MODULE 5: BOUNDARY VALUE PROBLEMS

UNIT 1: INTRODUCTION TO BOUNDARY VALUE PROBLEMS

CONTENTS

1.0 INTRODUCTION

What is a Boundary Value Problem?

This is a kind of problem that is related to differential equations. A differential equation can be an Ordinary Differential Equation (ODE) or a Partial Differential Equation (PDE). However, an ODE can be classified into two, viz:

(i.) Initial Value Problem (IVP) and
(ii) Boundary Value Problem (BVP).

These two classes of differential equations are of great importance and so their numerical solutions are equally studied under Numerical Analysis. We shall give the definition of the two.

2.0 OBJECTIVE

At the end of this lesson, you should be able to

- distinguish between Initial Value Problem and Boundary Value Problem;
- derive finite difference scheme for solving BVP; and
- solve BVP using a finite difference scheme.

3.0 DISTINCTION BETWEEN IVP AND BVP

There is a distinction between an Initial Value Problem and a Boundary Value Problem. As the name goes, one is prescribed with an initial condition while the other is prescribed with boundary conditions. For a better understanding we shall define both ordinary differential equation and partial differential equation before distinguish between IVP and BVP. We hereby give the following definitions.

Definition 1

A differential equation involving ordinary derivatives (or total derivatives) with respect to a single independent variable is called an ordinary differential equation. Some examples of ODEs are given below.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0$$

$$3\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = x - 7$$

$$\frac{dy}{dx} - 2xy = \cos x$$

Definition 2

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a Partial Differential Equation (PDE) Some examples of PDEs are given below.

***Definition 3*: (Initial Value Problem)**

An Initial Value Problem (IVP) can be defined as an Ordinary Differential Equation with a condition specified at an initial point.

Where x_0 and y_0 are initial point of the equation. This example is simple enough as this involves only a single first order ODE. It is possible to have a system of first order ODEs with initial conditions all through for each of the equations.

However we may have some other differential equations with some conditions specified either at the derivative or at the boundary of the problem being defined. This leads to the next definition.

***Definition 4:* (Boundary Value Problem)**

A Boundary Value Problem (BVP) is a differential equation either an Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE) with at least two specified conditions at the boundary points.

The boundary points often will contain an initial point and the other at the end point of the problem. The two serve as the boundary to the problem.
For example for an Ordinary Differential Equation a simple example of a BVP will be:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = d\ , \quad y(x_o) = \alpha\ ,\ y(x_n) = \beta\ , \quad x_o \le x \le x_n \tag{3.2}$$

The above equation is a second order differential equation which is solvable for the specified range of values of x. Two conditions are specified at the extremes or the boundaries. That is the conditions are given at $x = x_0$ and $x = x_n$.

Example of a BVP involving a PDE will be given later when discussing the methods of solving Partial Differential Equations.
There are several numerical methods available today for solving first order ODEs with an initial condition. This is a course on its own as the subject is wide, though not so tasking. However, the focus of this course and Module is to expatiate on methods of solving BVPs. Hence, we shall in limit our discussion to the numerical methods for solving the BVPs of ordinary differential equations.

3.2 Solution of BVP of ODE

The numerical solution of a second order Ordinary Differential Equation usually will involve solving system of equations. To do this, some approximations are put in place to replace the derivative function involved in the given differential equation.

Suppose we are to solve the differential equation (3.2) using a numerical method, Two popular methods among other methods of solving this equation are either by Finite Difference Method (FDM) or by Shooting method.
We shall in this unit discuss the Finite Difference Method for solving equation (3.2).
Consider the Taylor series expansion of the function y(x+h) where h is regarded as the step length to be used in the problem. Then we shall obtain

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + O(h^3) \qquad (3.3)$$

where $O(h^3)$ is called error of order 3 representing the truncation error of where the expansion is terminated.
We can obtain a first derivative approximation from this expansion by writing

$$y(x+h) - y(x) = hy'(x) + \frac{h^2}{2!}y''(x) + O(h^3)$$

$$y(x+h) - y(x) = hy'(x) + O(h^2)$$

Dividing through by *h* we obtain

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \qquad (3.4)$$

This shows that for small step length h, the error in approximating $y'(x)$ is proportional to h. Furthermore if we expand the function y(x – h) we equally get

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - O(h^3) \qquad (3.5)$$

We can also obtain a first derivative approximation from this expansion as

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \qquad (3.6)$$

Equations (3.4) and (3.6) are approximations to $y'(x)$ which can be used to replace the function as it may be required. Equation (3.4) is the forward difference representation while equation (3.6) is the backward difference representation.

