MODULE 1 APPROXIMATIONS

UNIT 1 POLYNOMIALS

CONTENTS

1.0 INTRODUCTION

Polynomials are very useful in the study of Mathematics especially in Numerical Analysis. Over the years polynomials have been used as approximation to several functions. Although polynomials are sometimes difficult to solve as equations yet they help in appreciating the value of certain functions. To this end, polynomials are of paramount importance when it comes to approximation theory.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define a polynomial;
- understand the degree of a polynomial;
- distinguish between polynomial as a function and a polynomial equation;
- express simple functions as polynomials; and
- name types of approximation methods.

3.0 MAIN CONTENT

3.1 WHAT IS A POLYNOMIAL?

From elementary Mathematics, you have come across polynomials in various forms. The commonest one is what is usually called the quadratic expression which can be written as

$ax^2 + bx + c$

Thus examples of polynomials may include:

$2x^2 – 3x+1$
$x^2 + 6x – 5$
$x^4 + 3x^3- x^2 +2x + 5$

and so on.

We shall therefore give a standard definition of what a polynomial is

Definition 1
A function P(x) of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \qquad (3.1)$$

is said to be a polynomial in x, where a_o, a_1, a_2, . . . , an are the coefficients of the function P(x).

These coefficients may be real or complex numbers.

3.2 The Degree of a Polynomial

Definition 2

The highest power to which the variable x is raised in a polynomial P(x) is the degree of the polynomial.

Hence, the polynomial function P(x) given by equation (3.1) is of degree n, For example any quadratic expression is a polynomial of degree 2.

If $P(x) = x^4 + 3x^3- x^2 +2x + 5$, then P(x) is of degree 4 $2x^2 – 3x+1$ is a polynomial of degree two.

3.3 Polynomial Equation

A polynomial is simply an expression whereas a polynomial can become an equation if the expression is equated to a quantity, often to zero.
Thus, when we write P(x) = 0, from equation (3.1) then equation (3.1) becomes a polynomial equation.

Although (3.1) is called an equation this is only because the polynomial is designated as P(x) on the left hand side. Apart from this, it is simply a polynomial. Thus

$a_0 + a_1x + a_2x^2 + \ldots + a_nx^2$

is a polynomial of degree n, whereas

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^2 = 0 \tag{3.2}$$

is a polynomial equation of degree n.

We must observe that if all the terms of a polynomial exist, then the number of coefficients exceed the degree of the polynomial by one. Thus a polynomial of degree n given by equation

(3.2) has n+1 coefficients.

Polynomial equation can be solved to determine the value of the variable (say x) that satisfies the equation. On the other hand there is nothing to solve in a polynomial. At best you may factorize or expand a polynomial and never to solve for the value of the variable.

Thus we can solve the polynomial equation

$x^3 + x^2 - 2x = 0$

But we can only factorize $x^3 + x^2 - 2x$

To factorize the expression $x^3 + x^2 - 2x$ we shall get: $x(x+2)(x - 1)$

But solving for x in the equation we get: $x = -2, 0, 1$

There are many things that we can do with polynomials. One of such things is to use polynomials to approximate non-polynomial functions.

3.4 Function Approximation

There are functions that are not polynomials but we may wish to represent such by a polynomial.

For example, we may wish to write ***cos x*** or ***exp(x)*** in terms of polynomials. How do we achieve this?

The learner should not confuse this with expansion of a seemly polynomial by Binomial expansion.

For example, we can expand $8\ (1 + x)^8$ using binomial expansion. Without expanding this, the expression is simply a polynomial of degree 8.

However, if we wish to write ex as a polynomial, then it can be written in the form:

$e^x = a_0\ a_1\ x + a_2x^2 + a_3x^3 + \ldots + a_n\ x^n$

This is only possible by using series expansion such as Taylor or Maclaurin series.
The learner is assumed to have studied Taylor or Maclaurin series of simple functions at the lower level. For example, the expansion of exp(x) is written as:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{2!} + \cdots$$

This shows that exp(x) can be expressed as a polynomial function. The degree of where the polynomial is truncated (terminated), say $\frac{x^k}{k!}$, is the approximation that is written for e^x. It may be reasonable to let k be large, say at least 2. Hence a fourth order approximation of exp(x) will be:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad (3.3)$$

That is, ex is written here as a polynomial of degree 4.

