MODULE 4

UNIT 1 BILEAR TRANSFORMATION

CONTENTS

1.0 INTRODUCTION

This is the most common method for transforming the system function H_a (s) of an analogue filter to the system function H(z) of an IIR discrete time filter. It is not the only possible transformation, but a very useful and reliable one.

Consider derivative approximation technique:

$$D(y[n]) = dy(t)/dt \text{ at } t = nT \approx (y[n] - y[n-1])/T.$$

$$D(x[n]) = dx(t)/dt \text{ at } t = nT \approx (x[n] - x[n-1])/T.$$

$$D'(y[n]) = d^2y(t)/dt^2 \text{ at } t = nT \approx D(D(y[n])) = (y[n] - 2y[n-1]+y[n-2])/T^2$$

$$D''(y[n]) = d^3y(t)/dt^3 \text{ at } t = nT \approx D(D'(y[n])) = (y[n]-3y[n-1]+3y[n-2]-y[n-3])/T^3$$

“Backward difference” approximation introduces delay which becomes greater for higher orders.

Try "forward differences”: $D[n] \approx [y[n+1] - y[n]] / T$, etc.

But this does not make matters any better.
Bilinear approximation:

$0.5(D[n] + D[n-1]) \approx (y[n] - y[n-1]) / T$ and similarly for $dx(t)/dt$ at $t=nT$.

Similar formulae may be derived for $d^2y(t)/dt^2$, and so on.

If D(z) is the z-transform of D[n] :

$$0.5(D(z) + z^{-1}D(z)) = (Y(z) - z^{-1}Y(z)) / T$$
$$\therefore D(z) = [2 (1 - z^{-1})/ [T(1+z^{-1})] Y(z)$$
$$= [(2/T) (z-1)/(z+1)] Y(z).$$

Applying y[n] to $[(2/T) (z-1)/(z+1)]$ produces an approximation to $dy(t)/dt$ at $t=nT$.

In an analogue circuit, applying y(t) to an LTI circuit with system function $H(s) = s$ produces $dy(t)/dt$ since the Laplace Transform of $dy(t)/dt$ is $sY(s)$.

Therefore, replacing s by $[(2/T) (z-1)/(z+1)]$ is the bilinear approximation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain bilinear transformation;
- explain design of an IIR low-pass filter by the bilinear transformation method;
- explain higher order IIR digital filters;
- discuss IIR discrete time high-pass band-pass and band-stop filter design; and
- compare IIR and FIR digital filters.

3.0 MAIN CONTENT

3.1 Bilinear Transformation Technique

Definition: Given analogue transfer function H $_a$ (s), replace *s* by:

$$\frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

to obtain H(z). For convenience we can take T=1.

SELF-ASSESSNMENT EXERCISE

If $H_a(s)=\dfrac{1}{(1+RCs)}$ then,

$$H(z)=\frac{z+1}{(1+2RC)z+(1-2RC)}=K\,\frac{1+z^{-1}}{1+b_1z^{-1}}$$

where $k=\dfrac{1}{(1+2RC)}$ and $b_1=\dfrac{(1-2RC)}{(1+2RC)}$

3.2 Properties of Bilinear Transformation

(i) This transformation produces a function H(z) such that given any complex number z,
H(z) = H_a(s) where s = 2 (z - 1) / (z + 1)
(ii) The order of H(z) is equal to the order of H_a(s)
(iii) If H_a (s) is causal and stable, then so is H(z).
(iv) H(exp(jΩ)) = H $_a$ (jω) where ω = 2 tan(Ω/2)

Proofs of properties (ii) and (ii) are straightforward but are omitted here.

<u>Proof of property (iv)</u>: When z = exp(jΩ), then

$$s=2\frac{e^{j\Omega}-1}{e^{j\Omega}+1}=\frac{2\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}{e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}=2j\tan\left(\frac{\Omega}{2}\right)$$