Now suppose we take the difference of equations (3.3) and (3.5) we shall obtain

$y(x + h) - y(x - h)=2hy'(x)+(h^3)$

This reduces to

$$y'(x) = \frac{y(x+h) - y(x-h)}{h} + O(h^2) \qquad (3.7)$$

Equation (3.7) is a central difference approximation to $y'(x)$. It would be observed that the error in the last equation is smaller than that of the two equations (3.4) or (3.6), since for small h, h^2 will be smaller than h.
On the other hand if we add equations (3.3) and (3.5) we shall obtain

$$y(x+h) + y(x-h) = 2y(x) + (2)\frac{h^2}{2!}y''(x) + O(h^4)$$

$$\Rightarrow \quad y(x+h) - 2y(x) + y(x-h) = h^2 y''(x) + O(h^4)$$

Divide through by h^2 we obtain

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2) \qquad (3.8)$$

This is a standard representation for the second derivative. Thus equations (3.4) and (3.8) can be substituted into scheme for solving that equation.

Recall the differential equation (3.2)

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = c$$

Substituting (3.4) and (3.8) we have

$$a\left(\frac{y(x+h) - 2y(x) + y(x-h)}{h^2}\right) + b\left(\frac{y(x+h) - y(x)}{h}\right) + cy(x) = d$$

evaluating at $x = x_n$,

we observe that $y(x+h) = y(x_n + h) = y(x_{n+1}) = y_{n+1}$

Also $y(x-h) = y(x_n - h) = y(x_{n-1}) = y_{n-1}$

hence, we obtain

$$a\left(\frac{y(x_n+h) - 2y(x_n) + y(x_n-h)}{h^2}\right) + b\left(\frac{y(x_n+h) - y(x_n)}{h}\right) + cy(x_n) = d$$

$$a\left(\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}\right)+b\left(\frac{y_{n+1}-y_n}{h}\right)+c\,y_n=d$$

$$a(y_{n+1}-2y_n+y_{n-1})+bh(y_{n+1}-y_n)+ch^2\,y_n=dh^2 \qquad (3.9)$$

On the other hand, we can use the central difference to replace the first derivative to get

$$a\left(\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}\right)+b\left(\frac{y_{n+1}-y_{n-1}}{2h}\right)+c\,y_n=d$$

$$a(y_{n+1}-2y_n+y_{n-1})+\tfrac{1}{2}bh(y_{n+1}-y_{n-1})+ch^2\,y_n=dh^2 \qquad (3.10)$$

Equations (3.9) and (3.10) are numerical schemes that can be used to solve equation (3.2). Either of these will yield the desire result with slight difference in accuracy. On applying the boundary conditions in (3.2) and writing the resulting equations for
n = 1, 2, . . . , k-1, we obtain a system of equations with equal number of unknowns. The above shall be illustrated by the next example.

3.2 Numerical Examples

Solution

Example 1

Solve the boundary value problem (BVP)

$$(1+x^2)y''+2xy'-y=x^2 \qquad (3.11)$$

Satisfying the boundary conditions

$y(0) = 1$ and $y(1) = 0$

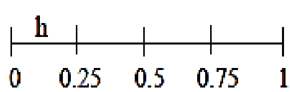
You may use a step length of 0.25.

Solution

To solve this problem we can apply the approximation of the derivatives to the given equation (11), we then obtain

$$\left(1+x_n^2\right)\left(\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}\right)+2x_n\left(\frac{y_{n+1}-y_{n-1}}{2h}\right)-y_n=x_n^2$$

Since $h=\frac{1}{4}$ it implies that the range of x is divided into four parts by 5 node points



