MODULE 2

UNIT 1 ANALYTIC FUNCTION

CONTENTS

1.0 INTRODUCTION

A function $f\ (z)$ is analytic at a point z_0 if its derivatives $f'(z)$ exist not only at z_0 but at every point z in a neighborhood of z_0.Suppose *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that $|f\ (z)|\ \leq\ M$ for all z.Note that the derivative of an analytic function is also analytic. Now suppose *f* is continuous on a domain D in which every point of D is an interior point and suppose that $\int_C\ f(z)\,dz =\ 0$ for every closecurve in D.

Even more exciting is the fact that if the real and imaginary parts of *f* satisfy these equations and if in addition, they have continuous first partial derivatives, then the function f *has* a derivative.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- investigate and explain analytic functions
- study Cauchy's integral formular
- look at solutions on Liouville's Theorem to see how |*f*(z)| must attain its maximum value somewhere in this domain D.

3.0 MAIN CONTENT

3.1 Cauchy's Integral Formula

Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its 'inside, and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz.$$

This is the famous Cauchy Integral Formula.

Let $\epsilon > 0$ be any positive number. We know that if f is continuous at $z_0$and so there is a number δ such that$|f(z) - f(z_0)| < \epsilon$whenever$|z - z_0| < \delta$.

Now let $p > 0$ be a number such that p <δ and the circle $C_0 = \{z : |z - z_0| = p\}$ is also inside C. Now, the function $\frac{f(z)}{z - z_0}$ is analytic in the region between C and C_o, thus

$$\int_C \frac{f(z)}{z - z_0}dz = \int_{C_0} \frac{f(z)}{z - z_0}dz.$$

We know that$\int_{C_0} \frac{1}{z - zo}dz = 2\pi i$, so we can write

$$\int_{C_0} \frac{f(z)}{z - z_0}dz - 2\pi i f(z_0) = \int_{C_0} \frac{f(z)}{z - z_0}dz - f(z_0)\int_{C_0} \frac{1}{z - z_0}dz$$

$$= \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}dz$$

For $z \in C_0$ we have

$$\left|\int_{C_0} \frac{f(z)}{z - z_0}dz - 2\pi i f(z_0)\right| = \left|\int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}dz\right|$$

$$\leq \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon.$$

Thus,

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| = \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

$$\leq \frac{\varepsilon}{\rho}.$$

Butε is any positive number, and so

$$\left|\int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0)\right| = 0.$$

Or,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz.$$

Which is exactly what we set out to show.

It says that if f is analytic on and inside a simple closed curve and we know the values f(z) for every z on the stipple closed curve, then we know the value for the function at every point inside the curve.

SELF-ASSESSMENT EXERCISE

Let C be the circle $|z| = 4$ traversed once in the counterclockwise direction.
Let's evaluate the integral

$$\int_{C} \frac{\cos z}{z^2 - 6z + 5} dz.$$

We simply write the integrand as

$$\frac{\cos z}{z^2 - 6z + 5} = \frac{\cos z}{(z - 5)(z - 1)} = \frac{f(z)}{z - 1},$$

where $f(z) = \frac{\cos z}{z-5}$.

Observe that f is analytic on and inside C, and so,

$$\int_C \frac{\cos z}{z^2 - 6z + 5} dz = \int_C \frac{f(z)}{z-1} dz = 2\pi i f(1) = -\frac{i\pi}{2}\cos(1).$$

3.2 Functions Defined by Integral

Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z} ds$$

for all $z \in C$. We shall show that G is analytic. Here we go.

Consider,

$$\frac{G(z+\Delta z) - G(z)}{\Delta z} = \frac{1}{\Delta z}\int_C \left[\frac{1}{s-z-\Delta z}\right] g(s)ds$$

$$= \int_C \frac{g(z)}{(s-z-\Delta z)(s-z)} ds.$$

Thus,

$$\frac{G(z+\Delta z) - G(z)}{\Delta z} - \int_C \frac{g(s)}{(s-z)^2} ds$$

$$= \int_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right] g(s)ds$$

$$= \int_C \left[\frac{(s-z)-(s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2}\right] g(s)ds$$

$$= \Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds.$$

Now we want to show that

$$\lim_{\Delta z - 0} \left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds \right] = 0.$$

To that end, let $M = \max\{|g(s)| : s \in C\}$, and let d be the shortest distance from z to C.

