

MODULE 3**UNIT 1 TAYLOR AND LAURENT SERIES****CONTENTS**

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1.0 INTRODUCTION

This unit introduces you to Taylor's Series as well as Laurent's Series.

2.0 OBJECTIVES

At end of this unit, you should be able to:

- explain Taylor's Series; and
- explain Laurent's Series

3.0 MAIN CONTENT**3.1 Taylor Series**

Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in

$$\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)} = \frac{1}{s - z_0} \left[\frac{1}{1 - \frac{z - z_0}{s - z_0}} \right] = \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(s - z_0)^{j+1}}$$

this disk and choose C to be the positively oriented circle of radius p , where $|z - z_0| < p < r$. Then for $s \in C$ we have $\left| \frac{z - z_0}{s - z_0} \right| < 1$. The convergence is uniform, so we may integrate.

$$\int_C \frac{f(s)}{s - z} ds = \sum_{j=0}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{j+1}} ds \right) (z - z_0)^j$$

or

$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(s)}{s-z} ds = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_c \frac{f(s)}{(s-z_0)^{j+1}} ds \right) (z-z_0)^j.$$

We have thus produced a power series having the given analytic function as a limit:

where

$$c_j = \frac{1}{2\pi i} \int_c \frac{f(s)}{(s-z_0)^{j+1}} ds.$$

This is the celebrated **Taylor Series** for f at $z = z_0$.

We know we may differentiate the series to get and this one converges uniformly where the series for f does. We can thus differentiate again and again to obtain

$$f'(z) = \frac{1}{2\pi i} \int_c j c_j (z-z_0)^{j-1} ds$$

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)\dots(j-n+1) c_j (z-z_0)^{j-n}.$$

Hence, $f^{(n)}(z_0) = n! c_n$ or $c_n = \frac{f^{(n)}(z_0)}{n!}$. But we also know that

$$c_n = \frac{1}{2\pi i} \int_c \frac{f(s)}{(s-z_0)^{n+1}} ds.$$

This gives us

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(s)}{(s-z_0)^{n+1}} ds,$$

for $n = 0, 1, 2, \dots$

This is the famous *Generalised Cauchy Integral Formula*.

Recall that we previously derived this formula for $n = 0$ and 1.

What does all this tell us about the radius of convergence of a power series?

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j,$$

Suppose we have and the radius of convergence is R . Then we know, of course, that the limit function f is analytic for $|z - z_0| < R$. We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0| < r$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z - z_0| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f is analytic.

SELF-ASSESSMENT EXERCISE

Let $f(z) = \exp(z) = e^z$. Then

$f(0) = f^{(n)}(0) = \dots = f^{(n)}(0) = \dots = 1$, and the Taylor series for f at

$z_0 = 0$ is

$$e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

and this is valid for all values of z since f is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

3.2 Laurent Series

Suppose f is analytic in the region $R_1 < |z - z_0| < R_2$, and let C be a positively oriented simple closed curve around z_0 in this region. (Note: we include the possibilities that R_1 can be 0, and $R_2 = \infty$.) We shall show that for $z \in C$ in this region

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=0}^{\infty} \frac{1}{(z - z_0)^{j+1}}$$

where

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds,$$

for $j = 0, 1, 2, \dots$

and

$$b_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{-j+1}} ds, \text{ for } j = 1, 2, \dots$$

The sum of the limits of these two series is frequently written

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j$$

where

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds,$$

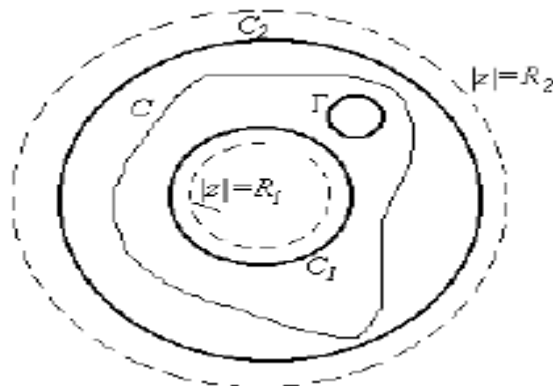
for $j = 0, \pm 1, \pm 2, \dots$

This recipe for $f(z)$ is called a **Laurent series**, although it is important to keep in mind that it is really two series.

Let us derive the above formula.

First, let r_1 and r_2 be so that $R_1 < r_1 \leq |z - z_0| \leq r_2 < R_2$ and so that the point z and the curve C are included in the region $r_1 \leq |z - z_0| \leq r_2$.

Also, let Γ be a circle centered at z and such that Γ is included in this region.