Thus $x_0 = 0$ and x4 = 1 (that is : $x_1 = 0.25$ $x_2 = 0.5$ $x_3 = 0.75$)
The boundary condition y(0) = 1 and y(1) = 0 simply transform to $y_0 = 1$ and y4 = 0
Since n = 0 will be invalid as we will not be able to evaluate y-1 then the reasonable thing to do as in the theory above is to substitute n = 1,2,3,4. Hence, we obtain
With n = 1, the formula above becomes

$$\left(1+x_1^2\right)\left(\frac{y_2-2y_1+y_o}{\left(\frac{1}{4}\right)^2}\right)+2x_1\left(\frac{y_2-y_o}{2\left(\frac{1}{4}\right)}\right)-y_1=x_1^2$$

$$\left(1+\left(\frac{1}{4}\right)^2\right)\left(\frac{y_2-2y_1+1}{\left(\frac{1}{4}\right)^2}\right)+2\left(\frac{1}{4}\right)\left(\frac{y_2-1}{2\left(\frac{1}{4}\right)}\right)-y_1=\left(\frac{1}{4}\right)^2$$

$$16\left(\frac{17}{16}\right)(y_2-2y_1+1)+(y_2-1)-y_1=\frac{1}{16}$$

$$288y_2-560y_1=-255 \qquad \text{(i)}$$

Also for n = 2, we have

$$\left(1+x_2^2\right)\left(\frac{y_3-2y_2+y_1}{\left(\frac{1}{4}\right)^2}\right)+2x_2\left(\frac{y_3-y_1}{2\left(\frac{1}{4}\right)}\right)-y_2=x_2^2$$

$$\left(1+\left(\frac{1}{2}\right)^2\right)\left(\frac{y_3-2y_2+y_1}{\left(\frac{1}{4}\right)^2}\right)+2\left(\frac{1}{2}\right)\left(\frac{y_3-y_1}{2\left(\frac{1}{4}\right)}\right)-y_2=\left(\frac{1}{2}\right)^2$$

$$16\left(\frac{5}{4}\right)(y_3-2y_2+y_1)+2(y_3-y_1)-y_2=\frac{1}{4}$$

$$88y_3-164y_2+72y_1=1 \qquad \text{(ii)}$$

And for n = 3, we have

$$\left(1+\left(\frac{3}{4}\right)^2\right)\left(\frac{y_4-2y_3+y_2}{\left(\frac{1}{4}\right)^2}\right)+2\left(\frac{3}{4}\right)\left(\frac{y_4-y_2}{2\left(\frac{1}{4}\right)}\right)-y_3=\left(\frac{3}{4}\right)^2$$

$$448y_4-784y_3+352y_2=9$$

$$784y_3-352y_2=-9 \qquad \text{(iii)}$$

since $y_4 = 0$.

Thus we have a system of 3 equations in three unknowns y_1 , y_2 , y_3. The matrix form of these three equations from (i), (ii), (iii), is written as:

$$\begin{pmatrix}-560 & 288 & 0\\ 72 & -164 & 88\\ 0 & -352 & 784\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}-255\\ 1\\ -9\end{pmatrix}$$

On solving, correct to four decimal places, we obtain

$$\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}381783/599984\\ 1885/5357\\ 7991/54544\end{pmatrix}=\begin{pmatrix}0.6363\\ 0.3519\\ 0.1465\end{pmatrix}$$

Thus the values corresponding to y_1, y_2, y_3 are the results of the differential equation at points x_1, x_2, x_3.

4.0 CONCLUSION

We have seen that the finite difference scheme is systematic and dynamic in producing solution to BVP. The resulting technique led to system of linear equations which can be solved by any available methods used for solving such system. The learner can also check other texts for other method of solving BVP in ODE, such as the shooting method earlier mentioned.

5.0 SUMMARY

In this Unit you have learnt

(i) distinction between BVP and IVP
(ii) how to derive the Finite Difference scheme for solving BVP
(iii) how to implement the Finite Difference Method on a BVP.

6.0 TUTOR MARKED ASSIGNMENT

Solve the boundary value problem $x^2y'' + xy' - y = 2x$, satisfying the boundary conditions y(0) = 1 and y(1) = 0, use a step length h = 0.25.

7.0 FURTHER READING AND OTHER RESOURCES

Francis Scheid. (1989) *Schaum's Outlines Numerical Analysis 2nd ed.* McGraw-Hill New York.

