(A\cup B') = A\cup\varnothing$	3. Associative Law
4.	$A \cup\varnothing = A$	4. Substitution
5.	$\therefore (A\cup B)\cap(A\cup B') = A$	5. Definition of subset

Example 1.2: Prove $A \subset B$ and $B \subset C$ implies $A \subset C$.

	Statement	*Reason*
1.	$A = A \cap B$ and $B = B \cap C$	1. Definition of subset
2.	$\therefore A = A \cap (B \cap C)$	2. Substitution
3.	$A = (A \cap B) \cap C$	3. Associative Law
4.	$\therefore A = A \cap C$	4. Substitution
5.	$\therefore A \subset C$	5. Definition of subset

Principle of Duality

If we interchange $\cap$ and $\cup$ and $\varnothing$ in any statement about sets, then the new statement is called the dual of the original one.

Example 2.1: The dual of

$$(U \cup B) \cap (A \cup \varnothing) = A$$

is $$(\varnothing \cap B)\cup(A \cap U) = A$$

Notice that the dual of every law in Table 1 is also a law in Table 1. This fact is extremely important in view of the following principle:

Principle of Duality: If certain axioms imply their own duals, then the dual of any theorem that is a consequence of the axioms is also a consequence of the axiom. For, given any theorem and its proof, the dual of the theorem can be proven in the same way by using the dual of each step in the original proof.

Thus, the principle of duality applies to the algebra of sets

Example 2.2: Prove: $(A \cap B) \cup (A \cap B') = A$

The dual of this theorem is proven in Example 1.1; hence this theorem is true by the Principle of Duality.

3.2 Indexed Sets

Consider the sets

$A_1 = \{1, 10\}$, $A_2 = \{2, 4, 6, 10\}$, $A_3 = (3, 6, 9\}$, $A_4 = \{4, 8\}$, $A_5 = \{5, 6, 10\}$

And the set

$$I = \{1, 2, 3, 4, 5\}$$

Notice that to each element $i \in I$ there corresponds a set A_i. In such a situation *I* is called the ***index set,*** the sets $(A_1, \ldots, A_5)$ are called the ***indexed sets,*** and the subscript i of A_i, i.e. each $i \in I$, is called an ***index***. Furthermore, such an indexed family of sets is denoted by

$$(A_i)_{i \in I}$$

We can look at an indexed family of sets from another point of view. Since to each element $i \in I$ there is assigned a set A_i, we state

Definition 7.1: An indexed family of sets (A_i) $i \in I$ is a function

$$f: I \rightarrow A$$

Where the domain of f is the index set I and the range of f is a family of sets.

Example 3.1: Define $Bn = \{x \mid 0 \leq x \leq (1/n)\}$, where $n \in N$, the natural numbers. Then

$B_1 = [0,1]$, $B_2 = [0,1/2]$, . . .

Example 3.2: Let/ be the set of words in the English Language, and let $i \in I$. Define

$W_1 = \{x \mid x \text{ is a letter in the word } i \in I \}$.

If i is the word “follow” , then $W_i = \{f, 1, o, w \}$.

Example 3.3: Define $D_n = \{x \mid x \text{ is a multiple of } n\}$, where $n \in N$, the natural numbers. Then

$D_1 = \{1, 2, 3, 4, . . . , D_2 = \{2, 4, 6, 8, . . . \}$,
$D_3 = \{3, 6, 9, 12, . . . \}$

Notice that the index set *N* is also D_1 and also the universal set fro the indexed sets.

Remark 7.1: Any family of B of sets can be indexed by itself. Specifically, the identify function i: B $\rightarrow$ B

is an indexed family of sets

$\{A_i\}_{1 \in B}$

Where $A_{i \in B}$ and where $i = A_i$. In other words, the indexed of any set in B is the set itself

3.3 Generalized Operations

The operation of union and intersection were defined for two sets. These definitions can easily be extended, by induction, to a finite number of sets. Specifically, for sets $A_1, . . . A_{n,}$

$$\bigcup^{u}_{i=I} A_1 \equiv A_1 \cup A_2 \cup \ldots \cup A_n$$
$$\bigcap^{n}_{i=I} A_i \equiv A_1 \cap A_2 \cap \ldots \cap A_n$$

In view of the associative law, the union (intersection) of the sets may be taken in any order; thus parentheses need not be used in the above.