Similarly the Taylor series for cos x is given by

$$\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad (3.4)$$

The illustration above leads us to the study of approximation theory.

3.5 Types of Functions Approximation

Before we go fully into the discussion of various approximations in Numerical Analysis, we need to state that there may arise two problems.
The first problem arises when a function is given explicitly, but we wish to find a simpler type of function such as a polynomial, that can be used to approximate values of the given function.

The second kind of problem in approximation theory is concerned with fitting functions to a given set of data and finding the **"best"** function in a certain class that can be used to represent the set of data.

To handle these two problems, we shall in this study discuss some of the basic methods of approximations.

Some of the approximation methods of functions, in existence, include:

(i.) Taylor's Approximation
(ii.) Lagrange polynomials
(iii.) Least-Squares approximation
(iv.) Hermite approximation
(v.) Cubic Spline interpolation
(vi.) Chebyshev approximation
(vii.) Legendre Polynomials
(viii.) Rational function approximation;

and some few others more.

Every approximation theory involves polynomials; hence, some methods of approximation are sometimes called polynomials. For example, Chebyshev approximation is often referred to as Chebyshev polynomials. We shall begin this discussion of these approximations with the Least Squares Approximation.

SELF-ASSESSMENT EXERCISE

1. How many non-zero coefficients has $(2x + 5)(x^{2-1})$

2. What is the degree of the polynomial involved in the equation:
   $(2x + 1)(x^2 - \frac{2}{x}) = 0$?
   hence obtain its solution.
3. Write a polynomial of degree 3 with only two coefficients
4. By following equation (3.3) write down the expansion of e^{-x}

4.0 CONCLUSION

Polynomials are basic tools that can be used to express other functions in a simpler form. While it may be difficult to calculate e3 without a calculator, because the exponential function e is approximately 2.718, but we can simply substitute 3 into the expansion given by (3.3) and simplify to get an approximate value of e3. Hence a close attention should be given to this type of function.

5.0 SUMMARY

In this Unit we have learnt that

- polynomials are expression involving various degrees of variable x which may be sum together.
- polynomial expression is different from polynomial equation.
- simple functions can be written through the expansion given by Taylor or Maclaurin series.
- there are various polynomial approximations which can be used to estimate either a function or a set of data.

6.0 TUTOR-MARKED ASSIGNMENT

1. Obtain the Taylor's expansion of sin x
2. Distinguish between Taylor series and Binomial expansion
3. Find the Maclaurin series for e^{-x} as far as the term involving x^4 and hence estimate e^{-2}

7.0 REFERENCES/FURTHER READING

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UNIT 2 LEAST SQUARES APPROXIMATION (DISCRETE CASE)

CONTENTS

1.0 INTRODUCTION

Sometimes we may be confronted with finding a function which may represent a set of data points which are given for both arguments x and y. Often it may be difficult to find such a function y = y(x) except by certain techniques. One of the known methods of fitting a polynomial function to this set of data is the Least squares approach. The least squares approach is a technique which is developed to reduce the sum of squares of errors in fitting the unknown function.

The Least Squares Approximation methods can be classified into two, namely the discrete least square approximation and the continuous least squares approximation. The first involves fitting a polynomial function to a set of data points using the least squares approach, while the latter requires the use of orthogonal polynomials to determine an appropriate polynomial function that fits a given function. For these reasons, we shall treat them separately.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- handle fitting of polynomial (for discrete case) by least squares method
- derive the least square formula for discrete data
- fit a linear polynomial to a set of data points
- fit a quadratic or parabolic polynomial to a set of data points

3.0 MAIN CONTENT

The basic idea of least square approximation is to fit a polynomial function P(x) to a set of data points (xi, yi) having a theoretical solution

$y = f(x)$ (3.1)

The aim is to minimize the squares of the errors. In order to do this, suppose the set of data satisfying the theoretical solut ion (3.1) are

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Attempt will be made to fit a polynomial using these set of data points to approximate the theoretical solution f(x).

The polynomial to be fitted to these set of points will be denoted by P(x) or sometimes $P_n(x)$ to denote a polynomial of degree n. The curve or line P(x) fitted to the observation $y_1, y_2, \ldots, y_n$ will be regarded as the best fit to f(x), if the difference between $P(x_i)$ and $f(x_i)$, $i = 1, 2, \ldots, n$ is least. That is, the sum of the differences

$e_i = f(x_i) – P(x_i), i = 1, 2, \ldots, n$ should be the minimum.