Kandassamy P., Thilagarathy K., & Gunevathi K. (1997): Numerical Methods, S. Chand & Co Ltd, New Delhi, India

Turner P. R. (1994) *Numerical Analysis Macmillan College Work Out Series* Malaysia

UNIT 2: BOUNDARY VALUE PROBLEMS INVOLVING PARTIAL DIFFERENTIAL EQUATIONS

CONTENTS

1.0 Introduction

As earlier stated, a Boundary Value Problem (BVP) could be a Partial Differential Equation (PDE) with two specified points at the initial point and at the boundary point.
In scientific computing many problems are governed by non linear differential equation which requires a solution in a region R subject to exact condition on the boundary.
Unlike the BVP involving an ODE, most BVPs usually occur from problems involving rate of change with respect to two or more independent variables. Such problems lead to PDEs. The two dimensional second order Partial Differential Equation is generally of the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g = 0 \qquad (1.1)$$

where u is a function of two variables x and y, that is, u = u(x, y). The solution of this equation subject to prescribed conditions is generally obtained through some analytical methods by separating the variables x and y. However, the numerical solution of equation (1.1) can be obtained either by the finite difference method or the finite element method.

2.0 OBJECTIVE

At the end of this lesson, you should be able to

- define a second order PDE;
- define a Boundary Value Problem (BVP) involving a partial differential equation;
- classify various types of PDEs;
- classify types of boundary conditions among PDEs; and
- derive finite difference schemes for PDEs.

3.0 Types of Partial Differential Equations

A number of mathematical models describing the physical system are the special cases of general second order PDE

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} - H(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \qquad (3.1)$$

Or $\quad L(u) = Au_{xx} + Bu_{xy} + Cu_{yy} - H(x, y, u, u_x, u_y) = 0$

The following definitions are given with respect to equation (3.1).
Equation (3.1) is said to be **semi-linear,** if A, B and C are functions of independent variables x and y only.

If A, B and C are functions of x, y, u, u_x and u_y, then (3.1) is termed to be **quasi-linear.** However, when A, B and C are functions of x and y and H is a linear function of u, u_x and u_y, then (3.1) is said to be **linear**.

Hence, the most general second order linear PDE in two independent variables can be written as

$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u + G(x, y) = 0$ (3.2)

When G(x, y) = 0, then equation (3.2) is known as a **linear homogenous second order PDE**. A solution of equation (3.1) or (3.2) will be of the form

u = u(x, y)

Which represents a surface in (x, y, u) space called the **integral surface**.

If on the integral surface, there exist curves across which the derivatives u_{xx}, u_{yy} and u_{xy} are discontinuous or indeterminate then the curves are called **"characteristics".**

For this, we assume the solution of equation (3.1) is passing through a curve C whose parametric Equations are:

x = x(s) , y = y(s) and u = u(s) (3.3)

Furthermore, let each point (x, y, u) of curve C and the partial derivates u_x and u_y be known since the solution is of the form (3.3) at each point of x, y of curve C.

3.1 Classification of Partial Differential Equations

Thus there are two families of curve which can be obtained from equation (3.1) along which the second order derivatives will not be determined in a definite or finite manner. There are called characteristics curves which are classified according to the following conditions. If

$B^2 - 4AC > 0$, then we have real and distinct roots
$B^2 - 4AC < 0$, then we have imaginary roots
$B^2 - 4AC = 0$, then we have real and coincidence or equal roots

Hence, the Partial Differential Equation (3.1) or (3.2) is said to be:

Parabolic, if $B^2 - 4AC = 0$
It is Elliptic if $B^2 - 4AC < 0$
and it is Hyperbolic, if $B^2 - 4AC > 0$

Few examples are given below to illustrate these classifications.

Examples

1. The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

This equation is a Hyperbolic equation, since A = 1, B = 0, C = -1 so that,
$B^2 - 4AC = 4 > 0$

2. The heat flow equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Comparison with equation (3.1), we note that: A = 0, B = 0, C = -1 so that $B^2 - 4AC = 0$,
Hence the heat flow equation is a Parabolic equation.

3. The Laplace equation is also given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Comparison with equation (3.1), shows that: A = 1, B = 0, C = 1 so that
$B^2 - 4AC = -4 < 0$. Thus the Laplace equation is an Elliptic equation.