These concepts are generalised in the following way. Consider the indexed family of sets

$$\{A_i\}_{i \in I}$$

And let $J \subset I$. Then $\cup_{i \in J} A_i$

Consists of those elements which belong to at least one A_i where $i \in J$. Specifically,

$\cup_{i \in J} A_i = \{ x \mid \text{there exists an } i \in J \text{. such that } x \in A_i\}$

In an analogous way $\cup_{i \in J} A_i$

consist of those elements which belong to every A_i for $i \in J$. In other words, $\cap_{i \in J} A_i = \{x \mid x \in A_i \text{ for every } i \in J\}$

Example 4.1: Let $A_1 = \{1, 10\}$, $A_2 = \{2, 4, 6, 10\}$, $A_3 = \{3, 6, 9\}$, $A_4 = \{4, 8\}$, $A_5 = \{5, 6, 10\}$; and let $J = \{2, 3, 5, \}$. Then $\cap_{i \in J} A_i = \{6\}$ and $\cup_{i \in J} A_i = \{2, 4, 6, 10, 3, 9, 5\}$

Example 4.2: Let $B_n = [0, i/n]$, where $n \in N$, the natural numbers. Then $\cap_{i \in N} B_i = \{0\}$ and $\cup_{i \in N} B_i = [0, 1]$

Example 4.3: Let $D_n = \{x \mid x \text{ is a multiple of } n\}$, where $n \in N$, the natural numbers.

Then $\cap_{i \in N} D_1 = \varnothing$

There are also generalised distributive laws for a set B and an indexed family of sets $\{A_i\}_{i \in I}$ be an indexed family of sets. Then for any set B,

$$B \cap (\cup_{i \in I} A_i) = \cup_{i \in I} (B \cap A_i)$$
$$B \cup (\cap_{i \in I} A_i) = \cap_{i \in I} (B \cup A_i)$$

3.4 Partitions

Consider the set A = { 1, 2, . . . 9, 10} and its subsets
$B_1 = \{1, 3\}$, $B_2 = \{7, 8, 10\}$, $B_3 = \{2, 5, 6\}$.$B_4 = 4, 9\}$
The family of sets $B = \{B_1, B_2, B_3, B_4,\}$ has two important properties.

1. A is the union of the sets in B, i.e.,
   $A = B_1 \cup B_2 \cup B_3 \cup B_4$

2. For any sets B_i and B_j,
   Either $B_i = B_j$ or $B_i \cap B_j = \varnothing$

Such a family of sets is called ***partition*** of A. Specifically, we say

Definition 7.2:

Let $\{B_i\}_{i \in I}$ be a family of non-empty subsets of A. Then $\{B_i\}_{i \in I}$ is called a ***partition*** of A if

P_1: $\cup_{i \in I} B_i = A$
P_2: For any B_i, B_j, either B_j or $B_i \cap B_j = \varnothing$

Furthermore, each B_i is then called an ***equivalence class*** of A.

Example 5.1: Let N = {1, 2, 3, . . .}, E = {2, 4, 6, . . .} and F = {1, 3, 5}. Then {e, F} is a partition of N.

Example 5.2: Let T = {1, 2, 3,. . ., 9, 10} and let A = {1, 3, 5}, B = {2, 6, 10} and C = {4, 8, 9}. Then {A, B, C} is not a partition of T since

$T \neq A \cup B \cup C$

i.e. since $7 \in T$ but $7 \neq (A \cup B \cup C)$.

Example 5.3: Let T = {1, 2, …, 9, 10}, and F = {1, 3, 5, 7, 9}, and G = {2, 4, 10} and H = {3, 5, 6, 8}. Then {f, G, H} is not a partition of T since

$F \cap H \neq \emptyset, F \neq H$

Example 5.4: let y_1, y_2, y_3 and y_4 be respectively the words "follow", "thumb", "flow" and "again", and let

A = {w, g, u, o, I, m, a, t, f, n, h, b}

Furthermore, define

$W_i = \{x \mid x \text{ is the letter in the word } y_i\}$

Then $\{W_i\ W_2\ W_3\ W_4\}$ is the partition of A. notice that W_1 and W_3 are not disjoint, but there is no contradiction since the sets are equal.

