MODULE 1 INTERPOLATION

UNIT 1 INTERPOLATION (LAGRANGE'S FORM)

CONTENTS

1.0 INTRODUCTION

Let f be a real-valued function defined on the interval [a, b] and we denote $f(x_k)$ by f_k. Suppose that the values of the function f(x) are given to be $f_0, f_1, f_2, \ldots, f_n$ when $x = x_0, x_1, x_2, \ldots, x_n$ respectively where $x_0 < x_1 < x_2 \ldots < x_n$ lying in the interval [a, b]. The function f(x) may not be known to us. The technique of determining an approximate value of f(x) for a non-tabular value of x which lies in the interval [a, b] is called interpolation. The process of determining the value of f(x) for a value of x lying outside the interval [a, b] is called extrapolation. In this unit, we derive a polynomial P(x) of degree n which agrees with the values of f(x) at the given (n + 1) distinct points, called nodes or abscissas. In other words, we can find a polynomial P(x) such that $P(x_j) = f_j$, j = 0, 1, 2, …, n. such a polynomial P(x) is called the interpolating polynomial of f(x).

In section 3.1 we prove the existence of an interpolating polynomial by actually constructing one such polynomial having the desired property. The uniqueness is proved by invoking the corollary of the fundamental theorem of Algebra. In section 3.2 we derive general expression for error in approximating the function by the interpolating polynomial at a point and this allows us to calculate a bound on the error over an interval. In proving this we make use of the general Rolle's theorem.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the Lagrange's form of interpolating polynomial interpolating f(x) at n + 1 distinct nodal points
- compute the approximate value of f at a non-tabular point
- compute the value of $\bar{x}$ (approximately) given a number $\bar{y}$ such that f($\bar{x}$) = ($\bar{y}$) (inverse interpolation)
- compute the error committed in interpolation, if the function is known, at a non-tabular point interest
- find an upper bound in the magnitude of the error.

3.0 MAIN CONTENT

3.1 Lagrange's Form

Let us recall the fundamental theorem of algebra and its useful corollaries.

Theorem 1

If P(x) is a polynomial of degree $n \geq 1$, that is $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \ldots, a_n$ real or complex numbers and $a_n \neq 0$, then P(x) has at least one zero, that is, there exists a real or complex number ξ such that $p(\xi) = 0$.

Lemma 1

If $z_1, z_2, \ldots, z_k$ are distinct zeros of the polynomial P(x), then

$$P(x) = (x - z_1)(x - z_2) \ldots (x - z_k)R(x)$$

for some polynomial R(x).

Corollary

If $P_k(x)$ and $Q_k(x)$ are the two polynomials of degree £ k which agree at the k + 1 distinct points $z_0, z_1, z_2, \ldots, z_k$ then $P_k(x) = Q_k(x)$ identically.

You have come across Rolle's theorem in the perquisite course. But we need a generalized version of this theorem . (General Error Term). This is stated below.

Theorem 2

(GeneralisedRolle's Theorem). Let f be a real-valued function defined on [a, b] which is n times differentiable on]a, b[. If f vanishes at the n + 1 distinct points $x_0, \ldots, x_n$ in [a, b], then a number c in]a, b[exists such that $f^{(n)}(c) = 0$.

We now show the existence of an interpolating polynomial and also show that it is unique. The form of the interpolating polynomial that we are going to discuss in this section is called the Lagrange's form of the interpolating polynomial. We start with a relevant theorem.

Theorem 3: Let x_0, x_1, … x_n be n + 1 distinct points on the real line and let f(x) be a real-valued function defined on some interval I = [a, b] containing these points. Then, there exists exactly one polynomial $P_n(x)$ of degree n, which interpolates f(x) at x_0, … x_n, that is, $P_n(x_j) = f(x_j)$, i = 0, 1, 2, …, n.

Proof: First we discuss the uniqueness of the interpolating polynomial, and then exhibit one explicit construction of an interpolating polynomial (Lagrange's Form).

Let $P_n(x)$ and $Q_n(x)$ be two distinct interpolating polynomials of degree n, which interpolate f(x) at (n + 1) distinct points x_0, x_1, … x_n. Let $h(x) = P_n(x) - Q_n(x)$. Note that h(x) is also a polynomial of degree $\leq$n. Also

$$h(x_j) = P_n(x_j) - Q_n(x_j) = f(x_j) - f(x_j) = 0, \; i = 0, 1, 2, \dots, n.$$

That is, h(x) has (n + 1) distinct zeros. But h(x) is of degree £ n and from the Corollary to Lemma 1, we have h(x) º 0. That is $P_n(x)$ $Q_n(x)$. This proves the uniqueness of the polynomial.

Since the data is given at the points (x_0, f_0), (x_1, f_1), …, (x_n, f_n) let the required polynomial be written as

$$P_n(x_j) = L_0(x)f_0 + L_1(x)f_1 + \dots + L_n(x)f_n = \sum_{i=0}^{n} L_i(x)f_i \qquad (1)$$

Setting $x = x_j$ in (1), we get

$$P_n(x_j) = \sum_{i=0}^{n} L_i(x_j)f_i \qquad (2)$$

Since this polynomial fits the data exactly, we must have

$$L_j(x_j) = 1$$

and $L_j(x_j) = 0, \; i \neq j$

or $L_j(x_j) = \partial_{ij}$ (3)

The polynomial $L_i(x)$ which are of degrees £ n are called the Lagrange fundamental polynomials. It is easily verified that these polynomial are given by

$$L_j(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots({}_i x - x_n)}$$

$$= \prod_{\substack{i=0 \\ i=j}}^{n} (x - x_j) / \prod_{\substack{i=0 \\ i=j}}^{n} (x_i - x_j) \qquad (4)$$

Substituting of (4) in (1) gives the required Lagrange form of the interpolating polynomial.

Remark

The Lagrange form (Eqn. (1)) of interpolating polynomial makes it easy to show the existence of an interpolating polynomial. But its evaluation at a point x_i involves a lot computation.

A more serious drawback of the Lagrange form arises in practice due to the following: One calculates a linear polynomial $P_1(x)$, a quadratic polynomial $P_2(x)$ e.t.c., by increasing the number of interpolation points, until a satisfactory approximationto f(x) has been found. In such a situation Lagrange form does not take any advantage of the availability of $P_{k-1}(x)$ in calculating $P_k(x)$. Later on, we shall see how in this respect, Newton form, discussed in the next unit, is more useful.

Let us consider some example to construct this form of interpolation polynomials.

Example 1

If f(1) = -3, f(3) = 9, f(4) = 30 and f(6) = 132, find the Lagrange's interpolation polynomial of f(x).

Solution

We have $x_0 = 1$, $x_1 = 3$, $x_2 = 4$, $x_3 = 6$ and $f_0 = -3$, $f_1 = 9$, $f_2 = 30$, $f_3 = 132$.

The Lagrange's interpolating polynomial P(x) is given by

$$P(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 + L_3(x)f_3 \qquad (5)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{(x - 3)(x - 4)(x - 6)}{(1 - 3)(1 - 4)(1 - 6)}$$

$$= \frac{1}{30}\ (x^3 - 13x^2 + 54x - 72)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$= \frac{(x - 1)(x - 4)(x - 6)}{(3 - 1)(3 - 4)(3 - 6)}$$

$$= \frac{1}{6}\ (x^3 - 11x^2 + 34x - 24)$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$= \frac{(x - 1)(x - 4)(x - 6)}{(4 - 1)(4 - 3)(4 - 6)}$$

$$= \frac{1}{6}\ (x^3 - 10x^2 + 27x - 18)$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= \frac{(x - 1)(x - 3)(x - 4)}{(6 - 1)(6 - 3)(6 - 4)}$$

$$= \frac{1}{30}\ (x^3 - 8x^2 + 19x - 12)$$

Substituting $L_j(x)$ and f_jj = 0, 1, 2, 3 in Eqn. (5), we get

$$P(x) = -\frac{1}{30}\ [x^3 - 13x^2 + 54x - 72]\ (-3) + \frac{1}{6}\ [x^3 - 11x^2 + 34x - 24] \quad (9)$$

$- \frac{1}{6} [x^3 - 10x^2 + 27x - 18] (30) + \frac{1}{30}[x^3 - 8x^2 + 19x - 12]$ (132)

$= \frac{1}{10}[x^3 - 13x^2 + 54x - 72] + \frac{2}{3} [x^3 - 11x^2 + 34x - 24]$

$-5 [x^3 - 10x^2 + 27x - 18] + \frac{22}{5}[x^3 - 8x^2 = 19x - 12]$

which gives on simplification

$P(x) = x^3 - 3x^2 = 5x - 6$

which is the Lagrange's interpolating polynomial of f(x).

Example 2

Using Lagrange's interpolation formula, find the value of f when x = 1.4 from the following table.

x	1.2	1.7	1.8	2.0
f	3.3201	5.4739	6.0496	7.3891

Solution

the Lagrange's interpolating formula with 4 points is

$$P(x)=\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f_1 +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f_3 \quad (6)$$

Substituting

$x_0 = 1.2$, $x_1 = 1.7$, $x_2 = 1.8$, $x_3 = 2.0$ and
$f_0 = 3.3201$, $f_1 = 5.4739$, $f_2 = 6.0496$, $f_3 = 7.3891$

in (6), we get

$$P(x) = \frac{(x - 1.7)(x - 1.8)(x - 2.0)}{(1.2 - 1.7)(1.2 - 1.8)(1.2 - 2.0)} * 3.3201 +$$

$$\frac{(x - 1.2)(x - 1.8)(x - 2.0)}{(1.7 - 1.2)(1.7 - 1.8)(1.7 - 2.0)} * 5.4739 +$$

$$\frac{(x - 1.2)(x - 1.7)(x - 2.0)}{(1.8 - 1.2)(1.8 - 1.7)(1.8 - 2.0)} * 6.0496 +$$

$$\frac{(x-1.2)(x-1.7)(x-1.8)}{(2.0-1.2)(2.0-1.7)(2.0-1.8)} * 7.3891 \qquad (7)$$

Putting x = 1.4 on both sides of (7), we get

$$f(1.4) = P(1.4) = \frac{(1.4-1.7)(1.4-1.8)(1.4-2.0)}{(-0.5)(-0.6)(-0.8)} * 3.3201 + \frac{(1.4-1.2)(1.4-1.8)(1.4-2.0)}{(0.5)(-0.1)(0.3)} * 5.4739 +$$

$$\frac{(1.4-1.2)(1.4-1.7)(1.4-2.0)}{(0.6)(0.1)(-0.2)} * 6.0496 +$$

$$\frac{(1.4-1.2)(1.4-1.7)(1.4-1.8)}{(0.8)(0.3)(0.2)} * 7.3891$$

$$= \frac{(-0.3)(-0.4)(-0.6)}{(-0.5)(-0.6)(-0.8)} * 3.3201 +$$

$$\frac{(0.2)(-0.4)(-0.6)}{(0.5)(-0.1)(-0.3)} * 5.4739 +$$

$$\frac{(0.2)(-0.3)(-0.6)}{(0.6)(0.1)(-0.2)} * 6.0496 +$$

$$\frac{(0.2)(-0.3)(-0.4)}{(0.8)(0.3)(0.2)} * 7.3891$$

$$= 0.99603 + 17.51648 - 18.1488 + 3.69455$$

$$= 4.05826$$

Therefore f(x) = 4.05826.

