# MODULE 1 LIMIT AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

Unit 1: Real Functions
Unit 2: Limit of Function of Several Variables.
Unit 3: Continuity of Function of Several Variables.

## UNIT 1 REAL FUNCTION

### CONTENTS



### 1.0 INTRODUCTION

A real-valued function, f, of x, y, z,... is a rule for manufacturing a new number, written f(x, y, z, ...), from the values of a sequence of independent variables (x,y,z, ...).

The function *f* is called a real-valued function of two variables if there are two independent variables, a real-valued function of three variables if there are three independent variables, and so on.

As with functions of one variable, functions of several variables can be represented numerically (using a table of values), algebraically (using a formula), and sometimes graphically (using a graph).

**Examples**

1. f(x, y) = $\tilde{x}$ y  Function of two variables
 f(1, 2) = $\tilde{1}$ 2 = $\tilde{1}$  Substitute 1 for x and 2 for y
 f($\tilde{2}$, 1) = $\tilde{2}$ ($\tilde{1}$) = 3  Substitute 2 for x and $\tilde{1}$ for y
 f(y, x) = $\tilde{y}$ x  Substitute y for x and x for y

2. h(x, y, z) = x + y + xz  Function of three variables
 h(2, $\tilde{2}$, 2) = 2 + 2 + $\tilde{2}$ (2)  Substitute 2 for x, 2 for y, and $\tilde{2}$ for z.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to define and explain:

- Domain;
- real function;
- value of functions;
- types of graph; and
- types of function.

## 3.0 MAIN CONTENT

f is a function from set A to a set B if each element x in A can be associated with a unique element in B.

**usually written as f : A → B**

The unique element in B which f associates with x in A denoted by f (x)

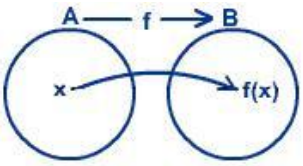



Figure 1.1

**Domain**

In the above definition of the function, set A is called domain.

**Co-domain**

In the above definition of the function, set B is called co-domain.

**Real Functions**

A real valued function $f^1$: A to B or simply a real function $f^1$ is a rule which associates to each possible real number $x \in A$, a unique real number $f(x) \in B$, when A and B are subsets of R, the set of real numbers.

In other words, functions whose domain and co-domain are subsets of R, the set of real numbers, are called real valued functions.

**Value of a Function**

If '$f^1$ ' is a function and x is an element in the domain of f, then image f(x) of x under f is called the value of '$f^1$ ' at x.

**Types of Functions and their Graphs**

Constant Function

A function f: A→ B Such that A, B∈R, is said to be a constant function if there exist K ∈B such that f(x) = k.

Domain = A

Range = {k}

The graph of this function is a line or line segment parallel to x-axis. Note that, if k>0, the graph B is above X-axis. If k<0, the graph is below the x-axis. If k = 0, the graph is x-axis itself. Graph of constant function is illustrated below:

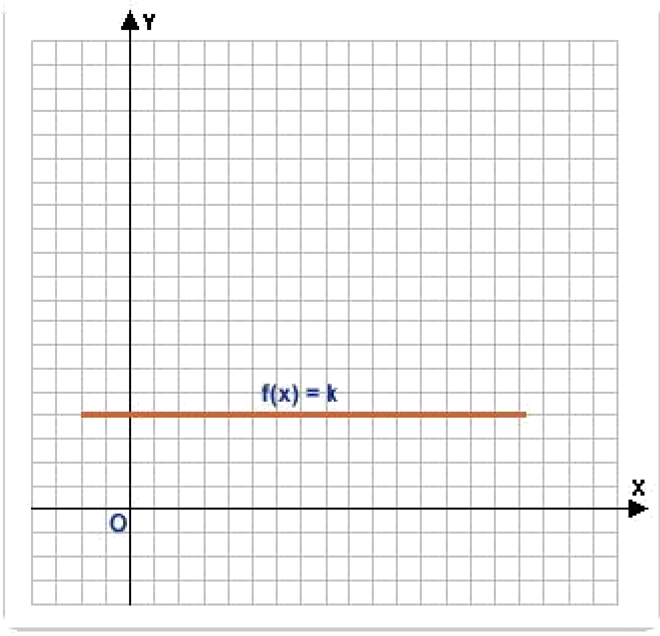



Figure 1.2

**Identity Function**

A function f : R→R is said to be an identity function if for all x∈R, f(x) = x.

Domain = R

Range = R

Graph of identity function is illustrated below

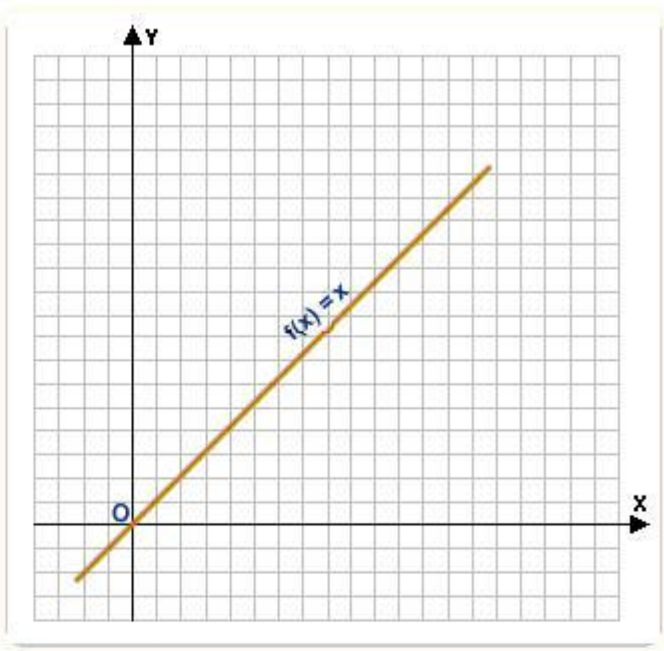



Figure: 1.3

**Polynomial Function**

A function $f : R \rightarrow R$ is said to be a polynomial function if for each $x \in R$, f(x) is a polynomial in x.

