MODULE 2 ORTHOGONAL POLYNOMIALS

Unit 1 Introduction to Orthogonal System
Unit 2 The Legendre Polynomials

UNIT 1 INTRODUCTION TO ORTHOGONAL SYSTEM

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1.0 INTRODUCTION

Orthogonal polynomials are of fundamental importance in many branches of mathematics in addition to approximation theory and their applications are numerous but we shall be mainly concerned with two special cases, the Legendre polynomials and the Chebyshev polynomials. More general applications are however easily worked out once the general principles have been understood.

2.0 OBJECTIVE

By the end of this unit, you should be able to:

- define what orthogonal polynomials are;
- formulate orthogonal and orthonormal polynomials; and
- handle inner product of functions.

3.0 ORTHOGONAL POLYNOMIALS

We begin this study by giving the definition of orthogonal functions:

Definition 1

A system of real functions $\phi_n(x)$, $\phi_1(x)$, ...defined in an interval [a, b] is said to be **orthogonal** in this interval if

$$\int_a^b \phi_m(x)\phi_n(x)\, dx = \begin{cases} 0 , & m \neq n \\ \lambda_n , & m = n \end{cases}$$

If $\lambda_0 = \lambda_1 = ... = 1$ the system is said to be normal. An orthogonal system which is also

normal is sometimes referred to as an **orthonormal system**.
Note that since $\varphi_n(x)$ is real, $\lambda n \geq 0$ and we shall assume that each $\varphi_n(x)$ is continuous and non-zero so that $\lambda n > 0$.

The advantages offered by the use of orthogonal functions in approximation theory can now be

made clear as follows. Suppose {φn,(x)} is an orthogonal system and that f(x) is any function and we wish to express f(x) in the form

$$f(x) = c_o\phi_o(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) + \ldots \qquad (3.1)$$

Then $\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx = c_n\lambda_n$

since all the other terms on the right-hand side are zero and so

$$c_n = \frac{1}{\lambda_n}\int_a^b f(x)\phi_n(x)\,dx \qquad (3.2)$$

Thus the coefficients c_n in equation (3.1) can be found. These coefficients c_n are called the Fourier coefficients of f(x), with respect to the system {φn(x)}

3.1 The Inner Products

Let w(x) be the weighting function and let the inner product of two continuous functions f(x) and g(x) be defined as

$$\langle f, g\rangle = \int_a^b w(x) f(x) g(x)\,dx$$

where f, g are continuous in [a, b], then f(x) and g(x) satisfy the following properties:

(i.) $<\alpha f, g> = <f, \alpha g> = \alpha <f, g>$, α is a scalar

(ii.) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$

(iii.) $\langle f, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$

(iv.) $\langle f, g\rangle = \langle g, f\rangle$

(v) $\langle f, f\rangle > 0$ for all $f \in C[a,b]$ and $\langle f, f\rangle = 0$ iff $f = 0$.

The functions f and g are said to be orthogonal if the inner product of f and g is zero, that is if $\langle f, g\rangle = 0$

In a similar manner we can define the inner product for the discrete case. The inner product of discrete functions f and g satisfy the orthogonality property given by

$$\langle f, g\rangle = \sum_{k=0}^{m} f(x_k) g(x_k)$$

where $\{x_k\}$ are the zeroes of the function.

We remark here that polynomial approximation is one of the best ways to fit solution to unknown function f(x).

A good polynomial Pn(x) which is an approximation to a continuous function f(x) in a finite range [a, b] must possess oscillatory property. Among such polynomial approximation functions include the Chebyshev Polynomials and the Legendre Polynomials. We shall examine these polynomials and their properties in our discussion in this course as we go along.

Definition 2 (Orthogonality with respect to a weight function)

A series of functions {φn,(x)} are said to be orthogonal with respect to the weight function w(x) over (a,b) if

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx = \begin{cases} 0 & , \quad m \neq n \\ \lambda_n & , \quad m = n \end{cases}$$

The idea and principle of orthogonality properties are now extended to two common polynomials in the next few sections.

3.2 Example

The best-known example of an orthogonal system is the trigonometric system

1, cos x, sin x, cos2x, sin2x,…

Over the interval [-π, π].
We shall define various combination of integral of product functions of sine and cosine as follows:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (m \neq n)$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \qquad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (m \neq n)$$

and $$\int_{-\pi}^{\pi} \cos nx . \sin nx \, dx = 0 \qquad (m = n \neq 0)$$

whereas $$\int_{-\pi}^{\pi} \cos nx \cos nx \, dx = 0 \qquad (m = n)$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}(1 + \cos 2nx) \, dx$$

$$= \tfrac{1}{2}(x + \tfrac{1}{2n} \sin 2nx)\Big|_{-\pi}^{\pi}$$

$$= \tfrac{1}{2}(\pi + \tfrac{1}{2n} \sin 2n\pi) - \tfrac{1}{2}(-\pi + \tfrac{1}{2n} \sin 2n(-\pi)) = \pi$$

Also $$\int_{-\pi}^{\pi} \sin nx \sin nx \, dx = 0 \qquad (m = n)$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \qquad (m = n \neq 0)$$

and finally for n = 0, $$\int_{-\pi}^{\pi} \cos^2 0 \, dx = \int_{-\pi}^{\pi} 1^2 \, dx = 2\pi \qquad (m = n = 0)$$

Comparing this with our Definition 1 above, we obtain from these integrals the following values

$$\lambda_1 = 2\pi \, , \; \lambda_2 = \lambda_3 = \ldots = \pi$$

It follows therefore that the system

$$\frac{1}{\sqrt{2\pi}}, \; \frac{\cos x}{\sqrt{\pi}}, \; \frac{\sin x}{\sqrt{\pi}}, \; \frac{\cos 2x}{\sqrt{\pi}}, \; \frac{\sin 2x}{\sqrt{\pi}}$$

is orthogonal and normal

4.0 CONCLUSION

The discussion above has simply illustrated the way to determine where a set of functions is orthogonal or otherwise. Other examples can be produced to show the orthogonality property.

5.0 SUMMARY

In this unit you have learnt that

(i.) a normal orthogonal system is an orthonormal system
(ii.) orthogonality of some functions can be obtained by integration
(iii.) inner product is written as an integral or a sum

6.0 TUTOR MARKED ASSIGNMENT

Verify whether the following functions are orthogonal or not

(i.) $1, e^x , e^{2x} , e^{3x} , \ldots .$
(ii.) ln x, ln2x , ln3x , ln4x, . . .

7.0 FURTHER READING AND OTHER RESOURCES

Francis Scheid. (1989) *Schaum's Outlines Numerical Analysis 2nd ed.* McGraw-Hill New York.

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Leadermann Walter (1981) (Ed.): Handbook of Applicable Mathematics, Vol 3, Numerical Analysis, John Wiley, N. Y.

UNIT 2 THE LEGENDRE POLYNOMIALS

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1.0 INTRODUCTION

Legendre polynomial is known to possess some oscillatory property among which makes it of importance in the field of numerical analysis. The polynomial has its root from the Legendre equation which is a second order differential equation. The first set of solutions of the Legendre equation is known as the Legendre polynomial.