3.2 Inverse Interpolation

In inverse interpolation for a table of values of x and y = f(x), one is given a number $\overline{y}$ and wishes to find the point $\overline{x}$ so that $f(\overline{x}) = \overline{y}$, where f(x) is the tabulated function. This problem can always be solved if f(x) is (continuous/and) strictly increasing or decreasing (that is, the inverse of f exists). This is done by considering the table of values x_i, $f(x_i)$, i = 0, 1, …, n to be a table of values y_i $g(y_i)$, i = 0, 1, 2, …, n for the inverse function $g(y) = f^{n-1}(y) = x$ by taking $y_i = f(x_i)$, $g(y_i) = x_i$, i = 0, 1, 2, …, n. Then we can interpolate for the unknown value $g(\overline{y})$ in this table.

$$P_n(\bar{y}) = \sum_{i=0}^{n} x_i \prod_{\substack{i=0\\ i=j}}^{n} \frac{(y - y_j)}{(y_i - y_j)}$$

and $\bar{x} = P_n(\bar{y})$. This process is called inverse interpolation.

Let us consider some examples.

Example 3

From the following table, find the Lagrange's interpolating polynomial which agrees with the values of x at the given values of y. Hence find the value of x when y = 2.

x	1	19	49	101
y	1	3	4	5

Solution

Let x = g(y). the Lagrange's interpolating polynomial P(y) of g(y) is given by

$$P(y) = \frac{(y-3)(y-4)(y-5)}{(1-3)(1-4)(1-5)} * 1 + \frac{(y-1)(y-4)(y-5)}{(3-1)(3-4)(3-5)} * 19$$

$$+ \frac{(y-1)(y-3)(y-5)}{(4-1)(4-3)(4-5)} * 49 + \frac{(y-1)(y-3)(y-4)}{(5-1)(5-3)(5-4)} * 101$$

$$= -\frac{1}{24}[y^3 - 12y^2 + 47y - 60] + \frac{19}{4}[y^3 - 10y^2 + 29y - 20]$$

$$-\frac{49}{3}[y^3 - 9y^2 + 23y - 15] + \frac{101}{8}[y^3 - 8y^2 + 19y - 12]$$

which, on simplification, gives

$P(y) = y^3 - y^2 + 1$.

The Lagrange's interpolating polynomial of x is given by P(y).

There fore, $x = P(y) = y^3 - y^2 + 1$

Therefore, when y = 2, x = P(2) = 5.

Example 4

Find the value of x when y = 3 from the following table of values.

x	4	7	10	12
y	-1	1	2	4

Solution

The Lagrange's interpolation polynomial of x is given by

$$P(y) = \frac{(y-1)(y-2)(y-4)}{(-2)(-3)(-5)}(4) + \frac{(y+1)(y-2)(y-4)}{2(1)(-3)}(7)$$

$$+ \frac{(y+1)(y-1)(y-4)}{(3)(1)(-2)}(10) + \frac{(y+1)(y-1)(y-2)}{(5)\ (3)\ (2)}(12)$$

$$\text{Therefore } P(3) = \frac{(2)\ (1)\ (-1)}{-(2)\ (3)\ (5)}(4) + \frac{(4)\ (1)\ (-1)}{(2)\ (3)}(7)$$

$$+ \frac{(4)\ (2)\ (-1)}{-(3)\ (2)}(10) + \frac{(4)\ (2)\ (1)}{(5)(3)\ (2)}(12)$$

$$= \frac{4}{15} - \frac{14}{3} + \frac{40}{3} + \frac{48}{15}$$

$$= \frac{182}{15} = 12.1333$$

Hence, x(3) = P(3) = 12. 1333.

Now we are going to find the error committed in approximating the value of the function by $P_n(x)$.

3.3 General Error Term

Let $E_n(x) = f(x) - P_n(x)$ be the error involved in approximating the function f(x) by an interpolating polynomial. We derive an expression for $E_n(x)$ in the following theorem. This result helps us in estimating a useful bound on the error as explained in an example.

Theorem 4

Let $x_0, x_1, \ldots, x_n$ be distinct numbers in the interval [a, b] and f has (continuous) derivatives upto order (n + 1) in the open interval]a, b[. if $P_n(x)$ is the interpolating polynomial of degree $\leq$ n, which interpolates f(x) at the points $x_0, \ldots, x_n$, then for each $x \in$ [a, b], a number $\xi(x)$ in]a, b[exists such that

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\ldots\ldots(x - x_n) \qquad (8)$$

Proof

If $x \neq x_k$ for any k = 0, 1, 2, …, n, define the function g for t in [a, b] by

$$g(t) = f(t) - P_n(t) - [f(x) - P_n(x)]\prod_{j=0}^{n}\frac{(t - x_j)}{(x - x_j)}.$$

since f(t) has continuous derivatives up to order (n + 1) and P(t) ha derivatives of all orders, g(t) has continuous derivatives upto (n + 1) order. Now, for k = 0, 1, 2, …, n, we have

$$g(x_k) = f(x_k) = P_n(x_k) - [f(x) - P_n(x)]\prod_{j=0}^{n}\frac{(x_k - x_j)}{(x - x_j)}.$$
$$= 0 - [f(x) - P_n(x)].0 = 0$$

Furthermore, $g(x) = f(x) - P_n(x) - [f(x) - P_n(x)]\prod_{j=0}^{n}\frac{(x - x_j)}{(x - x_j)}.$

$$= f(x) - P_n(x) - [f(x) - P_n(x)].\ 1 = 0$$

Thus g has continuous derivatives upto order (n + 1) and g vanishes at the (n + 2) distinct points x, x_0, …, x_n. By the generalized Rolle's Theorem (Theorem 2) there exists $\xi(x)$ in]a, b[for which $g^{(n+1)}\xi(x) = 0$. Differentiating g(t), (n + 1) times (with respect to t) and evaluating at $\xi(x)$ i, we get

$$0 = g^{(n+1)}\xi(x) = f^{(n+1)}\xi(x) - (n+1)!\frac{[f(x) - P_n(x)]}{\prod_{i=0}^{n}(x - x_i)}$$

Simplifying we get (error at $x = \bar{x}$)

$$E_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = \prod_{i=0}^{n}(\bar{x} - x_i)\frac{f^{(n+1)}\varsigma(\bar{x})}{(n+1)!} \qquad (9)$$

The error formula (Eqn. (9)) derived above, is an important theoretical results because Lagrange interpolating polynomials are extensively used in deriving important formulae for numerical differentiation and numerical integration.

It is to be noted that $\xi = \xi(\bar{x})$ depends on the point $\bar{x}$ at which the error estimate is required. This dependence need not even be continuous. This error formula is of limited utility since $f^{(n+1)}(x)$ is not known (when we are given a set of data at specific nodes) and the point x is hardly known. But the formula can be used to obtain a bound on the error of interpolating polynomial. Let us see how, by an example.

Example 5

The following table gives the values of f(x) = e^x. If we fit an interpolating polynomial of degree four to the data, find the magnitude of the maximum possible error in the computed value of f(x) when x = 1.25.

x	1.2	1.3	1.4	1.5	1.6
y	3.3201	3.6692	4.0552	4.4817	4.9530

Solution

From Eqn. (9), the magnitude of the error associated with the 4th degree polynomial approximation is given by

$$|E_4(x)| = |(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)|\frac{f^{(5)}(\xi)}{5!}$$

$$= |(x-1.2)(x-1.3)(x-1.4)(x-1.5)(x-1.6)|\frac{f^{(5)}(\xi)}{5!} \qquad (10)$$

Since f(x) = c^x, $f^{(5)}(x) = e^x$.

When x lies in the interval [1.2, 1.6],

Max $|f^{(5)}(x)| = e^{1.6} = 4.9530$ (11)

Substituting (11) in (10), and putting x = 1.25, the upper bound on the magnitude of the error

$$= |(0.05\ (-0.05)\ (-0.15)\ (-0.25)\ (-0.35)|*\frac{4.9530}{120}$$

= 0.00000135.

4.0 CONCLUSION

Let us take a brief look at what you have studied in this unit as the concluding path of this unit to the summary.

5.0 SUMMARY

In this unit, we have seen how to derive the Lagrange's form of interpolating polynomial for a given data. It has been shown that he interpolating polynomial for a given data is unique. Moreover the Lagrange form of interpolating polynomial can be determined for equally spaced or unequally spaced nodes. We have also seen how the Lagrange's interpolation formula can be applied with y as the independent variable and x as the dependent variable so that the value of x corresponding to a given value of y can be calculated approximately when some conditions are satisfied. Finally, we have derived the general error formula and its use has been illustrated to judge the accuracy of our calculation. The mathematical formulae derived in this unit are listed below for your easy reference.

1) Lagrange's Form

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

where

$$L_i(x) = \left[\prod_{\substack{j=0 \\ j\neq i}}^{n} (x - x_j)\right] / \left[\prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)\right]$$

2) Inverse Interpolation

$$P_n(y) = \sum_{i=0}^{n} x_i \left[\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)}\right]$$

3) Interpolation Error

$$E_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = \prod_{i=0}^{n} (\bar{x} - x_i) \frac{f^{(n+1)}\varsigma(\bar{x})}{(n+1)!}$$

6.0 TUTOR-MARKED ASSIGNMENT

i Show that

a) $\sum_{i=0}^{n} L_i(x) = 1$

b) $\sum_{i=0}^{n} L_i(x)\, x_i^k = x^k,\ k \leq n$

where $L_i(x)$ are Lagrange fundamental polynomials

ii Let $w(x) = \prod_{k=0}^{n} (x - x_k)$. Show that the interpolating polynomial of degree $\leq$ n with the nodes $x_0, x_1, \ldots, x_n$ can be written as

$$P_n(x) = w(x) \sum_{i=0}^{n} \frac{f(x_k)}{(x - x_k) w'(x_k)}$$

iii Find the Lagrange's interpolation polynomial of f(x) from the following data. Hence obtain f(2).

x	0	1	4	5
f(x)	8	11	68	123

iv Find the value of y when x = 6 from the following table:

x	1	2	7	8
y	4	5	5	4

v Using the Lagrange's interpolation formula, find the value of y when x = 10.

x	5	6	9	11
y	12	13	14	16

vi For the data of Example 5 with last one omitted, i.e., considering only first four nodes, if we fit a polynomial of degree 3, find an estimate of the magnitude of the error in the computed value of f(x) when x = 1.25. Also find an upper bound in the magnitude of the error.

vii Find the value of x when y = 4 from the table given below:

x	8	16	20	72
y	-1	1	3	5

viii Using Lagrange's interpolation formula, find the value of f(4) from the following data:

x	8	16	20	72
y	-1	1	3	5

7.0 REFERENCES/FURTHER READINGS

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus. McGraw – Hill N.Y.