Thus, for $s \in C$, we have $|s-z| \geq d > 0$ and also

$$|s-z-\Delta z| \geq |s-z| - |\Delta z| \geq d - |\Delta z|.$$

Putting all this together, we can estimate the integrand above:

$$\left| \frac{g(s)}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{M}{(d-|\Delta z|)d^2}$$

for all$s \in C$. Finally,

$$\left| \Delta z \int_C \frac{g(z)}{(s-z-\Delta z)(s-z)^2} ds \right| \leq |\Delta z| \frac{M}{(d-|\Delta z|)d^2} length(C).$$

And it is clear that

$$\lim_{\Delta z - 0} \left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds \right] = 0.$$

Just as we set out to show. Hence G has a derivative at z, and

$$G'^{(z)} = \int_C \frac{g(s)}{(s-z)^2} ds.$$

We see that G` has a derivative and it is just what you think it should be.

Consider

$$\frac{G'(z+\Delta z)-G'(z)}{\Delta z}=\frac{1}{\Delta z}\int_C \left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right] g(s)\,ds$$

$$=\frac{1}{\Delta z}\int_C \left[\frac{(s-z)^2-(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right] g(s)\,ds$$

$$=\frac{1}{\Delta z}\int_C \left[\frac{2(s-z)\Delta z-(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right] g(s)\,ds$$

$$=\int_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\right] g(s)\,ds.$$

Next,

$$\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}\,ds$$

$$=\int_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right] g(s)\,ds$$

$$=\int_C \left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right] g(s)\,ds$$

$$=\int_C \left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z)^2+4\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right] g(s)\,ds$$

$$=\int_C \left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right] g(s)\,ds$$

Hence,

$$\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}ds\right|$$
$$=\left|\int_C \left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)\,ds\right|$$

$$\leq |\Delta z|\frac{(|3m|+2|\Delta z|)M}{(d-\Delta z)^2 d^3}.$$

Where $m=\max\{|s-z|: s\in C\}$. It should be clear then that

$$\lim_{\Delta z\to 0}\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C \frac{g(s)}{(s-z)^3}ds\right|=0,$$

or in other words,

$$G''(z)=2\int_C \frac{g(s)}{(s-z)^3}ds.$$

Suppose f *is* analytic in a region *D* and suppose *C* is a positively oriented simple closed curve in *D*.

Suppose also the inside of C is in *D*. Then from the Cauchy Integral formula, we know that

$$2\pi i f(z)=\int_C \frac{f(s)}{s-z}ds$$

and so with $g=f$ in the formulas just derived, we have

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds$$

and

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds,$$

for all z inside the closed curve C. They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\int f(z)$dz = 0 for every closedcurve in D. Then we know that f has an anti-derivative in D─in other words f is the derivative of an analytic function. We now know this means that *f* is itself analytic. We thus have the celebrated Morera's Theorem:

If $f: D \rightarrow C$ is continuous and such that *f(z)*dz = 0 for every closed curve in *D*, then *f* is analytic in *D*.

SELF-ASSESSMENT EXERCISE

Let us evaluate the integral

$$\int_C \frac{e^z}{z^3}dz,$$

where C is any positively oriented closed curve around the origin. We simply use the equation

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds,$$

with z = 0 and *f*(s) = e^s. Thus,

$$\pi i e^0 = \pi i = \int_C \frac{e^z}{z^3}dz.$$

3.3 Liouville's Theorem

Suppose *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that $|f\ (z)| \leq M$ for all z. Then it must be true that $f\ (z)\ =\ 0$ identically. To see this, suppose that $f'(w) \neq\ 0$ for some w.

Choose Rlarge enough to insure that $\frac{M}{R} < \ |f\ (w)|$.Now let C be a circle centered at 0 and with radius$p\ > \max\{R, |w|\}$.

Then we have:

$$\frac{M}{R} < |f'(w)| \leq \left| \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-w)^2} ds \right| \leq \frac{M}{2\pi\rho^2} 2\pi\rho = \frac{M}{\rho},$$

a contradiction. It must therefore be true that there is now for which, $f'(w) = 0$; or, in other words, $f\ (z) = 0$ for all z. This, of course, means that f is a constant function. We have shown Liouville's Theorem:

The only bounded entire functions are the constant function.
Let us put this theorem to some good use.
Let$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$be a polynomial. Then

$$p(z) = \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n}\right) z^n.$$

Now choose R large enough to insure that for each $j = 1, 2, \ldots, n,$ we have $\left|\frac{a_{n-j}}{z^j}\right| < \frac{a_n}{2n}$whenever $|z| > R$. (We are assuming that $a_n \neq 0$.) Hence, for $|z| > R,$