The differences obtained from e_i could be negative or positive and when all these e_i are summed up, the sum may add up to zero. This will not give the true error of the approximating polynomial. Thus to estimate the exact error sum, the square of these differences are more appropriate. In other words, we usually consider the sum of the squares of the deviations to get the best fitted curve.

Thus the required equation for the sum of squares error is then written as

$$S = \sum_{i=1}^{n} [f(x_i)]^2 \quad (3.2)$$

which will be minimized.

where P(x) is given by

$$p(x)= a_0+a_1x+a_2x^2{}_{\ldots\ldots}+a_nx^n \quad (3.3)$$

The above approach can be viewed either in the discrete case or in the continuous case

3.1 Fitting of polynomials (Discrete case)

We shall now derive the formula in discrete form that fits a set of data point by Least squares technique. The aim of least squares method is to minimize the error of squares.

To do this we begin by substituting equations (3.1) and (3.3) in equation (3.2), this gives:

$$s = \sum_{i=1}^{n} [yi - (a_0 + a_1 x_i + a_2 x_{i+}^2 \quad (3.4)$$

To minimize S, we must differentiate S with respect to ai and equate to zero. Hence, if we differentiate equation (3.4) partially with respect to a_0, $a_1,\ldots,a_k$, and equate each to zero, we shall obtain the following:

$$\left.\begin{aligned}
\frac{\partial S}{\partial a_o} &= -2\sum_{i=1}^{n}\left[y_i - (a_o + a_1x_i + a_2x_i^2 + \ldots + a_kx_i^k)\right] = 0 \\
\frac{\partial S}{\partial a_1} &= -2\sum_{i=1}^{n}\left[y_i - (a_o + a_1x_i + a_2x_i^2 + \ldots + a_kx_i^k)\right]x_i = 0 \\
\frac{\partial S}{\partial a_2} &= -2\sum_{i=1}^{n}\left[y_i - (a_o + a_1x_i + a_2x_i^2 + \ldots + a_kx_i^k)\right].x_i^2 = 0 \\
&\ldots\ldots\ldots\ldots \\
\frac{\partial S}{\partial a_k} &= -2\sum_{i=1}^{n}\left[y_i - (a_o + a_1x_i + a_2x_i^2 + \ldots + a_kx_i^k)\right].x_i^k = 0
\end{aligned}\right\} \qquad (3.5)$$

These can be written as

$$\left.\begin{aligned}
\sum y_i &= na_o + a_1\sum x_i + a_2\sum x_i^2 + \ldots + a_k\sum x_i^k \\
\sum x_iy_i &= a_o\sum x_i + a_1\sum x_i^2 + a_2\sum x_i^3 + \ldots + a_k\sum x_i^{k+1} \\
\sum x_i^2y_i &= a_o\sum x_i^2 + a_1\sum x_i^3 + a_2\sum x_i^4 + \ldots + a_k\sum x_i^{k+2} \\
&\ldots\ldots\ldots \\
\sum x_i^ky_i &= a_o\sum x_i^k + a_1\sum x_i^{k+1} + a_2\sum x_i^{k+2} + \ldots + a_k\sum x_i^{2k}
\end{aligned}\right\} \qquad (3.6)$$

where $\sum$ is assumed as short form for $\sum_{i=1}^{n}$

Solving equation (3.6) to determine a_0, $a_1,\ldots a_k$ and substituting into equation (3.3) gives the best fitted curve to (3.1).

The set of equations in (3.6) are called the **Normal Equations** of the Least Squares Method Equation (3.6) can only be used by creating a table of values corresponding to each sum and the sum is found for each summation.
We shall now illustrate how to use the set of equations (3.6) in a tabular form.

SELF-ASSESSMENT EXERCISE

i. For a quadratic approximation, how many columns will be required, list the variables for the columns excluding the given x and y columns.
ii. Give one reason why we need to square the errors in a least square method.

3.2 Numerical Example

Example 1

By using the Least Squares Approximation, fit

(a) a straight line
(b) a parabola
to the given data below

x	1	2	3	4	5	6
y	120	90	60	70	35	11

Which of these two approximations has least error?