$f(x) = x^3 + x^2 + x$

$g(x) = x^4 + 3x^2 + 2\sqrt{3}x + \sqrt{5}$ are examples of polynomial functions.

$h(x) = 3x^2 + \frac{2}{x}$ is not a polynomial function.

**Modulus Function**

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$f : R \rightarrow R$ such that $f(x) = |x|$, is called the modulus function or absolute value function.

Domain = R

Range = $\begin{Bmatrix} x : x \geq 0 \\ x \in R \end{Bmatrix}$

Graph of modules function is illustrated below:

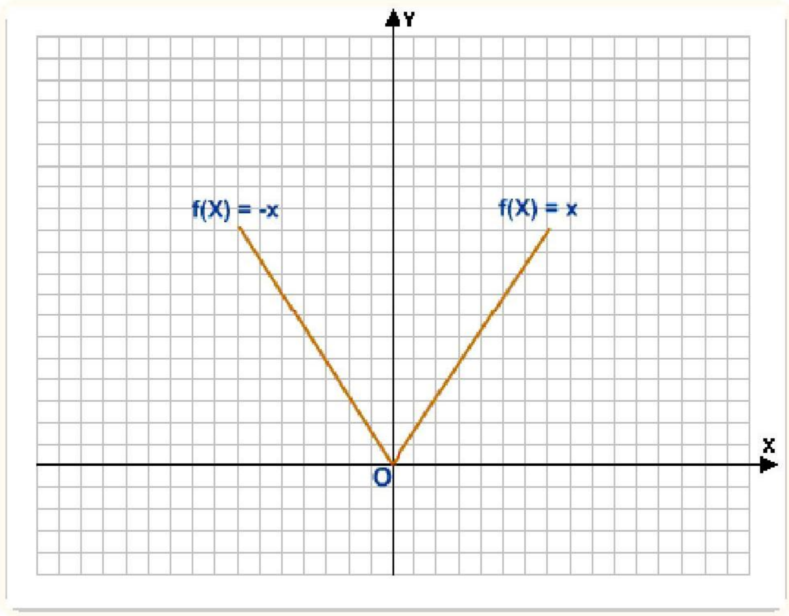



Figure: 1.4

**Square Root Function**

Since square root of a negative number is not real, we define a function f : $R^+ \rightarrow R$ such that

$f(x) = \sqrt{x}$

Domain of f = $R^+$ (set of all non – negative real numbers)

Range = $R^+$ (set of all non – negative real numbers)

Graph of square root function is illustrated below:

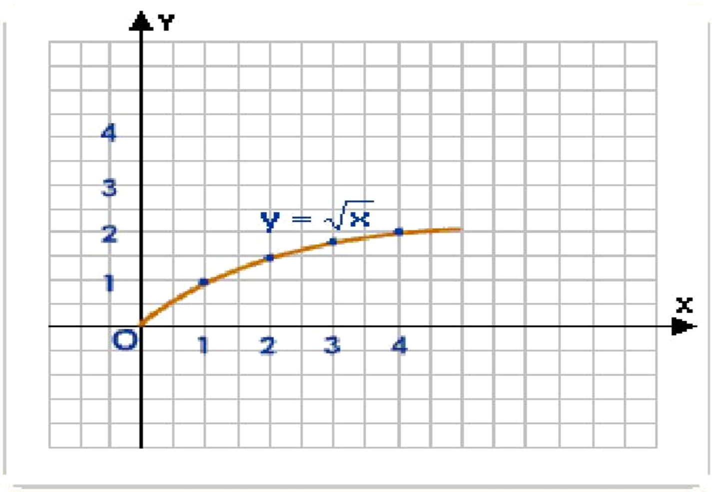



Figure 1.5

**Greatest Integer Function or Step Function (floor function)**

f (x) = [x] = greatest integer less than or equal to x

[x] = n, where n is an integer such that $n \leq x < n+1$

**Smallest Integer Function (ceiling function)**

For a real number x, we denote by [x], the smallest integer greater than or equal to x. For example, [5.2] = 6, [-5 . 2] = -5, etc. The function f:R $\rightarrow$ R defined by

f(x) = [x], x $\in$R

is called the smallest integer function or the ceiling function.

