MODULE 1

UNIT 1 COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

It has been observed that when the only number you know is the ordinary everyday integers, you have no trouble solving problems in which you are, for instance, asked to find a variable x such that $3x = 6$. You will be quick to answer '2'. Then, find a number x such that $3x = 8$.You become stumped—there was no such 'number'! You perhaps explained that 3(2)= 6 and 3(3)= 9, and since 8 is between 6 and 9, you would somehow need a number between 2 and 3, but there isn't any such number. Thus one is introduced to 'fractions'.

These fractions, or rational or quotient numbers, are defined to be ordered pairs of integers, for instance, (8, 3) is a rational number. Two rational numbers (n, m) and (p, q) are defined to be equal whenever $nq = pm$. (More precisely, in other words, a rational number is an equivalence class of ordered pairs, etc.) Recall that the arithmetic of these pairs was then introduced: the sum of (n, m) and (p, q) was defined by

$$(n, m)+(p,q) = (nq+ pm, mq),$$

and the product by

$$(n, m)(p, q) = (np, mq).$$

Subtraction and division are defined, as usual, simply as the inverses of the two operations.

You probably felt at first like you had thrown away the familiar integers and were starting over. But no, you noticed that $(n,1)+(p,1)=(n+p,1)$ and also $(n,1)(p,1)=(np,1)$. Thus, the set of all rational numbers whose second coordinate is one behaves just like the integers. If we simply abbreviate the rational number $(n,1)$by n, there is absolutely no danger of confusion: 2 + 3 = 5 stands for (2,1)+(3,1)=(5,1). The equation $3x = 8$ that started this all may then be interpreted as shorthand for the equation$(3,1)(u, v)=(8,1)$, and one easily verifies that $x=(u, v)=(8,3)$is a solution. Now, if someone runs at you in the night and hands you a note with 5 written on it, you do not know whether this is simply the integer 5 or whether it is shorthand for the rational number (5,1).

What we see is that it really does not matter. What we have really done is expanding the collection of integers to the collection of rational numbers. In other words, we can think of the set of all rational numbers as including the integers–they are simply the rationals with second coordinate 1.One last observation about rational numbers: it is, as everyone must know, traditional to write the ordered pair (n, m) as nm. Thus n stands simply for the rational number n_1, *etc.*

Now why have we spent this time on something everyone learned in the grade? Because this is almost a paradigm for what we do in constructing or defining the so-called complex numbers. Euclid showed us there is no rational solution to the equation $x^2= 2$. We are thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals. This is hard, and you likely did not see it done in elementary school, but we shall assume you know all about it and move along to the equation $x^2=-1$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Investigate complex numbers;
- Explain geometry on complex plane; and
- explain some polar co-ordinates.

3.0 MAIN CONTENT

3.1 Complex Numbers

A complex number z is simply defined as any number that can be expressed in the form $z = x + iy$ where x and y are real numbers and i is an imaginary number with the property that $i^2 = -1$. The real number x is called the *real part* of z and is denoted by $Re(z)$, while the real number y is called the *imaginary part* of z and is denoted by $Im(z)$. The set of all complex numbers is denoted by $\mathbb{C}$ Thus,

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i^2 = -1\}.$$

Note that ay real number x is also a complex number whose imaginary part is 0. Thus, the set of all real numbers is a subset of the set of all complex numbers. A complex number whose real part is 0, is called a *pure complex number*.

The operations of addition $(+)$ and multiplication $(\times)$ of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are given, respectively, by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Note that addition of complex numbers is done component-wise and multiplication is mealy by expansion while observing the property $i^2 = -1$. Subtraction of complex numbers is done as addition using the fact that $z_1 - z_2 = z_1 + (-z_2)$.

To define division of complex numbers, we need to introduce the concept of conjugate of a complex number. If $z = x + iy$, the *conjugate* of z, denoted by $\bar{z}$, is defined as $\bar{z} = x - iy$. The conjugate of a complex number has an interesting property that both $z + \bar{z}$ and $z\bar{z}$ are real numbers. In fact, $z + \bar{z} = 2x$ and $z\bar{z} = x^2 + y^2$.