3.5 Equivalence Relations and Partition

Recall the following

Definition: A relation in a set A is an equivalent relation if**:**

1. R is reflexive, i.e. for every $a \in A$, a is related to itself;
2. R is symmetric, i.e. if a is related to b then b is related to a;
3. R is transitive, i.e. if a is related to be and b is related to c then a is related to c.

Theorem 7.2: Fundamental Theorem of Equivalence Relations: Let R be an equilvalence relation in a set A and for, every $a \in A$, let

$B_\infty = \{x \mid (x, \alpha) \in R)$

i.e. the set of elements related to α. Then the family of sets#

$\{B_\alpha\}_{\alpha \in A}$

is a partition of A.

In order words, an equivalence relation R in a set A partition the set A by putting those elements which are related to each in the same equivalence class.

Moreover, the set B_α is called ***equivalence class*** determined by α, and the set of equivalence classes $\{B_\alpha\}_{\in\alpha A}$ is denoted by

$$A/R$$

And called ***quotient set.***

The converse of the previous theorem is also true. Specifically,

Theorem 7.3: Let $\{B_i\}_{i\in I}$ be a partition of A and let R be the relation in A defined by the open sentence "x is in the same set (of the family $\{B_i\}_{i\in I}$ as y". Then R is an equivalence relation in A.

Thus there is a one to one correspondence between all partitions of a set A and all equivalence relations in A.

Example 6.1: In the Euclidean plane, similarities of triangles is an equivalence relation. Thus all triangles in the plane are portioned into disjoint sets in which similar triangles are elements of the same set.

Example 6.2: Let R_5 be the relation in the integers defined by

$\mathbf{x \equiv y}$ (mod 5)

which reads "x is congruent to y modulo 5" and which means "x – y is divisible equivalence classes in Z/ R_5: E_0, E_1, E_2, E_3 and E_4. Since each integer x is uniquely expressible in the form $x = 5p + r$ where $0 \le r < 5$, then x is a member of the equivalence class E_r where r is the remainder. Thus

E_0 = {. . ., -10, -5, 0, 5, 10, . . .}
E_1 = {. . . , -9, -4, 1, 6, 11, . . .}
E_2 = {. . . , -8, -3, 2, 7, 12, . . . }
E_3 = {. . . , -7, -2, 3, 8, 13, . . }
E_3 = {. . . , -6, 1- , 4, 9, 14, . . .}

Add the quotient set $Z/R_5 = \{E_0, E_1, E_2, E_3, E_4\}$

4.0 CONCLUSION

You are gradually being introduced to using set notation rather than statements. Theorems in the algebra of sets are most useful in proving identities related to logic and reasoning, in most cases using the principle of duality.

5.0 SUMMARY

In investigating the algebra of sets you need to take note of the dual of the statements in table 1.

An indexed family of sets, $\{A_i\}_{i \in I}$ is such that for each index i = 1, 2, 3, 4, . . . we have sets A_1, A_2, A_3, . . .

Let $\{B_i\}_{i \in I}$ be a family of non- empty subsets of A. then $\{B_i\}_{i \in I}$ is called ***partition*** of A if $\cup_{i \in I} B_i = A$ and for any B_i B_j, either $B_i = B_j$, or $B_i, \cap B_j = \varnothing$

Furthermore, each B_i, is then called an ***equivalence class*** of A.

6.0 TUTOR MARKED ASSIGNMENTS

1. Write the dual of each of the following:
   i. $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
   ii. $A \cup (A' \cap B) = A \cup B$
   iii. $(A \cap U) \cap (\varnothing \cup A') = \varnothing$

2. Prove: $(A \cap B) \cup (A \cap B') = A$.

3. Let $B_i = [i, i + 1]$, where I ε Z, the integer. find

   1. $B_1 \cup B_2$
   2. $B_3 \cup B_4$
   3. $\cup^{18}_{i = 7} B_i$
   4. $\cup_{i \varepsilon Z} B_i$

4. Let A = {a, b, c, d, e, f, g}. State whether or not each of the following families of sets is a partition of A.

   1. $\{B_1 = \{a, c, e\}, B_2 = \{b\}, B_3 = \{d, g\}\}$
   2. $\{C_1 = \{a, e, g\}, C_2 = \{c, d\}, C_3 = \{b, e, f\}\}$
   3. $\{D_1 = \{a, b, e, g\}, D_2 = \{c\}, D_3 = \{d, f\}\}$
   4. $\{E_1 = \{a, b, c, d, e, f, g\}\}$