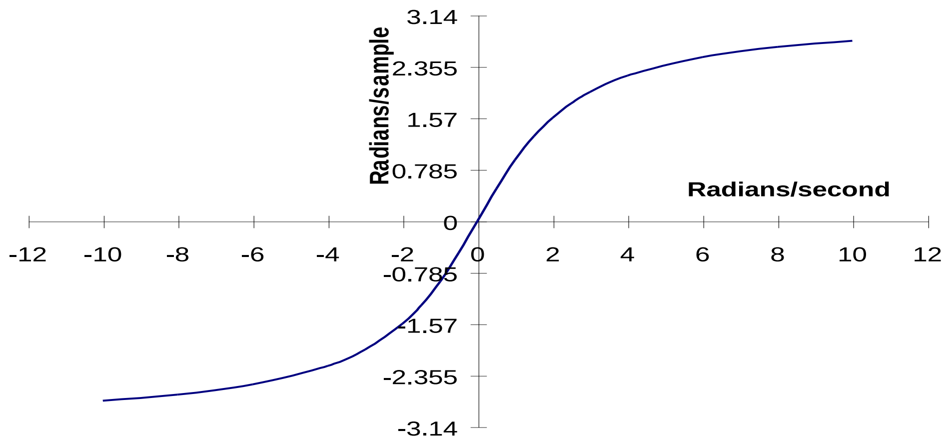

Figure 4.1: Frequency Warping

Frequency Warping:

By property (iv) the discrete time filter's frequency response H(exp(jΩ)) at relative frequency Ω will be equal to the analogue frequency response H_a (jω) with ω = 2 tan(Ω/2). The graph of Ω against ω in fig 6.1, shows how ω in the range -∞ to ∞ is mapped to Ω in the range -π to π. The mapping is reasonably linear for ω in the range -2 to 2 (giving Ω in the range -π/2 to π/2), but as ω increases beyond this range, a given increase in ω produces smaller and smaller increases in Ω. Comparing the analogue gain response shown in fig 6.2(a) with the discrete time one in fig. 6.2(b) produced by the transformation, the latter becomes more and more compressed as Ω → ± π. This "frequency warping" effect must be taken into account when determining a suitable $H_a(s)$ prior to the bilinear transformation.

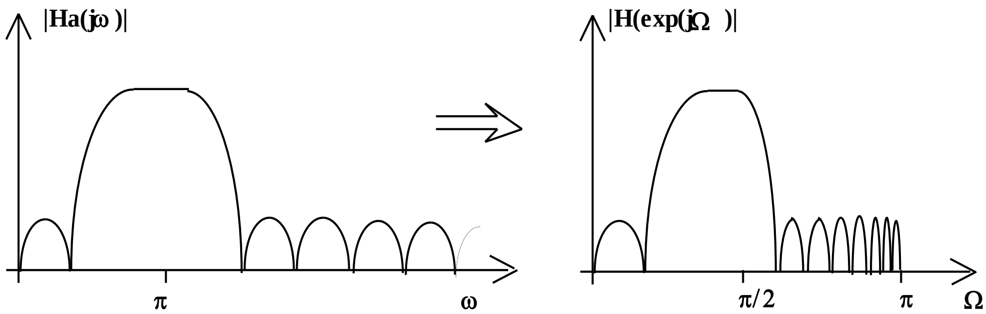

Fig 6.2(a): Analogue gain response

Fig 6.2(b): Effect of bilinear transformatio

Figure 4.2 (a): analogue Gain Response

Figure 4.2(b): Effect of Bilinear Transformation

3.3 Design of an IIR Low-pass Filter by the Bilinear Transformation Method

Given the required cut-off frequency Ω_c in radians/sample:-

(i) Find $H_a(s)$ for an analogue low-pass filter with cut-off $\omega_c = 2\tan(\Omega_c/2)$ radians/sec.
(ω_c is said to be the "pre-warped" cut-off frequency).
(ii) Replace s by 2(z - 1)/(z + 1) to obtain H(z).
(iii) Rearrange the expression for H(z) and realise by bi-quadratic sections.

SELF-ASSESSMENT EXERCISE

Design a second order Butterworth-type IIR low pass filter with
$\Omega_c = \pi / 4$.
Solution: Pre-warped frequency $\omega_c = 2\tan(\pi/8) = 0.828$

For an analogue Butterworth low-pass filter with cut-off frequency 1 radian/second:

$H_a(s) = 1/(1 + \sqrt{2}\,s + s^2)$

Replace *s* by *s* / 0.828, then replace s by 2(z - 1)/(z + 1) to obtain:

$$H(z) = \frac{z^2 + 2Z + 1}{10.3z^2 - 9.7z + 3.4} = 0.093\left(\frac{1 + 2z^{-1} + z^{-2}}{1 - 0.94z^{-1} + 0.33z^{-2}}\right)$$

which may be realised by the signal flow graph in fig 6.5. Note the extra multiplier scaling the input by 0.097.

