MODULE 3 FURTHER INTERPOLATION TECHNIQUES

UNIT 1 CUBIC SPLINE INTERPOLATION

CONTENTS

1.0 INTRODUCTION

One of the problems which frequently arises when we try to approximate a function by means of a polynomial of high degree is that the polynomial turns out to have closed placed maxima and minima, thus giving it an undulatory (or 'wiggly') character. This is very undesirable if the polynomial is to be used for interpolation, and disastrous if it is to be used for numerical differentiation.

In 1945, I. J. Schoenberg introduced the idea of approximation to functions by means of a series of polynomials over adjacent intervals with continuous derivatives at the end-points of the intervals. Such a set of polynomials he called 'splines', pointing out that architects and designers had been using mechanical devices of this kind for years. In his paper Schoenberg explains. A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic materials. The spline is placed on the sheet of graph paper and held in place at various points..

The mathematical equivalent of this 'flexible bar' is the cubic spline which has proved to be extremely useful for interpolation, numerical differentiation and integration and has been subject of many research papers.

2.0 OBJECTIVE

At the end of this unit, the learner should be able to:

- define a cubic spline;
- derive a method of fitting a cubic spline;
- fit a cubic spline to set of data points;
- interpolate a function from the fitted cubic spline; and
- find the error in the cubic spline.

3.0 MAIN CONTENT

To be able to understand this module we shall begin the discussion by defining what is meant by splines of degree k and then develop the special case of the cubic (k = 3)

Definition 1

A spline function S(x) of degree k with n nodes, $x_1 < x_2 < \ldots < x_n$ has the properties

(i.) S(x) is given in the interval $[x_i , x_{i+1}]$, i = 0, 1, . . . , n (where $x_o = - \infty$, $x_{n+1} = \infty$) by a polynomial of degree at most k (in general, a different polynomial is obtained in each interval);

(ii.) S(x) and all derivatives of orders 1, 2, …, k-1 are continuous on $(- \infty , \infty)$

In the case k = 3, the polynomials in each of the intervals are at most cubics and their first and second derivatives are continuous at the end points of the intervals. Such a set of polynomials form a cubic spline.

We can narrow down this definition to a cubic spline S(x) as follows:

Definition 2

A Cubic Spline S(x) is a function which satisfies the following properties S(x) is a polynomial of degree one for $x < x_0$ and $x > x_n$

S(x) is at most a cubic polynomial in (x_i, x_{i+1}) , i = 0, 1, 2,. . . , n - 1;
S(x) , Sc(x) and Ss(x) are continuous at each point (x_i , y_i) , i = 0, 1, 2,. . . , n; $S(x_i) = y_i$, i = 0, 1, 2,. . . , n.

3.1 Derivation of Cubic Spline

How can we construct a cubic spline? How many data points or set is required? We shall give simple and interesting, with step by step procedure of derivation of cubic splines. We proceed as follows:

Suppose that we are given a set of points $x_1 < x_2 < \ldots < x_n$, not necessary equally spaced, and a set of values $f(x_1)$, $f(x_2)$,…, $f(x_n)$ at these points. Take a particular interval $[x_i, x_{i+1}]$ and fit a cubic over the interval which satisfies the definition of a cubic spline. Since the cubic may differ from one interval to another let the cubic be
$S(x) = Pi(x)\ \ a_o + a_1x + a_2x^2 + a_3x^3$, $xi \le x \le x_{i+1}$ (3.1)

Equation (3.1) contains 4 unknowns. We impose 2 obvious conditions

$$P_i\,(x_i) = y_i,$$

and $P_{i+1}(x_i) = y_{i+1}$

The remaining 2 conditions are obtained by choosing the coefficients so that the first and second derivatives of $P_i(x)$ at x, are equal to the first and second derivatives of $P_{i+1}(x)$ at x_i , that is

$$P'_i(x_i) = P'_{i+1}(x_i)$$

$$P''_i(x_i) = P''_{i+1}(x_i)$$

There remain special problem at x_1 and x_n, but we will deal with these later. The conditions are now sufficient to determine the (n - 1) cubics which collectively constitute the cubic spline S(x) that is:

$$S(x) = P_i(x) \quad for \quad x_i \le x \le x_{i+1}$$

How can we solve these equations? The simplest method is to note that Ss(x) is linear in x and is also continuous over the whole interval $[x_1, x_n]$.