2.0 OBJECTIVE

By the end of this unit, you should be able to:

- state the necessary formulae for generating the Legendre polynomials;
- generate the Legendre polynomials; and
- define the Legendre polynomial as a class of orthogonal series.

3.0 LEGENDRE POLYNOMIAL APPROXIMATION

When we try to find good polynomial approximations to a given function f(x) we are trying to represent f(x) in the form

$$f(x) = \sum_{k=0}^{n} c_k x^k \qquad (3.1)$$

which is of the form of series equation (3.1) of the last unit with $\varphi_k(x) = x^k$.
Unfortunately the set 1, x, x^2,… is not orthogonal over any non-zero interval as may be seen at once since, for example

$$\int_a^b \phi_1(x)\phi_3(x)dx = \int_a^b x^4 dx > 0$$

which contradicts the assertion that $\{x^k\}$ is orthogonal. It is however possible to construct a set of polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$,… $P_n(x)$,… where $P_n(x)$ is of degree n which are orthogonal over the interval [-1, 1] and from these a set of polynomials orthogonal over any given finite interval [a, b] can be obtained.

The method for finding a set of polynomials which are orthogonal and normal over [-1, 1] is a relatively simple one and we illustrate it by finding the first three such polynomials. We shall at this junction give a definition of Legendre Polynomial which can be used to generate the set of polynomials required.

Definition 1

The Rodrigues' formula for generating the Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{1}{dx^n} [(x^2 - 1)^n] \qquad (3.2)$$

From the definition given above, you will observe that an n^{th} derivative must be carried out before a polynomial of degree n is obtained. Thus the first few set of Legendre polynomials can be obtained as follows:

$P_o(x)$ will not involve any derivative since n = 0, hence we have

$P_0(x) = 1$

Also for n = 1, we shall have

$$P_1(x) = \frac{1}{2^1.1!} . \frac{d}{dx}(x^2 - 1) = \frac{1}{2} .2x = x$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} \left[(x^2 - 1)^2\right]$$

$$P_2(x) = \frac{1}{8} \frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{8} \frac{d}{dx}(4x^3 - 4x)$$

$$P_2(x) = \frac{1}{8}(12x^2 - 4) = \frac{1}{2}(3x^2 - 1)$$

To obtain $P_3(x)$ it will require differentiating three times which will become cumbersome as *n* increases. With this difficulty that may be encountered with higher differentiation especially as n > 2 in $P_n(x)$ of Rodrigues' formula (3.2) above, a simpler formula for generating the

Legendre polynomials is given by its recurrence relation. This is given next.

3.1 Recurrence Formula for Legendre Polynomial

For n = 1 we have $P_2(x) = \left(\frac{3}{2}\right)x.P_1(x) - \left(\frac{1}{2}\right)P_o(x)$

$$P_2(x) = \left(\frac{3}{2}\right)x.x - \left(\frac{1}{2}\right).1 = \frac{3}{2}x^2 - \frac{1}{2}$$

Which is the same as the $P_2(x)$ earlier obtained using the Rodrigues' formula (3.2)

Furthermore, for n = 2, we have

$$P_3(x) = \left(\frac{5}{3}\right)x.P_2(x) - \left(\frac{2}{3}\right)P_1(x)$$

$$P_3(x) = \left(\frac{5}{3}\right)x.\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \left(\frac{2}{3}\right)x$$

$$\Rightarrow \quad p_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Similarly for n = 3 we have

$$P_4(x) = \left(\frac{7}{4}\right)x.P_3(x) - \left(\frac{3}{4}\right)P_2(x)$$

Substituting previous results we have

$$p_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

Also $P_5(x)$ gives $\quad p_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$ etc

The reader may like to generate the Legendre polynomials up to $p_{10}(x)$

One of the properties of the Legendre polynomial is its orthogonality property.

It is known that the Legendre Polynomial $P_n(x)$ satisfies the following property:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & , \; if \;\; m \neq n \\ \frac{2}{2n+1} & , \; if \;\; m = n; \end{cases} \qquad (3.4)$$

This is the orthogonality property which permits it to be a polynomial approximation to any continuous function within its range [-1, 1].

It follows at once from equation (3.4) that $\{P_n(x)\}$ forms an orthogonal, but not normal, set over [-1, 1] with respect to the weight function w(x) = 1 and that the set

$$\{q_n(x)\} = \left\{\sqrt{\frac{2n+1}{2}}\,P_n(x)\right\}$$

forms an orthonormal set.

4.0 CONCLUSION

You can observe that the Legendre Polynomials can be obtained from the Rodrigues' formula but much easier by using the recurrence formula generated from the Legendre differential equation.

5.0 SUMMARY

In this Unit you have learnt

(i) how to use the Rodrigue's formula to generate Legendre polynomials
(ii) how to use recurrence relation as alternative formula to derive the same Legendre polynomials by simple substitution of previously known polynomials
(iii) that the orthogonality property of the Legendre Polynomial permits it to be a polynomial approximation to an continuous function.

6.0 TUTOR MARKED ASSIGNMENT

Obtain the Legendre Polynomials $P_n(x)$ for n = 5, 6, . . . , 10 using both the Rodrigue's formula and the recurrence relation of the Legendre polynomials.

7.0 FURTHER READING AND OTHER RESOURCES

Francis Scheid. (1989) *Schaum's Outlines Numerical Analysis 2nd ed.* McGraw-Hill New York.

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UNIT 3 LEAST SQUARES APPROXIMATION BY LEGENDRE POLYNOMIALS

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1.0 INTRODUCTION

Legendre Polynomials are known to be applicable to least square approximation of functions. In this sense, we mean that we can follow the least square approximation technique and adapt this to Legendre polynomial.

2.0 OBJECTIVE

By the end of this unit, you should be able to:

- apply Legendre polynomial to least squares procedures; and
- obtain least square approximation using Legendre polynomial.