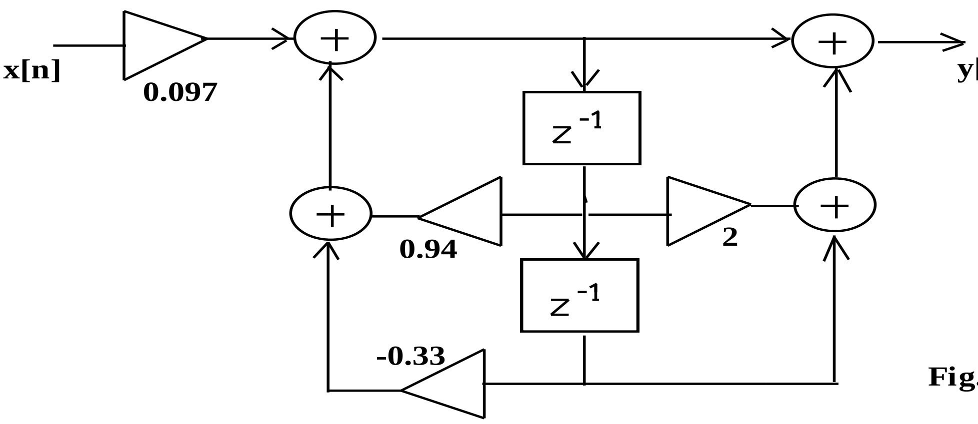

Fig. 6.3

Figure 4.3

3.4 Higher Order IIR Digital Filters

Recursive filters of order greater than two are highly sensitive to quantisation error and overflow. It is normal, therefore, to design higher order IIR filters as cascades of bi-quadratic sections.

SLEF-ASSESSMENT EXERCISE

Design a 4th order Butterworth-type IIR low-pass digital filter is needed with 3dB cut-off at one sixteenth of the sampling frequency $f_{s.}$

Solution: The relative cut-off frequency is $\Omega_C = \pi/8$ radians/sample
The pre-warped cut-off frequency is therefore $\omega_C = 2 \tan(\pi/16) = 0.4$ radians/sec.

Formula for 4th order Butterworth 1 radian/sec low-pass system function:

$$H_a(s) = \left(\frac{1}{1+0.77s+s^2}\right)\left(\frac{1}{1+1.85s+s^2}\right)$$

Scale the analogue cut-off frequency to ω_c by replacing s by s / 0.4.
Then replace *s* by 2 (z - 1)/(z +1) to obtain:

$$H(z) = 0.033\left(\frac{1+2z^{-1}+z^{-2}}{1-1.6z^{-1}0.74z^{-2}}\right)0.028\left(\frac{1+2z^{-1}+z^{-2}}{1-1.365z^{-1}+0.48z^{-2}}\right)$$

H(z) may be realised in the form of cascaded bi-quadratic sections as shown in fig 4.1