Ss (x) can therefore be represented in $[x_i, x_{i+1}]$ by a linear function which is only seen to be

$$S''(x) = S''(x_i) - \frac{x - x_i}{x_{i+1} - x_i}\left[S''(x_{i+1}) - S''(x_i)\right] \tag{3.2}$$

We rewrite (3.2) in the form

$$S''(x) = \frac{x - x_i}{x_{i+1} - x_i} S''(x_{i+1}) - \frac{x - x_{i+1}}{x_{i+1} - x_i} S''(x_i) \tag{3.3}$$

We integrate twice then, putting $h = h_i = x_{i+1} - x_i$ as usual, the result can be written as

$$S(x) = \frac{(x - x_i)^3}{6h} S''(x_{i+1}) - \frac{(x - x_{i+1})^3}{6h} S''(x_i) + a(x - x_i) + b(x - x_{i+1}) \qquad (3.4)$$

Hence any expression of the form Ax + B may be written as $a(x - x_i) + b(x - x_{i+1})$ for suitable choice of a,b provided $x_i \neq x_{i+1}$

We now impose the conditions that $S(x_i) = y_i$ and $S(x_{i+1})\ y_{i+1}$ so that on putting $x = x_i$, equation (3.4) becomes

$$y_i = \frac{-(x_i - x_{i+1})^3}{6h} S''(x_i) + b(x_i - x_{i+1}) \qquad (3.5)$$

$$y_i = \frac{h^3}{6h} S''(x_i) - bh$$

$$\Rightarrow \qquad bh = -y_i + \frac{h^2}{6} S''(x_i)$$

$$b = \frac{-y_i}{h} + \frac{h}{6} S''(x_i) \qquad (3.6)$$

Also putting $x = x_{i+1}$ in equation (3.4), we get

$$y_{i+1} = \frac{h^3}{6h} S''(x_{i+1}) + ah$$

$$\text{or} \qquad a = \frac{y_{i+1}}{h} - \frac{h}{6} S''(x_{i+1}) \qquad (3.7)$$

Substituting equations (3.6) and (3.7) in equation (3.4) gives:

$$S(x) = \frac{(x - x_i)^3}{6h} S''(x_{i+1}) - \frac{(x - x_{i+1})^3}{6h} S''(x_i) + (x - x_i)\left[\frac{y_{i+1}}{h} - \frac{h}{6} S''(x_{i+1})\right] + (x - x_{i+1})\left[\frac{-y_i}{h} + \frac{h}{6} S''(x_i)\right]$$

after slight re-arrangement of terms, we obtain

$$S(x) = \frac{S''(x_i)}{6}\left[\frac{(x_{i+1} - x)^3}{h} - h(x_{i+1} - x)\right] + \frac{S''(x_{i+1})}{6}\left[\frac{(x - x_i)^3}{h} - h(x - x_i)\right] + y_i\left[\frac{(x_{i+1} - x)}{h}\right] + y_{i+1}\left[\frac{(x - x_i)}{h}\right] \qquad (3.8)$$

This expression is valid for the interval $x_i < x < x_{i+1}$

It is worth noting that if in (3.8) we ignore the two terms involving $Ss(x_i)$ and $Ss(x_{i+1})$ we obtain the approximation to S(x)

$$S(x) \cong \frac{(x_{i+1} - x)y_i - (x - x_i)y_{i+1}}{h}\ i$$

which is the result for linear interpolation. We see therefore that the terms involving $Ss(x_i)$ and $Ss(x_{i+1})$ can be regarded as correction terms obtained by using a cubic instead of a linear approximation. Before we can use (3.8) to determine S(x) we must find the values $Ss(x_i)$ and this we do by using the conditions that the first derivatives are continuous at the nodes.