3.0 THE PROCEDURE

Let f(x) be any function defined over [-1, 1] and $L_n(x) = \sum_{k=0}^{n} a_k P_k(x)$

be a linear combination of Legendre polynomials. We shall now determine what values of the coefficients $\{a_k\}$ will make $L_n(x)$ the best approximations in f(x) in the least squares sense over the interval [-1, 1]. Our objective is to minimize

$$I(a_o, a_1, \ldots, a_n) = \int_{-1}^{1} [f(x) - L_n(x)]^2 \, dx \qquad (3.1)$$

and so as in the least squares method, we must set

$$\frac{\partial I}{\partial a_r} = 0 \quad , \quad r = 0, 1, \ldots, n \qquad (3.2)$$

Using equation (3.2) in (3.1), we obtain an equivalent term written as

$$\int_{-1}^{1} P_r(x)\left(f(x) - \sum_{k=0}^{n} a_k P_k(x) \right) dx = 0 \qquad (r = 0, 1, 2, \ldots, n)$$

$$\Rightarrow \quad \int_{-1}^{1} P_r(x) f(x) dx - \sum_{k=0}^{n} a_k \int_{-1}^{1} P_r(x) P_k(x)\, dx = 0$$

Recall from last unit that the Legendre Polynomial $P_n(x)$ satisfies the orthogonality property:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & , \ if \ \ m \neq n \\ \dfrac{2}{2n+1} & , \ if \ \ m = n; \end{cases} \qquad (3.3)$$

when k = r, and by the orthogonality property (3.3) we shall obtain

$$\int_{-1}^{1} P_r(x)f(x)dx = a_r\left(\frac{2}{2r+1}\right)$$

$$\Rightarrow \qquad a_r = \frac{2r+1}{2}\int_{-1}^{+1} f(x).P_r(x)\,dx \qquad (r = 0, 1, 2, \ldots, n) \qquad (3.4)$$

When the coefficients $\{a_r\}$ have been found $L_n(x)$ can be re-arranged as desired, as a polynomial in powers of x, that is,

$$\sum_{k=0}^{n} a_k P_k(x) = \sum_{k=0}^{n} b_k\, x^k$$

which provides a solution to the least squares polynomial approximation problem. The evaluation of the integrals on the right-hand side of (2.9) may have to be done numerically.

The following examples shall be used to illustrate the least squares approximation method using the Legendre polynomial.

3.1 Numerical Experiment

Example 1

Find the fourth degree least squares polynomial to |x| over [-1, 1] by means of Legendre polynomials.

Solution

Let the polynomial be $\sum_{k=0}^{n} a_k P_k(x)$

Then, from equation (2.9)

$$a_r = \frac{2r+1}{2}\int_{-1}^{+1} |x|.P_r(x)\,dx \qquad (r = 0, 1, 2, 3, 4)$$

Hence, $\qquad a_o = \frac{1}{2}\int_{-1}^{+1} x.P_0(x)\,dx = = \frac{1}{2}\int_{-1}^{+1} x.1\,dx = \frac{1}{2}$

$$a_1 = \frac{3}{2}\int_{-1}^{+1} x.P_1(x)\,dx = \frac{3}{2}\int_{-1}^{+1} x.(x)\,dx = 0$$

$$a_2 = \frac{5}{2}\int_{-1}^{+1} x.P_2(x)\,dx = \frac{5}{2}\int_{-1}^{+1} x.(\frac{3x^2-1}{2})\,dx = \frac{5}{2}\int_{0}^{1} (3x^3 - x)\,dx = \frac{5}{8}$$

$$a_3 = \frac{7}{2}\int_{-1}^{+1} x.P_3(x)\,dx = \frac{7}{2}\int_{-1}^{+1} x.(\frac{5x^3-3x}{2})\,dx = 0$$

$$a_4 = \frac{9}{2}\int_{-1}^{+1} x.P_4(x)\,dx = \frac{9}{2}\int_{-1}^{1} x.(\frac{35x^4-30x^2+3}{8})\,dx = -\frac{3}{16}$$

The required polynomial is therefore

$$\frac{1}{2}P_0(x) - \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) \qquad (3.5)$$

The expression (3.5) can be converted to normal polynomial form by substituting the polynomial form of $P_0(x)$, $P_2(x)$ and $P_4(x)$ as given in the last unit. This ends up giving the required polynomial as:

$$\mathbf{I} = \frac{1}{128}\left(15 + 210x^2 - 105x^4\right) \qquad (3.6)$$

Which is therefore the least squares polynomial for |x| over [-1, 1]

Verification

This result may be verified directly by using the least squares method given in the last module.

Now the least squares polynomial is

$$\sum_{k=0}^{4} a_k x^k$$

And by least square method we minimize

$$S = \int_{-1}^{1} \left[|x| - (a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\right]^2\,dx$$

Now, setting the respective partial derivatives to zero by equation (3.2), we shall obtain the normal equations as follows:

$$\frac{\partial S}{\partial a_o} = \int_{-1}^{1} \left[|x| - (a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\right]dx = 0$$

$$\frac{\partial S}{\partial a_1} = \int_{-1}^{1} x\left[|x| - (a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\right]dx = 0$$

$$\frac{\partial S}{\partial a_2} = \int_{-1}^{1} x^2\left[|x| - (a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\right]dx = 0$$

$$\frac{\partial S}{\partial a_3} = \int_{-1}^{1} x^3 \left[|x| - (a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)\right] dx = 0$$

$$\frac{\partial S}{\partial a_4} = \int_{-1}^{1} x^4 \left[|x| - (a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)\right] dx = 0$$

Integrating, we get the following equations:

$$\tfrac{1}{2}x^2 - (a_o x + \tfrac{1}{2}a_1 x^2 + \tfrac{1}{3}a_2 x^3 + \tfrac{1}{4}a_3 x^4 + \tfrac{1}{5}a_4 x^5)\Big|_{-1}^{1} = 0$$

$$\tfrac{1}{3}|x|^3 - (\tfrac{1}{2}a_0 x^2 + \tfrac{1}{3}a_1 x^3 + \tfrac{1}{4}a_2 x^4 + \tfrac{1}{5}a_3 x^5 + \tfrac{1}{6}a_4 x^6)\Big|_{-1}^{1} = 0$$

$$\tfrac{1}{4}x^4 - (\tfrac{1}{3}a_0 x^3 + \tfrac{1}{4}a_1 x^4 + \tfrac{1}{5}a_2 x^5 + \tfrac{1}{6}a_3 x^6 + \tfrac{1}{7}a_4 x^7)\Big|_{-1}^{1} = 0$$

$$\tfrac{1}{5}|x|^5 - (\tfrac{1}{4}a_0 x^4 + \tfrac{1}{5}a_1 x^5 + \tfrac{1}{6}a_2 x^6 + \tfrac{1}{7}a_3 x^7 + \tfrac{1}{8}a_4 x^8)\Big|_{-1}^{1} = 0$$

$$\tfrac{1}{6}|x|^6 - (\tfrac{1}{5}a_0 x^5 + \tfrac{1}{6}a_1 x^6 + \tfrac{1}{7}a_2 x^7 + \tfrac{1}{8}a_3 x^8 + \tfrac{1}{9}a_4 x^9)\Big|_{-1}^{1} = 0$$

Evaluating within the limits we obtain the following equations

$$a_0 + \frac{1}{3}a_2 + \frac{1}{5}a_4 = \frac{1}{2}$$

$$\frac{1}{3}a_1 + \frac{1}{5}a_3 = 0$$

$$\frac{1}{3}a_0 + \frac{1}{5}a_2 + \frac{1}{7}a_4 = \frac{1}{4}$$

$$\frac{1}{5}a_1 + \frac{1}{7}a_3 = 0$$

$$\frac{1}{5}a_0 + \frac{1}{7}a_2 + \frac{1}{9}a_4 = \frac{1}{6}$$

Solving these simultaneously, we deduce at once that $a_1 = a_3 = 0$ and that

$$a_o = \frac{15}{128}, \quad a_2 = \frac{210}{128}, \quad a_4 = \frac{105}{128}$$