3.3 Classification of Boundary Conditions for PDE

The parabolic and hyperbolic types of equations are either IVP or initial BVP whereas the elliptic equation is always a BVP. There are three types of boundary conditions. These are given below as follows:

i) *Dirichlet Conditions*
Here the function (say u(x,y)) is prescribed along the boundary. If the function

takes on zero value along the boundary, the conditions is **called homogenous dirchlet condition** otherwise it is called **inhomogenousdirichlet boundary conditions**.

ii) *The Neumann Boundary Condition*
Here the derivative of the function is specified along the boundary. We may also have homogenous or inhomogenous boundary conditions

iii) *Mixed Boundary Conditions*
Here the function and its derivatives are prescribed along the boundary. We may also have homogenous and inhomogenous conditions.

3.4 Finite Difference Scheme

Most PDEs are solved numerically by Finite Difference Method (FDM), although another known method is the Finite Element Method (FEM). Hence, there is the need to develop schemes of finite differences for derivatives of some functions.

In ODE of the second order which was discussed earlier, the function y is a function of a single variable x. The treatment of the finte difference method was easier. However a similar technique and analogy will be employed for the development of the finite difference schemes (FDS) of a second order PDE. The difference now is ***u*** being a function of two variables x and y.

In this regard, finite difference schemes or methods required that the (x, y) region of the problem to be examined be divided into smaller regions by rectilinear grid, mesh or lattice of discrete points with co-ordinates (x_i, y_j) given by

$$x_i = x_o + i\delta x$$

$$y_j = y_o + j\delta y$$

This shows that each axis is divided into set of equal intervals by node points.
Usually we shall represent $\delta x = h$,$\delta y = k$ as our step lengths in x and y directions

respectively. Hence,
$x_{n+r} = x_n + rh$, $y_{n+r} = y_n + rk$

Consider a function u(x, y) of two variables, with an increment δx in x yield $u(x + \delta x, y) = u(x + h, y)$

If this is expanded by Taylor series, we shall obtain

$$u(x+h,y) = u(x,y) + h\frac{\partial u(x,y)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x,y)}{\partial x^2} - \ldots \qquad \text{(i)}$$

Similarly $u(x-\delta x, y)$, the expansion yield

$$u(x-\delta x, y) = u(x-h, y)$$

$$= u(x,y) - h\frac{\partial u(x,y)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x,y)}{\partial x^2} - \ldots \qquad \text{((ii)}$$

Other expansions with increments on y give:

$$u(x, y+\delta y) = u(x, y+k)$$

$$= u(x,y) + k\frac{\partial u(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u(x,y)}{\partial y^2} + \ldots \qquad \text{(iii)}$$

$$u(x, y-\delta y) = u(x, y-k)$$

$$= u(x,y) - k\frac{\partial u(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u(x,y)}{\partial y^2} - \ldots \qquad \text{(iv)}$$

Truncating equation (i) at second term yields,

$$u(x+h,y) - u(x,y) = hu_x(x,y) + O(h^2) \cong hu_x$$

$$u_x = \frac{u(x+h,y) - u(x,y)}{h} \qquad (3.4)$$

at point (i, j) we have

$$u_x(x_i, y_j) = \frac{u(x_i + h, y_j) - u(x_i, y_j)}{h}$$

$$\frac{\partial u(x_i, y_j)}{\partial x} = \frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h}$$

This is then written for easy handling as:

$$\text{or} \quad u_x = \frac{u_{i+1,j} - u_{i,j}}{h} \qquad (3.5)$$

Similarly from (iii) by following the same procedure, we get

$$\frac{\partial u}{\partial y} = u_y = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad (3.6)$$

Equations (3.5) and (3.6) are forward difference approximation of u_x and u_y respectively. Similarly, truncating at third time we shall obtain the second derivative approximation. This can be achieved by taking the sum of equations (i) and (ii), to get

$$u(x+h,y)+u(x-h,y)=2u(x,y)+h^2\frac{\partial^2 u}{\partial x^2}$$

$$u(x_{i+1},y_j)+u(x_{i-1},y_j)=2u(x_i,y_j)+h^2\frac{\partial^2 u}{\partial x^2}$$

This simplified to

$$\frac{\partial^2 u}{\partial x^2}=u_{xx}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} \qquad (3.7)$$

Also adding equations (iii) and (iv), we obtain a similar result as

$$\frac{\partial^2 u}{\partial y^2}=u_{yy}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} \qquad (3.8)$$

Equations (3.7) and (3.8) are the finite difference approximation for the second derivatives u_{xx} and u_{yy}. They are sometimes called the second central difference approximations. These approximations are often used to develop the finite difference schemes which are tools for solving BVPs numerically.