Now if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}$$
$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{(x_2y_1-x_1y_2)}{x_2^2+y_2^2}\right).$$

SELF-ASSESSMENT EXERCISE

Show that for any two complex numbers $z_1$and z_2,
i. $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ and ii. $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$.

Solution: Let $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$. Then

i.
$$\overline{z_1+z_2} = \overline{(x_1+x_2)+\imath(y_1+y_2)}$$
$$= (x_1+x_2) - i(y_1+y_2)$$
$$= (x_1 - iy_1) + (x_2 - iy_2)$$
$$= \bar{z}_1 + \bar{z}_2.$$

ii.
$$\overline{z_1z_2} = \overline{(x_1x_1 - y_1y_2) + \imath(x_1y_2 + x_2y_1)}$$
$$= (x_1x_1 - y_1y_2) - i(x_1y_2 + x_2y_1)$$
$$= (x_1x_1 + i^2y_1y_2) - i(x_1y_2 + x_2y_1)$$
$$= x_1x_1 + i^2y_1y_2 - ix_1y_2 - ix_2y_1$$
$$= (x_1 - iy_1)(x_2 - iy_2)$$
$$= \bar{z}_1\bar{z}_2.$$

Each complex number $z = x + iy$ correspond to the ordered pair (x, y) in the real plane $\mathbb{R}^2$. Thus, elements of $\mathbb{R}^2$ are regarded as complex numbers. In this regard a complex number $z = x + iy$ can be plotted in the position of the point (x, y) in the plane $\mathbb{R}^2$ as in the next figure.

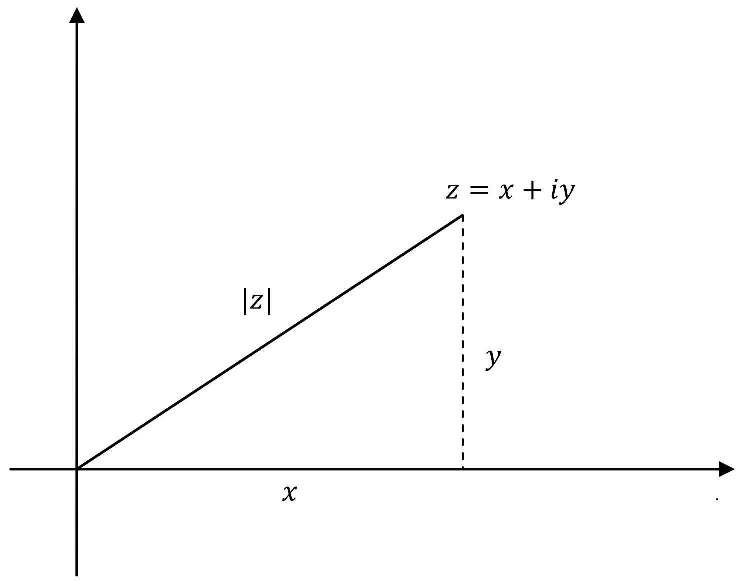


Fig 1:

The distance from the origin to the complex number $z = x + iy$ is called the modulus of z and is denoted and given by

$$|z| = \sqrt{x^2 + y^2} \quad = \sqrt{R_e(z)^2 + I_m(z)^2}$$

Note that $|z|^2 = z\bar{z}$. The angle between the line joining the origin and (x, y), and the positive x-axis is called the argument of z and is denoted by and given by

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right).$$

SELF-ASSESSMENT EXERCISE

Show that for any two complex numbers $z_1$and z_2, $|z_1 + z_2| \le |z_1| + |z_2|$.

Solution:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |\overline{z_1}|^2 + 2Re(z_1\overline{z_2}) + |\overline{z_2}|^2 \\
\le\ & |\overline{z_1}|^2 + 2|z_1||\overline{z_2}| + |\overline{z_2}|^2 \\
=\ & (|z_1| + |z_2|)^2.
\end{aligned}$$

Thus, $|z_1 + z_2| \le |z_1| + |z_2|$.