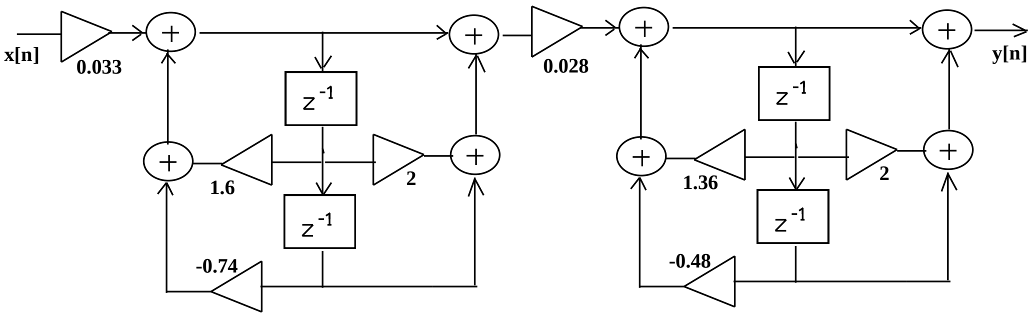

Fig. 6.4: Fourth order IIR Butterworth filter with cut-off fs/16

Figure 4.4: Fourth Order IIR Butterworth Filter with Cut-Off fs/16

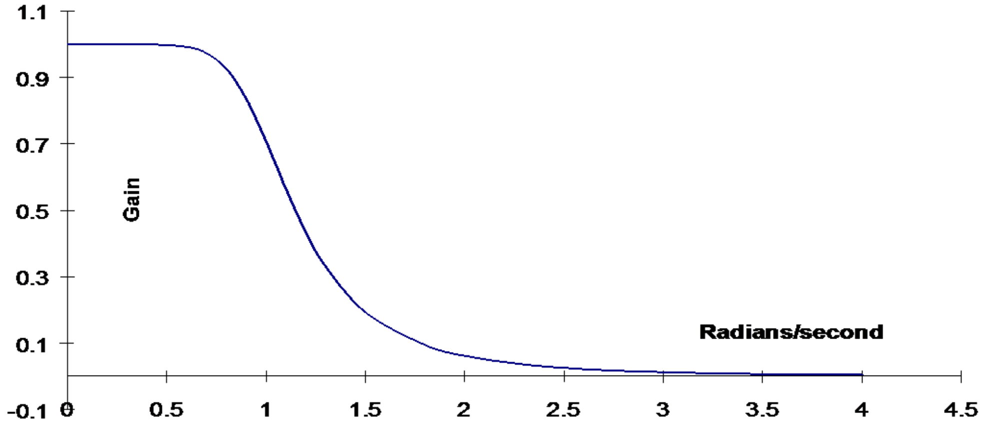

Figure 4.5(a): Analogue 4th Order Butterworth Gain Response

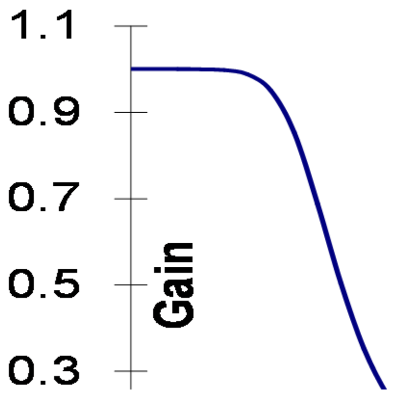

Figure 4.5(a) shows the gain response for the 4th order Butterworth low-pass filter whose transfer function was used here as a prototype. Fig 4.5(b) shows the gain response of the derived digital filter which, like the analogue filter, is 1 at zero frequency and 0.707 at the cut-off frequency. Note however that the analogue gain approaches 0 as $\omega \rightarrow \infty$ whereas the gain of the digital filter becomes exactly zero at $\Omega = \pi$. The shape of the Butterworth gain response is “warped " by the bilinear transformation. However, the 3dB point occurs exactly at Ω_c for the digital filter, and the cut-off rate becomes sharper and sharper as $\Omega \rightarrow \pi$ because of the compression as $\omega \rightarrow \infty$.

3.5 IIR Discrete Time High-pass Band-pass and Band-stop Filter Design

The bilinear transformation may be applied to analogue system functions which are high-pass, band-pass or band-stop. Such system functions may be obtained from an analogue low-pass 'prototype' system function (with cut-off 1 radian/second) by means of the frequency band transformations introduced in Section 2. Wide-band band-pass and band-stop filters ($f_U >> 2f_L$) may be designed by cascading low-pass and high-pass sections, thus avoiding the need to apply frequency band transformations, but 'narrow band' band-pass/stop filters (f_U not >> 2f_L) will not be very accurate if a cascading approach is used.

3.6 Comparison of IIR and FIR Digital Filters

IIR type digital filters have the advantage of being economical in their use of delays, multipliers and adders. They have the disadvantage of being sensitive to coefficient round-off inaccuracies and the effects of overflow in fixed point arithmetic. These effects can lead to instability or serious distortion. Also, an IIR filter cannot be exactly linear phase.

FIR filters may be realised by non-recursive structures which are simpler and more convenient for programming especially on devices specifically designed for digital signal processing. These structures are always stable, and because there is no recursion, round-off and overflow errors are easily controlled. A FIR filter can be exactly linear phase. The main disadvantage of FIR filters is that large orders can be required to perform fairly simple filtering tasks.

4.0 CONCLUSION

In this closing unit, you learnt how to explain bilinear transformation; design an IIR low-pass filter by the bilinear transformation method; explain higher order IIR digital filters; IIR discrete time high-pass band-pass and band-stop filter design and compare IIR and FIR digital filters..

5.0 SUMMARY

We defined bilinear transformation and its properties.