We know that

$$|p(z)| \geq \left| |a_n| - \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right| \right| |z|^n$$

$$\geq \left| |a_n| - \left| \frac{a_{n-1}}{z} \right| - \left| \frac{a_{n-2}}{z^2} \right| - \ldots - \left| \frac{a_0}{z^n} \right| \right| |z|^n$$

$$> \left| |a_n| - \frac{|a_n|}{2n} - \frac{|a_n|}{2n} - \ldots - \frac{|a_n|}{2n} \right| |z|^n$$

$$> \frac{|a_n|}{2} |z|^n.$$

Hence, for $|z| > R,$

$$\frac{1}{|p(z)|} < \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|R^n}.$$

Now, suppose that $p\ (z) \neq 0$ for all z. Then $r\,\frac{1}{|p(z)|}$is also bounded on the disk $|z| \leq R$. Thus,$\frac{1}{p(z)}$is a bounded entire function, and hence, by Liouville's Theorem, constant! Hence the polynomial is constant if it has no zeros. In other words, if $p(z)$ is of degree at least one, there must be at least one z_0 for which $p(z_0) = 0$.

This is, of course, the celebrated fundamental theorem of algebra.

3.4 Maximum Moduli

Suppose f is analytic on a closed domain D. Then, being continuous, |*f*(z)| must attain its maximum value somewhere in this domain. Suppose this happens at an interior point. That is, suppose |*f*(z)|≤ M for all $z \in D$and suppose that$|f(z_0)| = M$ for some z_0 in the interior of D. Now $z_0$is an interior point of D, so there is a number R such that the disk Λ centered at z_0 having radius R is included in D. Let C be a positively oriented circle of radius p ≤ R centered at z_0. From Cauchy's formula, we know

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s - z_0} ds.$$

Hence,

$$f(z_0) = \frac{1}{2\pi i}\int_C^{2\pi} f(z_0 + \rho e^{it})\, dt.$$

and so,

$$M = |f(z_0)| \leq \frac{1}{2\pi}\int_C^{2\pi} \left|f(z_0 + \rho e^{it})\right| dt \leq M.$$

Since$\left|f(z_0 + \rho e^{it})\right| \leq M$. This means

$$M = \frac{1}{2\pi}\int_C^{2\pi} \left|f(z_0 + \rho e^{it})\right| dt.$$

Thus,

$$M = \frac{1}{2\pi}\int_C^{2\pi} \left|f(z_0 + \rho e^{it})\right| dt = \frac{1}{2\pi}\int_C^{2\pi} \left(M - \left|f(z_0 + \rho e^{it})\right|\right) dt = 0.$$

This integrand is continuous and non-negative, and so must be zero. In other words, $|f(z)|$= M for all z ϵ C. There was nothing special about C except its radius p ≤ R, and so we have shown that *f* must be constant on the disk Λ.

I hope it is easy to see that if D is a region (=connected and open), then the only way in which the modulus $|f\ (z)|$of the analytic function *f*can attain a maximum on D is for *f* to be constant.

4.0 CONCLUSION

In this unit, the achievement resulting from this unit is highlighted in the summary.

5.0 SUMMARY

The famous Cauchy integral formula was well defined in the beginning of the unit. We have observed that if *f* is analytic on and inside a simple closed curve and we know the values f(z) for every z on the stipple closed curve, then we know the values for the function at every point inside the curve.

We also knew that the derivative of an analytic function is also analytic. Suppose *f* is continuous on a domain D in which every point of D is an interior point and suppose that $\int f(z)$dz = 0 for every closedcurve in D.

Then we knew that f has an anti-derivative in D─in other words *f* is the derivative of an analytic function. We said that *f* is itself analytic. We thus have the celebrated Morera's Theorem.

If *f* is entire and bounded; that is, *f* is analytic in the entire plane and there is a constant M such that |*f* (z)| ≤ M for all z.

Then it must be true that $f'(z) = 0$ identically.Suppose*f*is analytic on a closed domain D. Then, being continuous, |*f*(z)| must attain its maximum value somewhere in the domain. Suppose this happens at an interior point.

That is, suppose |*f*(z)| ≤ M for all z $\epsilon\ D$ and suppose that |*f*(z_0)| = M for somez_0 in the interior of D. Now z_0 is an interior point of D, so there is a number R such that the disk Λ centered at $z_0$having radius R is included in D. Let C be a positively oriented circle of radius p ≤ R centered at z_0.