In agreement with coefficients of equation (3.6)

Example 2

Given a continuous function e^x for $x \in [-1,1]$ fit a linear polynomial $c_0 + c_1 x$ to e^x and determine its root mean square error

Solution

Using Equation (3.1) we have

$$S = \int_{-1}^{1} [f(x) - P(x)]^2 \, dx$$

$$S = \int_{-1}^{1} \left[e^x - c_o - c_1 x\right]^2 \, dx$$

For $f(x) = e^x$, we can write the linear polynomial as

$$P(x) = a_o + a_1 x$$

By using equation (3.4) we have

$$a_o = \frac{1}{2}\int_{-1}^{1} e^x .1\, dx = \frac{1}{2} e^x \Big|_{-1}^{1} = 1.1752$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} x.e^x \, dx = \frac{3}{2}(xe^x - e^x) \Big|_{-1}^{1} = 1.1036$$

Therefore the linear polynomial is

P(x) = 1.1752 + 1.1036x

An average error of approximating f(x) by P(x) on the interval [a, b] is given by

$$E = \frac{1}{\sqrt{b-a}}\sqrt{\int_a^b |f(x)-P(x)|^2\, dx}$$

$$= \frac{\|f(x)-P(x)\|}{\sqrt{b-a}} \qquad (3.7)$$

Hence by (3.7), the least square approximation will give a small error on [a, b]. The quantity E is called the root mean square error in the approximation of f(x)

Now since $E = \frac{\|f(x)-P(x)\|}{\sqrt{b-a}}$

We can evaluate E using any of the k-, l- , or m- norm

Using the l-norm, we write

$$E = \frac{\left\| e^x - (1.1752 + 1.1036x) \right\|_l}{\sqrt{1+1}}$$

$$= \frac{\max\limits_{-1<x<1} \left| e^x \right| - \max\limits_{-1<x<1} \left|1.1752 + 1.1036x\right|}{\sqrt{2}}$$

$$= \frac{\|2.71828 - 2.2788\|}{\sqrt{2}} = 0.3108$$

Hence the error is as large as 0.3108. If higher approximating polynomial is used the error will be smaller that this.

4.0 CONCLUSION

The use of Legendre polynomial as a technique for least square approximation shows that the same result is achievable from the least square approximation method as well as the Legendre Polynomial approach.

5.0 SUMMARY

In this Unit you have learnt

(i) the technique of using Legendre polynomial to obtain and approximation using the least square method.

(ii) that both Legendre approach and the Least squares approach will often produce the same result.

6.0 TUTOR MARKED ASSIGNMENT

Obtain a fourth degree least squares polynomial for $f(x) = \frac{1}{|x|}$ over [-1, 1] by means of Legendre polynomials.

7.0 FURTHER READING AND OTHER RESOURCES

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UNIT 4: THE CHEBYSHEV POLYNOMIALS

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 3.2 Properties of Chebyshev Polynomials
 3.3 Derivation of the Recurrence Formula
4.0 Conclusion
5.0 Summary
6.0 Tutor Marked Assignment
7.0 Further Reading And Other Resources

1.0 INTRODUCTION

It is always possible to approximate a continuous function with arbitrary precision by a polynomial of sufficient high degree. One of such approach is by using the Taylor series method. However, the Taylor series approximation of a continuous function *f* is often not so accurate in

the approximation of *f* over an interval [*a*, *b*]. If the approximation is to be uniformly accurate over the entire interval. This may be due to the fact that:

(i) in some cases, the Taylor series may either converge too slowly or not at all.
(ii) the function may not be analytic or if it is analytic the radius of convergence of the Taylor series may be too small to cover comfortably the desired interval.

In addition, the accuracy of the Taylor series depends greatly on the number of terms contained in the series. However, a process that was based on the fundamental property of Chebyshev polynomial may be considered as alternative and it works uniformly over any given interval. We know that there are several special functions used for different purposes including approximation, polynomial fittings and solutions of differential equations. Some of these special functions include Gamma, Beta, Chebyshev, Hermite, Legendre, Laguerre and so on. However, not all these are good polynomial approximation to continuous functions.

However, Chebyshev polynomials have been proved to be very useful in providing good approximation to any continuous function.

To this end, the Chebyshev polynomial is usually preferable as polynomial approximation. The Chebyshev polynomial has equal error property and it oscillates between –1 and 1. Due to its symmetric property, a shifted form of the polynomial to half the range (0, 1) is also possible.

2.0 OBJECTIVE

By the end of this unit, you should be able to:

- state the necessary formulae for generating the Chebyshev polynomials;
- obtain Chebyshev polynomials $T_n(x)$ up to n = 10; and
- classify Chebyshev polynomial as a family of orthogonal series.

3.0 INTRODUCTION TO CHEBYSHEV POLYNOMIALS

As it was earlier stated, Chebyshev polynomials are often useful in approximating some functions. For this reason we shall examine the nature, properties and efficiency of the Chebyshev polynomial.

Chebyshev Polynomial is based on the function "cos$n\theta$" which is a polynomial of degree n in cosθ. Thus we give the following basic definition of the Chebyshev polynomial.

Definition 1
The Chebyshev polynomial is defined in terms of cosine function as

$$T_n(x) = \cos(n\,.\cos^{-1} x) \text{ for } -1 \le x \le 1, n \ge 0 \qquad (3.1)$$

This definition can be translated to polynomials of x as it would be discussed very soon. Before we do this, if we put x = cosθ , the Chebyshev polynomial defined above becomes
$T_n(x) = \cos(n\theta)$

$T_n(x)$ is of the orthogonal family of polynomials of degree n and it has a weighting function

$$w(x) = \frac{1}{\sqrt{1-x^2}} \quad , \qquad 1 \le x \le 1$$

It has an oscillatory property that in $0 \le \theta \le \pi$ the function has alternating equal maximum and minimum values of ± 1 at the n+1 points. Thus the orthogonality relation of the Chebyshev polynomial is given as:

$$\theta_r = \frac{r\pi}{n}, \qquad r = 0, 1, 2, \ldots, n$$

or $$x_r = \cos\left(\frac{r\pi}{n}\right) \quad , \quad r = 0, 1, 2, \ldots, n$$

Thus the orthogonality relation of the Chebyshev polynomial is given as:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\,dx = \begin{cases} 0 \ , & n \ne m \\ \pi \ , & n = m = 0 \\ \frac{1}{2}\pi \ , & n = m \ne 0 \end{cases} \qquad (3.2)$$

It also has a symmetric property given by

$$T_n(-x) = (-1)^n T_n(x) \qquad (3.3)$$

3.1 Generating Chebyshev Polynomials

Over the years the function $T_n(x)$ is the best polynomial approximation function known for f(x). In order to express $T_n(x)$ in terms of polynomials the definition can be used to some extent, but as n value increases, it becomes more difficult to obtain the actual polynomial except by some trigonometric identities, techniques and skill.
For the reason, a simpler way of generating the Chebyshev polynomials is by using the **recurrence formula** for $T_n(x)$ in [-1, 1].

