MODULE 2 PARTIAL DERIVATIVES OF FUNCTION OF SEVERAL VARIABLES

Unit 1: Derivative
Unit 2: Partial derivative.
Unit 3: Application of Partial derivative.

UNIT 1: DERIVATIVE

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1.0 INTRODUCTION

In calculus, a derivative is a measure of how the function changes as the input changes. Loosely speaking, a derivative can be thought of how much one quantity is changing in response to changes in some other quantity. For example, the derivative of the position of a moving object with respect to time, is the object instantaneous velocity.

The derivative of a function at a given chosen input value describe the best linear approximation of the function near that input value. For a real valued function of a single real variable. The derivative at a point equals the slope of the tangent line to the graph of the function at that point. In higher dimension, the derivative of a function at a point is linear transformation called the linearization. A closely related notion is the differential of a function. The process of finding a derivative is differentiation. The reverse is Integration.

The derivative of a function represents an infinitesimal change in the function with respect to one of its variables.

The “simple” derivative of a function f with respect to a variable x is denoted either $f'(x)$ or $^{df}/_{dx}$

2.0 OBJECTIVE

In this Unit, you should be able to:

- explain the derivative of a function;
- identify higher derivative;
- solve problems by computing derivative; and
- identify derivative of higher dimension

3.0 MAIN CONTENT

3.1 The Derivative of a Function

Let f be a function that has a derivative at every point a in the domain of f. because every point a has a derivative, there is a function that sends the point a to the derivative of f at a. This function is written $f'(x)$ and is called the *derivative function* or the *derivative* of f. The derivative of f collects all the derivatives of f at all the points in the domain of f.

Sometimes f has a derivative at most, but not all, points of its domain. The function whose value at a equals $f'(a)$ whenever $f'(a)$ is defined and elsewhere is undefined is also called the derivative of f. It is still a function, but its domain is strictly smaller than the domain of f.

Using this idea, differentiation becomes a function of functions: The derivative is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions. If we denote this operator by D, then $D(f)$ is the function $f'(x)$. Since $D(f)$ is a function, it can be evaluated at a point a. By the definition of the derivative function, $D(f)(a) = f'(a)$.

For comparison, consider the doubling function $f(x) = 2x$; f is a real-valued function of a real number, meaning that it takes numbers as inputs and has numbers as outputs:

The operator D, however, is not defined on individual numbers. It is only defined on functions:

$1 \to 2,$
$2 \to 4,$
$3 \to 6.$

The operator D, however, is not defined on individual numbers. It is only defined functions:

$D(x \to 1) = (x \to 0)$
$D(x \to x) = (x \to 1)$
$D(x \to x^2) = (x \to 2x).$

Because the output of *D* is a function, the output of *D* can be evaluated at a point. For instance, when *D* is applied to the squaring function,

$x \to x^2$

D outputs the doubling function,

$x \to 2x$

which we named *f* (*x*). This output function can then be evaluated to get *f* (1) = 2, *f* (2) = 4, and so on.

3.2 Higher derivative

Let *f* be a differentiable function, and let *f'(x)* be its derivative. The derivative of *f'(x)* (if it has one) is written *f''(x)* and is called the second derivative **of f**. Similarly, the derivative of a second derivative, if it exists, is written *f'''(x)* and is called the **third derivative** of ***f***. These repeated derivatives are called *higher-order derivatives*.

If *x*(*t*) represents the position of an object at time *t*, then the higher-order derivatives of *x* have physical interpretations. The second derivative of *x* is the derivative of *x'*(*t*), the velocity, and by definition this is the object's acceleration. The third derivative of *x* is defined to be the jerk, and the fourth derivative is defined to be the jounce.

A function *f* need not have a derivative, for example, if it is not continuous. Similarly, even if *f* does have a derivative, it may not have a second derivative. For example, let

$$f(x) = \begin{cases} +x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x \leq 0. \end{cases}$$

Calculation shows that *f* is a differentiable function whose derivative is

$$f'(x) = \begin{cases} +2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x \leq 0. \end{cases}$$

f'(x) is twice the absolute value function, and it does not have a derivative at zero. Similar examples show that a function can have *k* derivatives for any non-negative integer *k* but no *(k+ 1)*-order derivative.