7.0 REFERENCES/FURTHER READING

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UNIT 3 FURTHER THEORY OF FUNCTIONS, OPERATIONS

CONTENTS

1.0 INTRODUCTION

There are some further concepts you have to become familiar with now which you will come across in mathematical analysis and abstract mathematics. This unit introduces you to some of them. Pay attention not only to the definitions but also to the examples given.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state whether or not a diagram of functions is cumulative from the arrows connecting the functions.
- explain the terms Restriction and Extension of functions
- describe the following: Set functions, Real-valued functions and its algebra, Characteristic function
- apply the Rule of the Maximum Domain.
- explain operations on Cartesian products
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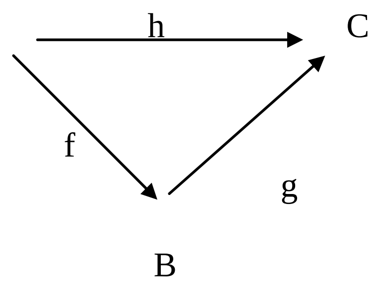
3.0 MAIN CONTENT

Functions and Diagrams

As mentioned previously, the symbol

$$A \longrightarrow B$$

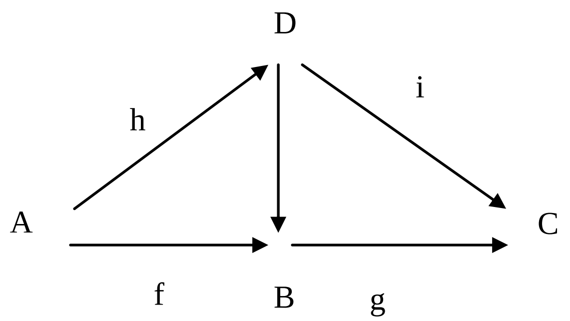
Denotes a function of A into B. In a similar manner, the diagram



Consists of letters A, B and C denoting sets, arrows f, g, h denoting functions f: A → B, g: B → C and h: A → C, and the sequence of arrows {f, g} denoting the composite function g o f: A → C. Each of the functions h: A →C and g o f: A → C, that is, each arrow or sequence of arrows connecting A to C is called a path from A to C.

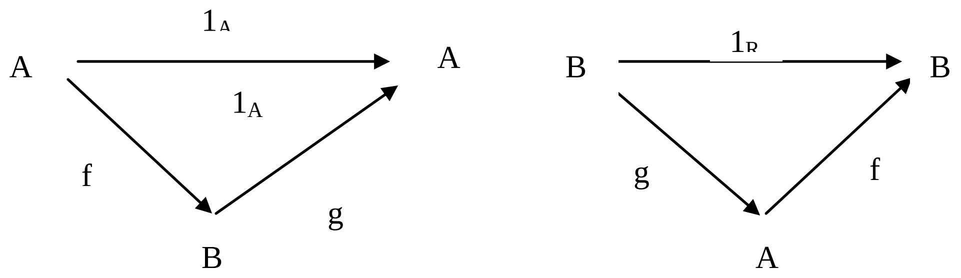
Definition 8. 1: A diagram of functions is said to be cumutative if for any set X and Y in the diagram, any two paths from X to Y are equal.

Example 1.1: Suppose the following diagram of functions is cumulative.



Then i o h = f, g o I = j and g o f = j o h = g o I o h.

Example 1.2: The functions f: A ⟶ B and g: B ⟶ A are inverses if and only if the following diagrams are cumulative:



Here 1_A and 1_B are the identity functions.

3.1 Restrictions and Extensions of Functions

Let f be a function of A into C, i.e. let f: A→ C, and let B be a subset of A. Then f induces a function f': B → C which is defined by

F(b) = f(b)

For any b∈B. the function f' is called the restriction of F to B and is denoted by
F| B

Example 2.1: Let f: ℜ → ℜ be defined by f(x) = $x^{2,}$. Then

F| N = {(1, 1), (2,4), (3,9), (4, 16), …}
Iis the restriction of to N, the natural numbers.