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach. 559 Nathan Abbott, Stanford, California, USA

UNIT 2 NEWTON FORM OF THE INTERPOLATING POLYNOMIAL

CONTENTS

1.0 INTRODUCTION

The Lagrange's form of the interpolating polynomial derived in Unit 1 has some drawbacks compared to Newton form of interpolating polynomial that we are going to consider now.

In practice, one is often not sure as to how many interpolation points to use. One often calculates $P_1(x)$, $P_2(x)$, … increasing the number of interpolation points, and hence the degrees of the interpolating polynomials till one gets a satisfactory approximation $P_k(x)$, no advantage is taken of the fact that one has already constructed $P_{k-1}(x)$, whereas in Newton form it is not so.

Before deriving Newton's general form of interpolating polynomial, we introduce the concept of divided difference and the tabular representation of divided differences. Also the error of the interpolating polynomial in this case is derived in terms of divided differences. Using the two different expressions for the error term we get a relationship between nth order divided difference and nth order derivative.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- obtain a divided difference in terms of function values
- form a table of divided differences and find divided differences with a given set of arguments from the table
- show that divided difference is independent of the order of its arguments

- obtain the Newton's divided differences interpolating polynomial for a given data
- find an estimate of f(x) for a given non-tabular value of x from a table of values of x and y [f(x)]
- relate the kth order derivative of f(x) with the kth order divided difference from the expression for the error term.

3.0 MAIN CONTENTS

3.1 Divided Differences

Suppose that we have determined a polynomial $P_{k-1}(x)$ of degree £ k – 1 which interpolates f(x) at the points $x_0, x_1, \ldots x_{k-1}$. In order to make use of $P_{k-1}(x)$ in calculating $P_k(x)$ we consider the following problem: What function g(x) should be added to $P_{k-1}(x)$ to get $P_k(x)$? Let $g(x) = P_k(x). - P_{k-1}(x)$. Now, g(x0 is a polynomial of degree £ k and $g(x_i) = P_k(x_i) - P_{k-1}(x_i) = f(x_i) - f(x_i) = 0$ for i = 0, 1, …, k – 1.

Suppose that $P_n(x)$ is the Lagrange polynomial of degree at most n that agrees with the function f at the distinct numbers $x_0, x_1, \ldots x_n$. $P_n(x)$ can have the following representation, called Newton form.

$$P_n(x) = A_0 + A_1 (x_1 - x_0) + A_1 (x_1 - x_0)(x - x_1) + \ldots + A_n (x - x_0)\ldots(x - x_{n-1}) \quad (1)$$

for appropriate constant $A_0, A_1, \ldots, A_n$.

Evaluating $P_n(x)$ (Eqn. (1)) at x_0 we get $A_0 = P_n(x_0)$. Similarly when $P_n(x)$ is evaluated at x_1, we get $A_1 = \dfrac{f(x) - f(x_1)}{x_1 - x_0}$. Let us introduce the notation for divided differences and define it at this stage: The zeroeth divided difference of the function f, with respect to x_i, is denoted by $f[x_i]$ and is simply the evaluation of f at x_i, that is, $f[x_i] = f(x_i)$. the first divided difference of f with respect to x_i and x_{i+1} is denoted by $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The remaining divided differences of higher orders are defined inductively as follows. The kth divided differences relative to $x_i, x_{i+1}, \ldots, x_{i+k}$ is defined as

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

where the (k – 1)st divided differences $f[x_i, \ldots, x_{i+k}]$ have been determined. This shows that kth divided difference is the divided differences of (k – 1)st divided differences justifying the name. The divided difference $f[x_i, x_2, \ldots, x_k]$ is invariant under all permutations of the arguments $x_i, x_2, \ldots, x_k$. To show this we proceed giving another expression for the divided difference.

For any integer k between 0 and n. let $Q_k(x)$ be the sum of the first k + 1 terms in form (1), i.e.

$Q_k(x) = A_0 + A_1 (x – x_0) + \ldots + A_k(x – x_0)\ldots(x – x_{k-1})$..

Since each of the remaining terms in Eqn. (1) has the factor $(x – x_0) (x – x_1)\ldots(x – x_k)$, Eqn. (1) can be rewritten as

$P_n(x) = Q_k(x) + (x – x_0)\ldots(x – x_k) R(x)$ for some polynomial R(x). as the term $(x – x_0)(x – x_1)\ldots(x – x_k)R(x)$ vanishes at each of the points $x_0, \ldots x_k$, we have $f(x_i) = P_n(x_i) = Q_k(x_i)$, i = 0, 1, 2, …, k. Since $Q_k(x)$ is a polynomial of degree £ k, by uniqueness of interpolating polynomial $Q_k(x) = P_k(x)$.

This shows that $P_n(x)$ can be constructed step by step with the addition of the next term in Eqn. (1), as one construct the sequence $P_0(x), P_1(x) \ldots$ with $P_k(x)$ obtained from $P_{k-1}(x)$ in the form

$P_k(x) = P_{k-1}(x) + A_k(x – x_0)\ldots(x – x_{k-1})$ (2)

That is, g(x) is a polynomial of degree £ k having (at least) the k distinct zeros $x_0, \ldots, x_{k-1}$.

\ $P_k(x) - P_{k-1}(x) = g(x) = A_k(x – x_0)\ldots(x – x_{k-1})$, for some constant A_k. this constant A_k is called the kth divided difference of f(x) at the points $x_0, \ldots, x_k$ for reasons discussed below and is denoted by $f[x_0, x_1, \ldots, x_k]$. this coefficient depends only on the values of f(x) at the point $x_0, \ldots, x_k$. thus Eqn. (2) can be written as
$P_k(x) =P_{k-1}(x) + f[x_0, \ldots,x_k] (x – x_0)\ldots(x – x_{k-1})$,

since $(x – x_0) (x – x_1)\ldots(x – x_{k-1}) = x^k$ + a polynomial of degree < k,
we can rewrite $P_k(x)$ s $P_k(x) = f[x_0, \ldots, x_k] x^k$ + apolynomial of degree < k (4)

(as $P_{k-1}(x)$ is a polynomial of degree < k).

But considering the Lagrange form of interpolating polynomial we have

$$P_k(x) = \sum_{i=0}^{k} f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{(x - x_j)}{(x_i - x_j)}$$

$$= \sum_{i=0}^{k} \left[\frac{f(x_i)^k}{\prod_{\substack{j=0 \\ i \neq j}}^{k} (x_i - x_j)} \right] x^k + \text{a polynomial of degree} < k.$$

Therefore, on comparison with Eqn. (4) we have

$$f[x_0, \ldots, x_k] = \sum_{i=0}^{k} \frac{f(x_i)}{(x_i - x_0)\ldots(x_1 - x_{i-1})\ldots(x_1 - x_{i+1})\ldots(x_i - x_k)} \quad (5)$$

This shows that

$$f[y_0, \ldots, y_k] = f[x_0, \ldots, x_k]$$

if $y_0, \ldots, y_k$ is a reordering of the sequence $x_0, \ldots, x_k$. We have defined the zeroeth divided difference of f(x) at x_0 by $f[x_0] = f(x_0)$ which is consistent with Eqn. (5).

For k = 1, we have from Eqn. (5)

$$f[x_0, x_k] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

This shows that the first divided difference is really a divided difference of divided differences.

We show below in Theorem 1 that for k> 2

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0} \quad (6)$$

This shows that the kth divided difference is the divided difference of (k – 1)st divided differences justifying the name. If $M = (x_0, \ldots, x_n)$ and N denotes any n – 1 elements of M and the remaining two elements are denoted by a and b, then

$$(f[x_0, \ldots, x_n = \frac{[(n - 1\text{st divided difference on N and a} - (n - 1)\text{st divided difference on N and b}]}{a - b} \quad (7)$$

Theorem 1:

$$f[x_0, \ldots, x_j] = \frac{f[x_1, \ldots, x_j] - f[x_0, x_1, \ldots, x_{j-1}]}{x_j - x_0} \quad (8)$$

Proof: Let $P_{i-1}(x)$ be the polynomial of degree £ $i - 1$ which interpolates f(x) at x_0, …, x_{i-1} and let $Q_{j-1}(x)$ be the polynomial of degree $\leq j - 1$ which interpolates f(x) at the points $x_1, \ldots, x_j$. Let us define P(x) as

$$P(x) = \frac{x - x_0}{x_j - x_0} Q_{j-1}(x) + \frac{x_j - x}{x_j - x_0} P_{j-1}(x).$$

This is a polynomial of degree £ j, and $P(x_i) = f(x_i)$ for i = 0, 1, …, j. By uniqueness of the interpolating polynomial we have $P(x) = P_j(x)$. Therefore

$$P_j(x) \frac{x - x_0}{x_j - x_0} Q_{j-1}(x) + \frac{x_j - x}{x_j - x_0} P_{j-1}(x).$$

Equating the coefficient of x^j from both sides of Eqn. (8), we obtain (leading) coefficient of

$$x^j \text{ in } P_j(x) = \frac{\text{leading coefficient of } Q_{j-1}(x)}{x_j - x_0} - \frac{\text{leading coefficient of } P_{j-1}(x)}{x_j - x_0}$$

That is $f[x_0, \ldots, x_j] = \dfrac{f[x_1, \ldots, x_j] - f[x_0, \ldots, x_{j-1}]}{x_j - x_0}$.

We now illustrate this theorem with the help of a few examples but before that we give the table of divided differences of various orders.

Table of divided differences

Suppose we denote, for convenience, a first order divided difference of f(x) with any two arguments by f[.,.], a second order divided difference with any three arguments by f[.,.,.] and so on. Then the table of divided difference can be written as follows

Table 1

x	f[.]	f[.,.]	f[.,.,.]	f[.,.,.,.]	f[.,.,.,.,.]
x_0	f_0				
		$f[x_0,x_1]$			
x_1	f_1		$f[x_0,x_1x_2]$		
		$f[x_1,x_2]$		$f[x_0,x_1x_2x_3]$	
x_2	f_2		$f[x_1,x_2x_3]$		$f[x_0,x_1x_2x_3x_4]$
		$f[x_2,x_3]$		$f[x_1x_2x_3x_4]$	
x_3	f_3		$f[x_2x_3x_4]$		
		$f[x_3,x_4]$			
x_4	f_4				

Example 1: If $f(x) = x^3$, find the value of f[a, b, c].