6.0 TUTOR- MARKED ASSIGNMENT

i. Suppose f and g are analytic on and inside the simple closed curve C, and suppose moreover that $f(z) = g(z)$ for all z on C. Prove that $f(z) = g(z)$ for all z inside C.

ii. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the Counter-clockwise direction. Define the function g by

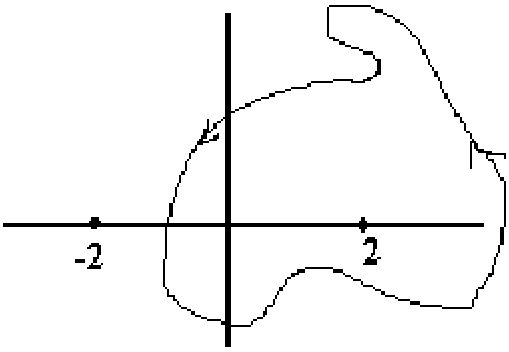
$$g(z) = \int_c \frac{s^2 + s + 1}{s - z} ds$$

Find (a) $g(i)$ (b) $g(4i)$

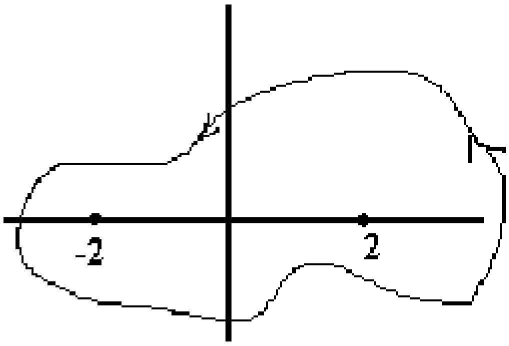
iii. Find

$$\int_c \frac{e^{2z}}{z^2 - 4} dz$$

Where C is the closed curve in the picture:



iv. Find where Γ is the contour in the picture:



v. Evaluate

$$\int_C \frac{\sin z}{z^2} dz$$

Where C is a positively oriented closed curve around the origin.

vi. Let C be the circle |z – *i*| = 2 with the positive orientation. Evaluate

a) $\int_c \frac{e^{2z}}{z^2-4} dz$ $\quad b) \int_c \frac{e^{2z}}{(z^2-4)^2} dz$

vii. Suppose f is analytic inside and on the simple closed curve C. Show that

$$\int_c \frac{f'(z)}{z-w} dz = \int_c \frac{f(z)}{(z-w)^2} dz$$

for every w є C.

viii. (a) Let *a* be a real constant, and let C be the circle y(t) = e^{it}, -π ≤ *t* ≤ π. Evaluate $\int_c \frac{e^{2z}}{z^2-4} dz$

b) Use your answer in part (a) to show that

$$\int_0 e^{a \cos t} \cos(\alpha \sin t)\, dt = \pi$$

ix. Suppose *f* is an entire function, and suppose there is an M such that Re*f* (z) ≤ M for all z. Prove that *f* is a constant function.

ix. Suppose w is a solution of 5z&z² -7z + 14 = 0. Prove that (w J < 3.

xi. Prove that if p is a polynomial of degree n, and if p(a) = 0, then p(z) = (z - a)q(z), where q is a polynomial of degree n - 1.

xii. Prove that if p is a polynomial of degree n> 1, then

$$p(z) = c(z - z_1)^{k_1} (z - z_2)^{k_2} \ldots (z - z_j)^{k_j}$$

xiii. Suppose p is a polynomial with real coefficients. Prove that p can be expressed as a product of linear and quadratic factors, each with real coefficients.

xiv. Suppose *f* is analytic and not constant on a region D and suppose *f* (z) ≠ 0 for all z є D. Explain why |*f*(z)| does not have a minimum in D.

xv. Suppose *f* (z) = u(x, y) + *iv*(x, y) is analytic on a region D. Prove that if u(x, y) attains a maximum value in D, then u must be constant.

7.0 REFERENCES/FURTHER READING

Schum Series, *Advance Calculus*.

Stroud, K. A. *Engineering Mathematics*.