Example 2.2: The set g = {(2, 5), (5, 1), (3, 7), (8, 3), (9, 5)} is a function from {2, 5, 3, 8, 9} into N. Then

{(2, 5), (3, 7), (9, 5)}

a subset of g, is the restriction of g to {2, 3, 9}, the set of first elements of the ordered pairs in g.

We can look at this situation from another point of view. Let f:
A → C and let B be a superset of a. Then a function F: B → C is called an **extension** of f if, for every a ∈ A,

F(a) = f(a)

Example 2.3: Let f be the function on the positive real number defined by f(x) = x, that is, let the identity function. Then the absolute value function

$$|X| = \begin{cases} X \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Is an extension of f to all real numbers.

Example 2.4: **C**onsider the function

F = {(1, 2), (3, 4), (7,)}

Whose domain is {1, 3, 7}. Then the function

F = {(1, 7), (3, 4), (5, 6), (7, 2)}

Which is a superset of the function f, is an extension of f.

3.2 Set Functions

Let f be a function of A into B and let T be a subset of A, that is, $A \xrightarrow{t} B$ and $T \subset A$.

Then

F(T)

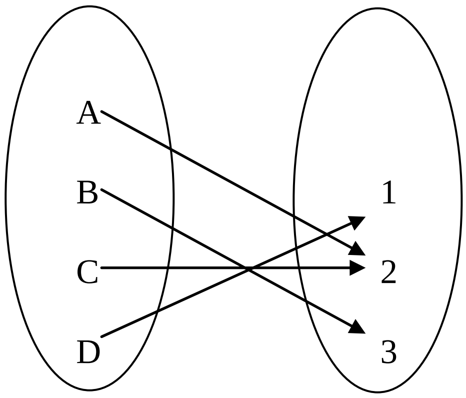
Which is read "f of T", is defined to be the set of image point of elements in T. In other words,

$$F(T) = \{x | f(a) = x, a \in T, x \in B\}$$

Notice that f(T) is a subset of B.

Example 3.1: Let A = {a, b, c, d}, T = {b, c} and B = {1, 2, 3}.

Define f: A ⟶ B by



Then f (T) = {2, 3}.

Example 3.2: Let g: $\Re \to \Re$ be defined by $g(x) = x^2$, and let T = [3, 4]. Then

G(T) = [9, 16] = {x | $9 \le x \le 16$}.

Now let A be the family of subsets of A, and let B be the family of subsets of B, if f: A → B, then f assigns to each set T ∈ A a unique set of f(T) ∈ B. In other words, the function f: A → B induces a function f: A → B. Although the same letter denotes each function, they are essentially two different functions. Notice that the domain of f: A → B consists of sets.

Generally speaking, a function is called a **set function** if its domain consists of sets.

3.3 Real-Valued Functions

A function f: A→ $\Re$, which maps a set into the real numbers, i.e. which assigns to each a ∈ A a real number f(a) ∈ $\Re$ is **a real-valued function.** Those functions which are usually studied in elementary mathematics, e.g.,

$P(x) = a_o x^n + a_{1x}{}^{n-1} x + a_n$
t(x) = sin x, cos x or tan x
$f(x) = \log x$ or e^x

That is, polynomials, trigonometric functions and logarithmic and exponential functions are special examples of real-valued functions.

3.4 Algebra and Real-Valued Functions

Let F_D be the family of all real-valued functions with the same domain D. Then many (algebraic) operations are defined in F_D. Specifically, let f: D $\Re$ and g: D $\rightarrow \Re$, and let k $\in \Re$. Then each of the following functions is defined as follows:

(f + k): D$\longrightarrow$ $\Re$ by (f + k)(x) = f(x) + k

(|f|): D$\longrightarrow$ $\Re$ by (f)(x) = | f(x) |

(f"): D$\longrightarrow$ $\Re$ by (f")(x) = (f(x))"

(f ± g): D$\longrightarrow$ $\Re$ by (f ± g)(x) =f(x) ± g(x)

(kf): D$\longrightarrow$$\Re$ by (kf)(x) = k(f(x))

(fg): D$\longrightarrow$ $\Re$ by (fg)(x) = f(x)g(x)

(f/g): D$\longrightarrow$ $\Re$ by (f/g)(x) =f(x)/g(x)

where g(x) ≠ 0

Note that (f/g): D $\longrightarrow$ $\Re$ is not the same as the composition function which was discussed previously.