The recurrence formula for generating the Chebyshev polynomial $T_n(x)$ in [-1, 1] is given by

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)\ ,\quad n\geq 1 \qquad (3.4)$$

Thus to obtain the Chebyshev polynomials, a combination of (3.1) and (3.4) can be used. Starting with the definition (3.1), that is

$$T_n(x)=\cos(n.\cos^{-1}x)$$

We obtain the least polynomial when n = 0 as

$$T_0(x)=\cos 0=1$$

Also when n = 1, we get $T_1(x)=\cos(\cos^{-1}x)=x$

When n = 2, $T_2(x)=\cos(2\cos^{-1}x)$

with x = cosθ

$$T_2(x)=\cos 2\theta$$
$$=2\cos^2\theta-1$$
$$=2x^2-1$$

For n = 3, 4, . . . it will be getting more difficult to obtain the polynomials. However if we use the recurrence formula (3.4) , we can obtain $T_2(x)$ by putting n = 1 so that

$$T_2(x)=2xT_1(x)-T_0(x)$$

Substituting $T_0(x)=1$, $T_1(x)=x$, (from the result earlier obtained), we have

$$T_2(x)=2x.(x)-1\ =2x^2-1$$

This is simpler than using the trigonometric identity.
Thus for n = 2, 3, . . . we obtain the next few polynomials as follows:
When n = 2, the recurrence formula gives

$$T_3(x)=2xT_2(x)-T_1(x)$$
$$=2x(2x^2-1)-x$$
$$=4x^3-3x$$

Similarly for n = 3, we obtain

$$T_4(x)=2xT_3(x)-T_2(x)$$
$$=2x(4x^3-3x)-(2x^2-1)$$
$$=8x^4-8x^2+1$$

In a similar manner

$T_5(x) = 16x^5 - 20x^3 + 5x$

We can now write all these polynomials out for us to see the pattern which they form.

$$\left.\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \end{aligned}\right\} \qquad (3.5)$$

You can now derive the next few ones, say up to $T_{10}(x)$, following the same technique.

Note that the recurrence formula is one step higher than the definition for the "n" value being used. In other words, when n = 2 in the definition we obtain $T_2(x)$, whereas to get the same $T_2(x)$ from the recurrence formula we use n = 1. The reason is obvious; the recurrence formula starts with subscript "n+1" as against "n" in the definition.
These polynomials are of great importance in approximation theory and in solving differential equations by numerical techniques.

3.2 Properties of Chebyshev Polynomials

In the interval $-1 \le x \le 1$ the Chebyshev Polynomial $T_n(x)$ satisfies the following properties:

(i.) $-1 \le T_n(x) \le +1$

(ii.) $T_n(x) = 1$ at (n + 1) points $x_0, x_1, \ldots, x_n$, where $x_r = \cos\left(\frac{r\pi}{n}\right)$, $r = 0, 1, 2, \ldots, n$

(iii.) $T_n(x) = (-1)^n$

(iv.) The leading coefficient in $T_n(x)$ is 2^{n-1}.

3.3 Derivation of the Recurrence Formula

Now that we have seen the usefulness of the recurrence formula (2.16), it might be necessary for us to derive this formula from certain definition. There are two ways to this. We can use some trigonometric functions to get this since Chebyshev polynomial is defined as a cosine function. However, we can also derive this formula by solving it as a **difference equation** which can be shown to produce the definition (2.13). For the purpose of this course, since we are not treating linear difference equation, we shall go via the first type, by using some trigonometric functions.

Equation (2.16) is given by

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $n \ge 1$

To obtain this formula, we can recall from trigonometric knowledge that

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

If we put $A = (n+1)\arccos x$ and $B = (n-1)\arccos x$

Then $\cos A + \cos B = \cos\{(n+1)\arccos x\} + \cos\{(n-1)\arccos x\}$

$$= 2\cos\frac{1}{2}[(n+1+n-1)\arccos x].\cos\frac{1}{2}[(n+1-n+1)\arccos x]$$

$$= 2\cos\frac{1}{2}[(2n)\arccos x].\cos\frac{1}{2}(2\arccos x)$$

$$= 2\cos(n\arccos x)\,.\,\cos(\arccos x)$$

$$\cos A + \cos B = 2\cos(n\arccos x).\,x$$

$$\cos A = 2x\cos(n\arccos x) - \cos B$$

That is $\cos[(n+1)\arccos x] = 2x\cos[n\arccos x] - \cos[(n-1)\arccos x]$

By definition, $T_n(x) = \cos(n.\cos^{-1}x)$,

we then have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Thus the recurrence formula is easily established.

4.0 CONCLUSION

The derivation of Chebyshev polynomials has been demonstrated and made simple by using the recurrence formula rather than using the basic definition (3.1). we have equally given the derivation of the recurrence formula by simply using some trigonometry identities, although this derivation can be established by solving the recurrence formula as a difference equation from which the basic definition (3.1) is obtained. Other methods of derivation equally exist.

5.0 SUMMARY

In this Unit we have learnt that:

(i) Chebyshev polynomials are special kind of polynomials that satisfy some properties

(ii) Chebyshev polynomials which are valid within [-1, 1] have either odd indices or even indices for $T_n(x)$ depending on whether n is odd or even.

(iii) Chebyshev polynomials can be obtained from the recurrence formula.

(iv) the recurrence formula for Chebyshev polynomials $T_n(x)$ is more suitable to generate the polynomials than its definition.

6.0 TUTOR MARKED ASSIGNMENT

Obtain the Chebyshev polynomials $T_n(x)$ for n = 5, 6, . . . , 10

7.0 FURTHER READING AND OTHER RESOURCES

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UNIT 5: SERIES OF CHEBYSHEV POLYNOMIALS

CONTENT

1.0 Introduction
2.0 Objective
3.0 Approximation by Chebyshev Polynomials
 3.1 Numerical Examples
4.0 Conclusion
5.0 Summary
6.0 Tutor Marked Assignment
7.0 Further Reading and other Resources

1.0 INTRODUCTION

Chebyshev polynomials can be used to make some polynomial approximations as against the use of least square method. The orthogonality properties of the Chebyshev polynomial permit the use of the polynomial as approximation to some functions. A case of cubic approximation will be considered in this study.

2.0 OBJECTIVE

By the end of this unit you should have learned:

- the form of the function f(x) which permits the use of Chebyshev polynomials as approximation to it; and
- how to apply Chebyshev polynomials to fit a cubic approximation to a function f(x).

3.0 APPROXIMATION BY CHEBYSHEV POLYNOMIALS

If we have a function f(x) which we wish to approximate with a series of Chebyshev polynomials

$$f(x) = \tfrac{1}{2}c_o + c_1T_1(x) + c_2T_2(x) + \dots + c_nT_n(x) \qquad (3.1)$$

How we can find the coefficients c_i?