Solution: $f[a, b] = \frac{f(b) - f(a)}{b - a} = \frac{b^3 - a^3}{b - a}$

$$= b^2 + ba + a^2 = a^2 + ab + b^2$$

Similarly,
$f[a, b] = c^2 + cb + b^2 = b^2 + bc + c^2$

$$f[a, b, c] = \frac{f[b, c] - f[a, b]}{c - a}$$

$$= \frac{(b^2 + bc + c^2) - (a^2 + ab + b^2)}{c - a}$$

$$= \frac{(c^2 - a^2) + b(c - a)}{c - a}$$

$$= \frac{(c - a)(c + a + b)}{(c - a)}$$

$$= a + b + c$$

$f[a, b, c] \quad = a + b + c.$

Example 2: If $f(x) = \frac{1}{x}$, show that

$$f[a, b, c, d] = -\frac{1}{abcd}$$

Solution: $f[a, b] = \dfrac{\frac{1}{b} - \frac{1}{a}}{b - a} = \dfrac{a - b}{ab(b - a)} = -\dfrac{1}{ab}$

Similarly,

$f[b, c] = -\dfrac{1}{bc}$, $f[c, d] = -\dfrac{1}{cd}$

$f[a, b, c] = \dfrac{\frac{1}{bc} + \frac{1}{ab}}{c - a} = \dfrac{\frac{1}{ab} - \frac{1}{bc}}{c - a}$

$= \left[\dfrac{\frac{c-a}{abc}}{c-a}\right] = \dfrac{1}{abc}$

Similarly,

$f[b, c, d] = \dfrac{1}{bcd}$

however$f[a, b, c, d] = \left[\dfrac{\frac{c-a}{abc}}{c-a}\right] = \dfrac{1}{abc}$

$= \left[\dfrac{\frac{a-d}{abcd}}{d-a}\right]$

$= -\dfrac{1}{abcd}$

Consequently, $f[a, b, c, d] = -\dfrac{1}{abcd}$

In next section we shall make use of the divided difference to derive Newton's general form of interpolating polynomial.

3.2 Newton's General Form of Interpolating Polynomial

In section 3.1 we have shown how $P_n(x)$ can be constructed step by step as one construct the sequence $P_0(x)$, $P_1(x)$, $P_2(x)$, ..., with $P_k(x)$ obtained from $P_{k-1}(x)$ with the addition of the next term in Eqn. (3), that is,

$P_k(x) = P_{k-1}(x) + (x - x_0)(x - x_1)...(x - x_{k-1}) f[x_0, ..., x_k]$
Using this Eqn. (1) ca be rewritten as

$$P_n(x) = f[x_0] + (x - x_0) f[x_0,x_1] + (x - x_0)(x - x_1)f[x_0,x_1,x_2] + ... + (x - x_0)(x - x_1)...(x - x_{n-1}) f[x_0,x_1,....,x_n]. \quad (9)$$

This can be written compactly as follows:

$$P_n(x) = \sum_{i=0}^{n} f[x_0,...,x_i] \prod_{j=0}^{j-1} (x - x_j) \quad (10)$$

This is the Newton's form of interpolating polynomial.

Example 3: From the following table of values, find the Newton's form of interpolating polynomial approximating (x).

x	-1	0	3	6	7
f(x)	3	-6	39	822	1611

Solution: We notice that the values of x are not equally spaced. We are required to find a polynomial which approximates f(x). We form the table of divided differences of f(x).

Table 2

x	f[.]	f[.,.]	f[.,.,.]	f[.,.,.,.]	f[.,.,.,.,.]
-1	3				
		9			
0	- 6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

Since the divided difference upto order 4 are available, the Newton's interpolating polynomial $P_4(x)$ is given by

$$P_4(x) = f(x_0) + (x - x_0) f[x_0,x_1] + (x - x_0)(x - x_1)f[x_0,x_1,x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0,x_1,x_2,x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f[x_0,x_1,x_2,x_3,x_4] \quad (11)$$

where $x_0 = -1$, $x_1 = 0$, $x_2 = 3$, $x_3 = 6$ and $x_4 = 7$.

The divided differences $f(x_0)$, $f[x_0,x_1]$, $f[x_0,x_1,x_2]$, $f[x_0,x_1,x_2,x_3]$ and $f[x_0,x_1,x_2,x_3,x_4]$ are those which lie along the diagonal at $f(x_0)$ as shown by the dotted line. Substituting the values of x_i and the values of the divided differences in Eqn. (11), we get

$$P_4(x) = 3 + (x + 1) (-9) + (x + 1) x (6) + (x + 1) x (x - 3) (5) + (x + 1) x (x - 3) (x - 6) (1)$$

which on simplification gives

$$P_4(x) = x^4 - 3x^3 + 5x^2 - 6$$

Therefore, $f(x) = P_4(x) = x^4 - 3x^3 + 5x^2 - 6$

We now consider an example to show how Newton's interpolating polynomial can be used to obtain the approximate value of the function f(x) at any non-tabular point.

Example 4: Find the approximate values of f(x) at x = 2 and x = 5 in Example 3.

Solution: Since $f(x) = P_4(x)$, from Example 3, we get
$f(2) = P_4(2) = 16 - 24 + 20 - 6 = 6$

and
$f(5) = P(5) = 625 - 375 + 125 - 6 = 369$

Note 1: When the values of f(x) for given values of x are required to be found, it is not necessary to find the interpolating polynomial $P_4(x)$ in its simplified form given above. We can obtain the required values by substituting the values of x in Eqn. (11) itself. Thus,

$$P_4(2) = 3 + (3) (-9) + (3) (2) (6) + (3) (2) (-1) (5) + (3) (2) (-1) (-4) 1$$

Therefore, $P_4(2) = 3 - 27 + 36 - 30 + 24 = 6$.

Similarly,
$$P_4(5) = 3 + (6) (-9) + (6) (5) (6) + (6) (5) (2) (5) + (6) (5) (2) (-1) (1)$$
$$= 3 - 54 + 180 + 300 - 60 = 369.$$

Then $f(2) = P_4(2) = 6$
And
$f(5) = P(5) = 369$.

Example 5: Obtain the divided differences interpolation polynomial and the Lagrange's interpolating polynomial of f(x) from the following data and show that they are same.

x	0	2	3	4
f(x)	-4	6	26	64

Solution:

(a) Divided differences interpolation polynomial:

Table 3

x	f[x]	f[.,.]	f[.,.,.]	f[.,.,.,.]
0	-4			
		5		
2	6		5	
		20		1
3	26		9	
		38		
4	64			

$$P(x) = -4 + x(5) + x(x-2)(5) + x(x-2)(x-3)(1)$$

$$= x^3 + x - 4$$

\ $P(x) = x^3 + x - 4$

b) Lagrange's interpolation polynomial:

$$P(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)}(-4) + \frac{x(x-3)(x-4)}{(2)(-1)(-2)}(6)$$

$$+ \frac{x(x-2)(x-4)}{(3)(1)(-1)}(26) + \frac{x(x-2)(x-3)}{(4)(2)(1)}(64)$$

$$= \frac{1}{6}(x^3 - 9x^2 + 26x - 24) + \frac{3}{2}(x^3 - 7x^2 + 12x)$$

$$-\frac{26}{3}(x^3 - 6x^2 + 8x) + 8(x^3 - 5x^2 + 6x).$$

On simplifying, we get

$P(x) = x^3 + x - 4.$

Thus, we find that both polynomials are same.

In Unit 1 we have derived the general error term i.e. error committed in approximating f(x) by $P_n(x)$. In next section we derive another expression for the error term in term of divided difference.

3.3 The Error of the Interpolating Polynomial

Let $P_n(x)$ be the Newton form of interpolating polynomial of degree £ n which interpolates f(x) at x_0 ...., x_n.
The interpolating error $E_n(x)$ of $P_n(x)$ is given by

$$E_n(x) = f(x) - P_n(x) \qquad (12)$$

Let $\bar{x}$ be any point different from x_0, ...,x_n. If $P_n(x)$ is the Newton form of interpolating polynomial which interpolates f(x) at x_0, ...,x_n and $\bar{x}$, then $P_{n+1}(\bar{x}) = f(\bar{x})$. Then by (10) we have

$$P_{n+1}(x) = P_n(x) + f[x_0, ..., x_n, \bar{x}]\prod_{j=0}^{n}(x - x_j)$$

Putting $x = \bar{x}$ in the above, we have

$$f(\bar{x}) = P_{n+1}(\bar{x}) = P_n(\bar{x}) + f[x_0, ..., x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j)$$

i.e. $$E_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = f[x_0, ..., x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j) \qquad (13)$$

This shows that the error is like the next term in the Newton form.

3.4 Divided Difference and Derivative of the Function

Comparing Eqn. (13) with the error formula derived in Unit 1 Eqn. (9), we can establish a relationship between divided difference and the derivatives of the function

$$E_n(\bar{x}) = \frac{f^{(n+1)}[x(\bar{x})]}{(n+1)!}\prod_{j=0}^{n}(\bar{x} - x_j)$$

$$= f[x_0, x_1, ..., x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j)$$

Comparing, we have $f[x_0, x_1, ..., x_{n+1}] = \frac{f^{(n+1)}\varsigma}{(n+1)!}$

(considering $\bar{x} = x_{n+1}$)
Further it can be shown that x Î]min x_i, max x_i[.
We state these results in the following theorem.

Theorem 2: Let f(x) be a real-valued function, defined on [a, b] and n times differentiable in]a, b[. If x_0, ...,x_n are n + 1 distinct points in [a, b], then there exists $\varsigma \in$]a, b[such that

$$f[x_0, ..., x_n] = \frac{f^{(n+1)}\varsigma}{n!}$$

Corollary 1:
If $f(x) = x^n$, then

$$f[x_0, ..., x_n] = \frac{n!}{n!} = 1.$$

Corollary 2:
If $f(x) = x^k$, $k < n$, then

$$f[x_0, ..., x_k] = 0$$

since nth derivative of x^k, $k < n$, is zero.

For example, consider the first divided difference

$$f[x_0,x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

by Mean Value Theorem $f(x_1) = f(x_0) + (x_1 - x_0) f'(\varsigma)$, $x_0 < \varsigma < x$,

substituting, we get
$f[x_0,x_1] = f'(\varsigma)$, $x_0 < \varsigma < x_1$.

Example 6: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, then find $f[x_0, x_1, ..., x_n] = a_n * \frac{n!}{n!} + 0 = a_n$.

Let us consider another example.

Example 7: If $f(x) = 2x^3 + 3x^2 - x + 1$, find

f[1, -1, 2, 3], f[a, b, c, d], f[4, 6, 7, 8].

Solution: Since f(x) is a cubic polynomial, the 3rd order divided differences of f(x) with any set of argument are constant and equal to 2, the coefficient of x^3 in f(x).

Thus, it follows that f[1, -1, 2, 3], f[a, b, c, d], and f[4, 6, 7, 8] are each equal to 2.

In the next section, we are going to discuss about bounds on the interpolation error.