Then is an analytic function (of s) on the region bounded by C_1 , C_2 , and Γ , where C_1 is the circle $|z|=r_1$ and C_2 is the circle $|z|=r_2$. Thus,

$$\int_{C_2} \frac{f(s)}{s-z} ds = \int_{C_1} \frac{f(s)}{s-z} ds + \int_{\Gamma} \frac{f(s)}{s-z} ds.$$

$$\int_{\Gamma} \frac{f(s)}{s-z} ds = 2\pi i f(z).$$

(All three circles are positively oriented, of course.)
and so we have

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds.$$

Look at the first of the two integrals on the right-hand side of this equation. For $s \in C_2$,

We have $|z - z_0| < |s - z_0|$, and so

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{(s-z_0) - (z-z_0)} = \frac{1}{s-z_0} \left[\frac{1}{1 - \frac{z-z_0}{s-z_0}} \right] \\ &= \frac{1}{s-z_0} \sum_{j=0}^{\infty} \left(\frac{z-z_0}{s-z_0} \right)^j. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{C_2} \frac{f(s)}{s-z} ds &= \sum_{j=0}^{\infty} \left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}} ds \right) (z-z_0)^j \\ &= \sum_{j=0}^{\infty} \left(\int_C \frac{f(s)}{(s-z_0)^{j+1}} ds \right) (z-z_0)^j \end{aligned}$$

For the second of these two integrals, note that for $s \in C_1$ we have $|s - z_0| < |z - z_0|$, and so

$$\begin{aligned}
\frac{1}{s-z} &= \frac{-1}{(z-z_0)(s-z_0)} = \frac{-1}{(z-z_0)} \left[\frac{1}{1 - \frac{s-z_0}{z-z_0}} \right] \\
&= \frac{-1}{(z-z_0)} \sum_{j=0}^{\infty} \left(\frac{s-z_0}{z-z_0} \right)^j = - \sum_{j=0}^{\infty} (s-z_0)^j \frac{-1}{(z-z_0)^{j+1}} \\
&= - \sum_{j=0}^{\infty} (s-z_0)^{j-1} \frac{1}{(z-z_0)^j} = \sum_{j=0}^{\infty} \left(\frac{1}{(s-z_0)^{-j+1}} \right) \frac{1}{(z-z_0)^j}
\end{aligned}$$

As before,

$$\begin{aligned}
\int_{c_1} \frac{f(s)}{s-z} ds &= - \sum_{j=1}^n \left(\int_{c_1} \frac{f(s)}{(s-z_0)^{-j-1}} ds \right) \frac{1}{(z-z_0)^j} \\
&= \sum_{j=1}^n \left(\int_c \frac{f(s)}{(s-z_0)^{-j-1}} ds \right) \frac{1}{(z-z_0)^j}
\end{aligned}$$

Putting this altogether, we have the Laurent series:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{c_2} \frac{f(s)}{s-z_0} ds - \frac{1}{2\pi i} \int_{c_1} \frac{f(s)}{s-z_0} ds \\
&= \sum_{j=0}^{\infty} \frac{1}{2\pi i} \left(\int_c \frac{f(s)}{(s-z_0)^{j-1}} ds \right) (z-z_0)^j + \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_c \frac{f(s)}{(s-z_0)^{-j-1}} ds \right) \frac{1}{(z-z_0)^j}
\end{aligned}$$

SELF-ASSESSMENT EXERCISE

Let f be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that f is analytic in the region $0 < |z| < 1$.

Let us find the Laurent series for f valid in this region.

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let us find another Laurent series for f , the one valid for the region $1 < |z| < \infty$.

$$\frac{1}{z-1} = \frac{1}{z} \left[\frac{1}{1 - \frac{1}{z}} \right].$$

Now since $|\frac{1}{z}| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{z} \left[\frac{1}{1 - \frac{1}{z}} \right] = \frac{1}{z} \sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

and so
$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit, we have produced a power series having the given analytic function as a limit.

We have differentiated the series to get and this one converges uniformly where the series for f does. We showed that if f is analytic in $|z - z_0| < r$, then the series converges for $|z - z_0| < r$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z - z_0| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f' is analytic. Finally, we find another Laurent series for f , the one valid for the region $1 < |z| < \infty$.

6.0 TUTOR-MARKED ASSIGNMENT

i. Show that for all z ,
$$e^z = e \sum_{j=0}^{\infty} \frac{1}{j!} (z-1)^j.$$

ii. What is the radius of convergence of the Taylor series for $\tanh z$? $\left(\sum_{j=0}^n c_j z^j \right)$

iii. Show that
$$\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$$

For $|z-i| < \sqrt{2}$.

- iv. If $f(z) = \frac{1}{1-z}$, what is $f^{(0n)}(i)$?
- v. Suppose f is analytic at $z = 0$ and $f(0) = f'(0) = f''(0) = 0$. Prove there is a function g analytic at 0 such that $f(z) = z^3g(z)$ in a neighborhood of 0.
- vi. Find the Taylor series for $f(z) = \sin z$ at $z_0 = 0$.
- vii. Show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at $z = 0$, and find $f(0)$.

- viii. Find two Laurent series in powers of z for the function f defined by
and specify the regions in which the series converge to $f(z)$.

$$f(z) = \frac{1}{z^2(1-z)}$$

- ix. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z(1+z^2)}$$

and specify the regions in which the series converge to $f(z)$.

- x. Find the Laurent series in powers of $z - 1$ for $f(z) = \frac{1}{z}$ in the region $1 < |z - 1| < \infty$.

7.0 REFERENCES/FURTHER READING

Schum Series. *Advance Calculus*.

Stroud, K. A. *Engineering Mathematics*.