Domain: R

Range: Z

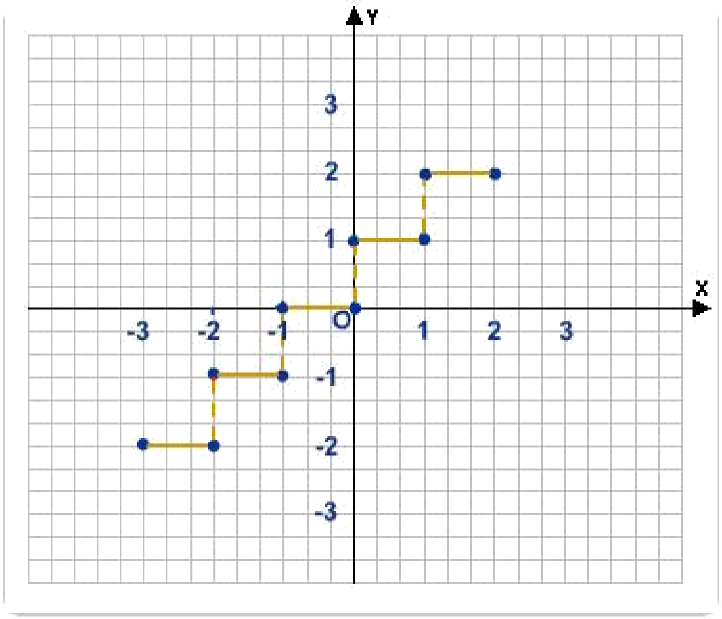
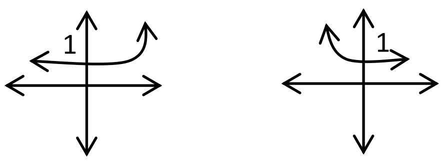



Figure 1.6

**Exponential Function**

The exponential function is defined as f(x) = $e^x$. Its graph is illustrated below:



**Logarithmic Function**

Logarithmic function is f (x) = log x. Its graph is illustrated below:

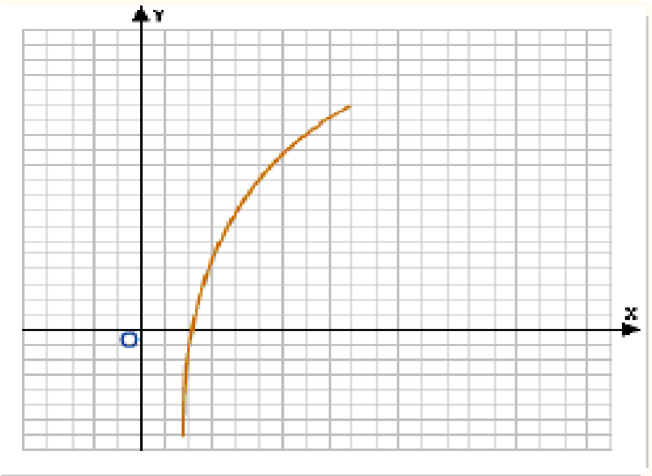



Figure: 1.7

**Trigonometric Functions**

Trigonometric functions are sinx, cosx, tanx, etc. The graph of these functions has been done in class XI.

**Inverse Functions**

Inverse functions are $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. The graph of these functions has been done in class XI.

**Signum Functions**

$$f(x) = \begin{Bmatrix} \frac{|x|,}{x} & x \neq 0 \\ 0, & x = 0 \end{Bmatrix}$$

$$\text{i.e. } f(x) = \begin{Bmatrix} 1 & x > 0 \\ 0 & x = 0 \\ & x < 0 \end{Bmatrix}$$

The graph of signum functions is illustrated below

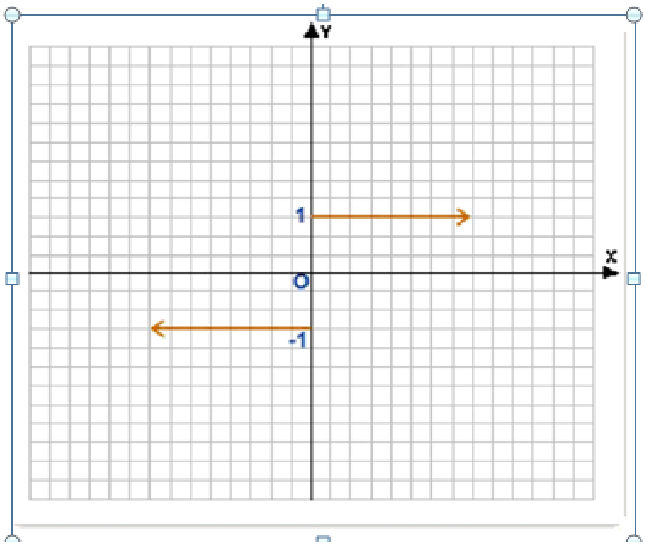



Figure: 1.8

**Odd Function**

A function f : A →B is said to be an odd function if f(x) = - f(-x) for all x ∈A

The domain and range of f depends on the definition of the function. Examples of odd function are
y = sinx, y = $x^3$, y = tanx

**Even Function**

A function f : A →B is said to be an even function if
f(x) = f(-x) for all x ∈A.

The domain and range of f depends on the definition of the function. Examples of even function are

y = cosx, y = $x^2$, y = secx

A polynomial with only even powers of x is an even function.