The theoretical method is to multiply f(x) by $\dfrac{T_m(x)}{\sqrt{1-x^2}}$ and integrate over [-1, 1], thereby making use of the orthogonality property of $T_n(x)$. Thus, if we multiply both sides by this factor and integrate over [-1, 1], we can write

$$\int_{-1}^{1}\frac{f(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \tfrac{1}{2}c_o\int_{-1}^{1}\frac{T_m(x)}{\sqrt{1-x^2}}\,dx + \sum_{m=1}^{n} c_m\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx$$

The only term on the right which doesn't vanish is the one where $m = n \neq 0$. In other words if we use the orthogonality property given by equation (3.2) of the last unit, we have

$$\int_{-1}^{1} \frac{f(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \tfrac{1}{2}\pi\, c_m$$

$$c_m = \tfrac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_m(x)}{\sqrt{1-x^2}}\,dx \qquad (3.2)$$

The evaluation of the integral for *cm* given by (3.2) will in general have to be done numerically and in such cases it is obviously important to ensure that the truncation error is sufficiently small or the accuracy available via the Chebyshev approximation to f(x) will be reduced. In a few special cases, the integral can be evaluated analytically and the problem of truncation error does not arise; the most important of such case is when $f(x) = x^n$ $(n \geq 0)$ and we shall deal with this case below; but first we look at an example where evaluation of (3.2) is computed numerically.

3.1 Numerical Examples

Example 1

Find a cubic approximation to e^x by using Chebyshev polynomials

Solution

Let the approximation be

$$e^x = \tfrac{1}{2}c_o + c_1T_1(x) + c_2T_2(x) + \ldots + c_nT_n(x)$$

Then, from (2.17)

$$c_r = \frac{2}{\pi}\int_{-1}^{1} \frac{e^x\,T_r(x)}{\sqrt{1-x^2}}\,dx \qquad (r = 0, 1, 2, 3)$$

Using the substitution $x = \cos\theta$, we transform this integral as follows:

$$x = \cos\theta \quad \Rightarrow \quad dx = -\sin\theta\, d\theta = -\sqrt{1-\cos^2\theta}\, d\theta = -\sqrt{1-x^2}\, d\theta$$

when $x = 1$, $\theta = 0$ and when $x = -1$, $\theta = \pi$

Substituting into the integral above, we have

$$c_r = \frac{2}{\pi}\int_{\pi}^{0} \frac{e^{\cos\theta}\cos(r\theta)}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)d\theta$$

Canceling out the common terms and reversing the limits which eliminates the (-) sign we obtain

$$c_r = \frac{2}{\pi}\int_{0}^{\pi} e^{\cos\theta}\cos(r\theta)\, d\theta \qquad (3.3)$$

This is better from a numerical point of view since the integrand no longer contains a singularity. In evaluating integrals containing a periodic function as a factor in the integrand it is usually best to make use of the simplest quadrature formulae, such as the midpoint rule, Simpson rule or trapezium rule. By using any of these methods the coefficients cj can be evaluated for a series of decreasing step-sizes and the results compared. This will establish some confidence in the accuracy of the results. Thus using the trapezoidal (or simply trapezium) rule with step-sizes $\pi/2^k$ $(k = 1,2,3,4)$

$$f(x) = \frac{h}{2}(y_o + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n)$$

where h is the step size. From equation (3.3), we obtain the following estimates for c_0

$$c_o = \frac{2}{\pi}\int_0^{\pi} e^{\cos\theta}\, d\theta$$

With k = 1 we have $h = \frac{\pi}{2}$, and for interval (0, π) we have three points $0, \frac{\pi}{2}, \pi$

Thus we take $y = e^{\cos\theta}$

x	0	$\frac{\pi}{2}$	π
y	e	1	e^{-1}

This integral by trapezium rule will give

$$f(x) = \frac{h}{2}(y_o + 2y_1 + y_2)$$
$$= \frac{2}{\pi}\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\left[e + 2(1) + e^{-1}\right]$$
$$= 2.543081$$

With k = 2, we have $h = \frac{\pi}{4}$, and for interval (0, π) we have five points $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	2.718282	2.028115	1	0.493069	0.367879

$$f(x) = \frac{h}{2}(y_o + 2y_1 + 2y_2 + 2y_3 + y_4)$$
$$= \frac{2}{\pi}\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)[2.718282 + 2(2.028115 + 1 + 0.493069) + 0.367879]$$
$$= 2.532132$$

K	Estimate	
1	2.543081	(6 d.p)
2	2.532132	(6 d.p)
3	2.53213176	(8 d.p)
4	2.53213176	(8 d.p)

And we conclude that $c_0 = 2.53213176$ to 8d.p

The other coefficients are evaluated similarly and we find (to 8 d.p)

$c_1 = 1.13031821$, $c_2 = 0.27149534$, $c_3 = 0.04433685$

So that the required approximation is

$$e^x \cong 1.26606588T_o(x) + 1.13031821T_1(x) + 0.27149534T_2(x) + 0.04433685T_3(x) \qquad (3.4)$$

It is not necessary to re-order (3.4) in powers of x for this formula may be used directly for the computation of approximations to e^x by using the Chebyshev polynomials $T_n(x)$ earlier obtained in the last unit. Thus, taking x= 0.8 for an example, we have

$T_o(0.8) = 1$, $\quad T_1(0.8) = 0.8$

Also

$$T_2(0.8) = 2(0.8)(0.8) - 1 = 0.28$$

and $T_3(0.8) = 2(0.8)(0.28) - 0.8 = -0.352$

and equation (2.19) then gives rounded to (4d.p)

$$e^{0.8} \cong 2.2307$$

The correct value to 4d.p is 2.2255

By comparison the cubic approximation obtained by truncating the Taylor series for e^x after 4 terms gives

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

$$= 1 + 0.8 + \frac{1}{2}(0.8)^2 + \frac{1}{6}(0.8)^3 = 2.2053$$

When we consider the errors in the two approximations we note that the error from the Chebyshev approximation is

$E_{Cheby} = |2.2255 - 2.2307| = 0.00052$

While that of the Taylor series is

$E_{Tay} = |2.2255 - 2.2307| = 0.0202$

The error of the Taylor series is almost 4 times as large as that of Chebyshev approximation. For small values of x however the Taylor series cubic will give better results e.g. at x = 0.2, The Chebyshev series gives $e^{0.2} = 1.2172$ (4 dp)
While the Taylor series cubic gives $e^{0.2} = 1.2213$ and in fact the exact value is
$e^{0.2} = 1.2214$ which illustrates the point that Chebyshev approximations do not necessarily produce the best approximations at any given point in the interval [-1, 1] but they do guarantee to minimize the greatest error in the interval.

In general it frequently happens that several approximation formulae are available and each will have its own advantages and disadvantages. In particular, different formulae may give the best results over different parts of the interval of approximation and it may require considerable analysis to decide which to use at any point.