3.2 Polar Coordinates

Now let us look at polar coordinates (r, θ) of complex numbers. From figure 1 above, we immediately see that, by applying Pythagoras Theorem, we have that

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{x}{y}$$

and

$$x = r\cos\theta, \qquad y = r\sin\theta$$

The enable us to write a complex number $z = x + iy$ as

$$z = r(\cos\theta + i\sin\theta),$$

which is called the *polar form* of the complex number z. Note that the earlier form $z = x + iy$ is called the *Cartesian form* of z.

In the polar form of a complex number, we do not allow negative values for r. The number θ is the argument of z, and there are, of course, many different possibilities for θ. Thus a complex number has an infinite number of arguments, any two of which differ by an integral multiple of 2π. The principal argument of z is the unique value of θ that lies in the interval $(-\pi, \pi)$.

SELF-ASSESSMENT EXERCISE

If 1- i, we have

$$1 - i = \sqrt{2}(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right))$$
$$= \sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right))$$
$$= \sqrt{2}(\cos\left(\frac{399\pi}{4}\right) + i\sin\left(\frac{399\pi}{4}\right)).$$

Each of the numbers $\frac{7\pi}{4}$, $-\frac{\pi}{4}$, and $\frac{399\pi}{4}$ is an argument of 1 - i, but the principal argument is $-\frac{\pi}{4}$.

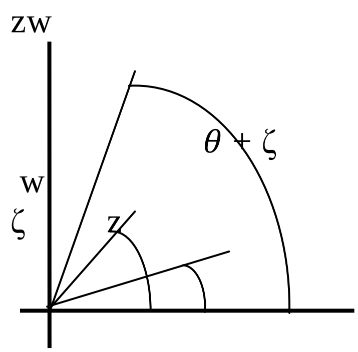
Suppose

$$z = r(\cos\theta + i\sin\theta) \text{ and } w = s(\cos\zeta + i\sin\zeta).$$

Then

$$zw = r(\cos\theta + i\sin\theta) \times s(\cos\zeta + i\sin\zeta)$$
$$= rs[(\cos\theta\cos\zeta - \sin\theta\sin\zeta) + i(\sin\theta\cos\zeta + \sin\zeta\cos\theta)]$$
$$= rs(\cos(\theta + \zeta) + i\sin(\theta + \zeta).$$

We have the nice result that the product of two complex numbers is the complex number whose modulus is the product of the module of the two factors and an argument is the sum of arguments of the factors. A picture:



We now define exp $(i\theta)$, or $e^{i\theta}$by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We shall see later as the drama of the term unfolds that this very suggestive notation is an excellent choice. Now, we have in polar form

$$z = re^{i\theta},$$

where$r = |z|$and θ is any argument of z. Observe that we have just shown that

$$e^{i\theta}e^{i\zeta} = e^{i(\theta+\zeta)}.$$

It follows from this that $e^{i\theta}e^{-i\theta} = 1$.Thus

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

It is easy to see that

$$\frac{z}{w} = \frac{\text{re}\, i\,\theta}{\text{sei}\,\zeta} = \frac{r}{s}(\cos(\theta - \zeta) + i\sin(\theta - \zeta)).$$

3.0 CONCLUSION

The achievements resulting from this unit are highlighted in the summary.

4.0 SUMMARY

The summary of the work carried out in this unit are highlighted below.

- we introduced you to fraction or rational or quotient numbers, which was defined to be ordered pairs of integers
- we showed you that there is no rational solution to the equation $x^2 = 2$. We were thus led to defining even more new numbers, the so-called real numbers, which, of course, include the rationals.
- complex numbers were also defined on modules, length conjugate, triangle inequality, argument and principal argument using examples to illustrate these definitions.