Solution

(a) In order to fit a straight line to the set of data above, we assume the equation of the form

$$y = a_o + a_1 x$$

The graph of the set of data above is given by in figure 1

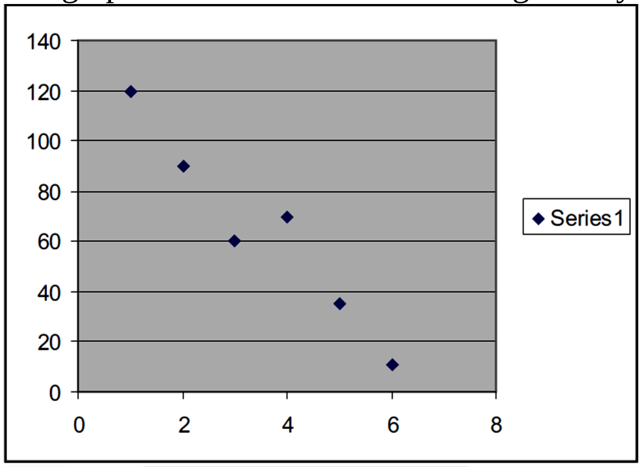

Figure 1

By inspection a straight line may be fitted to this set of data as the line of best fit, since most of the points will lie on the fitted line or close to it. However, some may want to fit a curve to this but the accuracy of the curve fitted is a thing for consideration.

Now from the straight line equation above, we have to determine two unknowns a_o, and a_1 , the normal equations necessary to determine these unknowns can be obtained from equation (1.7) as:

$$\sum y_i = na_o + a_1 \sum x_i$$

$$\sum x_i y_i = a_o \sum x_i + a_1 \sum x_i^2$$

Hence we shall need to construct columns for vales of xy and x2 in addition to x and y values already given.

Thus the table below shows the necessary columns:

Table 1

x	**y**	**x^2**	**xy**
1	120	1	120
2	90	4	180
3	60	9	180
4	70	16	280
5	35	25	175
6	11	36	66
$\sum$	**386**	**91**	**1001**

$386 = 6a_0 + 21a_1$

$1001 = 21a_0 + 91a_1$

Solving these two equations, we obtain

$a_0 = 134.33$ - $1\ 20_x$

Therefore the straight line fitted to the given data is
$y = 134.33 – 20x$

(b) In a similar manner, the parabola can be written as $y = a_o + a_1x + a_2\ x^2$. Hence the required normal equations to determine the unknowns a_o, a_1 and a_2 are:

$$\sum y_i = na_o + a_1 \sum x_i + a_2 \sum x_i^2$$

$$\sum x_i y_i = a_o \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3$$

$$\sum x_i^2 y_i = a_o \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4$$

Thus, we construct the necessary columns in Table 2 given below:

Table 2

x	y	x^2	x^3	x^4	xy	x^2y
1	120	1	1	1	120	120
2	90	4	8	16	180	360
3	60	9	27	81	180	540
4	70	16	64	256	280	1120
5	35	25	125	625	175	875
6	11	36	216	1296	66	396
21	**386**	**91**	**441**	**2275**	**1001**	**3411**

Substituting into the normal equations, we have

$6a_0 + 21a_1 + 91a_2 = 386$

$21a_0 + 91a_1 + 441a_2 = 1001$

$91a_0 + 441a_1 + 2275a_2 = 3411$

On carefully solving these equations, we obtain

$a_0 = 136$, $a_1 = -\frac{85}{4}$, $a_2 = \frac{5}{28}$

As a learner, you are expected to solve these equations by any simple method known to you.

Hence the fitted curve is 2

$Y = 136 - \frac{85}{4} x + \frac{5}{28} x^2$

The graphs of the two approximate functions are shown below:

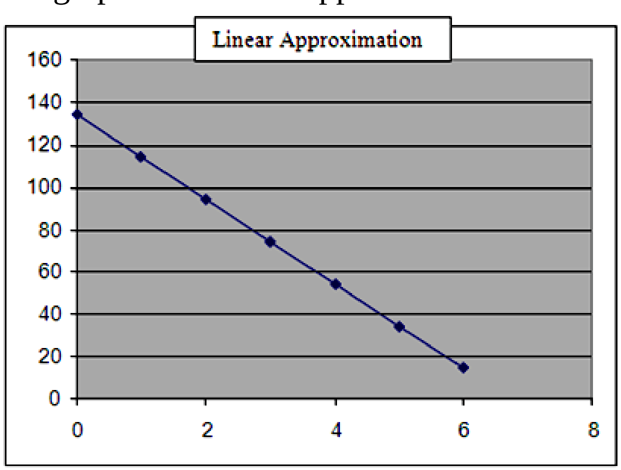

Figure 2

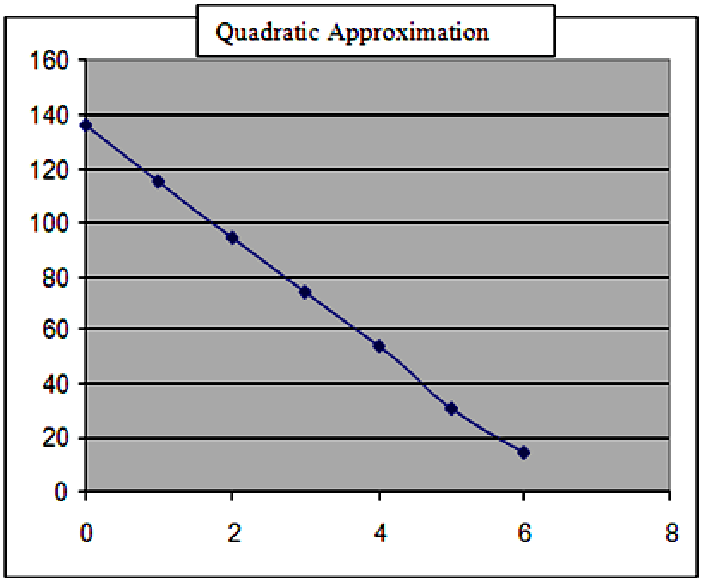

Figure 3

The two lines look like a straight line but the errors of the two lines will depict the better approximation.

We can now consider the accuracy of the two fitted functions to the data and decide on which one of the two approximations is better. To do this we may retain values of x for 1, 2, . . . , 6 and evaluate the corresponding y values.

For example when x = 1, the linear approximation gives y = 134.33 - 20 = 114.33, where as for the same x = 1 the quadratic approximation gives: y 136 $-\frac{85}{4}+\frac{5}{28}=$ 1149. For the set of values of x tabulated we have the corresponding values for y in both approximations. The squares of the errors are considered as $(y - y_i)^2$. The table below describes the error table.

Table 3

x	y	$y_1 = y(L)$	$y_2 = y(Q)$	$(y-y_1)^2$	$(y-y_2)^2$
1	120	114.33	114.9	32.1489	26.01
2	90	94.33	94.2	18.7489	17.64
3	60	74.3	73.9	204.49	193.21
4	70	54.3	53.9	246.49	259.21
5	35	34.3	31.3	0.49	13.69
6	11	14.3	14.9	10.89	15.21
				513.2578	524.97

The sums of squares of the error of fitted lines by linear function y(L) and quadratic function y(Q) are shown above in Table 3. The comment here is that the sum of squares of error of the linear (513.26) is slightly lower than that of the quadratic (524.97). Invariably the linear function is a better approximation to the data above.

4.0 CONCLUSION

Fitting a polynomial to discrete data by least squares methods is easily handled by creating tables of values and generating all necessary columns that will enable one to obtain the normal equations. The normal equations are then solved simultaneously to determine the unknowns which are then substituted into the required approximate polynomial equation.

5.0 SUMMARY

In this Unit we have learnt

- how to derive the normal equations of the least squares method
- that only necessary terms of the normal equations are computed;
- that the set of normal equations can be used to obtain any polynomial approximation by the least square method.
- that the choice of the degree of polynomial chosen in fitting the data by least squares method may not necessarily be the best fit;
- that computation by least squares method is simple and straight forward to apply.

6.0 TUTOR MARKED ASSIGNMENT

1. The table below gives the readings from a laboratory experiment.

Time t	2	3	5	6	9
Reading y	7	17	49	71	161

Fit (a) a linear function and (b) a quadratic polynomial to the above data by method of least squares and determine which of the two is a better approximation.

2. How many normal equations will be needed to fit a cubic polynomial? Hence list the entire necessary variables for such fitting.