Differentiating (3.8) and putting $x = x_i$ we have

$$S'(x_i) = \frac{S''(x_i)}{6}\left[-\frac{3(x_{i+1}-x_i)^2}{h}+h\right]-\frac{hS''(x_{i+1})}{6}-\frac{y_i}{x_i}+\frac{y_{i+1}}{h} \qquad (3.9)$$

If we now replace i by (i – 1) in equation (3.8), differentiate and again put $x = x_i$, we obtain

$$S'(x_i) = \frac{S''(x_i)}{6}\left[-\frac{3(x_i-x_{i-1})^2}{h}-h\right]+hS''(x_{i-1})-\frac{y_{i-1}}{h}+\frac{y_i}{h} \qquad (3.10)$$

where in this last equation $h = x_i - x_{i-1} = h_{i-1}$

The continuity of Ss(x) at x_i now requires that the expressions on the right of (3.9) and (3.10) are equal and this leads to the equation:

$$S''(x_{i-1})h_{i-1} + S''(x_i)[2(x_{i+1}-x_{i-1})] + S''(x_i)h = 6\left[\frac{y_{i+1}-y_i}{h}-\frac{y_i-y_{i-1}}{h_{i-1}}\right] \qquad (3.11)$$

In the case where the x_i are evenly spaced (3.11) is simplified to

$$S''(x_{i-1}) + 4S''(x_i) + S''(x_{i+1}) = 6\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)$$

Since $h = h_o$, $x_i - x_{i-1} = h$, $x_{i+1} - x_i = h \Rightarrow x_{i+1} - x_{i-1} = 2h$

We can simply replace $S''(x_i) = M_i$ so as to get

$$M_{i-1} + 4M_i + M_{i+1} = 6\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right) \qquad (3.12)$$

The sets (n - 1) equations (3.11) and (3.12) contain (n+1) unknowns $Ss(x_i)$, (i = 0,1,…,n) and in order to obtain a unique solution we must impose conditions on $S''(x_0)$ and $S''(x_n)$ and this is usually done by taking the spline in the intervals $(-\infty, x_o)$ and (x_n, f) (that is, $x < x_o$ and $x > x_n$) to be a straight line, so that

$$S''(x_0) = 0, \quad S''(x_n) = 0.$$

This corresponds, in physical terms to allowing the spline to assume its natural straight shape outside the intervals of approximation. The spline S(x) so determined under this condition is called the **natural cubic spline.**

Given these extra two conditions the equations (3.12) are now sufficient to determine the Ss(x_j) and so Ss(x). The system of linear equations which is usually generated from this equation is of tri-diagonal form and such systems can be solved either by direct methods, such as Gaussian elimination or, if n is large, by indirect methods such as the Gauss-Seidel. Often foe a small system, simple way of solving simultaneous equation is used.

The above procedure is the usual mathematical principle of fitting a cubic spline to a set of data points. However, there exist an alternative method and this is given as follows:

3.1.1 Alternative Method of Deriving Cubic Spline

In the interval (x_{i-1} , x_i), let S(x) be such that

$$S(x)\ P_i(x)\ a_i x^3 + b_i x^2 + c_i x + di \quad i = 1,2,\ldots,n \tag{3.13}$$

Since each of equation (3.13) has 4 unknowns, then we have *4n* unknowns a_i, b_i , c_i , d_i ¸ i = 1, 2, . . ., n

Using continuity of S(x), Sc(x) , Ss(x0, we get

$$P_i(x_i) = y_i = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \tag{3.14}$$

$$P_{i+1}(x_i) = y_i = a_{i+1} x_i^3 + b_{i+1} x_i^2 + c_{i+1} x_i + d_{i+1} \tag{3.15}$$

$$\text{for } \quad i = 1,2,\ldots,(n-1)$$

equations (3.14) and (3,15) give 2(n - 1) conditions

$$\left.\begin{aligned} 3a_i x_i^2 + 2b_i x_i + c_i &= 3a_{i+1} x_i^2 + 2b_{i+1} x_i + c_{i+1} \\ 6a_i x_i + 2b_i &= 6a_{i+1} x_i + 2b_{i+1} \\ &\text{for } \quad i = 1,2,\ldots,(n-1) \end{aligned}\right\} \tag{3.16}$$

Hence, totally we have 4n - 2 conditions.

Furthermore, $y_0 = a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1$

and $y_n = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n$

Hence, totally we have 4n – 2 conditions.