3.5 Further Results on Interpolation Error

We have derived error formula

$E_n(x) = f(x) - P_n(x) = \prod_{i=0}^{n} (\bar{x} - x_i) \frac{f^{(n+1)} \varsigma(\bar{x})}{(n+1)!}$.,

We assume that f(x) is (n + 1) times continuously differentiable in the interval of interest [a, b] = I that contains $x_0, ..., x_n$ and x. since $\varsigma(x)$ is known we may replace $f^{(n+1)}(\varsigma(x))$ by $\max_{x \in I} |f^{(n+1)}(x)|$. If we denote $(x - x_0)(x - x_1)...(x - x_n)$ by $\psi_n(x)$ then we have

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{\max_{x \in I} |f^{(n+1)}(t)|}{(n+1)!} \max_{x \in I} |\psi_n(t)| \quad (14)$$

Consider now the case when the nodes are equally spaced, that is (m $x_j = x_0 + jh$), j = 0,.....,N, and h is the spacing between consecutive nodes. For the case n = 1 we have linear interpolation. If $x \in [x_{i-1}, x_i]$, then we approximate f(x) by $P_1(x)$ which interpolates at

x_{i-1}, and x_i. From Eqn. (14) we have $|E_n(x)| \le \frac{1}{2} \max_{t \in I} |f''(t)| \max_{t \in I} |\psi_1(t)|$

where $\psi_1(x) = (x - x_{i-1})(x - - x_i)$.

Now,

$\frac{d\psi_1}{dx} = x - x - - = 0$

gives $x = (x_{i-1} - x_i)/2$.

Hence, the maximum value of $(x - x_{i-1})(x - - x_i)|$ occurs at
$x = x^* = (x_{i-1} - x_i)/2$.

The maximum value is given by

$|\psi_1(x^*)| = \frac{(x_i - x_{i-1})^2}{4} = \frac{h^2}{4}$.

Thus, we have for linear interpolation, for nay $x \in I$

$$|E_1(x)| = |f(x) - P_1(x)| \le \frac{(x_i - x_{i-1})^2}{4} \frac{1}{2} \max_{x \in I} |f''(x)|$$

$$= \frac{h^2}{8} M. \quad (15)$$

where $|f''(x)| \le M$ on I.

For the case n = 2, it can be shown that for any $x \in [x_{i-1}, x_{i+1}]$.

$$|E_2(x)| \le \frac{h^2 M}{9\sqrt{3}} \text{ where } |f'''(x)| \le M \text{ on I.} \qquad (16)$$

Example 8: Determine the spacing h in table of equally spaced values of the function of $f(x) = \sqrt{x}$ between 1 and 2, so that interpolation with a first degree polynomial in this table will yield seven place accuracy.

Solution: Here

$f''(x) = -\frac{1}{4}x^{-3/2}$

$\max_{1 \le x \le 2} |f''(x)| = \frac{1}{4}$.

and $|E_1(x)| \le \frac{h^2}{32}$.

For seven place accuracy, h is to be chosen such that

$\frac{h^2}{32} < 5.10^{-8}$.

or $h^2 < (160)10^{-8}$ that is $h < .0013$.

4.0 CONCLUSION

This unit shall be concluded by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have derived a form of interpolating polynomial called Newton's general form, which has some advantage over the Lagrange's form discussed in Unit 1. This form is useful in deriving some other interpolating formulas. We have introduced the concept of divided differences and discussed some of its important properties before deriving Newton's general form. The error term has also been derived and utilizing the error term we have established a relationship between the divided difference and the derivative of the function f(x) for which the interpolating polynomial has been obtained. The main formula derived are listed below:

- $f[x_0,\ldots,x_j] = \dfrac{f[x_1,\ldots,x_j] - f[x_0,\ldots,x_{j-1}]}{x_j - x_0}$

- $P_n(x) = \sum_{i=0}^{n} f[x_0,...,x_i]\prod_{j=0}^{j-1}(x - x_j)$
- $E_n(x) = f[x_0,...,x_n,x]\prod_{j=0}^{n}(x - x_j)$
- $f[x_0,....,x_n] = \frac{f^{(n)}(\xi)}{n!}$, $\xi \in$]min x_i, max$_i$[

6.0 TUTOR-MARKED ASSIGNMENT

i Find the Lagrange's interpolating polynomial of f(x) from the table of values given below and show that it is the same as the Newton's divided differences interpolating polynomial.

x	0	1	4	5
f(x)	8	11	68	123

ii Form the table of values given below, obtain the value of y when x = 1.5 using
a) divided differences interpolation formula.
b) Lagrange's interpolation formula.

x	0	1	2	4	5
f(x)	5	14	41	98	122

iii Using Newton's divided difference interpolation formula, find the values of f(8) and f(15) from the following table.

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

iv If $f(x) = 2x^3 - 3x^2 + 7x + 1$, what is the value of f[1, 2, 3, 4]?
v If $f(x) = 3x^2 - 2x + 5$, find f[1, 2], f[2, 3] and f[1, 2, 3].
vi If f(x) takes the values -21, 15, 12 and 3 respectively when x assumes the values -1, 1, 2 and 3, find the polynomial which approximates f(x).
vii Find the polynomial which approximate f(x), tabulated below

x	-4	-1	0	2	5
f(x)	1245	33	5	9	1335

Also find an approximate value of f(x) at x = 1 and x = -2.
viii From the following table, find the value of y when x = 102

x	93.0	96.2	100.0	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

7.0 REFERENCES/FURTHER READINGS.

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus. McGraw – Hill N.Y.

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UNIT 3 INTERPOLATION AT EQUALLY SPACED POINTS

CONTENTS

1.0 INTRODUCTION

Suppose that y is a function of x. The exact functional relation y = f(x) between x and y may or may not be known. But, the values of y at (n + 1) equally spaced of x are supposed to be known, i.e., (x_i, y_i); i = 0, ..., n are known where $x_i – x_{i-1}$ = h (fixed), i = 1, 2, ..., n. Suppose that we are required to determine an approximate value of f(x) or its derivative f'(x) for some values of x in the interval of interest. The methods for solving such problems are based on the concept of finite differences. We have introduced the concept of forward, backward and central differences and discussed their interrelationship in the previous unit

We have already introduced two important forms of the interpolating polynomial in Units 1 and 2. These forms simply when the nodes are equidistant. For the case of equidistant nodes, we have derived the Newton's forward, backward difference forms and Stirling's central difference form of interpolating, each suitable for use under a specific situation. We have derived these methods in the previous unit and also given the corresponding error term.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- write a forward difference in terms of function values from a table of forward differences and locate a difference of given order at a given point

- write a backward difference in terms of function values from a table of backward differences and identify differences of various orders at any given point from the table
- expand a central difference in terms of function values and form a table of central differences
- establish relations between V, ∇, d and divided difference
- obtain the interpolating polynomial of f(x) for a given data by applying any one of the interpolating formulas
- compute f(x) approximately when x lies near the beginning of the table and estimate the error
- compute f(x) approximately when x lies near the end of the table and estimate the error
- estimate the value of f(x) when x lies near the middle of the table and estimate the error.

3.0 MAIN CONTENTS

3.1 Differences

Suppose that we are given a table of values (x_i, y_i), i = 0, 1, 2, ..., N where $y_i = f(x_i) = f_j$.

Let the nodal points be equidistant. That is

$$x_i = a + ih, \; i = 0, ..., N, \text{ with } N = (b - a)/h \qquad (1)$$

For simplicity we introduce a linear change of variables

$$s = s(x) = \frac{x - x_0}{h}, \text{ so that } x = x(s) = x_0 + sh \qquad (2)$$

and introduce the notation

$$f(x) = f(x_0 + sh) = f_s \qquad (3)$$

The linear change of variables in Eqn. (2) transforms polynomials of degree n in x into polynomials of degree n is s. we have already introduced the divided-difference table to calculate a polynomial of degree £ n which interpolates f(x) at $x_0, x_1, ..., x_n$. For equally spaced nodes, we shall deal with three types of differences, namely, forward, backward and central and discuss their representation in the form of a table. We shall also derive the relationship of these differences with divided differences and their interrelationship.

3.1.1 Forward Differences

We denote the forward differences of f(x) if ith order at $x = x_0 + sh$ by $\Delta^i f_s$ and define it as follows:

$$\Delta^i f_s = \begin{cases} f_s & i = 0 \\ V(V^{i-1} f_s) = V^{i-1} f_{s+1} - V^{i-1} f_s, & i > 0. \end{cases}$$

Where V denotes forward difference operator.

When s = k, that is, $x = x_k$, we have

for i = 1 $\quad \Delta f_k = f_{k+1} - f_k$

for i = 2 $\quad \Delta^2 f_k = f_{k+1} - f_k$

$$= f_{k+2} - f_{k+1} - [f_{k+1} - f_k]$$

$$= f_{k+2} - f_{k+1} + f_k$$

Similarly $\quad \Delta^3 f_k = f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k$

We recall the binomial theorem

$$(a + b)^s = \sum_{j=0}^{n} \begin{bmatrix} s \\ j \end{bmatrix} a^j b^{r-j} \qquad (4)$$

where s is a real non-negative integer.

We give below in Lemma 1 the relationship between the forward and divided differences. This relation will be utilized to derive the Newton's forwarddifference formula which interpolates f(x) at $x_k + ih$, i = 0, 1, ..., n.

Lemma 1: For all $i \geq 0$

$$f[x_k, ..., k_{k+1}] = \frac{1}{i!h^i} \Delta^i f_k \qquad (5)$$

Proof: We prove the result by induction.
For i = 0, both sides of relation (5) are same by convention, that is,

$$f[x_k] = f(x_k) = f_k = \Delta^0 f_k.$$

Assuming that relation (5) holds for $i = n \geq 0$, we have for i = n + 1

$$f[x_k, x_{k+1}, ..., k_{k+n+1}] = \frac{f[x_{k+1}, ..., x_{k+n+1}] - f[x_k, ..., x_{k+n}]}{x_{k+n+1} - x_k}$$

$$= \frac{\left[\Delta^n f_{k+1} / n!h^n\right] - \left[\Delta^n f_k / n!h^n\right]}{x_0 + (k+n+1)h - x_0 - kh}$$

$$= \frac{\Delta^n f_{k+1} - \Delta^n f_k}{(n+1)!h^{n+1}} = \frac{\Delta^{n+1} f_k}{(n+1)!h^{n+1}}$$

This shows that relation (5) holds for i = n + 1 also. Hence (5) is proved. We now give a result which immediately follows from this theorem in the following corollary.

Corollary: If $P_n(x)$ is a polynomial of degree n with leading coefficient a_n, and x_0 is an arbitrary point, then

$\Delta^n P_n(x_0) = a_n n!\, h^n$

and $\Delta^{n+1} P_n(x_0) = 0$, i.e., all higher differences are zero.

Proof: Taking k = 0 in relation (5) we have

$$f[x_0, ..., x_i] = \frac{1}{1!h^i} \Delta^i f_0. \qquad (6)$$

Let us recall that

$$f[x_0, ..., x_i] = \frac{f^{(1)}(\xi)}{1!} \qquad (7)$$

where f(x) is a real-valued function defined on [a, b] and i times differentiable in]a, b[and $\xi \in$]a, b[.