UNIT2 LIMITANDCONTINUITY OF SEQUENCES

CONTENTS

1.0 INTRODUCTION

The basic definitions for complex sequences and series are essentially the same as for the real case. A sequence of complex numbers is a function g: $Z_+ \rightarrow C$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus, we write g(n) = z_n and an explicit name for the sequence is seldomly used; we write simply (z_n) to stand for the sequence g for which $g(n) = z_n$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define limit and continuity of sequence of complex numbers;
- explain what series means; and
- explain the concept of power series.

3.0 MAIN CONTENT

3.1 Limit

The number L is a **limit** of the sequence (z_n) if given an $\epsilon > 0$, there is an integer N_ϵ such that $|z_n - L| < \epsilon$ for all $n \geq N_\epsilon$. If L is a limit of(z_n), we sometimes say that (z_n) converges to L.

We frequently write lim (z_n) = L. It is relatively easy to see that if the complex sequence (z_n) = $(u_n + iv_n)$ converges to L, then the two real sequences (u_n) and *(v_n)* each have a *limit: (u_n)* converges to $Re(L)$ and (v_n) converges to $Im(L)$. Conversely, if thetwo real sequences (u_n) and (v_n) each have a limit, then so *also* does the complex sequence ($u_n + iv_n$).

All the usual properties of *limits of* sequences hold such as

$\lim(z_n \pm w_n) = \lim(z_n) \pm \lim(w_n)$;$\lim(z_n w_n) = \lim(z_n)\lim(w_n)$ and$\lim\left(\frac{z_n}{w_n}\right) = \frac{\lim z_n}{\lim w_n}$.

provided that lim (z_n) and lim (w_n)exist. (And in the last equation, we must, of course, insist that lim (w_n) $\neq 0$.).

A necessary and sufficient condition for the convergence of a sequence (a_n) is the celebrated **Cauchy criterion:** given $\epsilon > 0$, there is an integer N_ϵ so that $|a_n - a_m| < \epsilon$ whenever n, m >$N\epsilon$.

A sequence (f_n) of functions on a domain D is the obvious thing a function from the positive integers into the set of complex functions on D. Thus, for each $z \epsilon D$, we have an ordinary sequence ($f_n(z)$). If each of the sequences ($f_n(z)$) converges, then we say the sequence of functions (f_n) converges to the function *f* defined by $f(z) = \lim(f_n(z))$. The sequence (f_n) is said to converge to *f* uniformly on a set S if given an $\epsilon > 0$, there is an integer N_ϵ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_\epsilon$ and all $z \epsilon$ S.

Note that it is possible for a sequence of continuous functions to have a limit function that is not continuous. This cannot happen if the convergence is uniform. To see this, suppose the sequence (f_n)of continuous functions converges uniformly to *f* on a domain D, let $z_0 \epsilon D$, and let $\epsilon > 0$. We need to show that there is a δ so that $|f(z_0)-f(z)| < \epsilon$ whenever$|z_0 - z)| < \delta$.

Choose *N* so that $|f_N(z) - f(z)| < \frac{\epsilon}{3}$. We can do this because of the uniform convergence of the sequence (f_n). Next, choose δ so that$|f_N(z_0) - f_N(z)| < \frac{\epsilon}{3}$ whenever $|z_0 - z| < \delta$. This is possible because, f_N is continuous.
Now then, when $|z_0 - z| < \delta$, we have

$$|f(z_0) - f(z)| = |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)|$$

$$\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now suppose we have a sequence (f_n)of continuous functions which converges uniformly on a contour C to the function f.

Then the sequence$\int_C f_n(z)\,dz$ converges to$\int_C f(z)\,dz$.

This is easy to see. Let $\epsilon > 0$. Now let N be so that $|f_n(z)-f(z)| < \frac{\epsilon}{A}$for n >$N$, where A is the length of C. Then,

$$\left|\int_C f_n(z)\,dz - \int_C f(z)\,dz\right| = \left|\int_C \left(f_n(z) - f(z)\right)dz\right| < \frac{\varepsilon}{A}A = \varepsilon$$

whenever $n > N$.

Now suppose (f_n)is a sequence of functions each analytic on some region
D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic. This result is in marked contrast to what happens with real functions — examples of uniformly convergent sequences of differentiable functions with a non-differentiable limit abound in the real case. To see that this uniform limit is analytic, let $z_0 \epsilon D$, and let S = {z: $|z - z_0| < r$} ⊂ D. Now consider any simple closed curve C ⊂ S. Each f_n, is analytic, and so$\int_C f_n(z)\,dz = 0$ for every n.

From the uniform convergence of (f_n), we know that $\int_C f(z)\,dz$is the limit and so $\int_C f(z)\,dz = 0$.