Example 4.1: Let D = {a, b}, and let f: D $\longrightarrow$ $\Re$ and g: D$\longrightarrow$ $\Re$ be defined by :

F(a) = 1, f(b) = 3 and g(a) = 2, g(b) = -1
In other words,
F = {(a, 1), (b, 3) and g = {a, 2), (b, -1)}

Then
(3f – 2g)(a) = 3f(a) – 2g(a) = 3(1) -2(2) =-1
(3f – 2g)(b) = 3f(b) – 2g(b) 3(3) – g(-1) = 11

that is, 3f – 2g = {(a, 1), (b, 11)}

Also since |g|(x) = g(x) and (|g|+3)(x) = g(x) + 3,

G ={(a, |2|), (b, 1)} AND G + 3 = {{(A, 5), (b,2)}

Example 4.2: Let f: $\Re \rightarrow \Re$ and g: $\Re \rightarrow \Re$ be defined
By the formulas

F(x) = 2x = - 1 and g(x) =^{2}

The formulas which define the function (3f – 2g):
$\Re \rightarrow \Re$ and (fg): $\Re \rightarrow \Re$ are found as follows:

$(3f – 2g)(x) = 3(2x – 1) -2(x^2) = - 2x^2 + 6x – 3$
$(fg)(x) = (2x – 1)(x^2) = 2x^2 - x^2$

3.5 Rule Of The Maximum Domain

A formula of the form
F(x) = 1/x, g(x) = sin x, h(x) = $\sqrt{x}$

Does not in itself, define a function unless there is given, explicitly or implicitly, a domain, i.e. a set of numbers, on which the formula then defines a function. Hence the following expressions appear:

Let f(x) = x^2 be define on [- 2, 4].
Let g(x) = sin x be defined for $0 \leq x \leq 2x$

However, if the domain of a function defined by a formula is the maximum set of real numbers for which the formula yield a real number, e.g.,

Let f(x) = 1/X for $X \neq 0$

Then the domain is usually not stated explicitly. This convention is sometimes called the rule of the maximum domain.

Example 5.1: Consider the following functions

$F_1(x) = x^2$ for $x \geq 0$
$F_2(x) = 1/(x – 2)$ for $x \neq 0$
$F_3(x) = \cos x$ for $0 \leq x \leq 2\pi$
$F_4(x) = \tan x$ for $\neq \pi/2 + n\pi, n \in N$

The domains of f_2 and f_4 need not have been explicitly stated since each consists of all those numbers for which the formula has meaning, that is, the functions could have been defined by writing

$F_1(x) = x^2$ and $f_4(x) = \tan x$

Example 5.2: Consider the function $f(x) = \sqrt{1 - x^2}$; its domain, unless otherwise stated, is [-1,1]. Here we explicitly **Assume that the co-**domain is $\Re$.

3.6 Characteristic Functions

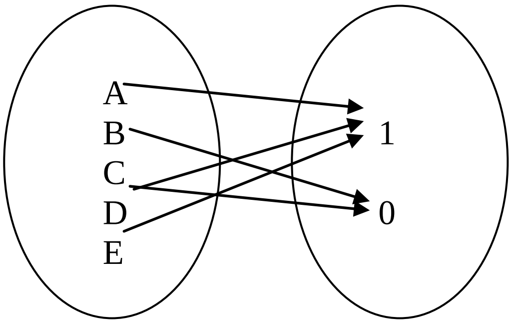
Let A be any subset of a universal set U. Then the real-valued function.

$$X_A : U \longrightarrow \{1, 0\}$$

defined by $$x_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \neq A \end{cases}$$

is called the characteristic function of A.

Example 6.1: Let U = {a, b, c, d, e} and A = {s, f, e}. Then the function of U into {1, 0} defined by the following diagram



Is the characteristic function $x_{A \text{ of }}$ A

Note further that any function f: U $\longrightarrow$ {1, 0} defines a subset

$$A_f = \{x \ x \in U, | f(x) = 1\}$$

Of U and that the characteristic function X_{Af} of A_f is the original function f. Thus there is a one to-one correspondence between all subsets of U, i.e. the power set of U, and the set of all functions of U into {1, 0}.