Taking i = n and $f(x) = P_n(x)$ in Eqns. (6) and (7), we get

$$\Delta^{in} P_n(x_0) = n!h^n P_n[x_0, ..., x_n] = n!h^n \frac{P_n^{(n)}(x)}{n!}$$
$$= h^n n! a_n.$$

Since $\Delta^{in+1} P_n(x_0) = \Delta^{in} P_n(x_1) - \Delta^{in} P_n(x_0)$

$$= h^n n! a_n - h^n n! a_n = 0.$$

This completes the proof

The shift operator E is defined as

$Ef_i = f_{i+1}$ (8)

In general $Ef(x) = f(x + h)$.

We have $E^s f_i = f_{i+s}$

For example,

$E^3 f_i = f_{i+3}$, $E^{1/2} f_i = f_{i+1/2}$ and $E^{-1/2} f_i = f_{i-1/2}$

Now,

$\Delta^i f_i = f_{i+1} - Ef_i - f_i = (E - 1)f_i$

Hence the shift and forward difference operations are related by

$\Delta = E - 1$

or $E = 1 + \Delta$

Operating s times, we get

$$\Delta^s = (e - 1)^s = \sum_{j=0}^{n} \binom{s}{j} E^j (-1)^{r-1} \quad (9)$$

Making use of relation (8) in Eqn. (9), we get

$$\Delta^s f_i = \sum_{j=0}^{n} (-1)^{r-1} \binom{s}{j} f_{j+1}$$

We now give in Table 1, the forward differences of various orders using 5 values.

Table 1: Forward Difference Table

x	f(x)	$\Delta^1 f$	$\Delta^2 f \Delta^3 f \Delta^4 f$
x_0	f_0		
Δf_0			
x_1	$f_1 \Delta^2 f_0$		
$\Delta f_1 \Delta^3 f_0$			
x_2	$f_2 \Delta^2 f_1 \Delta^4 f_0$		
$\Delta f_2 \Delta^3 f_1$			
x_3	$f_3 \Delta^2 f_2$		
x_4	$f_4 \Delta f_3$		

Note that the forward diffirence $\Delta^k f_0$ lie on a straight line sloping downward to the right.

3.1.2 Backward Differences

Let f be a real-valued function of x. let the values of f(x) at n + 1 equally spaced points $x_0, x_1, \ldots, x_n$ be $f_0, f_1, \ldots, f_n$ respectively.

The backward differences of f(x) of ith order at $x_k = x_0 + kh$ are denoted by $\nabla^i f_k$. They are defined as follows:

$$\nabla^i f_k = \begin{cases} f_k, & 1 = 0 \\ \nabla^{i-1}(\nabla f_k) = \nabla^{i-1}[f_k - f_{k-1}], & i \geq 1 \end{cases} \qquad (10)$$

where ∇ denotes backward difference operator.

Using (10), we have for

$i = 1;\ \nabla f_k = f_k - f_{k-1}$

$$\begin{aligned} i = 2;\ \nabla^2 f_k &= \nabla(f_k - f_{k-1}) \\ &= \nabla f_k - f_{k-1} \\ &= f_k - 2f_{k-1} + f_{k-2} \end{aligned}$$

$$\begin{aligned} i = 3;\ \nabla^3 f_k &= \nabla^2[f_k - f_{k-1}] = \nabla^2 f_k - \nabla^2 f_{k-1} = \nabla[f_k] - \nabla[f_{k-1}] \\ &= \nabla[f_k - f_{k-1}] - \nabla[f_{k-1} - f_{k-2}] \\ &= \nabla f_k - \nabla f_{k-1} - \nabla f_{k-1} + f_{k-2} \\ &= f_k - f_{k-1} - 2[f_{k-1} + f_{k-2}] + f_{k-2} - f_{k-3} \\ &= f_k - 3f_{k-2} + 3f_{k-2} - f_{k-3} \end{aligned}$$

By induction we can prove the following lemma which connects the divided difference with the backward difference.

Lemma 2: The following relation holds

$$f[x_{n-k}, \ldots, x_n] = \frac{1}{k!h^k}\nabla^k f(x_n) \qquad (11)$$

The relation between the backward difference operator ∇ and the shift operator E is given by

$$\nabla = 1\ E^{-1} \text{ or } E = (i - \nabla)^{-1}$$

Since $\quad \nabla f_k = f_k - f_{k-1} = f_k - E^{-1} f_k = [1 - E]f_k.$

Operating s times, we get

$$\nabla^s f_k = [1 - E]^s f_k = \left[\sum_{j=0}^{n} \binom{s}{m} E^{-m} (-1)^m \right] f_k$$

$$= \sum_{j=0}^{n} \binom{s}{m} (-1)^m f_{k-m} \qquad (12)$$

We can extend the binomial coefficient notation to include negative numbers, by letting

$$\begin{bmatrix} s \\ i \end{bmatrix} = \frac{-s(-s-1)(-s-2)\ldots(-s-i+1)}{i!} = (-1)^i \frac{s(s+1)\ldots.(s+i-1)}{i!}$$

The backward differences of various orders with 5 nodes are given in Table 2.

Table 2: Backward Difference Table

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
∇f_1					
x_1	$f_1 \nabla^2 f_2$				
$\nabla f_2 \nabla^3 f_3$					
x_2	$f_2 \nabla^2 f_3 \nabla^4 f_4$				
$\nabla f_3 \nabla^3 f_4$					
x_3	$f_3 \nabla^2 f_4$				
∇f_4					
x_4	f_4				

Let us consider the following example:

Example 1: Evaluate the differences

(a) $\nabla^3[a_2x^2 + a_1x + a_0]$

(b) $\nabla^3[a_3x^3 + a_2x^2 + a_3x + a_0]$.

Solution:

(a) $\nabla^3[a_2x^2 + a_1x + a_0] = 0$

(b) $\nabla^3[a_3x^3 + a_2x^2 + a_3x + a_0]$.
$= a_3\nabla^3(x^3) + \nabla^3[a_2x^2 + a_1x + a_0]$
$= a_3.3\,!\,h^2$

Note that the backward differences $\nabla^k f_4$ lie on a straight line sloping upward to the right.

Also note that $V f_k = \nabla f_{k+1} = f_{k+1} - f_k$.

Try to show that $V^4 f_0 = \nabla^4 f_4$.

Let us now discuss about the central differences.

3.1.3 Central Differences

The first order central difference of f at x_k, denoted by $d f_k$, is defined as

$$d f = f(x + h/2) - f(x - h/2) = f_{k+1/2} - f_{k-1/2}.$$

Operating with d, we obtain the higher order central differences as

$$d^s f_k = f_k \text{ when } s = 0.$$

The second order central difference is given by

$$\begin{aligned} d^2 f_k &= d[f_{k+1/2} - f_{k-1/2}] = d[f_{k+1/2}] - d[f_{k-1/2}] \\ &= f_{k+1} - f_k - f_k + f_{k-1} \\ &= f_{k+1} - 2f_k + f_{k-1} \end{aligned}$$

Similarly,

$d^3 f_k = f_{k+3/2} - 3f_{k+1/2} + 3f_{k-1/2} - f_{k-3/2}$

and $d^4 f_k = f_{k+2} - 4f_{k+1} + 6f_k - 4f_{k-1} + f_{k-2}$.

Notice that the even order differences at a tabular value x_k are expressed in terms of tabular values of f and odd order differences at a tabular value x_k are expressed in terms of non-tabular value of f. also note that the coefficients of $d^s f_k$ are the same as those of the binomial expansion of $(1 - x)^s$, s = 1, 2, 3, ...

Since

$d f_k \;\; f_{k+1/2} - f_{k-1/2} = (E^{1/2} - E^{-1/2}) f_k$

We have the operation relation

$$d = E^{1/2} - E^{-1/2} \qquad (14)$$

The central differences at a non-tabular point $x_{k+1/2}$ can be calculated in a similar way.

For example,

$df_{k+1/2} = f_{k+1} - f_k$

$$
\begin{aligned}
d^2f_{k+1/2} &= f_{k+3/2} - 2f_{k+1/2} + f_{k-1/2} \\
d^3f_{k+1/2} &= f_{k+2} - 3f_{k+1} + 3f_k - f_{k-1} \\
d^4f_{k+1/2} &= f_{k+3/2} - 4f_{k+3/2} + 6f_{k+1/2} - 4f_{k-1/2} + f_{k-3/2}
\end{aligned} \qquad (15)
$$

Relation (15) can be obtained easily by using the relation (14)

We have

$d^sf_k = [E^{1/2} - E^{-1/2}]^s f_k$

$$= \left[\sum_{i=0}^{n} \binom{s}{i} E^{-i/2} E^{(n-i)/2} (-1)^i \right] f_k$$

$$= \left[\sum_{i=0}^{n} \binom{s}{i} (-1)^i \right] f_{k+(n/2)-1} \qquad (16)$$

The following formulas can also be established:

$$f[x_0, \ldots, x_{2m}] = \frac{1}{(2m)!h^{2m}} d^{2m} f_m \qquad (17)$$

$$f[x_0, \ldots, x_{2m+1}] = \frac{1}{(2m+1)!h^{2m+1}} d^{2m+1} f_{m+1/2} \qquad (18)$$

$$f[x_{-m}, \ldots, x_0, \ldots, x_m] = \frac{1}{(2m)!h^{2m}} d^{2m} f_0 \qquad (19)$$

$$f[x_{-m}, \ldots, x_0, \ldots, x_{m+1}] = \frac{1}{(2m+1)!h^{2m+1}} d^{2m+1} f_{1/2} \qquad (20)$$

$$f[x_{-(m+1)}, \ldots, x_0, \ldots, x_m] = \frac{1}{(2m+1)!h^{2m+1}} d^{2m+1} f_{-1/2} \qquad (21)$$

We now give below the central difference table with 5 nodes.

Table 3: Central Difference Table

x	f	df	d^2f	d^3f	d^4f
x_{-2}	f_{-2}				
		$df_{-3/2}$			
x_{-1}	f_{-1}		d^2f_{-1}		
		$df_{-1/2}$		$d^3f_{-1/2}$	
x_0	f_0		d^2f_0		d^4f_0
		$df_{1/2}$		$d^3f_{1/2}$	
x_1	f_1		d^2f_1		
		$df_{3/2}$			
x_2	f_2				

Note that the difference $d^{2m}f_0$lie on a horizontal line shown by the dotted lines.

Table 4: Central Difference Table

x	f	$df\,d^2f$	d^3f	d^4f
x_0	f_0			
$df_{1/2}$				
x_1	$f_1\, d^2f_1$			
$df_{3/2}\, d^3f_{3/2}$				
x_2	$f_2\, d^2f_2\, d^4f_2$		....	....
$df_{5/2}\, d^3f_{5/2}$				
x_3	$f_3\, d^2f_3$			
$df_{7/2}$				
x_4	f_4			

Note that the difference $d^{2m}f_2$ lie on a horizontal line.