Morera's theorem now tells us that f is analytic on S, and hence at z_0.

3.2 Series

A series is simply a sequence (s_n) in which $s_n = a_1 + a_2 + \ldots\ldots + a_n$, In other words, there is sequence (a_n) so that $s_n = s_n + a_n$. The $s_n$are usually called the partial sums.

if the series $\sum_{j=1}^{n} a_j$has a limit, then it must be true that

lim(a_n) = 0.

Consider $f_n \rightarrow \infty$of functions. Chances are this series will converge for some values

$$\sum_{j=1}^{n} f_j(z)$$

of$z$ and not converge for others. A useful result is the celebrated

Weierstrass M-test: Suppose (M_j) is a sequence of real numbers such that $M_j \geq 0$ *for all j>J,* where J is some numbers, and suppose that the series converges. If for all $z \in D$, we have $|f_j(z)| \leq M_j$ for all j > J, then, the series converges uniformly on D.

$$\sum_{j=1}^{n} f_j(z) \leq \sum_{j=1}^{n} M_j.$$

To prove this, begin by letting ϵ > 0 and choosing N > J so that

$$\sum_{j=m}^{n} M_j < \varepsilon$$

for all $n, m > N$ (We can do this because of the famous Cauchy criterion.) Next weobserve that

$$\left|\sum_{j=m}^{n} f_j(z)\right| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j < \varepsilon.$$

This shows that$\sum_{j=1}^{n} f_j(z)$ converges. To see the uniform convergence, observe tha

$$\left|\sum_{j=m}^{n} f_j(z)\right| = \left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| < \varepsilon$$

for all *zϵD* and n > m > N. Thus,

$$\lim_{n\to\infty}\left|\sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| = \left|\sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{m-1} f_j(z)\right| \leq \varepsilon$$

for $m > N$. The limit of aseries$\sum_{j=0}^{n} a_j$is almost always written as $\sum_{j=0}^{\infty} a_j$

3.3 Power Series

We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:

$$s_n(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots + c_n(z-z_0)^n.$$

(We start with n = 0 for aesthetic reasons).

These are the so-called **power series.** Thus, a power series is a series of functions of the form

$$\sum_{j=0}^{n} c_j (z - z_0)^j.$$

Let us look first at a very special power series, the so-called *Geometric series.*

$$\sum_{j=0}^{n} c_j z^j.$$

Here $s_n = 1 + z + z^2 + \dots + z^n$, and $s_{n+1} = z + z^2 + z^3 + \dots + z^{n+1}$. Subtracting the second of these from the first gives us $(1 - z)s_n = 1 - z^{n+1}$

If z = 1, then we cannot go any further with this, but I hope it is clear that the series does not have a limit in case z = 1. Suppose now z ≠ 1. Then we have

$$S_n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}.$$

Now if $|z| < 1$, it should be clear that $\lim(z^{n+1}) = 0$, and so

$$\lim\left(\sum_{j=0}^{n} z^j\right) = \lim S_n = \frac{1}{1-z}$$

or,

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \qquad for\ |z| < 1.$$

Note that if |z| > 1, then the *Geometric Series* does not have a limit. Next, note that if |z| ≤ p < 1, then the Geometric series converges uniformly to $\frac{1}{1-z}$. To see this, note that

$$\sum_{j=0}^{n} \rho^j$$

has a limit and appeal to the Weierstrass M-test.

Clearly a power series will have a limit for some values of z and perhaps not for others. First, note that any power series has a limit when z = z_0. Let us see what else we can say. Consider a power series

$$\sum_{j=0}^{n} c_j (z - z_0)^j.$$

Let $\lambda = \lim\sup\left(\sqrt[j]{|c_j|}\right)$.(Recall that limsup ($a_k$) = lim(sup{$a_k$: k ≥ n}.)

Now let R =$\frac{1}{\lambda}$. (We shall say R = 0 ifλ=∞, and R = ∞, if λ = 0). We are going to show that the series converges uniformly for all (z – z_0| ≤ p < R and diverges for all | z – z_0| > R.

First, let us show that the series does not converge for |z – z_0| > R. To begin, let k be so that$\frac{1}{|z-z_0|} < k < \frac{1}{R} = \lambda$.