A function that has *k* successive derivatives is called ***k* times differentiable**. If in addition the *k*th derivative is continuous, then the function is said to be of differentiability class C^k. (This is a stronger condition than having *k* derivatives.) A function that has infinitely many derivatives is called **infinitely differentiable**.

On the real line, every polynomial function is infinitely differentiable. By standard differentiation rules, if a polynomial of degree *n* is differentiated *n* times, then it becomes a constant function. All of its subsequent derivatives are identically zero. In particular, they exist, so polynomials are smooth functions.

The derivatives of a function f at a point *x* provide polynomial approximations to that function near *x*. For example, if f is twice differentiable, then

$$f(x+h) \approx f(x) + f'(x)h + \frac{1}{2}f''(x)h^2$$

in the sense that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) + f'(x)h + \frac{1}{2}f''(x)h^2}{h^2} = 0$$

If f is infinitely differentiable, then this is the beginning of the Taylor series for f.

Inflexion Point

A point where the second derivative of a function changes sign is called an **inflexion point**. At an inflexion point, the second derivative may be zero, as in the case of inflection point $x=0$ of the function $y=x^3$, or it may fail to exist, as in the case of the inflexion point $x=0$ of the function $y= x^{1/3}$. At an inflexion point, a function switches from being a convex function to being a co ncave function or vice versa.

3.3 Computing the derivative

The derivative of a function can, in principle, be computed from the definition by considering the difference quotient, and computing its limit. In practice, once the derivatives of a few simple functions are known, the derivatives of other functions are more easily computed using *rules* for obtaining derivatives of more complicated functions from simpler ones.

Derivative of Elementary Function

Most derivative computations eventually requires taking the derivative of some common functions. The following incomplete list gives some of the most frequently used functions of a single real variable and their derivatives.

- Derivative power: if

$f(x) = x^r$

where*r* is any real number, then
$f'(x) = rx^{r-1}$,

wherever this function is defined. For example, if $f(x) = x^{1/4}$, then

$f'(x) = (1/4)x^{\frac{-3}{4}}$,

and the derivative function is defined only for positive x, not for $x = 0$. When $r = 0$, this rule implies that $f'(x)$ is zero for $x \neq 0$, which is almost the constant rule (stated below).

Exponential *and* logarithm *functions*:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = \ln(a)a^x$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}\log_a x = \frac{1}{xln(a)}$$

Trigonometric Functions:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(\mathrm{x}) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = sec^2(x) = \frac{1}{cos^2(x)} = 1 + tan^2(x)$$

Inverse Trigonometric Function:

$$\frac{d}{dx}\text{arc}\sin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\text{arc}\cos(\mathrm{x}) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\text{arc}\tan(x) = \frac{1}{\sqrt{1+x^2}}$$

Rules for finding the derivative

In many cases, complicated limit calculations by direct application of Newton's difference quotient can be avoided using differentiation rules. Some of the most basic rules are the following.

Constant rule: if $f(x)$ is constant, then

$f' = 0$

Sine rule:
$(af + bg)' = af' + bg'$ for all functions f and g and all real numbers a and b.

Product rule :

$(fg)' = f'g + fg'$ for all functions f and g.

Quotient rule:

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ for all functions f and g where $g \neq 0$.

Chain rule: If $f(x) = h(g(x))$, then

$f'(x) = h'(\text{g(x)}).g'(x)$

Example computation

The derivative of $f(x) = x^4 + \sin(x^2) - \ln(x)\, e^x + 7$

$$f(x) = 4x^{(4-1)} + \frac{d(x^2)}{dx}\cos(x^2) - \frac{d(\ln x)}{dx}e^x - \ln x\frac{d(e^x)}{dx} + 0$$

$$= 4x^3 + 2x\cos\left(x^2\right) - \frac{1}{x}e^x - \ln(x)\, e^x$$

Here the second term was computed using the chain rule and third using the product rule. The known derivatives of the elementary functions x^2, x^4, $\sin(x)$, $\ln(x)$ and $\exp(x) = e^x$, as well as the constant 7, were also used.