6.0 TUTOR-MARKED ASSIGNMENT

i. Write the following complex numbers in the form $x + iy$:

a) $(4 - 7i)(-2 + 3i)$
b) $(1 - i)^3$
c) $\frac{5+2i}{1+i}$
d) $\frac{1}{i}$

ii. Find all complex z = (x, y) such that$z^2 + z + 1 = 0$

iii. Prove that if $wz = 0$, then $w = 0$ or $z = 0$.

iv. Prove the following for any tow complex numbers z and w.

a) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$
b) $||z| - |w|| \leq |w|$.

v. Prove that $|zw| = |z||w|$and that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

vi. Sketch the set of points satisfying

a) $|z - 2 + 3i| = 2$
b) $|z + 2i| \leq 1$
c) $\text{Re}(\bar{z} + i) = 4$
d) $|z - 1 + 2i| = |z + 3 + i|$
e) $|z + 1| + |z - 1| = 4$
f) $|z + 1| - |z - 1| = 4$

vii. Write in polar form $re^{i\theta}$:

a) i b) $1 + i$
c) -2 d) $-3i$
e) $\sqrt{3} + 3i$

viii. Write in rectangular form-no decimal approximations, no trig functions:

a) $2e^{i3\pi}$ b) $e^{i100\pi}$
c) $10e^{3\pi/6}$ d) $\sqrt{2}\, e^{i5\pi/4}$

ix. a) Find a polar form of $(1 + i)(1 + i\sqrt{3})$.
b) Use the result of a) to find cos $(\frac{7\pi}{12})$ and sin $(\frac{7\pi}{12})$.

x. Find the rectangular form of $(-1 + i)^{100}$

xi. Find all z such that $z^3 = 1$. (Again, rectangular form, no trig functions.)

xii. Find all z such that $z^4 = 16i$. (Rectangular form etc.).

7.0 REFERENCES/FURTHER READING

Schum Series, *Advance Calculus*.

Stroud, K. A. *Engineering Mathematics*.

UNIT 2 COMPLEX FUNCTIONS

CONTENTS

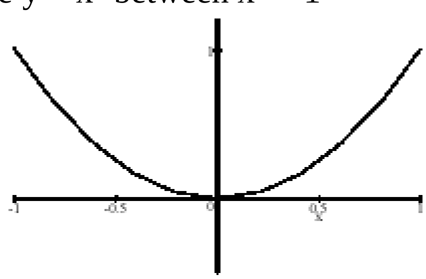
1.0 INTRODUCTION

A function y: I → **C** from a set I of real's into the complex numbers C is actually a familiar concept from elementary calculus. It is simply a function from a subset of the reals into the plane, what we sometimes call a vector-valued function.

Assuming the function y is nice, it provides a vector, or parametric, description of a curve. Thus, the set of all $\{y(t) : y(t) = e^{it} = \cos t + i \sin t = (\cos t, \sin t), 0 \leq t \leq 2\pi\}$ is the circle of radius one, centered at the origin. We also already know about the derivative of such functions. If $y(t) = x(t) + iy(t)$, then the derivative of y is simply $y'(t) = x'(t) + iy'(t)$, interpreted as a vector in the plane, it is tangent to the curve described by y at the point y(t)

SELF-ASSESSMENT EXERCISE 1

Let $y(t) = t + it^2, -1 < t < 1$. One easily sees that this function describes that part of the curve $y = x^2$ between x = -1



2.0 OBJECTIVES

At the end **of** this unit, you should be able to:

- investigate and study functions of a complex variable;
- explain derivatives of a function; and
- explain at Cauch-Riemann Equation.

3.0 MAIN CONTENT

3.1 Functions of a Complex Variable

In this unit we consider function *f: D*- **C** in which the domain *D* is a subset of the complex numbers. In some sense, these too are familiar to us from elementary calculus. They are simply functions from a subset of the plane into the plane:

$$f(z) = f(x, y) = u(x, y) + iv(x, y) = ((x, y), v(x, y))$$

Thus, $f(z) = z^2$looks like$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$. In other words, u(x, y) = $x^2 - y^2$ and $v(x, y) = 2xy$. The complex perspective, as we shall see, generally provides richer and more profitable insights into these functions.
The definition of the limit of a function *f*at a point z = z_0 is essentially the same as that which we learned in elementary calculus:

$$\lim_{z \to z_0} f(z) = L$$

means that given $\varepsilon > 0$, there is a δ such that $|f(z) - L| < \varepsilon$ whenever $|z - z_0| < \delta$. As you could guess, we say that f is continuous at z_0 if it is true that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at each point of its domain, we simply say that f is continuous.