Furthermore, let $S''(x_o) = M_o$, $S''(x_n) = M_n$

Now we have 4n conditions to solve for the 4n unknowns. This will give the cubic spline in each subinterval.

If $M_0 = 0$, $M_n = 0$, we call this cubic spline as **natural spline.**

These two approaches can be used to obtain a cubic spline.

It is necessary to emphasis that the interval may be uniform or non uniform. When the step is uniform, h is constant, but when the interval is uneven, then our step is taken as $x_i - x_{i-1} = hi$ for each interval.

Some examples are given below to illustrate this method.

3.2 Numerical Examples

Example 1

From the following table

X	1	2	3
Y	-8	-1	18

Obtain the cubic spline and hence by using your cubic spline, compute y(1.5) and yc(1),.

Solution

Here h = 1, and n = 2 . also assume $M_0 = 0$ and $M_2 = 0$, we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$, for i = 1,2,…,(n – 1)

From this,

$$M_0 + 4M_1 + M_2 = 6\left[y_0 - 2y_1 + y_2\right]$$
$$\therefore \quad 4M_1 = 6\left[-8 - 2(-1) + 18\right] = 72$$
$$\therefore \quad M_1 = 18$$

From equation (6) , for 1 d x d 2 putting I = 1, we get

$$S(x) = \frac{1}{6}\left[18(x-1)^3\right] + (2-x)(-8) - 4(x-1)$$
$$= 3(x-1)^3 + 4x - 12$$
$$= 3x^3 - 9x^2 + 13x - 15$$

$$y(1.5) = S(0.5) = 3(0.5)^3 + 4(1.5) - 12 = -\frac{45}{8}$$

$$y' = S'(x) = 9(x-1)^2 + 4$$
$$y(1) = 4$$

Remark

1. We can also find S(x) in the interval (2, 3) using the equation (3.8) for i = 2
2. Since y(1.5) is required, we have not cared to find S(x) in (2,3)
3. Note that $y = x^3 - 9$ also gives the above table values in the range (1,3).

Method 2

We will use the second method and work out the above problem. Let the cubic spline be

$$P_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1 \quad \text{in } [1, 2]$$
$$P_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2 \quad \text{in } [2, 3]$$
$$P_1(1) = a_1 + b_1 + c_1 + d_1 = -8 \qquad \text{(i)}$$
$$P_1(2) = 8a_1 + 4b_1 + 4c_1 + d_1 = -1 \qquad \text{(ii)}$$
$$P_2(2) = 8a_2 + 4b_2 + 2c_2 + d_2 = -1 \qquad \text{(iii)}$$
$$P_2(3) = 27a_2 + 9b_2 + 3c_2 + d_2 = 18 \qquad \text{(iv)}$$
$$P_1'(x_1) = P_2'(x_1) \ \text{gives}$$
$$3a_1(2)^2 + 2b_1(2) + c_1 = 3a_2(2)^2 + 2b_2(2) + c_2 \qquad \text{(v)}$$
$$6a_1(2) + 2b_1 = 6a_2(2) + 2b_2 \qquad \text{(vi)}$$
$$P_1''(x_0) = S''(x_0) = 0 \ \text{gives}$$
$$6a_1(1) + 2b_1 = 0 \qquad \text{(vii)}$$
$$P_2''(x_2) = S''(x_2) = 0 \ \text{gives}$$
$$6a_2(3) + 2b_2 = 0 \qquad \text{(viii)}$$

Solving (i) to (viii), we obtain

$$a_1 = 3 \ , \ b_1 = -9 \ , \ c_1 = 13 \ , \ d_1 = -15$$
$$a_2 = -3 \ , \ b_2 = 27 \ , \ c_2 = -59 \ , \ d_2 = 33$$

Hence the cubic splines are:

$$P_1(x) = 3x^3 - 9x^2 + 13x - 15 \qquad \text{in } [1, 2]$$
$$P_2(x) = -3x^3 + 27x^2 - 59x + 33 \qquad \text{in } [2, 3]$$

The learner is expected to verify this result by solving the 8 equations. Now we can interpolate at x = 1.5 from our polynomial, we then obtain

$$P_1(1.5) = 3(1.5)^3 - 9(1.5)^2 + 13(1.5) - 15 = -\frac{45}{8}$$

$$P_1'(x) = 9x^2 - 18x + 13$$

$$\therefore \quad P_1'(1) = 9 - 18 + 13 = 4$$

$$P_2(2) = -3(2^3) + 27(2^2) - 59(2) + 33 = -1 = P_1(2)$$

$$P_2(3) = -3(3^3) + 27(3^2) - 59(3) + 33 = 18$$

All these values tally with tabular values as x= 1, 2, 3.