**Reciprocal Function**

F(x) = $\frac{1}{x} x \neq 0$

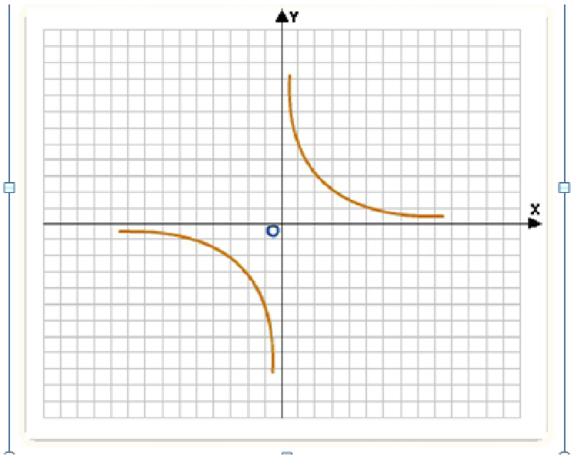



Figure: 1.9

**CONCLUSION**

In this unit, you have defined domain and types of domain. You have known real functions and have also learnt value of functions. You have also known types of graph and type of function.

**SUMMARY**

In this unit, you have defined and explained domain real function, value of functions, types of graph and function.

- domain
- real function
- value of functions
- types of graph
- types of function

**TUTOR – MARKED ASSIGNMENT**

1. Function f is defined by $f(x) = - 2x^2 + 6x – 3$, find f(- 2).
2. Function h is defined by $h(x) = 3x^2 - 7x – 5$, find h(x - 2).
3. Functions f and g are defined by f(x) = - 7x - 5 and g(x) = 10 x – 12, find (f + g)(x)
4. Functions f and g are defined by f(x) = 1/x + 3x and g(x) = -1/x + 6x – 4, find (f + g)(x) and its domain
5. Functions f and g are defined by $f(x) = x^2 -2x + 1$ and g(x) = (x - 1)(x + 3),find (f/g)(x) and its domain.

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# UNIT 2: LIMIT OF FUNCTION OF SEVERAL VARIABLES

**CONTENTS**



## 1.0: INTRODUCTION

Let f be a function of two variables defined on a disk with center (a,b), except possibly at (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y)=L$$

If for every number $\varepsilon > 0$ there is a corresponding number δ > 0 such

/f(x,y) – L/<ε whenever 0< $\sqrt{(x-a)^2+(y-b)^2} < \delta$

Other notations for the limit are

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = < and \; f(x,y)\to L \;\; as(x,y)\to(a,b)$$

Since $\frac{|f(x,y)-L|}{\sqrt{(x-a)^2+(y-b)^2}}$ is the distance between the numbers f(x,y) and L, and is the distance between the point (x,y) and the point (a,b). Definition 12.5 says that the distance between f(x,y) and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small (but not 0). Figure 12.15 illustrates Definition 12.5 by means of an arrow diagram. If any small interval (L - ε, L+ ε) is given around L, then we can find a disk $D_\delta$ with center (a,b) and radius $\delta < 0$ such that f maps all the points in $D_\delta$ [except possibly (a,b)] into the interval (*L* - ε, *L* + ε).

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain the definition of terms
- Find limit of function of several variable.

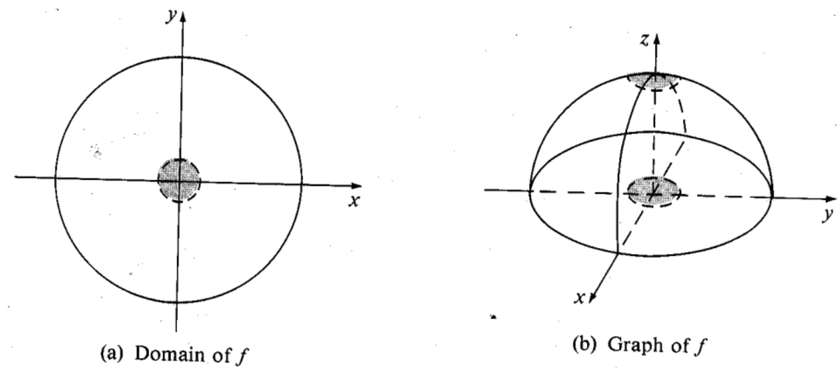
## 3.0 MAIN CONTENTS

Consider the function f(x,y) = $\sqrt{9^2 - x^2 y^2}$ whose domain is the closed disk D = $\{(x,y)|x^2 + y^2 \leq 9\}$ shown in Figure 12.14(a) and whose graph is the hemisphere shown in Figure 12.14(b)

If the point (x,y) is close to the origin, then x and y are both close to 0, and so f(x,y) is close to 3. In fact, if (x,y) lies in a small open disk $\sqrt{x^2 - y^2 \delta^2}$, then

f(x,y) = $\sqrt{9^2 - (x^2 + y^2)} > \sqrt{9^2 - \delta^2}$

Figure 12.14




Thus we can make the values of f(x,y) as close to 3 as we like by taking (x,y) in a small enough disk with centre (0,0). We describe this situation by using the notation

$$\lim_{(x,y)\to(a,b)} \sqrt{9-(x^2+y^2)} = 3$$

In general, the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Means that the values of f(x,y) can be made as close as we wish to the number L by taking the point (x,y) close enough to the point (a,b). A more precise definition follows.