4.0 CONCLUSION

We have seen that the subject of BVP is wide. Partial differential equations with boundary conditions differ depending on the type of boundary conditions. This will invariably affect the scheme which will be developed for its solution. We remark here that the basic differentiation formula as we have in analysis is the same used here for the development of the finite differences for the partial derivatives.

5.0 SUMMARY

In this Unit you have learnt

(i) the definition for various types of PDEs,

(ii) about types of boundary conditions

(iii) how to derive the finite differences for first and second partial derivatives.

6.0 TUTOR MARKED ASSIGNMENT

Write a finite difference for $\frac{\partial^2 u}{\partial x \partial y}$

7.0 FURTHER READING AND OTHER RESOURCES

Conte S. D. and Boor de Carl *Elementary Numerical Analysis an Algorithmic Approach2nd ed.* McGraw-Hill Tokyo.

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UNIT 3: SOLUTION OF LAPLACE EQUATION IN A RECTANGLE

CONTENTS

1.0 INTRODUCTION

There are various technique required when developing finite difference schemes for partial differential equations. The type of PDE depends on the type of scheme that will be obtained, whether it is parabolic, elliptic or hyperbolic in nature. One PDE that is simple to develop a finite difference scheme for is the Laplace equation. We shall in this unit provide a Finite Difference Method for the Laplace equation and its method of solution.

2.0 OBJECTIVE

At the end of this lesson, you should be able to

- define a second order PDE;
- define a Boundary Value Problem (BVP) involving a partial differential equation;
- classify various types of PDEs;
- classify types of boundary conditions among PDEs; and
- derive finite difference schemes for PDEs.

3.0 LAPLACE EQUATION IN A RECTANGULAR BOUNDARY

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3.1)$$

where D is the domain in (x, y) plane, and C is its boundary. For simplicity, the domain D is chosen to be a rectangle such that

$D \equiv \{(x, y): 0 < x < a,\ 0 < y < b\}$

with its boundary composed by

$C \equiv \{(x, y): x = 0, a;\ y = 0, b\}$

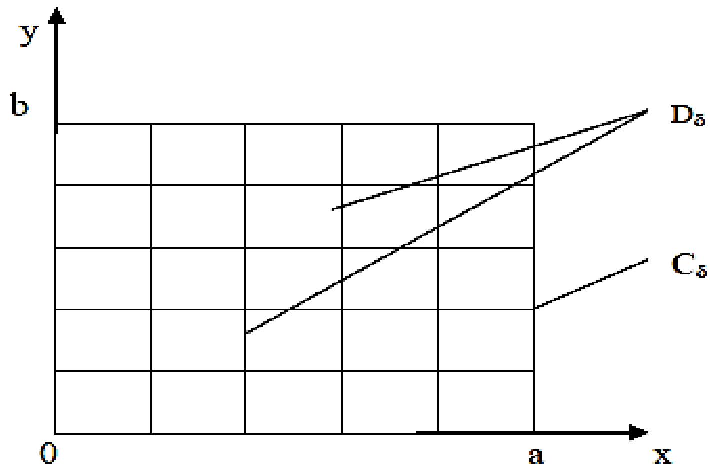

Figure 1

To obtain the numerical solution of (1) we introduce the net spacing

$$\delta x = h = \frac{a-0}{n+1} \quad , \quad \delta y = k = \frac{b-0}{m+1}$$

with uniformly net points

$$x_i = i.\delta x = ih \quad , \quad y_j = j.\delta y = jk \quad , \quad i, j = 0, \pm 1, \pm 2, \ldots$$

Hence, the interior points to D are called

$D_\delta D_\delta = \{(x_i, y_j) : 1 \le i \le n,\ 1 \le j \le m\}$

The net points on boundary C with exception of the 4 corners of the rectangle are called C_δ

$$C_\delta = \left\{ (x_i, y_j) : \begin{cases} i = (0, n+1)\ ,\ 1 \le j \le m+1 \\ j = (0, m+1)\ ,\ 1 \le i \le n+1 \end{cases} \right\}$$

We shall seek an approximate solution $u(x_i, y_j)$ of (1) at the net points $D_\delta + C_\delta$. The PDE (3.1) is replaced by a central second difference quotients obtained in the last unit. This will be illustrated by the following example.