Example 2

Find the cubic Spline given the table

x	0	2	4	6
y	1	9	41	41

where $M_0 = 0$, $M_3 = -12$

Solution

Here h = 2

$$M_0 + 4M_1 + M_2 = \tfrac{6}{4}[y_0 - 2y_1 + y_2]$$
$$= \tfrac{3}{2}(1 - 18 + 41)$$
$$= 36$$
$$M_1 + 4M_2 + M_3 = \tfrac{6}{4}[y_0 - 2y_1 + y_2]$$
$$= \tfrac{3}{2}(9 - 82 + 41)$$
$$= -48$$

Using $M_0 = 0$, $M_3 = -12$, we get

$$4M_1 + M_2 = 36$$

and $\quad M_1 + 4M_2 = -36$

Solving we obtain,

$M_1 = 12$, $M_2 = -12$

$$S(x) = \tfrac{1}{12}[(2-x)^3(0) + (x-0)^3(12)] + \tfrac{1}{2}(2-x)[1 - \tfrac{2}{3}(0)] + \tfrac{1}{2}[9 - \tfrac{2}{3}(12)]$$
$$= x^3 + 1 \qquad \text{in } [0, 2]$$

Similarly $S(x) = 25 - 36x + 18x^2 - 2x^3 \qquad$ in [2,4]

and $\quad S(x) = -103 + 60x - 6x^2 \qquad$ in [4, 6]

Example 3

Fit a natural cubic spline to the data below and use it to estimate f(55).

x	25	36	49	64	81
y = f(x)	5	6	7	8	9

Solution

We use (3.11) to form a set of linear equations for Ss(36), Ss(49), Ss(64) and we take Ss(25) = Ss(81) = 0. The equations are

$$(0).11 + 2(24)S''(36) + 13S''(49) = 6(\frac{1}{13} - \frac{1}{11})$$
$$13.S''(36) + 2(28)S''(49) + 15S''(64) = 6(\frac{1}{15} - \frac{1}{13})$$
$$15.S''(49) + 2(32)S''(64) + 17(0) = 6(\frac{1}{17} - \frac{1}{15})$$

(0).11 2(24)Scc(36) 13S c(49)

Let S"(36) = a , S"(49) = b , S"(64) = c, we re-write the equations in terms of a, b, c as

$$48a + 13b = \frac{-12}{143}$$
$$13.a + 56b + 15c = -\frac{12}{195}$$
$$15.b + 64c = -\frac{12}{255}$$

Solving these equations simultaneously we obtain

a = S"(36) = - 0.001594 b = S"(49) = - 0.000568, c = S"(64) = - 0.000602

The point at which we wish to interpolate, x = 55, lies in the interval [49, 64] and we must use the cubic appropriate to that interval, i.e. we use equation (3.8) when x_{i+1} = 64, x_i = 49, x = 55 and so we obtain

$$S(55) = \frac{S''(49)}{6}\left[\frac{(64-55)^3}{15} - 15(64-55)\right] + \frac{S''(64)}{6}\left[\frac{(55-49)^3}{15} - 15(55-49)\right]$$
$$+ 7\left[\frac{(64-55)}{15}\right] + 8\left[\frac{(55-49)}{15}\right]$$

i.e. $S(55) = 0.008179 - 0.007585 + 7.4 = 7.415764$

So our estimate for f(55) is 7.415764

As remarked above the last two terms constitute the linear approximation which therefore has the value

$$7\left(\frac{9}{15}\right)-8\left(\frac{6}{15}\right)=7.4$$

Since the function, f(x) is in fact $\sqrt{x}$ we can check on the accuracy of the estimate.