## 12.5 **Definition**

Let f be a function of two variables defined on a disk with centre (a,b), except possibly at (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If for every number $\varepsilon < 0$ there is a corresponding number $\delta < 0$ such that

$|f(x,y) - L < |\varepsilon$ whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$

Other notations for the limit are

$\lim_{\substack{x\to a\\ y\to b}}$ f(x,y) = L and f(x,y) → L as (x,y) → (a,b)

Since $|f(x,y) - L|$ is the distance between the numbers f(x,y) and L, and

$\sqrt{(x-a)^2+(y-b)^2}$ is the distance between the point (x,y) and the point (a,b), Definition12.5 says that the distance between f(x,y) and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small (but not 0). Figure 12.15 illustrates Definition 12.5 by means of an arrow diagram. If any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, then we can find a disk $D_\delta$ with center (a,b) and radius $\delta > 0$ such that f maps all the points in $D_\delta$ [except possibly (a,b)] into the interval $(L-\varepsilon, L+\varepsilon)$.

Another illustration of Definition 12.5 is given in Figure 12.16 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x,y) is restricted to lie in the disk $D_\delta$ and $(x,y) \neq (a,b)$, then the corresponding part of S lies between the horizontal planes $(z = L - \varepsilon)$ and $(z = L + \varepsilon)$. For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or right. Recall from Chapter 2 that *if* $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x,y) approach (a,b) from an infinite number of directions in any manner whatsoever (see Figure 12.7).

**Definition 12.5** refers only to the *distance* between (x,y) and (a,b). It does not refer to the direction of approach. Therefore if the limit exists, then f(x,y) must approach the same limit no matter how (x,y) approaches (a,b). Thus if we can find two different paths of approach along which f(x,y) has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

**Figure 12.15**

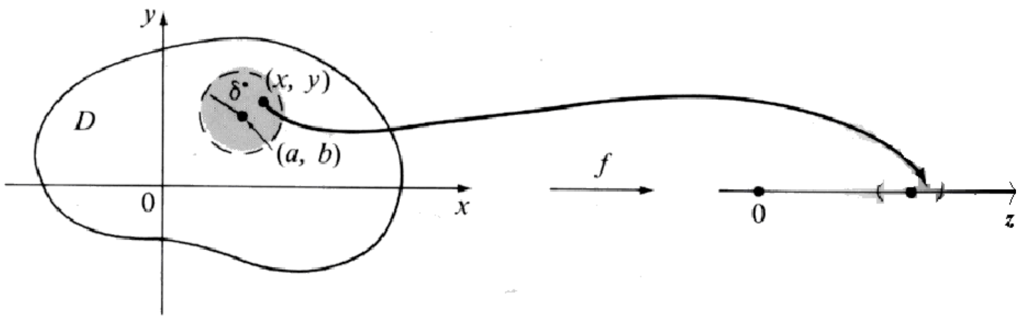



**Figure 12.16**

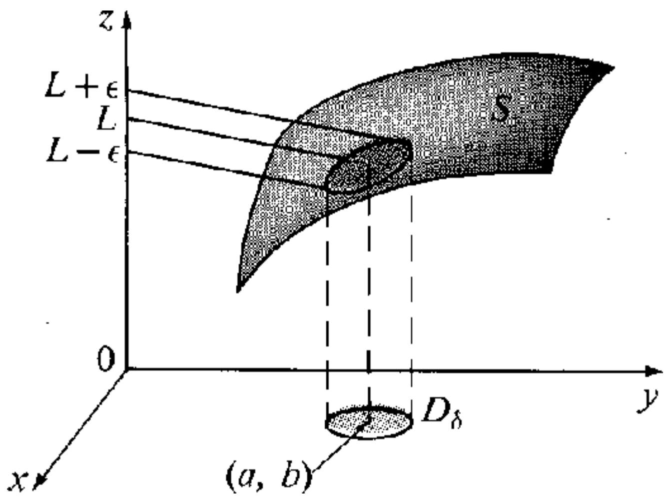
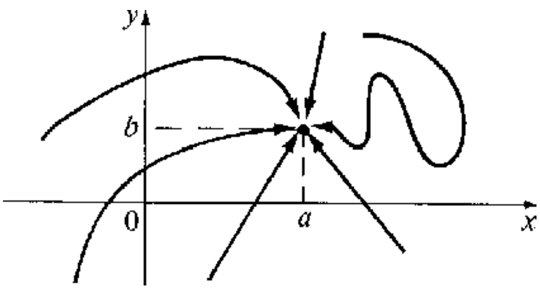


**Figure 12.17**




If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path $C_1$, and f(x,y) $\to L_2$ as (x,y) $\to (a,b)$ along a path $C_2$, where $L_1 \neq L_2$ , then $\lim_{(x,y)\to(a,b)} f(x,y)$

**Example 1**

Find $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ if it exists.