3.1 Numerical Example

Solve Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3.1)$$

Solution

For simplicity we shall choose the meshes to be uniform and equal on both the x- and y- axes; that is, let $\delta x = \delta y = h$

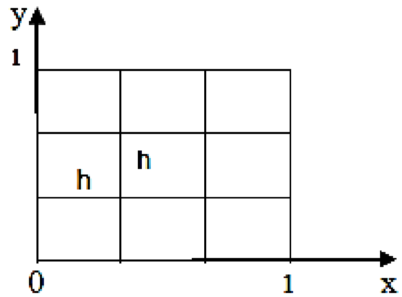

Figure 2

Replace equation (3.1) by second central differences, to have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Then we obtain on substituting into equation (3.1)

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

$$\Rightarrow \quad u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 0$$

$$\Rightarrow \quad u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) \qquad (3.2)$$

That is the average of 4 points supporting any point u_{ij} produces the result at point (x_i, y_j)

Using 3 internal meshes, that is, n = 3,

Then $h = \frac{1-0}{4-1} = \frac{1}{3}$

Hence, there are 4 internal points in domain D to be determined, since other points are on the boundary (Figure 2)

$u(x,0) = 1$, for all values of x, at $y = 0, j = 0$
$u = 1$
$u(0,y) = 0$ for all y
$u(1,y) = 0$, $u(x,1) = 1$
Let $u_{11} = u_1$, $u_{21} = u_2$, $u_{12} = u_3$, $u_{22} = u_4$

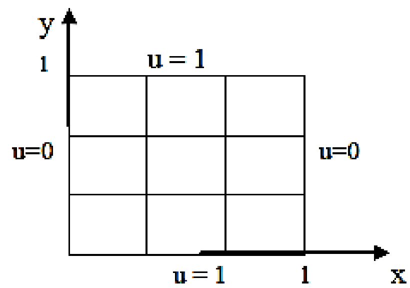

Figure 3

Then by equation (2)

$$u_1 = \frac{1}{4}\left(u_{0,1} + u_{2,1} + u_{1,o} + u_{1,2}\right)$$
$$= \frac{1}{4}(0 + u_2 + 1 + u_3)$$
$$u_2 = \frac{1}{4}\left(u_{1,1} + u_{3,1} + u_{2,o} + u_{2,2}\right)$$
$$= \frac{1}{4}(u_1 + 0 + 1 + u_4)$$
$$u_3 = \frac{1}{4}(u_4 + 0 + 1 + u_1)$$
$$u_4 = \frac{1}{4}(u_3 + 0 + 1 + u_2)$$

Solving for the four u's, we obtain

$$u_1 = u_2 = u_3 = u_4 = ½$$

Thus the four internal points for this problem are ½ each.
The internal points may be increased by increasing the number of meshes and different result will be obtained. Note that the results obtained are numerical values which serve as the solution to the BVP (3.1) at the node points.

4.0 CONCLUSION

It is expected that the learner should be able to use the simple approach given above to solve elementary BVPs with simple boundary conditions.

5.0 SUMMARY

In this Unit you have learnt how to

(i) develop finite different scheme for the Laplace equation,
(ii) solve Laplace equation using the finite difference scheme.

6.0 TUTOR MARKED ASSIGNMENT

Solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Subject to the boundary conditions:

$u(x, 0) = 1$, $u(0, y) = 0$, $u(1, y) = 0$, $u(x,1)=1$; $0 \leq x \leq 1$, $0 \leq y \leq 1$

Use h = ¼ on both axes

7.0 FURTHER READING AND OTHER RESOURCES

Francis Scheid. (1989) *Schaum's Outlines Numerical Analysis 2nd ed.* McGraw-Hill New York.

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Leadermann Walter (1981) (Ed.): Handbook of Applicable Mathematics, Vol 3, Numerical Analysis, John Wiley, N. Y.

Turner P. R. (1994) *Numerical Analysis Macmillan College Work Out Series* Malaysia