The error E of our cubic spline is obtained by taking the difference from the exact value

Now $\sqrt{55}$ 7.416198

Hence E = 7.416198 - 7.415764 = 0.000434

And the error of the linear approximation is E_1 = 7.416198 - 7.4 = 0.016198

Thus the linear estimate is correct to only 1 d.p. while the cubic spline turns out to be correct to 3 d.p. with little error of 4.34×10^{-4}

The result is satisfactory because we are working near the middle of a range of a smooth function with a small (absolute) value for its second derivative. Remember that we have taken S”(25) = S”(81) = 0.

SELF-ASSESSMENT EXERCISE

1. Can you find S(x) in the interval (2,3) for i = 2 in Example 1 above, using the same method.

4.0 CONCLUSION

Cubic spline has a great advantage of fitting polynomial of degree three simply by using the above techniques. It will be cumbersome to think of fitting a polynomial of higher degree as this will require deriving a set of formula as it was done in section 3.1. However, it is known that a cubic spline gives a good accuracy to several functions that permits polynomial fitting.

5.0 SUMMARY

In this Unit we have learnt

- how to derive the cubic spline formula involving set of linear equations
- how to fit a cubic spline or polynomial of degree three to a set of data points using cubic spline technique.
- that cubic spline have good accuracy with minimum error when used to fit a function.

6.0 TUTOR-MARKED ASSIGNMENT

Fit a natural cubic spline to the data below and use it to estimate f(24).

x	10	15	20	25	30
y = f(x)	8	10	12	15	18

7.0 REFERENCES/FURTHER READING

Abramowitz M., Stegun I. (eds) ,(1964): Handbook of Mathematical functions, Dover, N. Y.

De Boor C. (1978) : A Practical Guide to Splines, Springer Verlag, N. Y.

Francis Scheid. (1989) Schaum's Outlines Numerical Analysis 2nd ed. McGraw-Hill New York.

Kandassamy P., Thilagarathy K., & Gunevathi K. (1997) : Numerical Methods, S. Chand & Co Ltd, New Delhi, India

Leadermann Walter (1981) (Ed.): Handbook of Applicable Mathematics, Vol 3, Numerical Analysis, John Wiley, N. Y.

Turner P. R. (1994) Numerical Analysis Macmillan College Work Out Series Malaysia

UNIT 2 HERMITE APPROXIMATIONS

CONTENTS

1.0 INTRODUCTION

For approximation, or interpolation, of a function defined analytically a method due to Hermite is often useful. The method is superficially related to the spline method but in fact the two methods are very different because fitting of spline involves solving a system of linear equations to obtain numerical values for the second derivatives. $Ss(x_i)$, whereas for Hermite interpolation the values of the first derivatives are given, and the second derivatives are not relevant Spline are mainly used for fitting polynomials to data, Hermite polynomials are mainly used for approximating to functions

The most commonly used Hermite approximation polynomial is the cubic, as in the case of spline and we shall discuss only this case in detail, the more general case can be analyzed in a similar manner.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- distinguish between cubic spline and Hermite polynomial;
- figure out the Hermite approximation formula; and
- fit polynomial by Hermite approximation technique and find an estimate.

3.0 MAIN CONTENT

Suppose we have an analytic function y = f(x) with values $f(x_i)$ and derivatives $f'(x_i)$ given at n points x_i (i = 1,…,n). Across each pair of adjacent points x_m, x_{m+1} we fit a cubic $p_m(x)$ such that

$$p_m(x_m) = f(x_m)\ ,\ p'_m(x_m) = f'(x_m)$$
$$p_m(x_{m+1}) = f(x_{m+1})\ ,\ p'_m(x_{m+1}) = f'(x_{m+1})$$

Since $p_m(x)$ contains 4 coefficients, then four necessary equations will determine it uniquely, and indeed the formula for $p_m(x)$ can be explicitly stated as:

$$p_m(x) = \left(1+2\frac{x-x_m}{x_{m+1}-x_m}\right)\left(\frac{x_{m+1}-x}{x_{m+1}-x_m}\right)^2 f(x_m) + \left(1+2\frac{x_{m+1}-x}{x_{m+1}-x_m}\right)\left(\frac{x-x_m}{x_{m+1}-x_m}\right)^2 f(x_{m+1})$$
$$+\frac{(x-x_m)(x_{m+1}-x)^2}{(x_{m+1}-x_m)^2} f'(x_m) + \frac{(x-x_m)^2(x_{m+1}-x)}{(x_{m+1}-x_m)^2} f'(x_{m+1}) \qquad (3.1)$$

A cubic Hermite approximation thus consists of a set of cubic polynomials; each defined over one interval, with continuity of the cubics and their first derivatives at all the nodes. An example will be given to illustrate how this is used. This formula is not difficult to know, all it required is the placement of each subscript of x.