There is an infinite number of c_j. For which > k, otherwise lim sup $\sqrt[j]{|c_j|} <\ k$. For each of these c_j we have$\sqrt[j]{|c_j|}$

$$|c_j(z - z_0)^j| = \left(\sqrt[j]{|c_j|}|z - z_0|\right)^j > (k|z - z_0|)^j > 1.$$

It is thus not possible for $\lim|c_n(z - z_0)^n| = 0$, and so the series does not converge.

We show that the series does converge uniformly for
| z - z_0| ≤ p < R. Let k be so that

$$\lambda = \frac{1}{R} < k < \frac{1}{\rho}.$$

Now, for *j* large enough, we have < k. Thus for |z – z_0| ≤ p, we have

$$\left|c_j(z-z_0)^j\right| = \left(\sqrt[j]{|c_j|}|z-z_0|\right)^j < (k|z-z_0|)^j < (k\rho)^j.$$

The geometric series $\sum_{j=0}^{n}(k\rho)^j$ converges because $k\rho < 1$andthe uniform convergence of $\sum_{j=0}^{n} c_j(z-z_0)^j$follows from the M-test.

SELF-ASSESSMENT EXERCISE

Consider the series$\sum_{j=0}^{n}\frac{1}{j!}z^j$. Let us compute $R = \frac{1}{\limsup\left(\sqrt[j]{|c_j|}\right)} =$ $\limsup\left(\sqrt[j]{j!}\right)$. Let K be any positive integer and choose an integer m large enough to ensure that $2^m > \frac{n!}{K^n}$. Now consider $\frac{n!}{K^n}$, where $n = 2K + m$:

$$\frac{n!}{K^n} = \frac{(2K+m)!}{K^{2K+m}} = \frac{(2K+m)!\,(2K+m-1)\ldots(2K+1)(2K)!}{K^m K^{2K}}.$$

Thus, $\sqrt[n]{n!} > K$. Reflect on what we have just shown: given any number K, there is a number *n* such that $\sqrt[n]{n!}$ is bigger than it. In other words, $R = \limsup\left(\sqrt[j]{j!}\right) = \infty$, and so the series$\sum_{j=0}^{n}\frac{1}{j!}z^j$converges for all z.

Let us summarise what we have. For any power series $\sum_{j=0}^{n} c_j(z-z_0)^j$ there is a number $R = \frac{1}{\limsup\left(\sqrt[j]{|c_j|}\right)}$such that the series converges uniformly for $|z - z_0| \leq p < R$ and does not converge for $|z - z_0| > R$. (Note that we may have R = 0 or R = ∞.) The number R is called the **radius of convergence** of the series, and the set $|z - z_0| = R$ is called the **circle of convergence**. Observe also that the limit of a power series is a function analytic inside the circle of convergence.

3.4 Integration of Power Series

Inside the circle of convergence, the limit

$$S(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j$$

is an analytic function. We shall show that this series may be integrated "term-by-term"─that is, the integral of the limit is the limit of the integrals. Specifically, if *C* is any contour inside the circle of convergence, and the function g is continuous on *C*,

then

$$\int_C g(z)S(z)\,dz = \sum_{j=0}^{\infty} c_j \int_C g(z)(z-z_0)^j\,dz.$$

If ϵ > 0 Let M be the maximum of |g(z)| on *C* and let *L* be the length of *C*. Then there is an integer N so that

$$\left|\sum_{j=n}^{\infty} c_j(z-z_0)^j\right| < \frac{\varepsilon}{ML}.$$

For all $n > N$. Thus,

$$\left|\int_C \left(g(z)\sum_{j=n}^{\infty} c_j(z-z_0)^j\right)dz\right| < ML\frac{\varepsilon}{ML} = \varepsilon.$$

Hence,

$$\left|\int_C g(z)S(z)\,dz - \sum_{j=0}^{n-1} c_j \int_C g(z)(z-z_0)^j\,dz\right|$$
$$= \left|\int_C \left(g(z)\sum_{j=n}^{\infty} c_j(z-z_0)^j\right)dz\right| < \varepsilon$$

and we have shown what we promised.