3.7 Operations

You are familiar with the operations of addition and multiplication of numbers, union and intersection of sets, and composition of functions. These operations are denoted as follows:

A + b = c, a, b = c, $A \cup B = C$ $A \cap B = C$, $g^o f = h$

In each situation, an element (c, C or h) is assigned to an original pair of elements. In other words, there is a function that assigns an element to each ordered pair of elements. We now introduce

Definition 8.1: An operation α on a set A is a function of the Cartesian product A x A into A, i.e.,

Remark 8.2: The operation α A x A $\rightarrow$ A is sometimes referred to as a **binary operation** and an n-ary operation is defined to be a function

$$\alpha : A \ldots A \rightarrow A$$

We shall continue to use the word operation instead of binary operation

3.7.1 Cumulative Operations

The operation α: A x A $\rightarrow$A is called cumulative if, for every a, b $\in$ A,

$$\alpha\ (a, b) = \alpha\ (b, a)$$

Example 7.1: Addition and multiplication of real numbers are cumulative operations since

A + b = + a and ab =ba

Example 7.2: Let α: $\Re$ x $\Re \longrightarrow \Re$ be the operation of subtraction defined by α: (x, y) x – y

Then

$\alpha(5,1) = 4$ and $\alpha(1, 5) = - 4$

Hence subtraction is not a cumulative operations, since

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

3.7.2 Associative Operations

The operation α: A x A $\rightarrow$ A is called associative if, for every a, b, c $\in$ A.

$$\alpha(\alpha(a, b), c) = \alpha(a, \alpha(b, c))$$

In other words, if α(a, b) is written a $*$ b, then α is associative if

$$(a * b) * c = a * (b * c)$$

Example 8.1: Addition and multiplication of real numbers are associative operations, since

$$(a + b) + c = a + (b + c) \text{ and } \quad (ab)c = a(bc)$$

Example 8.2: Let $\alpha:\Re \times \Re \rightarrow \Re$ be the operation of division defined by α: (x, y) x/y
Then

$$\alpha\ (\alpha(12, 6), 2) = \alpha(2, 2) = 1$$
$$\alpha(12, \alpha(6, 2)) = \alpha12, 3) = 4$$

Hence division is not an associative operation.

Example 8.3: Union and intersection of sets are associative operations, since

$$(A \cup B\ ) \cup C = A \cup (B \cup C) \text{ and}$$
$$(A \cap B)\ \cap C = A \cap (B \cap C)$$

3.7.3 Distributive Operations

Consider the following two operations

$$\alpha: A \times A \rightarrow A$$
$$\beta: A \times A \rightarrow A$$

The operation α is said to be distributive over the operation β if, for every a, b, c $\in$ A.

$$\alpha(a\beta(b, c) = \beta(z, b), \alpha(a, c))$$

In other words, if α(a, b) is written a $*$ b. and β(a, b) is written a Δ b, then α distributes over β if

$$A * (b \Delta c) = (a\ * b) \Delta (a * c)$$

Example 9.1: The operation of multiplication of real numbers distributes over the operation of addition, since

$$A(b + c) = ab + ac$$

But the operation of addition of real numbers does not distribute over the operation of multiplication, since

$$A + (bc) \neq (a + b)(a + c)$$

Example 9.2: The operation of Union and intersection of sets distribute over each other since

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$$

3.7.4 Identity Elements

Let α: A x A $\longrightarrow$ A be an operation written $\alpha(a, b) = a * b$. An element $c \in A$ is called an identity element for the operation α if, for every $a \in A$.

$$C * a = a * c = a$$

Example 10.1: Let $\alpha:\Re \times \Re \to \Re$ be the operation of addition. Then 0 is an identity element for addition since, for every real number $a \in \Re$.
$0 * a = a * 0 = a$, that is, $0 + a = a + 0 = a$

Example 10;2: Consider the operation of intersection of sets. Then U, the universal set is an identity element, since for every set A (which is a subset of U),

U $*$ A = A $*$ U = Am that is,
$U \cap A = \mathbf{A} \cap U = A$

Example 10.3: Consider the operation of multiplication of
Real numbers. Then the number 1is an identity element since, for every real number a,

$1 * a = a * 1 = a$, that is, $1 \bullet a = a \bullet 1 = a$

Theorem 8. 1: If an operation α: A x A $\to$ A has an identity element.
Thus we can speak of the identity element for an operation instead of an identity element.