We replaced *s* by 2(z - 1)/(z + 1) to obtain H(z) and rearranged the expression for H(z) and realised by bi-quadratic sections. Therefore, we design higher order IIR filters as cascades of bi-quadratic sections.

You also learnt that wide-band band-pass and band-stop filters (f_U >> 2f_L) may be designed by Cascading low-pass and high-pass sections, thus avoiding the need to apply frequency band Transformations, but 'narrow band' band-pass/stop filters (f_U not >> 2f_L) will not be very accurate if a cascading approach was used. These effects can lead to instability or serious distortion. Also, an IIR filter cannot be exactly linear phase.

6.0 TUTOR-MARKED ASSIGNMENT

i. By referring to the general formula, show that the system function of a third order analogue Butterworth low-pass filter with 3 dB cut-off frequency at 1 radian/second is:

$$H_a(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

ii. Confirm from the general formula that the system function for a 3rd order Butterworth type low-pass analogue filter with cut-off frequency ω_C radians per second is:

$$H_a(s) = \frac{1}{\left[1 + 2\, s/\omega_c + 2\, s^2/\omega_c^2 + \left(s/\omega_c\right)^3\right]}$$

Give the corresponding differential equation.

Apply the derivative approximation technique to derive from this differential equation a third Order IIR Butterworth-type digital filter with cut-off frequency 500 Hz where the sampling Frequency is 10 kHz.

iii. A third order low-pass IIR discrete time filter is required with a 3dB cut-off frequency of one quarter of the sampling frequency, f_s. If the filter is to be designed by the bilinear transformation applied to a Butterworth low-pass transfer function, design the IIR filter and give its signal flow graph in the form of a second order and a first order section in serial cascade.

iv. Give a computer programme to implement the third order IIR filter designed above on a processor with floating point arithmetic. How would it be implemented in fixed point arithmetic?

v. A low-pass IIR discrete time filter is required with a cut-off frequency of one quarter of the sampling frequency, f_s, and a stop-band attenuation of at least 20 dB for all frequencies greater than $3f_s/8$ and less than $f_s/2$. If the filter is to be designed by the bilinear transformation applied to a Butterworth low-pass transfer function, show that the minimum order required is three. Design the IIR filter and give its signal flow graph.

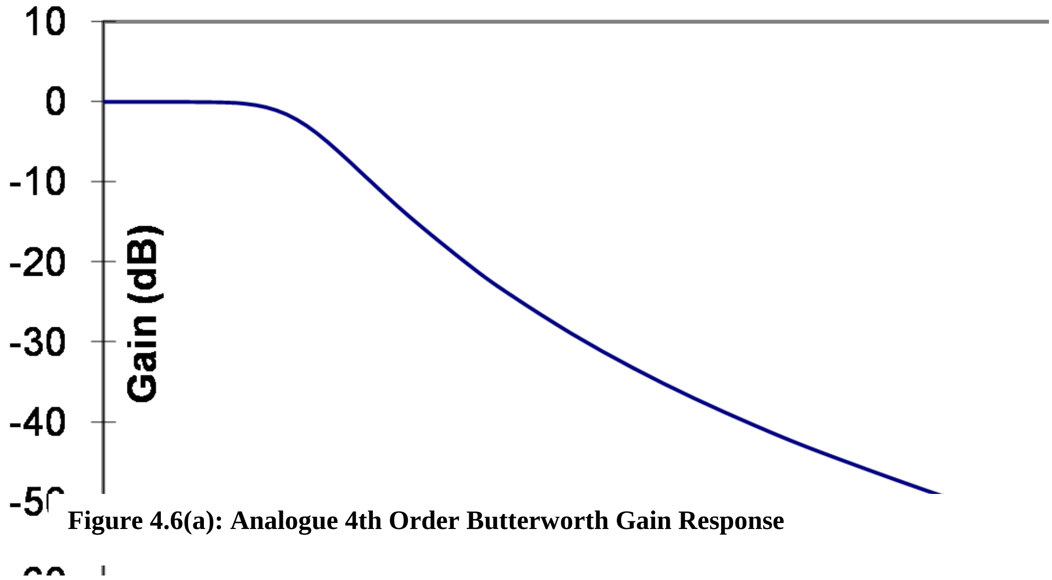

Figure 4.6(a): Analogue 4th Order Butterworth Gain Response

REFERENCES/FURTHER READING

Schum Series. *Advance Calculus*.

Stroud, K. A. *Engineering Mathematics*.