We now define the mean operator mas follows

$$mf_k = \frac{1}{2}[f_{k+1/2} + f_{k-1/2}]$$

$$= \frac{1}{2}[E^{1/2} + E^{-1/2}]f_k.$$

Hence

$$m = \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

Relation Between the Operators V, ∇, d and m

We have expressed V, ∇, d and m in terms of the operator E as follows

$$V = E - 1$$
$$\nabla = 1 - E^{-1}$$

$$d = E^{1/2} - E^{-1/2}$$

$$m = \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

$$V = E(1 = E^{-1}) = E\nabla$$

$$= E^{1/2}(E^{1/2} - E^{-1/2})\ E^{1/2}d$$

Also $E^{1/2} = m + \frac{d}{2}$

$E^{-1/2} = m - \frac{d}{2}$

Example 2:

(a) Express $V^3 f_1$ as a backward difference.

(b) Express $V^3 f_1$ as a central difference.

(c) Express $d^2 f_2$ as a forward difference.

Solution:

(a) $\Delta^3 f_1 = (E\nabla)^3 f_1 = E^3 \nabla^3 f_1 = \nabla^3 E^3 f_1 = \nabla^3 f_4$ $\qquad (\Delta = E\nabla)$

(b) $\Delta^3 f_1 = [E^{1/2}\partial]^3 f_1 = E^{3/2}\partial^3 f_1 = \partial^3 E^{3/2} f_1 = \partial^3 f_{5/2}$ $\qquad (\nabla = E^{1/2}\partial)$

(c) $\partial^2 f_2 = [E^{-1/2}\partial]^2 f_2 = E^{-1}\Delta^2 f_2 = \Delta^2 E^{-1} f_2 = \Delta^2 f_1$ $\qquad (\partial = E^{-1/2}\Delta)$

Example 3: Prove that

(a) $m^2 = 1 + \frac{\partial^2}{4}$

(b) $md = \frac{1}{2}(\Delta + \nabla)$

(c) $\sqrt{1 + m^2 d^2} = 1 + \frac{\partial^2}{2}$

Solution:

(a) We have $m = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

$$m^2 = \frac{(E^{1/2} + E^{-1/2})^2}{4} = \frac{(E^{1/2} - E^{-1/2})^2 + 4}{4}$$

$$= 1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}$$

$$= 1 + \frac{\partial^2}{4}$$

(b) L.H.S.

$md = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$

R.H.S.

$$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}[(E\text{-}1) + (1\text{-}E^{-1})] = \frac{1}{2}(E - E^{-1}).$$

Hence, the result.

(c) We have

$$md = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

$$\backslash \quad 1 + m^2 d^2 = 1 + \frac{(E - E^{-1})^2}{4} = \frac{(E - E^{-1})^2 + 4}{4} = \frac{(E + E^{-1})^2}{4}$$

$$\backslash \quad \sqrt{1 + m^2 d^2} = \frac{E + E^{-1}}{2} = \frac{(E^{1/2} - E^{-1/2})^2 + 2}{2}$$

$$= \frac{d^2 + 2}{2} = 1 + \frac{\partial^2}{4}$$

3.2 Difference Formulas

We shall now derive different difference formulas using the results obtained in the preceding section (Section 3.2).

3.2.1 Newton's Forward-Difference Formula

In Unit 2, we have derived Newton's form of interpolating polynomial (using divided differences). We have also established in Section 3.2 1, the following relationship between divided differences and forward differences

$$f[x_k, \ldots, x_{k+n}] = \frac{1}{n!h^n} V^n f_k \qquad (21)$$

Substituting the divided differences in terms of the forward differences in the Newton's form, and simplifying we get Newton's forward-difference form. The Newton's form of interpolating polynomial interpolating at $x_k, x_{k+1}, \ldots, x_{k+n}$ is

$$P_n(x) = \sum_{i=0}^{n} (x - x_k)(x - x_{k+1})\ldots\ldots\ldots.(x - x_{k+i-1}) f[x_k, \ldots\ldots.x_{k+1}]$$

Substituting (22), we obtain

$$P_n(x) = \sum_{i=0}^{n} (x - x_k)(x - x_{k+1})\ldots\ldots\ldots.(x - x_{k+i-1}) \frac{1}{1!h^1} \Delta^i f_k \qquad (23)$$

Setting k = 0, we have the form

$$P_n(x) = \sum_{i=0}^{n} \frac{1}{1!h^1}(x-x_0)(x-x_1)..........(x-x_{i-1})\Delta^i f_0$$

$$= f_0 + \frac{(x-x_0)}{1!}\frac{\Delta f_0}{h} + \frac{(x-x_0)(x-x_1)}{h^2}\frac{\Delta^2 f_0}{h^2} + ... + \frac{(x-x_0)....(x-x_{n-1})}{n!}\frac{\Delta^n f_0}{h^n} \qquad (24)$$

Using the transformation (2), we have

$x - x_{k+j} = x_0 + sh - [x_0 + (k + j)h] = (s - k - i + 1)\mathrm{V}^i f_k$

$$= \sum_{i=0}^{n} \Delta^1 f_k \begin{bmatrix} s-k \\ i \end{bmatrix}$$

$$= f_k + (s-k)\Delta f_k + \frac{(s-k)(s-k-1)}{2!}\Delta^2 f_k + ... + \frac{(s-k)(s-n-1)}{n!}\Delta^n f_k \qquad (25)$$

of degree £ n.

Setting k = 0 in (25) we get the formula

$$P_n(x_0 + sh) = \sum_{i=0}^{n} \Delta^i f_0 \begin{bmatrix} s \\ i \end{bmatrix} \qquad (26)$$

The form (23), (24), (25) or (26) is called the Newton's forward-difference formula.

The error term is now given by

$$E_n(x) = \begin{bmatrix} s \\ n+1 \end{bmatrix} h^{n+1} f^{n+1}(x)$$

-

Example 4: Find the Find the Newton's forward-difference interpolating polynomial which agrees with the table of values given below. Hence obtain the value of f(x) at x = 1.5.

x	1	2	3	4	5	6
f(x)	10	19	40	79	142	235

Solution: We form a table of forward differences of f(x).

Table 5: Forward differences

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$
1	10			
		9		
2	19		12	
		21		6
3	40		18	
		39		6
4	79		24	
		63		6
5	142		30	
		93		
6	235			

Since the third order differences are constant, the higher order differences vanish and we can infer that f(x) is a polynomial of degree 3 and the Newton's forward-differences interpolation polynomial exactly represents f(x) and is not an approximation to f(x). The step length in the data id h = 1. Taking $x_0 = 1$ and the subsequent values of x as $x_1, x_2, ..., x_5$ the Newton's forward-differnces interpolation polynomial.

$$f(x) = f_0 + (x-1)V f_0 + \frac{(x-1)(x-2)}{2!} V^2 f_0 + \frac{(x-1)(x-2)(x-3)}{3!} V^3 f_0$$

becomes

$$f(x) = 10 + (x-1)(9) + \frac{(x-1)(x-2)}{2}(12) + \frac{(x-1)(x-2)(x-3)}{6}(6)$$

$$l = 10 + (x-1) + 6(x-1)(x-2) + (x-1)(x-2)(x-3)$$

which on simplification gives

$f(x) = x^3 + 2x + 7$

\ $f(1.5) = (1.5)^3 + 2(1.5) + 7$

$= 3.375 + 3 + 7 = 13.375$

Note:

If we want only the value of (1.5) and the interpolation polynomial is not needed, we can use the formula (26). In this case,

$$s = \frac{x - x_0}{h} = \frac{1.5 - 1}{1} = 0.5$$

and

$$f(1.5) = 10 + (0.5)(9) + \frac{(0.5)(-0.5)}{2}(12) + \frac{(0.5)(-0.5)(-1.5)}{6}(6)$$

= 10 + 4.5 – 1.5 + 0.375

= 13.375.

Example 5: From the following table, find the number of students who obtained less than 45 marks.

Marks	30 - 40	40 - 50	50 – 60	60 - 70	70 - 80
No. of students	31	42	51	35	31

Solution: We form a table of the number of students f(x) whose marks are less than x. In other words, we form a cumulative frequency table.

Table 6: Frequency Table

x	f(x)	V f	V^2f	V^3f	V^4f
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We have x_0 = 40, x = 45 and h = 10, s = 0.5

$$\backslash \ f(45) \ ; \quad 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2}(9) + \frac{(0.5)(-0.5)(-1.5)}{6}(-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(37)$$

= 31 + 21 – 1.125 – 1.5625 – 1.4453

= 47. 8672 ; 48

\ The number of students who obtained less than 45 marks is approximately 48.

3.2.2 Newton's Backward-Difference Formula

Reordering the interpolating nodes as x_n, x_{n-1}, ..., x_0 and applying the Newton's divided difference form, we get

$P_n(x) = f[x_n] + (x – x_n) f[x_{n-1}, x_n] + (x – x_{n-1}) f[x_{n-2}, x_{n-1}, x_n]$
$+ .... + (x – x) ... (x –x_n) f[x_0, ..., x_n]$ (27)

We may also write

$$P_n(x) = P_n\left[x_n + \frac{x - x_n}{h} h\right]$$
$$= P_n[x_n + sh] = \sum_{i=0}^{n} (x - x_n)(x - x_{n-1})\ldots(x - x_{n-i+1}) f[x_n, ..., x_{n-1}]$$
$$= \sum_{i=0}^{n} \frac{1}{i!h^i}(x - x_n)(x - x_{n-1})\ldots(x - x_{n-i+1}) \nabla^i f_n \qquad (28)$$

Set $x = x_n + sh$, then

$x – x_i = x_n + sh – [x_n – (n – i)h] = (s + n – i)h$

$x – x_{n-j} = (s + n – n + j)h = (s + j)h$

and
$(x – x_n) (x – x_{n-1}) ... (x – x_{n-i+1}) = s(s + 1) ... s(s + i – 1)h^i$

Equation (28) becomes

$$P_n(x) = \sum_{i=0}^{n} \frac{1}{i!!} s(s+1)........(s+i-1) f_n$$

$$= f_n + s\nabla f_n + \frac{s(s + 1)}{2!}\nabla^2 f_n + \frac{s(s + 1)\ldots(s + n - 1)}{n!}\nabla^n f_n \qquad (29)$$

We have seen already that
$$\begin{bmatrix} s \\ k \end{bmatrix} = (-1)^k \frac{s(s + 1)\ldots(s + k - 1)}{k!}$$

Hence, equation (29) ca be written as
$$P_n(x) = f(x_n) + (-1)\begin{bmatrix} s \\ 1 \end{bmatrix}\nabla f(x_n)\ (-1) + (-1)^2\begin{bmatrix} s \\ 2 \end{bmatrix}\nabla^2 f(x_n)\ (-1)^2$$

$$+ ... + (-1)^k\begin{bmatrix} s \\ k \end{bmatrix}\nabla^k f(x_n)$$

or

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix} \nabla^k f(x_n) \qquad (30)$$

Equation (27), (28) or (29) is called the Newton's backward-difference form.