**Solution**

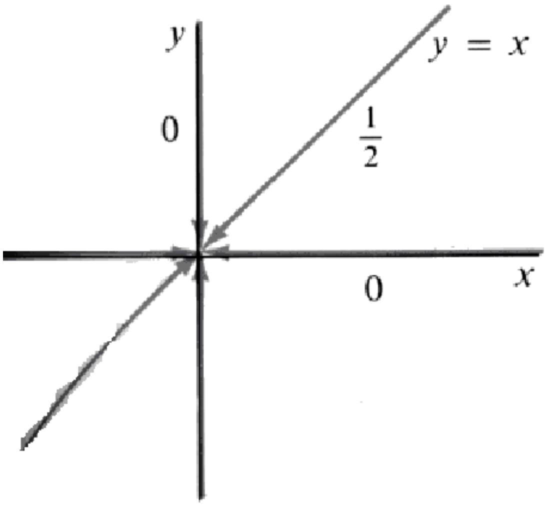
Let f(x,y) = $(x^2 - y^2)/(x^2 + y^2)$. First let us approach (0,0) along the x-axis. Then y = 0 gives f(x,0) = $x^2/x^2$ = 1 for all $x \neq 0$ , so

(x, y) → 1 as (x, y) → (0, 0) along the x-axis

We now approach along the y-axis by putting x = 0. Then f(0,y) = $-y^2/y^2$ = -1 for all $y \neq 0$ , so $(x,y) \to 1$ as $(x,y) \to (0,0)$ along the y-axis (see Figure 12.18.) Since f has two different limits along two different lines, the given limit does not exist.

**Figure 12.18**

**Figure 12.19**




**Example 2**

If f(x,y) = xy/($x^2$ + $y^2$), does $\lim_{(x,y) \to (0,0)}$ f(x,y) exist?

**Solution**

If y = 0, then f(x,0) = 0/$x^2$ = 0. Therefore
If (x,y) → 0 as (x,y) → (0, 0) along the x-axis
If x = 0, then f(0,y) = 0/$y^2$ = 0, so
If (x,y) → 0 as (x, y) → (0, 0) along the y-axis

Although we have obtained identical limits along the axes, which does not show that the given limit is 0. Let us now approach (0, 0) along another line, say y = x.
For all $x \neq 0$ .

$$f(x,y) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore f(x,y) → $\frac{1}{2}$ as (x,y) → (0,0) along y = x

(See Figure 12.19.) Since we obtained different limits along different paths, the given limit does not exist.

**Example 3**

If $f(x,y) = \frac{xy^2}{x^2 + y^2}$ , does $\lim_{(x,y) \to (0,0)}$ f(x,y) exist?

**Solution**

With the solution of Example 2 in mind, let us try to save time by letting $(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})$ along any line through the origin. Then y = mx, where m is the slope, and if $m \neq 0$ ,

$$f(x,y) = f(x,m\,x)\ \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^2}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2}$$

So $f(\mathbf{x},\mathbf{y}) \to \mathbf{0}$ as $(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})$ along y = mx

Thus f has the same limiting value along every line through the origin. But that does not show that the given limit is 0, for if we now let $(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})$ along the parabola $x = y^2$ we have

$$f(x,y) = f(y^2,y)\ \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist.

**Example 4**

Find $\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^2}$ if it exists.

**Solution**

As in Example 3, one can show that the limit along any line through the origin is 0. This does not prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist.

Let $\varepsilon < 0$ We want to find $\delta < 0$ such that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

This is $\frac{3x^2|y|}{x^2+y^2} < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

But $x^2 \leq x^2 + y^2$ since $y^2 \leq ,\ 0$

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$$

Thus if we choose $\delta = \frac{\varepsilon}{3}$ and let 0 $\sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 12.5

$$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^2} = 0$$

## 4.0: CONCLUSION

In this unit, you have known several definitions and have worked various examples.

## 5.0: SUMMARY

In this unit, you have studied the definition of terms and have solved various examples.

## 6.0: TUTOR-MARKED- ASSIGNMENT

1. Find the limit $\lim_{x\to1^-} \frac{x^2 2x - 3}{|x-1|}$
2. Find the limit $\lim_{x\to5} \frac{x^2-25}{x^2+\ x-30}$
3. Calculate the limit $\lim_{x\to2} \frac{x^2+4x-12}{|x-2|}$
4. Calculate the limit $\lim_{x\to1^+} \sqrt[3]{x+1}\ \mathbf{In}(\mathbf{x}+\mathbf{1})$
5. Find the limit $\lim_{x\to\theta} \frac{\sqrt{x}-3}{x-9}$
6. Find the limit $\lim_{t\to0} \frac{\sin t - t}{\tan t}$
7. Find the limit $\lim_{t\to\infty} \frac{3x}{\sqrt{16x^2+\ 1}}$

## 7.0: REFERENCES / FU RTHER READING

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# UNIT 3: CONTINUITY OF FUNCTION OF SEVERAL VARIABLES

## CONTENT



## 1.0 INTRODUCTION

Just as for functions of one variable, the calculation of limits can be greatly simplified by the use of properties of limits and by the use of continuity.

The properties of limits listed in Tables 2.14 and 2.15 can be extended to functions of two variables. The limit of a sum is the sum of the limits, and so on.

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} \mathbf{f(x)} = \mathbf{f(a)}$. Continuous functions of two variables are also defined by the direct substitution property.

**Definition**

Let f be a function of two variables defined on a disk with center (a,b). Then f is called **continuous at** (a,b) if f(x,y) = f(a,b)

## 2.0 OBJECTIVE

By the end of this unit, you should be able to the terns:

i. define continuity of function of several variables
ii. find whether a function is continuous or not
iii. evaluate unit of continuous function of several variable.

## 3.0 MAIN CONTENTS

Let f be a function of two variables defined on a disk with center (a,b). Then f is called **continuous at** (a,b) if $\lim_{(x,y) \to (a,b)}$ f(x,y) = f(a,b)

If the domain of f is a set $D \subset \mathrm{R}^2$, then Definition 12.6 defines the continuity of f at an **interior point** (a,b) of D, that is, a point that is contained in a disk $D_\partial \subset D$ [see Figure12.20(a)]. But D may also contain a **boundary point**, that is, a point (a,b) such that every disk with center (a,b) contains points in D and also points not in D [see Figure 12.20(b)].