3.7.5 Inverse Elements

Let α: A x A $\rightarrow$ A be an operation written α(a, b) = a $*$ b, and let e $\in$ A be the identity element for α. Then the inverse of an element a $\in$ A, denoted by a^{-1}
is an element in A with the following property:

$$^{-1} * a = a * a^{-1} = e$$

Example 11.1: Consider the operation of addition of real numbers for which 0 is the identity element. Then, for any real number a, its negative (-a) is its additive inverse since

- a $*$ a = a $*$ - a = 0m that is, (- a) + a = a + (- a) = a –a = 0

Example 11.2 Consider the operation of multiplication of rational numbers, for which 1 is the identity element. Then for any non-zero rational number p/q, where p and q are integers, its reciprocal q/p is its multiplicative inverse, since

$$(q/p)(p/q) = (p/q)(q/p) = 1$$

Example 11.3 Let α: N x N$\longrightarrow$N be the operation of multiplication for which 1 is the identity element; here N is the set of natural numbers. Then 2 has no multiplicative inverse, since there is no element x $\in$ N with the property

$$X \bullet 2 = 2 \bullet x = 1$$

In fact, no element in N has a multiplicative inverse except 1 which has itself as an inverse.

3.8 Operations and Subsets

Consider an operation α: A x A $\rightarrow$ A and a subset B of A. Then B is said to be closed under the operation of α if, for every b, b' $\in$ B,

α(b, b') $\in$ B

that is, if α(B x B) $\subset$ B

Example 12.1 Consider the operation of addition of natural numbers. Then the set of even numbers id closed under the operation of addition since the sum of any two even numbers is always even. Moreover, the set of odd numbers is not closed under the operation of addition since the sum of two odd numbers is not odd.

Example 12.2: The four complex numbers 1, - 1, I, - I are closed under the operation of multiplication.

4.0 CONCLUSION

In mathematical analysis and abstract mathematics, the unit is a prerequisite knowledge. Functions and diagrams go hand in hand. Set functions, real-valued functions, characteristic functions are basic functions in analysis

5.0 SUMMARY

See if you recall the following:

- The function f' is called a restriction of f to B (f B) of f:| A $\rightarrow$ C, if, given B a subset of A, f induces a function a function f': B $\rightarrow$ C which is defined by

  F'(b) = f(b)
  For any b$\in$ B.

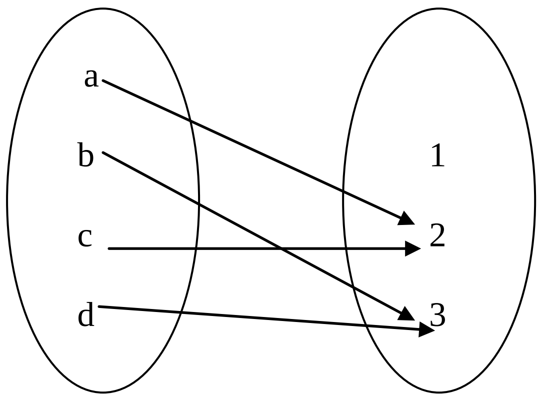
- Let f: A $\rightarrow$ C and let B be a superset of A. Then a function F: B $\rightarrow$ C is called an extension of f if, for every a $\in$ A.

  f(a) = f(a)

- A function is called a set function if its domain consists of sets
- A function on a a set that maps the elements of that set into the real numbers is called a real-valued function.
- The rule of Maximum domain is used to define domains which need not be stated explicitly since it is the maximum set of real numbers for which a function yields a real number.

6.0 TUTOR-MARKED ASSIGNMENTS

1 Let W = {a, b, c, d}; v = {1, 2, 3}, and let f: W $\rightarrow$ V be defined by the adjoining diagram. Find: (1) f({a, b, d}), (2) f({a, c}).



2 Find the domain of each of the following real-valued functions:

1. $f_1(x) = 1/x$ where $x > 0$

2 $f_2(x) = \sqrt{3 - x}$

3. $f_3(x) = \log(x - 1)$

4 $f_4(x) = x^2$ where $0 \leq x \leq 4$

7.0 REFERENCES/FURTHER READING

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