We now consider the special case when $f(x) = x^n$ ($n \geq 0$). The importance of this case lies in its role in the method of economization. It is possible to express the Chebyshev the term x^n , n = 1, 2, 3, … , in terms of $T_n(x)$. These Chebyshev representations for x_n are easily obtained by solving the Chebyshev polynomials successively as follows:

$T_0(x) = 1$ hence $x^0 = 1 = T_0(x)$

$T_1(x) = x$ hence $x = T_1(x)$

$T_2(x) = 2x^2 - 1 = 2x^2 - T_0(x)$, hence $x^2 = ½ [T_2(x) + T_0(x)]$

$T_3(x) = 4x^3 - 3x = 4x^3 - 3T_1(x)$, hence $x^3 = ¼ [T_3(x) + 3T_0(x)]$

Similarly, $x^4 = \frac{1}{8}(T_4(x) + 4T_2(x) + 3T_0(x))$

and so on, Higher powers of x can equally obtained in terms of Tn(x) and the learner is encouraged to obtain as far as x^8 as an exercise.

Now, since we can express x^k as a linear combination of Tk(x), Tk – 1(x),…,T0(x) we can as well any power series expansion of an arbitrary function f(x) into an expansion in a series of Chebyshev polynomials. An example is given next.

Example 2
Convert the first 5 terms of the Taylor series expansions for e^x into Chebyshev polynomials

Solution

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

$$= T_0(x) + T_1(x) + \frac{1}{4}(T_2(x) + T_0(x)) + \frac{1}{24}(T_3(x) + 3T_1(x)) + \frac{1}{192}(T_4(x) + 4T_2(x) + 3T_0(x))$$

$$= (1 + \frac{1}{4} + \frac{1}{64})T_0(x) + (1 + \frac{1}{8})T_1(x) + (\frac{1}{4} + \frac{1}{48})T_2(x) + \frac{1}{24}T_3(x) + \frac{1}{192}T_4(x)$$

$$e^x = \frac{81}{64}T_0(x) + \frac{9}{8}T_1(x) + \frac{13}{48}T_2(x) + \frac{1}{24}T_3(x) + \frac{1}{192}T_4(x)$$

If we truncate this result after the term $T_3(x)$ we shall obtain

$$e^x = \frac{81}{64}T_0(x) + \frac{9}{8}T_1(x) + \frac{13}{48}T_2(x) + \frac{1}{24}T_3(x) \qquad (3.4)$$

with the principal error as $\frac{1}{192}T_4(x) + \ldots$

This approximation can as well be regarded as the cubic expansion for e^x.
If we convert the coefficients of equation (3.3) to decimal form we have

$$e^x \cong 1.26562500T_0(x) + 1.125T_1(x) + 0.2708333T_2(x) + 0.041667T_3(x) \qquad (3.5)$$

Thus we can compare equations (3.4) and (3.5) since both are cubic approximations to e^x .obtained by the use of Chebyshev polynomials. The coefficients from the two equations are in the table below.

	$T_0(x)$	$T_1(x)$	$T_2(x)$	$T_3(x)$
Equation (3.4)	1.26606588	1.13031821	0.27149534	0.04433685
Equation (3.5)	1.26562500	1.12500000	0.27083333	0.04166667

Since both cubic approximations provide some kind of good approximations to e^x we would expect them to have similar coefficients but they are not identical because equation (3.4) is the approximation to e^x using the first 4 Chebyshev polynomials whereas equation (3.5) is based upon the Chebyshev equivalent of the first 5 terms of the Taylor series for e^x **'economized'** to a cubic

4.0 CONCLUSION

It would be observed that as it was done with Legendre polynomials we have similarly obtained an approximate functions to f(x) using the Chebyshev polynomials.
The technique of economization is a very useful one and can lead to significant improvements in the accuracy obtainable from a polynomial approximation to a power series. In the next section we present the technique in the general case and in passing see how (3.5) may be more easily obtained.

5.0 SUMMARY

In this Unit the reader have learnt that:

(i.) Chebyshev polynomials is a technique for approximation using the least square technique.

(ii.) Chebyshev polynomial approach to fitting of approximation to a function is similar to that of Taylor series for the same function.

6.0 TUTOR MARKED ASSIGNMENT

(1) Obtain a cubic polynomial to $f(x) = 1/x$ over $[-1, 1]$ by means of Chebyshev polynomials.

(2) Convert the first 5 terms of the Taylor series expansions for e^{-x} into Chebyshev polynomials

7.0 FURTHER READING AND OTHER RESOURCES

Abramowitz M., Stegun I. (eds), (1964): Handbook of Mathematical functions, Dover, N. Y.

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UNIT 6: CHEBYSHEV INTERPOLATION

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2.0 Objective
  3.0 Interpolation Technique
  3.1 Numerical Example
4.0 Conclusion
5.0 Summary
6.0 Tutor Marked Assignment
7.0 Further Reading and other Resources

1.0 INTRODUCTION

Often we use Lagrange's methods to interpolate some set of points defined by f(x). The technique is interesting when we involve the use of Chebyshev polynomials. The approach will be discussed in this unit with emphasis on terms such as Lagrange and Chebyshev polynomials.

2.0 OBJECTIVE

By the end of this unit the learner you should be able to:

- use Lagrange's formula;
- interpolate using Chebyshev polynomials; and
- Compute the error table from the approximation

3.1 INTERPOLATION TECHNIQUE

If the values of a function f(x) are known at a set of points $x_1 < x_2 < \ldots < x_n$ we can construct a polynomial of degree (n – 1) which takes the values $f(x_i)$ at x_i $(i = 1,2,\ldots,n)$. The polynomial is unique and can be found in various ways including the use of Newton's divided difference formula or Lagrange's method. Lagrange's formula is more cumbersome to use in practice but it has the advantage that we can write down the required polynomial explicitly as:

$$p(x) = \sum_{j=1}^{n} f(x_j) \prod_{\substack{i=1 \\ j \neq i}}^{n} \frac{x - x_i}{x_j - x_i} \qquad (3.1)$$

The reader should note that $\prod_{i=1}^{n}(x - x_i)$ is a product of function $(x - x_i)$, i = 1, 2, . . ., n, just as $\sum_{i=1}^{n}(x - x_i)$ is a summation function

Thus $\prod_{i=1}^{n}(x - x_i)$ is evaluated or expanded as:

$$\prod_{i=1}^{n}(x - x_i) = (x - x_1)(x - x_2) \ldots (x - x_n)$$

If f(x) is not a polynomial of degree ≤ (n – 1) the error when we use p(x) for interpolation can be shown to be

$$E(x) = \prod_{i=1}^{n}(x - x_i) \frac{f^{(n)}(\alpha)}{n!}$$

Where α is some number between x_1 and x_n. If the values x_1, x_2,…x_n have been fixed we can do nothing to minimize E(x) but if we can choose any n points within a specified interval it may be worthwhile choosing them in a particular way as we now show

Suppose, for simplicity, that we are interested in values of x lying in the interval $-1 \leq x \leq 1$ and that we are free to choose any n points x_1, …,x_n in this interval for use in the interpolation formula (3.1). Now

$$\prod_{i=1}^{n}(x - x_i)$$

is a polynomial of degree *n* with leading coefficient 1 and of all such polynomials the one with the minimum maximum value is $2^{-(n-1)}$. It follows therefore that if we wish to minimize (3.1) we should choose the xi so that

$$\prod_{i=1}^{n}(x - x_i) = 2^{-(n-1)} T_n(x) = \frac{T_n(x)}{2^{n-1}}$$

And this is equivalent to saying that we should choose x_1, x_2…,x_n to be the n roots of $T_n(x)$, that is, we should take

$$x_m = \cos\left(\frac{2m-1}{2n}\right)\pi \quad , \quad (m = 1, 2, \ldots, n) \qquad (3.2)$$

The main disadvantage of Chebyshev interpolation is the need to use the special values of x_i given by (3.2) rather than integral multiples of a step (such as 0.1, 0.2, . . . ,etc). The values, however, are easy to compute for a given n.