If (a,b) is a boundary of D, then Definition 12.5 is modified so that the last line reads

$|f(x,y) - L| < \varepsilon$ whenever $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

With this convention, Definition 12.6 also applies when f is defined at a boundary point (a,b) of D.

Finally, we say *f* is **continuous on** D if f is continuous at every point (a,b) in D.

The intuitive meaning of continuity is that if the point (x,y) changes by a small amount, then the value of f(x,y) changes by a small amount. This means that a surface that is the graph of a continuous function has no holes or breaks.

Using the properties of limits, you can see that sums, differences, products, and -quotients of continuous functions are continuous on their domains. Let us use this fact to give examples of continuous functions.

**A polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are non-negative integers. A **rational function** is a ratio of polynomials. For instance,

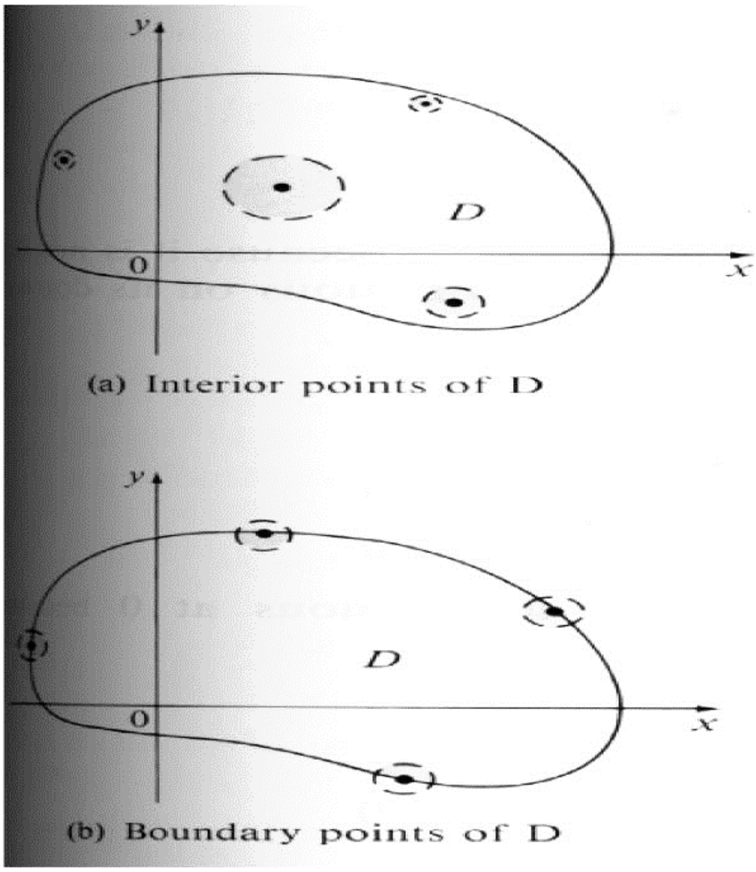
$$f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x,y) = \frac{2xy\ \ 1}{x^2 + y^2}$$

is a rational function.

**Figure 12.20**




From Definition it can be shown that

$$\lim_{(a,y)\to(a,b)} x = a \qquad \lim_{(a,y)\to(a,b)} y = b \qquad \lim_{(a,y)\to(a,b)} c = c$$

These limits show that the functions f(x,y) = x, g(x,y) = y, and h(x,y) = c are continuous. Since any polynomial can be built up out of the simple functions f, g and h by multiplication and addition, it follows that all polynomials are continuous on $R^2$. Likewise, any rational function is continuous on its domain since it is a quotient of continuous functions.

**Example 5**

Evaluate $\lim_{(x,y)-(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

**Solution**

Since $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so the limit can be found by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2.2^3 - 1^3.2^2 + 3.1 + 2.2 = 11$$

**Example 6**

Where is the function

f(x,) = $\frac{x^2 + y^2}{x^2 + y^2}$ Continuous?

**Solution**

The function f is discontinuous at (0,0) because it is not defined there. Since f is a rational function it is continuous on its domain D = $\{(x, y)|(x, y) \neq (0.0)\}$.

**Example 7**

Let

g(x,y) = $\left\{\frac{x^2 - y^2}{x^2 - y^2}\right\}$ if $(x, y) \neq (0.0)$

Here g is defined at (0,0) but g is still discontinuous at 0 because

$\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist (see Example 1).

**Example 8**

Let

f(x,y) = $\left\{\begin{matrix}\frac{3x^2 + y}{x^2 + y^2} \\ 0\end{matrix}\right\}$ If $(x, y) \neq (0,0)$

We know f is continuous $(x, y) \neq (0,0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim = \lim_{(x,y)\to(a,b)} \frac{3x^2 - y}{x^2 - y^2} = 0 \text{ f(0,0)}$$

Therefore f is continuous at (0,0), and so it is continuous on $R^2$.