In this case error is given by

$$E_n(x) = (-1)^{n+1} \frac{s(s+1)...(s+n)}{(n+1)!} h^{n+1} f^{n+1}(x). \qquad (31)$$

The backward-difference form is suitable for approximating the value of the function at x that lies towards the end of the table.

Example 6: Find the Newton's backward differences interpolating polynomial for the data of Example 4.

Solution: We form the table of backward differences of f(x).

Table 7: Backward Difference Table

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$
1	10			
		9		
2	19		12	
		21		6
3	40		18	
		39		6
4	79		24	
		63		6
5	142		30	
	93			
6	235			

Tables 5 and 7 are the same except that we consider the differences of Table 7 as backward differences. If we name the abscissas as $x_0, x_1, ..., x_5$, then $x_n = x_5 = 6$, $f_n = f_5 = 235$. with h = 1, the Newton's backward differences polynomial for the given data is given by

$$P(x) = f_5 + (x - x_5)\nabla f_5 + \frac{(x - x_5)(x - x_4)}{2!}\nabla^2 f_5 + \frac{(x - x_5)(x - x_4)(x - x_3)}{3!}\nabla^3 f_5$$

$$= 235 + (x - 6)(93) + \frac{(x - 6)(x - 5)}{2}(30) + \frac{(x - 6)(x - 5)(x - 4)}{6}(6)$$

$$= 235 + 93(x - 6) + 15(x - 6) + (x - 4)(x - 5)(x - 6)$$

which on simplification gives

$P(x) = x^3 + 2x + 7$,

which is the same as the Newton's forward differences interpolation polynomial in Example 4.

Example 7: Estimate the value of (1.45) from the data given below:

x	1.1	1.2	1.3	1.4	1.5
f(x)	1.3357	1.5095	1.6984	1.9043	2.1293

Solution: We form the backward differences table for the data given.

Table 8: Backward Differences Table

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
1.1	1.3357				
		0.1738			
1.2	1.5095		0.0151		
		0.1889		0.0019	
1.3	1.6984		0.0170		0.0002
		0.2059		0.0021	
1.4	1.9043		0.0191		
		0.2250			
1.5	2.1293				

Here $x_n = 1.5$, $x = 1.45$, $h = 0.1$

Hence, $s = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.5} = -0.5$

The Newton's backward differences interpolation formula gives

$$f(x) = f_n + s\nabla f_n + \frac{s(x+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_n$$

$$= 2.1293 + (-0.5)\ (0.2250) + \frac{(-0.5)(0.5)}{2}\ (0.0191) + \frac{(-0.5)(0.5)(1.5)}{6}(0.0021) + \frac{(-0.5)(0.5)(2.5)}{24}\ (0.0002)$$

$= 2.1293 – 0.1125 – 0.00239 – 0.00013 – 0.0000078$

$= 2.01427$ » 2.0143

3.3.3 Stirling's Central Difference Form

A number of central difference formulas are available which can be used according to a situation to maximum advantage. But we shall consider only one such method known as Stirling's method. This formula is used whenever interpolation is required of x near the middle of the table of values.

For the central difference formulas, the origin x_0, is chosen near the point being approximated and points below x_0 are labeled as x_1, x_2, ... and those directly above as x_{-1}, x_{-2}, ... (as in Table 3). Using this convention, Stirling's formula for interpolation is given by

$$P_n(x) = f(x_0) + \frac{s}{2}[df_{1/2} + df_{-1/2}] + \frac{s^2}{2!}d^2f_0$$
$$+ \frac{s(s^2 - 1^2)}{3!}\frac{1}{2}[d^3f_{1/2} + d^3f_{-1/2}] + ...$$
$$+ \frac{s(s^2 - 1^2)s(s^2 - 2^2)...[s^2 - (p - 1)^2]}{(2p - 1)!}\frac{1}{2}[d^{2p-1}f_{1/2} + d^{2p-1}f_{-1/2}]$$
$$+ \frac{s(s^2 - 1^2)...[s^2 - (p - 1)^2]}{(2p)!}d^{2p}f_0$$
$$+ \frac{s(s^2 - 1^2)...s(s^2 - p^2)}{(2p + 1)!}\frac{1}{2}[d^{2p+1}f_{1/2} + d^{2p+1}f_{-1/2}] \qquad (32)$$

where $s = (x - x_0)/h$ and if $n = 2p + 1$ is odd.

If $n = 2p$ is even, then the same formula is used deleting the last term.

The Stirling's interpolation is used for calculation when x lies between $x_0 - \frac{1}{4}h$ and $x_0 + \frac{1}{4}h$.

It may be noted from the Table 3, that the odd order differences at $x_{-1/2}$ are those which lie along the horizontal line between x_0 and x_{-1}. Similarly, the odd order differences at $x_{1/2}$ are those which lie along the horizontal line between x_0 and x_1. even order differences at x_0 are those which lie along the horizontal line through x_0.

Example 8: Using Stirling's formula, find the value of (1.32) from the following table of values.

x	1.1	1.2	1.3	1.4	1.5
f(x)	1.3357	1.5095	1.6984	1.9043	2.1293

Solution:

Table 9: Central Difference

x	f(x)	df	d^2f	d^3f	d^4f
1.1	1.3357				
		0.1738			
1.2	1.5095		0.0151		
		0.1889		0.0019	
1.3	1.6984		0.0170		0.0002
		0.2059		0.0021	
1.4	1.9043		0.0191		
		0.2250			
1.5	2.1293				

Choose $x_0 = 1.3$

Therefore $s = \frac{(x - x_0)}{h} = \frac{1.32 - 1.3}{0.1} = 0.2.$

From Eqn. (32), we have

$$f(x) \text{ » } f_0 + \frac{s}{2}[df_{-1/2} + df_{1/2}] + \frac{s^2}{2!}d^2f_0 + \frac{s(s^2 - 1^2)}{3!}\frac{1}{2}[d^3f_{-1/2} + d^3f_{1/2}] + \frac{s^2(s^2 - 1^2)}{4!}d^4f_0.$$

Now,

$$\frac{1}{2}[df_{-1/2} + df_{1/2}] = \frac{1}{2}(0.1889 + 0.2059) = 0.1974$$

$$\frac{1}{2}[d^3f_{-1/2} + d^3f_{1/2}] = \frac{1}{2}(0.0019 + 0.0021) = 0.0020$$

Also $d^2f_0 = 0.0170$, $d^4f_0 = 0.0002$.

Substituting in the above equation, we get

$$f(x) = 1.6984 + (0.2)(0.1974) + \frac{0.04}{2}(0.0170) + \frac{(0.2)(-0.96)}{6}(0.0020) + \frac{(0.04)(-0.96)}{24}(0.0002)$$

$= 1.6984 + 0.03948 + 0.00034 - 0.00006 - 0$

$= 1.73816 \; ; \quad 1.7382.$

4.0 CONCLUSION

As in the summary

5.0 SUMMARY

In this unit, we have derived interpolation formulas for data with equally spaced values of the argument. We have seen how to find the value of f(x) for a given value of x by applying an appropriate interpolation formula derived in this section. The application of the formulas derived in this section is easier when compared to the application of the formulas derived in Units1 and 2. However, the formulas derived in this unit can only be applied to data with equally spaced arguments whereas the formulas derived in Units 1 and 2 can be applied for data with equally spaced or unequally spaced arguments. Thus, the formulas derived in Units 1 and 2 are of a more general nature than those of Unit 3. The interpolation polynomial which fits a given data can be determined by using any of the formulas derived in this section which will be unique whatever be the interpolation formula that is used.

The interpolation formulas derived in this unit are listed below:

1) Newton's forward difference formula:

$$P_n(x) = P_n(x_0 + sh) = \sum_{i=0}^{n} \begin{bmatrix} s \\ k \end{bmatrix} \nabla^i f_0$$

$$f_0 + sVf_0 + \frac{s(s-1)}{2!} V^2 f_0 + \ldots + \frac{s(s-1)\ldots s(s-n+1)}{n!} V^n f_0$$

where $s = (x - x_0)/h$.

2) Newton's backward difference formula:

$$P_n(x) = P_n(x_n + sh) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix} \nabla^k f_n \quad \text{where } s = (x - x_0)/h$$

3) Stirling's central difference formula:

$$P_n(x) = P_n(x_0 + sh) = f_0 + \frac{s}{2}[df_{1/2} + df_{-1/2}] + \frac{s^2}{2!} d^2 f_0 + \frac{s(s^2 - 1^2)}{3!} \frac{1}{2}[d^3 f_{1/2} + d^3 f_{-1/2}] + \ldots + \frac{s^2(s^2 - 1^2)\ldots(s^2 - (p-1)^2)s^2 f_0}{(2p)!} + \frac{s^2(s^2 - 1^2)\ldots(s^2 - p^2)}{(2p+1)!}[d^{2p+1} f_{1/2} + d^{2p+1} f_{-1/2}]$$

if n = 2p + 1 is odd. If n = 2p is even, the same formula is used deleting the last term.

6.0 TUTOR-MARKED ASSIGNMENT.

i Express $\nabla^4 f_5$ in terms of function values.

ii. Show that (E + 1) d = 2(E – 1) m.

iii The population of a town in the decimal census was given below. Estimate population for the year 1915.

Year x	1911	1921	1931	1941	1951
Population: y (in thousands)	46	66	81	93	101

iv from the following table, find the value of y (0.23):

x	0 .20	0.22	0.24	0.26	0.28	0.30
y	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

v Find the number of men getting wages between Rs. 10 and Rs. 15 from the following table.

Wages in Rs. x	0 - 10	10 - 20	20 - 30	30 -40
No. of men y	9	30	35	42

vi The area A of a circle of diameter d is given in the following table. Find the area of the circle when the diameter is 82 units.

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

vii From the table of values of 3a, find the value of y when x = 0.29.

viii Using the backward differences interpolation, find the polynomial which agree with the values of y(x) where

y(0) = 1, y(1) = 0, y(2) = 1 and y(3) = 10.

ix In 3c, find the number of candidates whose marks are less than or equal to (i) 70, (ii) 89.

x Find f(1.725) from the following table.

x	1.5	1.6	1.7	1.8	1.9
f(x)	4.4817	4.9530	5.4739	6.0496	6.6859

xi Evaluate f(4.325) from the following.

x	4.1	4.2	4.3	4.4	4.5
f(x)	30.1784	33.3507	36.8567	40.7316	45.0141

xii Find the approximate value of y(2.15) from the table

x	0	1	2	3	4
f(x)	6.9897	7.4036	7.7815	8.1281	8.4510

7.0 REFERENCES/FURTHER READINGS.

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