3.1 Numerical Example

Example 1

Use Hermite cubic interpolation to estimate the value of $\sqrt{55}$taking *f(x)* = $\sqrt{x}$, $x_1 = 49$, $x_2 = 64$

Solution

Given $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$

From (3.1) with $x_m = 49$, $x_{m+1} = 64$, $f(x_m) = \sqrt{49} = 7, f'(x_m) = \frac{1}{2\sqrt{49}} = \frac{1}{14}$,

Similarly, $f(x_{m+1})\ \sqrt{64 = 8}$, $f'(x_{m+1}) = \frac{1}{2\sqrt{49}} = \frac{1}{14}$

we have the Hermite cubic approximation as

$$f(x) \cong \left(1+2\times\frac{x-49}{64-49}\right)\left(\frac{64-x}{64-49}\right)^2 (7) + \left(1+2\times\frac{64-x}{64-49}\right)\left(\frac{x-49}{64-49}\right)^2 (8)$$
$$+\frac{(x-49)(64-x)^2}{(64-49)^2}\left(\frac{1}{14}\right) + \frac{(x-49)^2(64-x)}{(64-49)^2}\left(\frac{1}{16}\right)$$

$$= \left(\frac{2x-83}{15}\right)\left(\frac{64-x}{15}\right)^2 (7) + \left(\frac{143-2x}{15}\right)\left(\frac{x-49}{15}\right)^2 (8) + \frac{(x-49)(64-x)^2}{225\times 14} + \frac{(x-49)^2(64-x)}{225\times 16}$$

This gives the required Hermite polynomial approximation to f (x) =

$\sqrt{x}$. We may simplify this as much as possible. However, since we are only to estimate the square root of 55, simplifying this expression may not be all that necessary until we have substituted the value for x. Hence, putting x = 55 in the last equation obtained yields the estimate

$$\sqrt{55} \cong 7.416286$$

The correct value of $\sqrt{55}$ to 6 d.p is 7416198, so the error is 0.000088 compared to an error of - .000434 when we used the natural cubic spline in Example 3 of Unit 1

4.0 CONCLUSION

In general the errors when we use the Hermite cubic and the natural cubic spline on the same problem will not be very different for in both cases the error is proportional to h^4 where h is the step between the two relevant nodes.

5.0 SUMMARY

In this Unit we have learnt that:

- Hermite approximation differs from cubic.
- cubic spline fit a polynomial to a set of data points while Hermite approximate a function as a polynomial.
- Hermite approximation may be more accurate than the cubic spline.

6.0 TUTOR-MARKED ASSIGNMENT

Obtain an Hermite approximation of polynomial of degree 3 to the function f(x) = ln x for $x_1 = 2$, $x_2 = 5$ and hence estimate the value of ln 3.5

7.0 REFERENCES/FURTHER READING

Abramowitz M., Stegun I. (eds) ,(1964): Handbook of Mathematical functions, Dover , N. Y.

Atkinson K. E. (1978): An Introduction to Numerical Analysis, 2nd Edition, John Wiley & Sons, N. Y

Conte S. D. and Boor de Carl Elementary Numerical Analysis an Algorithmic Approach 2nd ed. McGraw-Hill Tokyo.

Francis Scheid. (1989) Schaum's Outlines Numerical Analysis 2nd ed. McGraw-Hill New York.

Henrici P. (1982): Essential of Numerical Analysis, Wiley, N. Y

Kandassamy P., Thilagarathy K., & Gunevathi K. (1997): Numerical Methods, S. Chand & Co Ltd, New Delhi, India

Turner P. R. (1994) Numerical Analysis Macmillan College Work Out Series Malaysia