7.0 REFERENCES/FURTHER READING

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UNIT 3 LEAST SQUARES APPROXIMATION (CONTINUOUS CASE)

CONTENTS

1.0 INTRODUCTION

We have seen in the last unit how to fit a polynomial to a set of data points by using the least squares approximation technique. We shall in this unit consider the situation in which a function f(x) is to be approximated by the least squares method in terms of a polynomial. In this case since data are no longer given, it would not be necessary to create tables by columns but rather by carrying out some integration. The candidate is therefore required to be able to integrate simple functions as may be required.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- distinguish between discrete data and continuous function; and
- fit polynomials to continuous functions by least squares approach.

3.0 MAIN CONTENT

If we wish to find a least square approximation to a continuous function f(x), our previous approach must be modified since the number of points (x_i, y_i) at which the approximation is to be measured is now infinite (and non-countable). Therefore, we cannot use a summation as $\sum_{i=1}^{n}[f(x_i) - P(x_i)]^2$, but we must use a continuous measure, that is an integral. Hence if the interval of the approximation is [a, b], so that a $\leq$ x $\leq$ b for all points under consideration, then we must minimize

$\int_a^b [f(x_i) - P(x_i)]^2 \, dx$3ba

where y = f(x) is our continuous function and P(x) is our approximating function.

3.1 Derivation

Let f(x) be a continuous function which in the interval (a, b) is to be approximated by a
polynomial linear combination

$$P(x) \quad a_0 + a_1x + a_2x^2 + \ldots + a_kx^k \qquad (3.1)$$

$$S = \int_a^b [y_i - (a + a_1x_i + a_2x_i^k)]^2 \, dx$$

of n+1 given functions $\varphi_o, \varphi_1, \varphi_2. .., \varphi_n$. Then, $c_0, c_1, \ldots, c_n$ can be determined such that a weighted Euclidean norm of the error function f(x) - p(x) becomes as small as possible

That is, $$||f(x) \;\; P(x)||^2 = \int_a^b |f(x) - P(x)|^2 \, w(x)dx \qquad (3.2)$$

where w(x) is a non-negative weighting function. Equation (3.2) is the continuous least square approximation problem.

The minimization problem of f(x) by continuous function P(x) of(3.1) is given by

$$S = \int_a^b [f(x) - P(x)]^2 dx \qquad (3.3)$$

where the interval [a, b] is usually normalized to [-1,1] following Legendre polynomial or Chebyshev function approximation, hence, substituting equation (3.2) in (3.3), we obtain

$$S = \int_{-1}^1 [f(x) - \{c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x)\}]^2 w(x)dx \quad (3.4)$$ where $\varphi_o, \varphi_1, \varphi_2. .., \varphi_n$ are some chosen polynomials.

To minimize (3.3) or (3.4), one could consider two alternative methods to obtain the coefficients or the unknown terms.

3.1.1 First Method

The first approach for minimizing equation (3.3) is by carrying out an expansion of the term $[f(x)\text{-}P(x)]^2$, next carry out the integration and then by means of calculus, obtain the minimum by setting

$$\frac{\partial S}{\partial c_k} = 0\ , k = 0, 1, 2, \dots, n$$

This approach will be illustrated by the next example.

Example

Find the least square straight line that provides the best fit to the curve y $\sqrt{x}$ over the interval 0$\leq$ x$\leq$ 1.

Solution

Let the line be y = ax+b, we must minimize

$$S = \int_0^1 \left[\sqrt{x} - ax - b\right]^2\, dx$$

Expand the integrand, we obtain

$$S = \int_0^1 \left[x - 2ax^{3/2} - 2b\sqrt{x} + a^2x^2 + 2abx + b^2\right] dx$$

And integrating we get

$$S = \left[\tfrac{1}{2}x^2 - \tfrac{4}{5}ax^{5/2} - \tfrac{4}{3}bx^{3/2} + \tfrac{a^2}{3}x^2 + abx^2 + b^2x\right]_0^1$$

Evaluating we get

$$S = \frac{a^2}{3} + b^2 + ab - \frac{4}{5}a - \frac{4}{3}b + \frac{1}{2}$$

For a minimum error, we must set $\frac{\partial S}{\partial a} = 0$, *and* $\frac{\partial S}{\partial b} = 0$