Suppose that both$\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then the following properties are easy to establish:

$$\lim_{z \to z_0}\big(f(z) \pm g(z)\big) = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$$

$$\lim_{z\to z_0} f(z)\, g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)$$

and

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$$

provided that $\lim_{z\to z_0} g(z) \neq 0$.

It now follows at once from these properties that the sum, difference, product, and quotient of two functions continuous at z_0 are also continuous at z_0.

It should not be too difficult to convince yourself that if $z = x + iy$, $z_2 = x_0 + iy_0$ and $f\ (z) = u\ (x, y) + iv\ (x, y)$, then

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x, y) + i \lim_{(x,y)\to(x_0,y_0)} v(x, y)$$

Thus, f is continuous at $z_0 = (x_0, y_0)$ precisely when u and v are.

Our next step is to define the derivative of a complex function f. It is the obvious thing. Suppose f is a function and z_o is an interior point of the domain of f the derivative f (z_o) of f at z_0 is

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

SELF-ASSESSMENT EXERCISE 1

Suppose $f(z) = z^2$. Then, letting $\Delta z = z - z_0$, we have

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z_0 + \Delta z)^2 - (z_0)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z_0 \Delta z - (\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z)$$

$$= 2z_0.$$

No surprise here-the function *f (z)* =z^2has a derivative at every *z*, and it's simply 2z.

SELF-ASSESSMENT EXERCISE 2

Let $f(z) = z\bar{z}$. Then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - (z_0 \overline{z_0})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \overline{z_0} \Delta z + \overline{\Delta z} \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z}).$$

Suppose this limit exists,and choose$\Delta z = \Delta x$. Then,

$$\lim_{\Delta z \to 0} (z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z}) = \lim_{\Delta x \to 0} (z_0 \frac{\Delta x}{\Delta x} + \overline{z_0} + \Delta x) = \overline{z_0} + z_0.$$

Now, choose$\Delta z = i\Delta y$. Then

$$\lim_{\Delta z \to 0} (z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z}) = \lim_{\Delta x \to 0} (-z_0 \frac{i\Delta y}{i\Delta y} + \overline{z_0} - i\Delta y) = \overline{z_0} - z_0.$$

Thus, we must have$\overline{z_0} - \overline{z}_0 = \overline{z_0} + z_0$. In other words, there is nochance of this limit's existing, except possibly at $z_0 = 0$. So, this function does not have a derivative at all points.

Now, take another look at the first of these two examples. Meditate on this and you will be convinced that all the "usual" results for real-valued functions also hold for these new complex functions: the derivative of a constant is zero, the derivative of the sum of two functions is the sum of the derivatives, the "product" and "quotient" rules for derivatives are valid, the chain rule for the composition of functions holds, etc. For proofs, you only need to go back to your elementary calculus book and change $x's$to $z's$.

If *f* has a derivative at z_o, we say that *f* is **differentiable** at z_o. If *f* is differentiable at every point of a neighborhood of z_o, we say that is **analytic** at z_0. (A set *S* is a neighborhood of z_0 if there is a disk D = {: $|z—z_0| <r, r> 0$} so that $D \subseteq S$.
(If *f* is analytic at every point of some set *S*, we say that *f* is analytic on *S*. A function that is analytic on the set of all complex numbers is said to be an entire function.