3.4 Derivatives in higher dimensions

Derivative of vector valued function

A vector valued function $\mathbf{y}(t)$ of a real variable sends real numbers to vectors in some vector space $\mathbf{R}^n$. A vector-valued function can be split up into its coordinate functions $y_1(t), y_2(t), \ldots, y_n(t)$, meaning that $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))$. This includes, for example, parametric curve in$\mathbf{R}^2$or $\mathbf{R}^3$. The coordinate functions are real valued functions, so the above definition of derivative applies to them. The derivative of $\mathbf{y}(t)$ is defined to be the vector, called the tangent vector, whose coordinates are the derivatives of the coordinate functions. That is,

$$y'(t) = (y_1'(t), \dots, y_n'(t))$$

Equivalently,

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h},$$

if the limit exists. The subtraction in the numerator is subtraction of vectors, not scalars. If the derivative of **y** exists for ever y value of t, then $\mathbf{y'}$ is another vector valued function.

If $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis for $\mathbf{R}^n$, then $\mathbf{y}(t)$ can also be written as $y_1(t)\mathbf{e}_1 + \dots + y_n(t)\mathbf{e}_n$. If we assume that the derivative of a vector-valued function retains the linearity property, then the derivative of $\mathbf{y}(t)$ must be

$$y_1'(t)e_1 + \cdots + y_n'(t)e_n$$

because each of the basis vectors is a constant.

This generalization is useful, for example, if $\mathbf{y}(t)$ is the position vector of a particle at time t; then the derivative $\mathbf{y}'(t)$ is the velocity vector of the particle at time t.

Partial derivative

Suppose that f is a function that depends on more than one variable. For instance,

$$f(x, y) = x^2 + xy + y^2$$

f can be reinterpreted as a family of functions of one variable indexed by the other variables:

$$f(x, y) = f_x(y) = x^2 + xy + y^2$$

In other words, every value of x chooses a function, denoted f_x, which is a function of one real number. That is,

$$x \to f_x$$

$$f_x(y) = x^2 + xy + y^2$$

Once a value of x is chosen, s ay a, then $f(x,y)$ determines a function f_a that sends y to $a^2 + \text{ay} + y^2$:

$f_a(y) = a^2 + ay + y^2$

In this expression, a is a *constant*, not a *variable*, so f_a is a function of only one real variable. Consequently the definition of the derivative for a function of one variable applies:

$f_a'(y) = a + 2y$

The above procedure can be performed for any choice of a. Assembling the derivatives together into a function gives a function that describes the variation of f in the y direction:

$$\frac{df}{dy}(x, y) = x + 2y$$

This is the partial derivative of f with respect to y. Here ∂ is a rounded d called the **partial derivative symbol**. To distinguish it from the letter$d$,$\partial$is sometimes pronounced "der","del", or "partial" instead of "dee".

In general, the **partial derivative** of a function $f\ (x_1, \ldots, x_n)$ in the direction x_i at the point $(a_1 \ldots, a_n)$ is defined to be:

$$\frac{df}{dy}(a_1, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

In the above difference quotient, all the variables except x_i are held fixed. That choice of fixed values determines a function of one variable

$$f_{a_1, \ldots, a_i - 1, a_i + 1, \ldots, a_n}(x_i) = f(a_1, \ldots, a_{i-1,} x_i, a_{i+1,}, \ldots, a_n$$

and, by definition,

$$\frac{df_{a_1, \ldots, a_i - 1, a_i + 1, \ldots, a_n}}{dx_i}(a_i) = \frac{df}{dx_i}(a_1, \ldots, a_n)$$

In other words, the different choices of a index a family of one-variable functions just as in the example above. This expression also shows that the computation of partial derivatives reduces to the computation of one-variable derivatives.