**Example 9**

Let

$$f(x,y) = \begin{cases} \dfrac{3x^2 - y}{x^2 - y^2} & (x,y) \neq (0.0) \\ 17 \end{cases}$$

Again from Example 4, we have

$$\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} \frac{3x^2 - y}{x^2 - y^2} = 0 \neq 17 = g(0,0)$$

And so g is discontinuous at (0,0). But g is continuous on the set S = $\{(x,y)|(x,y) \neq (0.0)\}$ since it is equal to a rational function on S.

Composition is another way of combining two continuous functions to get at third. The proof of the following theorem is similar to that of Theorem 2.27.

**Theorem 12.7**

If f is continuous at (a,b) and g is a function of a single variable that is continuous at f(a,), then the composite function h = g o f defined by h(x,y) = g(f(x,y)) is continuous at (a,b).

**Example 10**

On what set is the function h(x,y) = ln($x^2$ + $y^2$ – 1) continuous?

**Solution**

Let f(x,y) = $x^2$ + $y^2$ – 1 and g(t) = ln t. Then

g(f(x,y)) = ln($x^2$ + $y^2$ – 1) = h(x,y)

So h = g o f. Now f is continuous everywhere since it is a polynomial and g is continuous on its domain {t$|t$ > 0}. Thus, by Theorem 12.7, h is continuous on its domain

D = {(x,y) $\left| x^2 + y^2 - 1 > 0 \right. = \{(x,y) | x^2 + y^2 > 1\}$

Which consists of all points outside the circle $x^2 + y^2 = 1$.

Everything in this section can be extended to functions of three or more variables. The distance between two points (x,y,z) and (a,b,c) in $R^3$ is

$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, so the definitions of limit and continuity of a function of three variables are as follows.

**Definition**

Let f: $D \subset R \to R$

(a) $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$

Means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

|f(x,y,z) - L| < ε whenever f(x,y,z)∈ D and

$0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$

(b) f is **continuous** at (a,b,c) if

$\lim_{(x,y)\to(a,b)} f(x,y,z) = f(a,b,c)$

If we use the vector notation introduced at the end of Section 12.1, then the definitions of a limit for functions of two or three variables can be written in a single compact form as follows.

If f: $D \subset R \to R$, then $\lim_{x\to a} \mathbf{f}(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

|f(x) – L| < ε whenever 0 < |x – a| < δ

Notice that if n = 1, then **x** = x and **a** = a, and (12.9) is just the definition of a limit for function of a single variable. If n = 2, then x = (x,y), **a** = (a,b), and $|x-a| = \sqrt{(x-a)^2+(y-b)^2}$, so (12.9) becomes Definition 12.5. If n = 3, then **x** = (x,y,z), **a** (a,b,c), and (12.9) becomes part (a) of Definition 12.8. In each case the definition of continuity can be written as

$$\lim_{x\to a} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$$

## 4.0 CONCLUSION

In this unit, you have known several definitions and have worked various examples.

## 5.0 SUMMARY

In this unit, you have studied the definition of terms and have solved various examples. The following limits

$$\lim_{(x,y)\to(a,b)} x = a, \quad \lim_{(x,y)\to(a,b)} y = b \text{ and } \lim_{(x,y)\to(a,b)} C = c$$

Show that the functions f(x,y) = x, g(x,y) = y, and h(x,y) = c are continuous. Obviously any polynomial can be built up out of the simple functions f, g and h by multiplication and addition, it follows that all polynomials are continuous on $R^2$. Likewise, any rational function is continuous on its domain since it is a quotient of continuous functions.

## 6.0 Tutor-Marked Assignment

In Exercises 1 – 3 determine the largest set on which the given function is continuous

1. $\mathbf{F(x,y) = \dfrac{x^2+y^2+1}{x^2+y^2-1}}$

2. $\mathbf{F(x,y) = \dfrac{x^6 + x^3y^3 + y^6}{x^3 + y^3}}$

3. $\mathbf{G(x,y) = \sqrt{x+y}} - \sqrt{x-y}$

4. For what values of the number r is the function

$$\mathbf{F(x,y)} = \begin{cases} \dfrac{\mathbf{x + y + z^r}}{\mathbf{x^2 + y^2 + z^2}} & \text{If}(x,y,z) \neq (0,0,0) \\ 0 & \end{cases}$$

Continuous on $R^3$?

5. If $c \in V_n$, show that the function f:$R^n \to R$ given by f(**x**) = **c**.**x** is continuous on $R^n$.

6. Show that function f defined below is not continuous at x = - 2.

*f(x) = 1 / (x + 2)*

7. Show that function f is continuous for all values of x in R.

$$f(x) = 1 / ( x^4 + 6)$$

8. Show that function f is continuous for all values of x in R.

$$f(x) = | x -5 |$$

9. Find the values of x at which function f is discontinuous.

$$f(x) = (x - 2) / [ (2 x^2 + 2x - 4)(x^4 + 5) ]$$

10. Evaluate the limit $\lim_{x \to a} \sin(2x + 5)$

11. Show that any function of the form $e^{ax + b}$ is continuous everywhere, *a* and *b* real numbers.

## 7.0 REFERENCES/FURTHER READING

Bartle, R. G. and Sherbert, D. *Introduction to Real Analysis.* New York: Wiley, p. 141, 1991.

Kaplan, W. "Limits and Continuity." §2.4 in *Advanced Calculus, 4th ed.* Reading, MA: Addison-Wesley, pp. 82-86, 1992

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