3.3. Derivatives

Suppose the function f *given* by $f\ (z) = u(x,y) + iv(x,y)$ has a derivative at$z_0 = x_0 + iy_0$. We know this means there is a number $f\,(z_0)$ so that

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$$

Choose $\Delta z = \Delta x$. Then,

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta x\to 0} \frac{\big(u(x_0+\Delta x,y_0)+i\,v(x_0+\Delta x,y_0)\big)-\big(u(x_0,y_0)+i\,v(x_0,y_0)\big)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \left(\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}\right)$$

$$= \frac{\partial u(x_0,y_0)}{\partial x} + i\,\frac{\partial v(x_0,y_0)}{\partial x}.$$

Next, choose $\Delta z = i\Delta y$. Then,

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta y \to 0} \frac{\big(u(x_0, y_0 + \Delta y) + i\, v(x_0, y_0 + \Delta y)\big) - \big(u(x_0, y_0) + i\, v(x_0, y_0)\big)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left(\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right)$$

$$= \frac{\partial v(x_0, y_0)}{\partial y} - i \frac{\partial u(x_0, y_0)}{\partial y}.$$

We have two different expressions for the derivative $f(z_0)$, and so

$$\frac{\partial u(x_0, y_0)}{\partial x} + i \frac{\partial v(x_0, y_0)}{\partial x} = \frac{\partial v(x_0, y_0)}{\partial y} - i \frac{\partial u(x_0, y_0)}{\partial y}.$$

Or

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{\partial v(x_0, y_0)}{\partial y}$$

$$\frac{\partial u(x_0, y_0)}{\partial y} = -\frac{\partial v(x_0, y_0)}{\partial x}.$$

These equations are called the **Cauchy Riemann Equations**.

We have shown that if *f* has a derivative at a point z_0, then its real and imaginary parts satisfy these equations. Even more exciting is the fact that if the real and imaginary parts of *f* satisfy these equations and if in addition, they have continuous first partial derivatives, then the function *f* has a derivative. Specifically, suppose $u(x, y)$ and $v(x, y)$ have partial derivatives in a neighborhood of $z_0 = (x_0, y_0)$, suppose these derivatives are continuous at z_0, and suppose

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{\partial v(x_0, y_0)}{\partial y}$$

$$\frac{\partial u(x_0, y_0)}{\partial y} = -\frac{\partial v(x_0, y_0)}{\partial x}.$$

We shall see that f is differentiable at z_0.

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} =$$

$$\frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}.$$

Observe that

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)=$$

$$[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)]$$

$$+[u(x_0,y_0+\Delta y)-u(x_0,y_0)].$$

Thus,

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)=\Delta x\frac{\partial u(\xi,y_0+\Delta y)}{\partial x}$$

and, $\frac{\partial u(\xi,y_0+\Delta y)}{\partial x}=\frac{\partial u(x_0,y_0)}{\partial x}+\varepsilon_1$ where $\lim_{\Delta z\to 0}\varepsilon_1=0$.

Thus

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0+\Delta y)=\Delta x\left[\frac{\partial u(x_0,y_0)}{\partial x}+\varepsilon_1\right].$$

Proceeding similarly, we get

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=$$

$$\frac{\Delta x\left[\frac{\partial u(x_0,y_0)}{\partial x}+\varepsilon_1+i\frac{\partial v(x_0,y_0)}{\partial x}+i\varepsilon_1\right]+\Delta y\left[\frac{\partial u(x_0,y_0)}{\partial y}+\varepsilon_3+i\frac{\partial v(x_0,y_0)}{\partial y}+i\varepsilon_4\right]}{\Delta x+i\Delta y},$$

where$\varepsilon\to 0$ $\Delta z\to 0$. Now, unleash the Cauchy-Riemann equation on this quotient and obtain

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\Delta y\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{Stuff}{\Delta x+i\Delta y}$$

$$=\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+\frac{Stuff}{\Delta x+i\Delta y},$$

where, $Stuff = \Delta x(\varepsilon_1 + i\varepsilon_2) + \Delta y(\varepsilon_3 + i\varepsilon_4)$.