An important example of a function of several variables is the case of a scalar valued function $f\ (x_1, \ldots x_n)$ on a domain in Euclidean space $\mathbf{R}^n$(e.g., on$\mathbf{R}^2$ or$\mathbf{R}^3$). In this case f has a partial derivative $\partial f\ /\partial x_j$ with respect to each variable x_j. At the point a, these partial derivatives define the vector

$$\nabla f(a) = \left(\frac{df}{dx_1}(a), \dots, \frac{df}{dx_n}(a)\right)$$

This vector is called the gradient of f at a. If f is differentiable at every point in some domain, then the gradient is a vector-valued function ∇f that takes the point *a* to the vector $\nabla f(a)$. Consequently the gradient determines a vector field.

Generalizations

The concept of a derivative can be extended to many other settings. The common thread is that the derivative of a function at a point serves as a linear approximation of the function at that point.

4.0 CONCLUSION

In this unit, you have explain the derivative of a function .Through the derivative of functions, you have identified higher derivative, and you have solved problems by computing derivative through the use of this functions. You have also identified derivative of higher dimension.

5.0 SUMMARY

In this unit, you have studied the following:

- the derivative of a function
- identify higher derivative
- solve problems by Computing derivative
- identify derivative of higher dimension

6.0 TUTOR MARKED ASSIGNMENT

1. Find the derivative of F(x,y) = 3sin(3xy)
2. Find the derivative of F(x,y)= (x^3+ln 6)($\sqrt{y}$)
3. Evaluate the derivative F(x,y) = x^2+ 3xy– 2 tan(y)
4. Find the derivative of F(x,y) =$\frac{y \sin x}{e^{cosx}}$

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UNIT 2: PARTIAL DERIVATIVES

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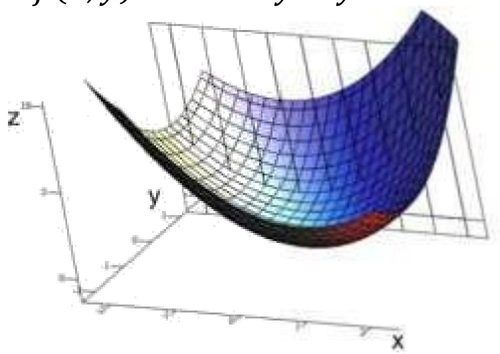
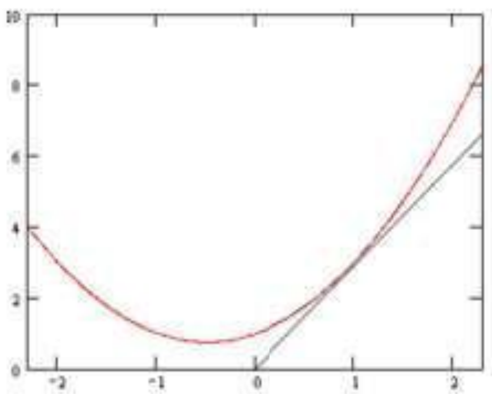
1.0 INTRODUCTION

Suppose that f is a function of more than one variable. For instance,

$$z = f(x,y) = x^2 + xy + y^2$$



A graph of $z = x^2 + xy + y^2$. For the partial derivative at (1, 1, 3) that leaves y constant, the corresponding tangent line is parallel to the xz-plane.



A slice of the graph above y=1

The graph of this function defines a surface in Euclidean space. To every point on this surface, there are infinite number of tangent lines. Partial differentiation is the act of choosing one of these lines and finding its slope. Usually, the lines of most interest are those that are parallel to the *xz*-plane, and those that are parallel to the *yz*-plane.