3.5 Differentiation of Power Series

Let

$$S(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j.$$

Now we are ready to show that inside the circle of convergence,

$$S'(z) = \sum_{j=0}^{\infty} jc_j(z - z_0)^{j-1},$$

Let z be a point inside the circle of convergence and let *C* be a positive oriented circle centered at z and inside the circle of convergence. Define

$$g(s) = \frac{1}{2\pi i(s - z)^2}$$

and apply the result of the previous section to conclude that

$$\int_C g(s)S(s)\,ds = \sum_{j=0}^{\infty} c_j \int_C g(s)(s - z_0)^j\,ds$$

or

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s - z)^2}ds = \sum_{j=0}^{\infty} c_j \int_C g(s)(s - z_0)^j\,ds.$$

Thus,

$$S'(z) = \sum_{j=0}^{\infty} jc_j(z - z_0)^{j-1}.$$

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

A sequence (f_n) of functions on a domain D is a function from the positive integers into the set of complex functions on D. Thus, for each $z \epsilon D$, we have an ordinary sequence ($f_n(z)$). If each of the sequences ($f_n(z)$) converges, then we say the sequence of functions (f_n) converges to the function f defined by $f(z) = \lim(f_n(z))$. The sequence (f_n) is said to converge to f uniformly on a set S if given an $\epsilon > 0$, there is an integer N_ϵ so that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N_\epsilon$ and all $z \epsilon S$.

Note that it is possible for a sequence of continuous functions to have a limit function that is not continuous.

If *(f_n)* a sequence of functions, each analytic on some region D, and suppose the sequence converges uniformly on D to the function f. Then f is analytic.

The number R is called the **radius of convergence** of the series, and the set $|z - z_0| = R$ is called the **circle of convergence**. We observed that the limit of a power series is a function analytic inside the circle of convergence. We showed that the series may be integrated "term-by-term"— that is, the integral of the limit is the limit of the integrals. Specifically, if *C* is any contour inside the circle of convergence, and the function g is continuous on *C*,

Then

$$\int_c g(z)\, s(z) = \sum_{j=o}^{\infty} c_j \int_c g(z)\,(z - z_0)^j\, dz$$

We showed that inside the circle of convergence,

$$s'(z) = \sum_{j=1}^{\infty} j c_j (z - z_0)^{j-1}$$

if z be a point inside the circle of convergence and let *C* be a positive oriented circle centered at z and inside the circle of convergence.

6.0 TUTOR-MARKED ASSIGNMENT

i. Prove that a sequence cannot have more than one limit (We thus speak of the limit of a sequence.)

ii. Give an example of a sequence that does not have a limit, or explain carefully why there is no such sequence.

iii. Give an example of a bounded sequence that does not have a limit, or explain carefully why there is no such sequence.

iv. Give a sequence (f_n) of functions continuous on a set D with a limit that is not continuous.

v. Give a sequence of real functions differentiable on an interval which converges uniformly to a non-differentiable function

vi. Find the set D of all z for which the sequence has a limit. Find the limit.

$$\frac{z^n}{z^n - 3^n}$$

vii. Prove that the series $\sum_{j=1}^{\infty} a_j$converges if and only if both the

Series $\sum_{j=1}^{n} R_e a_e (z - z_0)^{j-1}$and $\sum_{j=1}^{n} Im\, a_j$converge.

viii. Explain how you know that the series converges uniformly on the set |z| ≥ 5.$\left(\sum_{j=1}^{n} \left(\frac{1}{2}\right)^j\right)$

ix. Suppose the sequence of real number (a_{j}) has a limit. Prove that

lim sup(a_{j}) = lim(a_{j}).

For each of the following, find the set D of points at which the series converges:

x. $\left(\sum_{j=1}^{n} j!\, z'\right)$

xi. $\left(\sum_{j=1}^{n} jz'\right)$

xii.$\left(\sum_{j=1}^{n} \frac{j^2}{3^j} z^j\right)$

xiii. $\left(\sum_{j=1}^{n} \frac{-1^j}{2^{2j}(j!)} z^{2j}\right)$

xiv. Find the limit of

$$\left(\sum_{j=0}^{n} (j+1) z^j\right)$$

xv. For what values of z does the series converge?

xvi. Find the limit of

$$\left(\sum_{j=0}^{n} \frac{z^j}{j}\right)$$

For what values of z does the series converge?

Find a power series

$$\left(\sum_{j=0}^{n} c_j (z-1)^j\right)$$

such that

$$\frac{1}{z} = \sum_{j=0}^{\infty} c_j (z-1)^j, for\ |z-1| < 1$$

xvii. Find a power series

$$\left(\sum_{j=0}^{n} c_j (z-1)^j\right)$$

such that

$$Log\ z = \sum_{j=0}^{\infty} c_j (z-1)^j, for\ |z-1| < 1$$

7.0 REFERENCES/FURTHER READING

Schum Series, *Advance Calculus*.

.

Stroud, K. A. *Engineering Mathematics*.