Doing this, we get

$$\frac{\partial S}{\partial a} = \frac{2}{3}a + b - \frac{4}{5} = 0$$

$$\frac{\partial S}{\partial b} = 2b + a - \frac{4}{3} = 0$$

Thus, we solve the equations

$$\frac{2}{3}a + b = \frac{4}{5}$$

$$a + 2b = \frac{4}{3}$$

Solving, we get $a = \frac{4}{5}$, $b = \frac{4}{15}$

Hence the Least squares approximation is

$$y = \frac{4}{5}x + \frac{4}{15}$$

WE observe that The line $y = \frac{4}{5}x + \frac{4}{15}$ meets the curve y = $\sqrt{x}$ in two points P(0.1487, 0.3856) and Q(0.7471, 0.8643) as shown below.

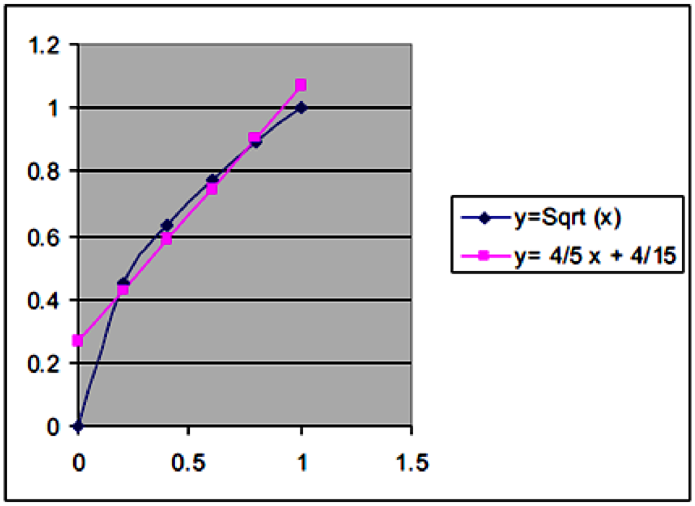

Figure 1

This straight line is only a linear approximation to the given curve. We could have as well found other polynomial approximation using the same least squares technique.

It will be observed that if P(x) is a polynomial of higher degree say n = 3 or more, the expression $[f(x) - P(x)]^2$ may not be easy to expand before integrating, so we must seek another approach for the minimization.

SELF-ASSESSMENT EXERCISE

Give one disadvantage of the technique used above.

3.1.2 Alternative Method

Now, suppose we wish to find the least squares approximation, using a polynomial of degree k to a continuous function y over [a,b]. In such a case, we must minimize the integral

$$S = \int_a^b [y_i - (a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_k x_i^k)]^2 \, dx$$

If we do not want to expand the expression in the squared bracket, then we must first get the normal equations. In other words, we derive the normal equation by obtaining integrating the resulting function and evaluating the result.

Doing this we obtain

$$\frac{\partial S}{\partial a_o} = \int_a^b -2\left(y_i - a_o - a_1 x - a_2 x^2 - \ldots - a_k x^k\right)\, dx = 0$$
$$\frac{\partial S}{\partial a_1} = \int_a^b -2x\left(y_i - a_o - a_1 x - a_2 x^2 - \ldots - a_k x^k\right)\, dx = 0$$
$$\frac{\partial S}{\partial a_2} = \int_a^b -2x^2\left(y_i - a_o - a_1 x - a_2 x^2 - \ldots - a_k x^k\right)\, dx = 0$$

and in general we can write

$$\frac{\partial S}{\partial a_r} = \int_a^b -2x^r\left(y_i - a_o - a_1 x - a_2 x^2 - \ldots - a_k x^k\right)\, dx = 0 \quad \text{for} \quad (r = 0, 1, \ldots, k)$$

The factor(-2) that appeared first in the integral can be ignored since the right hand side of the equation is zero. Hence, the normal equations can be written as

$$\int_a^b x^r\left(y_i - a_o - a_1 x - a_2 x^2 - \ldots - a_k x^k\right)\, dx = 0 \qquad \text{for} \quad (r = 0, 1, \ldots, k)$$

This will give (k+1) linear equations in the (k+1) unknowns $a_0, a_1, \ldots, a_k$ which can be solved simultaneously by any algebraic process.

This approach may be simpler than the first one and we therefore suggest this second method. However any of the two techniques may be used and are both valid.