3.1 Numerical Example

Example 1

Use Chebyshev interpolation to find a cubic polynomial approximation to $(1+x)^2$ over [-1, 1]

Solution

For a cubic polynomial approximation, we need four interpolating points. Hence, the four Chebyshev interpolation points from equation (3.2) are

$$x_1 = \cos(\tfrac{\pi}{8}), \quad x_2 = \cos(\tfrac{3\pi}{8}), \quad x_3 = \cos(\tfrac{5\pi}{8}), \quad x_4 = \cos(\tfrac{7\pi}{8})$$

and these values are $x_1 = 0.92388$, $x_2 = 0.382683$, $x_3 = -0.382683$, $x_4 = -0.92388$

We note that $x_3 = -x_2$ and $x_4 = -x_1$. The cubic can therefore be simplified by combining terms involving (x_1 and x_4) and (x_2 and x_3). Thus, from equation (3.1) we shall obtain

$$p(x) = \sum_{j=1}^{4} f(x_j) \prod_{\substack{i=1 \\ j\neq i}}^{4} \frac{x - x_i}{x_j - x_i}$$

$$= f(x_1)\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + f(x_2)\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$
$$+ f(x_3)\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + f(x_4)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

But $x_3 = -x_2$ and $x_4 = -x_1$, using this we get

$$P(x) = f(x_1)\frac{(x-x_2)(x+x_2)(x+x_1)}{(x_1-x_2)(x_1+x_2)(x_1+x_1)} + f(x_2)\frac{(x-x_1)(x+x_2)(x+x_1)}{(x_2-x_1)(x_2+x_2)(x_2+x_1)}$$
$$+ f(-x_2)\frac{(x-x_1)(x-x_2)(x+x_1)}{(-x_2-x_1)(-x_2-x_2)(-x_2+x_1)} + f(-x_1)\frac{(x-x_1)(x-x_2)(x+x_2)}{(-x_1-x_1)(-x_1-x_2)(-x_1+x_2)}$$

Now putting $f(x) = (1+x)^{\frac{1}{2}}$, we have

$$P(x) = (1+x_1)^{1/2}\frac{(x^2-x_2^2)(x+x_1)}{(x_1^2-x_2^2)(2x_1)} + (1+x_1)^{1/2}\frac{(x^2-x_1^2)(x+x_2)}{(x_2^2-x_1^2)(2x_2)}$$
$$+ (1-x_2)^{1/2}\frac{(x^2-x_1^2)(x-x_2)}{(x_2^2-x_1^2)(-2x_2)} + (1-x_2)^{1/2}\frac{(x^2-x_2^2)(x-x_1)}{(x_1^2-x_2^2)(-2x_1)}$$

$$P(x) = (1.92388)^{1/2}\frac{(x^2-0.382683^2)(x+0.92388)}{(0.92388^2-0.382683^2)(2\times 0.92388)}$$

$$+(1.92388)^{1/2}\frac{(x^2-0.92388^2)(x+0.382683)}{(0.382683^2-0.92388^2)(2\times 0.382683)}$$

$$+(1-0.382683)^{1/2}\frac{(x^2-0.92388^2)(x-0.382683)}{(0.382683^2-0.92388^2)(-2\times 0.382683)}$$

$$+(1-0.382683)^{1/2}\frac{(x^2-0.382683^2)(x-0.92388)}{(0.92388^2-0.382683^2)(-2\times 0.92388)}$$

$$P(x) = (1.38702)\times\frac{(x^2-0.146446)(x+0.92388)}{(0.853554-0.146446)(1.84776)}$$

$$+(1.38702)\times\frac{(x^2-0.853554)(x+0.382683)}{(0.146446-0.853554)(0.765366)}$$

$$+(0.785695)\times\frac{(x^2-0.853554)(x-0.382683)}{(0.146446-0.853554)(-0.765366)}$$

$$+(0.785695)\times\frac{(x^2-0.146446)(x-0.92388)}{(0.853554-0.146446)(-1.84776)}$$

$$P(x) = (1.0615769)\times(x^2-0.146446)(x+0.92388)$$

$$+(-2.5628773)\times(x^2-0.853554)(x+0.382683)$$

$$+(2.6383371)\times(x^2-0.853554)(x-0.382683)$$

$$+(-0.6013436)\times(x^2-0.146446)(x-0.92388)$$

Opening the brackets and simplifying we shall obtain the cubic polynomial approximation (to 5 decimal places) as:

$$P(x) = 1.01171 + 0.49084x + 0.21116x^2 + 0.12947x^3 \qquad (3.3)$$

Comparison of P(x) in equation (3.3) with $f(x) = (1+x)^{\frac{1}{2}}$ at x = – ½ (¼) ½ with the absolute error E = | f(x) – P(x) | is given in Table 1 below:

Table 1

x	– 0.5	– 0.25	0	0.25	0.50
P(x)	0.69732	0.87378	1.01171	1.12325	1.22052
$f(x) = (1+x)^{\frac{1}{2}}$	0.70711	0.86603	1.00000	1.11803	1.22475
Abs. Error E	0.00979	0.00775	0.00171	0.00522	0.00423

The above table displays the accuracy of the Chebyshev approximation to the given example.

4.0 CONCLUSION

We have been able to demonstrate the use of Lagrange's method in our interpolation technique. We have also seen that Chebyshev polynomials are of great usefulness in the interpolation of simple functions.

5.0 SUMMARY

In this Unit you have learnt that:

(i.) interpolation technique is possible by using Chebyshev polynomials.

(ii.) Lagrange's method of interpolating is basic and very useful.

(ii) the difference of the actual and the approximate value is the error

6.0 TUTOR MARKED ASSIGNMENT

Use Chebyshev interpolation technique to find a cubic polynomial approximation to $(1-x)^{2}$ over [-1, 1]

7.0 Further Reading and Other Resources

Atkinson K. E. (1978). An Introduction to Numerical Analysis, 2nd Edition, John Wiley & Sons, N. Y

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