It is easy to show that $\lim_{\Delta z \to 0} \frac{Stuff}{\Delta z} = 0$ and so,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

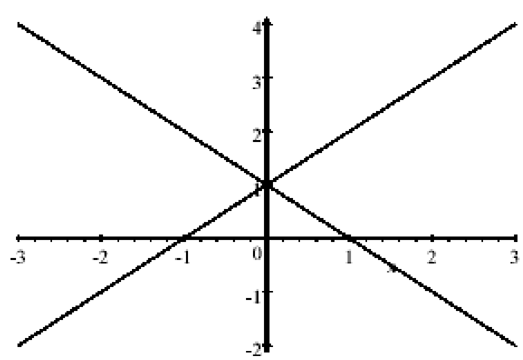
In particular, we have, as promised, shown that *f* is differentiable at z_0.

SELF-ASSESSMENT EXERCISE 3

Let us find all points at which the function *f* given by $f(z) = x^3 - i(1 - y)^3$ is differentiable. Here we have $u = x^3$ and $v = -(1 - y)^3$. The **Cauchy-Riemann** equations thus look like $3x^2 = 3(1 - y)^2$ and $0 = 0$.

The partial derivatives of *u* and *v* are nice and continuous everywhere, so *f* will be differentiable everywhere the C-R equations are satisfied.

This is simply the set of all points on the cross formed by the two straight lines.



4.0 CONCLUSION

To end the unit, we now give the summary of what we have covered in it.

5.0 SUMMARY

We can summarise this unit as follow:

We discussed complex number on functions of a real variable as a function f : I → C from a set of real numbers into the complex number C while functions of a complex variable was defined as a function f : D → C in which the domain D is a subset of the complex number.

We also showed that if f has a derivative at a point z_0, then its real and imaginary parts satisfy the following equations

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{\partial v(x_0, y_0)}{\partial y}$$

$$\frac{\partial u(x_0, y_0)}{\partial y} = -\frac{\partial v(x_0, y_0)}{\partial x}.$$

These equations are called the Cauchy-Riemann equations.
If the real and imaginary parts of f satisfy these equations and if in addition, they have continuous first partial derivatives, then the function f has a derivative.

6.0 TUTOR-MARKED ASSIGNMENT

i. (a). What curve is described by the function
y(t) = (3t + 4) + i(t - 6), $0 \le t \le 1$?

b). Suppose z and w are complex numbers. What is the curve described byy(t) =(1- t)w+tz,$0 \le t \le 1$

ii. Find a function y that describes that part of the curve $y = 4x^3 + 1$ between x = 0 and x = 10.

iii. Find a function y that describes the circle of radius 2 centered at z = 3 - 2*i*.

iv. Note that in the discussion of the motion of a body in a central gravitational force field, it was assumed that the angular momentum a is non-zero. Explain what happens in case α =0

v. Suppose f (z_0 = *3xy* + *i(x-y2)*. Find lim *f(z)*. Or explain carefully why it does not exist Z-3+2i

vi. Prove that if *f* has a derivative at *z*, then *f* is continuous at *z*

vii. Find all points at which the valued function *f* defined by *f(z)* = *z* has a derivative.

viii. Find all points at which the valued function *f* defined by *f*

$$f(z) = (2+i)z^3 - iz^2 + 4z - (1+7i)$$

has a derivative

ix. Is the function f given by

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

differentiable at z = 0? Explain.

x. At what points is the function *f* given by *f* (z) = x^3 + i(1 -y)3 analytic? Explain.

xi. Do the real and imaginary parts of the function *f* in question 9 satisfy the Cauchy-Riemann equations at z = 0? What do you make of your answer?

xii. Find all points at which *f* (z) = *2y* - *ix* is differentiable.

xiii. Suppose *f is* analytic on a connected open set *D*, and *f* (z) = 0 for all *zϵD*. Prove that *f* is constant.

xiv . Find all points at which

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

is differentiable. At what points is*f* analytic?

xv. Suppose *f* is analytic on the set D, and supposes Re *f* is constant on *D.* Is *f* necessarily constant on D*?* Explain.

xvi. Suppose *f is* analytic on the set D, and suppose *f* (z) I is constant on *D.* Is *f* necessarily constant on *D?* Explain.

7.0 REFERENCES/FURTHER READING

Schum Series, *Advance Calculus*.

Stroud, K.A. *Engineering Mathematics*.