Example 2

Find the least squares quadratic $ax^2 + bx + c$, which best fits the curve $y = \sqrt{x}$ over the interval $0 \leq x \leq 1$.

Solution

We need to minimize

$$S = \int_0^1 \left[\sqrt{x} - ax^2 - bx - c\right]^2\, dx$$

By this new approach, we shall first of all obtain the normal equations. Thus we have:

$$\frac{\partial S}{\partial c} = \int_0^1 \left(\sqrt{x} - ax^2 - bx - c\right) dx = 0$$
$$\frac{\partial S}{\partial b} = \int_0^1 x\left(\sqrt{x} - ax^2 - bx - c\right) dx = 0$$
$$\frac{\partial S}{\partial a} = \int_0^1 x^2\left(\sqrt{x} - ax^2 - bx - c\right) dx = 0$$

Integrating, we get the three equations as follows:

$$\frac{2}{3}x^{3/2} - \frac{1}{3}ax^3 - \frac{1}{2}bx^2 - cx\Big|_0^1 = 0$$
$$\frac{2}{5}x^{5/2} - \frac{1}{4}ax^4 - \frac{1}{3}bx^3 - \frac{1}{2}cx^2\Big|_0^1 = 0$$
$$\frac{2}{7}x^{7/2} - \frac{1}{5}ax^5 - \frac{1}{4}bx^4 - \frac{1}{3}cx^3\Big|_0^1 = 0$$

Evaluating within the limits we obtain three simultaneous equations

$$\frac{1}{3}a + \frac{1}{2}b + c = \frac{2}{3}$$
$$\frac{1}{4}a + \frac{1}{3}b + \frac{1}{2}c = \frac{2}{5}$$
$$\frac{1}{5}a + \frac{1}{4}b + \frac{1}{3}c = \frac{2}{7}$$

Solving these equations simultaneously we get

$$a = -\frac{4}{7}\ , \quad b = \frac{48}{35}\,, \quad c = \frac{6}{35}$$

Thus the least squares quadratic function is

$$f(x) = -\frac{4}{7}x^2 + \frac{48}{35}x + \frac{6}{35}$$

or $$f(x) = -\frac{2}{35}\left(10x^2 - 24x - 3\right)$$

The earlier approach of expanding the integrand is also valid. However the number of terms in the expansion may not be friendly. The higher the degree of the polynomial

we are trying to fit by least squares approximation, the more difficult it is to obtain the coefficients of the polynomial by expansion. Hence this last approach or method may be more appropriate for easy handling.
The error table of the above result is given below

x	$y=\sqrt{x}$	$f(x) = -\frac{2}{35}(10x^2 - 24x - 3$	E= |y - f(x) |
0	0.0000	0.1714	0.1714
0.1	0.316228	0.302857	0.013371
0.2	0.447214	0.422857	0.024357
0.3	0.547723	0.531429	0.016294
0.4	0.632456	0.628571	0.003885
0.5	0.707107	0.714286	0.007179
0.6	0.774597	0.788571	0.013974
0.7	0.836660	0.851429	0.13666
0.8	0.894427	0.902857	0.094427
0.9	0.948683	0.942857	0.048683
1.0	1.0000	0.971429	0.0000

4.0 CONCLUSION

To conclude this unit, it would be observed using the least squares approximation technique to fit a polynomial to a continuous function f(x) could be done in two ways. The reader is allowed to use either of the methods. However, expanding the integrand before carrying out the integration may be sometimes more difficult than to first of all find the normal equations. The error obtained in using the least square method will depend on how well a polynomial of certain degree is suitable in approximating the continuous function.

5.0 SUMMARY

In this unit we have learnt that

- it is possible to fit a polynomial to a continuous function by Least Squares Method (LSM)
- fitting a polynomial by LSM for a continuous function will necessarily require integration
- there are two approaches of fitting a polynomial by LSM for a continuous function
- one approach will require expansion of $[f(x) - P(x)]^2$ for a given polynomial P(x) while the other approach will go by the way of normal equation.

6.0 TUTOR-MARKED ASSIGNMENT

Find the least squares quadratic $ax^2 + bx + c$, which best fits the curve $y = \sqrt{2x1}$ over the interval $0 \leq x \leq 1$.

7.0 REFERENCES/FURTHER READING

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