To find the slope of the line tangent to the function at (1, 1, 3) that is parallel to the *xz*-plane, the y variable is treated as constant. The graph and this plane are shown on the right. On the graph below it, we see the way the function looks on the plane $y = 1$. By finding the derivative of the equation while assuming that y is a constant, the slope of f at the point (x, y, z) is found to be:

$$\frac{dz}{dx} = 2x + y$$

So at (1, 1, 3), by substitution, the slope is 3. Therefore

$$\frac{dz}{dx} = 3$$

at the point. (1,1,3). That is, the partial derivative of z with respect to x at (1,1,3) is 3

2.0 OBJECTIVES

After studying this, you should be able to:

- define Partial derivative;
- explain the geometric interpretation of partial derivatives;
- identify anti derivative analogue;
- solve problems on partial derivative for function of several variables; and
- identify higher order derivatives.

3.1 MAIN CONTENT

Let us consider a function

1) $u = f(x, y, z, p, q, \ldots)$

of several variables. Such a function can be studied by holding all variables except one constant and observing its variation with respect to one single selected variable. If we consider all the variables except x to be constant, then

$$\frac{du}{dx} = \frac{d\, f(x, \hat{y}, \hat{z}, \hat{p}, \hat{q}, \ldots)}{dx}$$

represents the partial derivative of f(x, y, z, p, q, ...) with respect to x (the hats indicating variables held fixed). The variables held fixed are viewed as parameters.

Definition of Partial derivative

The partial derivative of a function of two or more variables with respect to one of its variables is the ordinary derivative of the function with respect to that variable, considering the other variables as constants.

Example 1: The partial derivative of $3x^2y + 2y^2$ with respect to x is 6xy. Its partial derivative with respect to y is $3x^2 + 4y$.

The partial derivative of a function z = f(x, y,...) with respect to the variable x is commonly written in any of the following ways:

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad \frac{\partial f(x,y,\ldots)}{\partial x}, \quad D_x f(x,y,\ldots), \quad D_x f, \quad f_x(x,y,\ldots), \quad f_x, \quad f_1(x,y,\ldots)$$

Its derivative with respect to any other variable is written in a similar fashion.

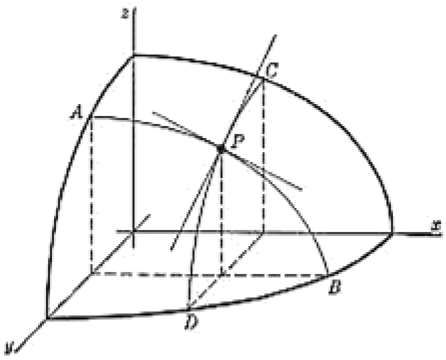


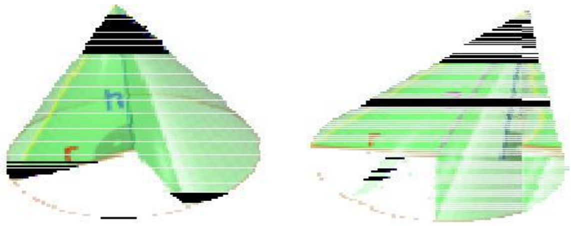
Figure: 2.1

Geometric Interpretation: The geometric interpretation of a partial derivative is the same as that for an ordinary derivative. It represents the slope of the tangent to that curve represented by the function at a particular point P. In the case of a function of two variables

z = f(x, y)

Fig. 2.1 shows the interpretation of df/dx and df/dy. df/dx corresponds to the slope of the tangent to the curve APB at point P (where curve APB is the intersection of the surface with a plane through P perpendicular to the y axis). Similarly, df/dy corresponds to the slope of the tangent to the curve CPD at point P (where curve CPD is the intersection of the surface with a plane through P perpendicular to the x axis).

Examples 2



The volume of a cone depends on height and radius

The volume V of a cone depends on the cone's height h and its radius r according to the formula

$$V(r,h) = \frac{\pi r^2 h}{3}.$$

The partial derivative of V with respect to r is

$$\frac{\partial V}{\partial r} = \frac{\pi r^2 h}{3},$$

which represents the rate with which a cone's volume changes if its radius is varied and its height is kept constant. The partial derivative with respect to h is

$$\frac{\partial V}{\partial h} = \frac{\pi r^2}{3},$$

which represents the rate with which the volume changes if its height is varied and its radius is kept constant.

By contrast, the *total* derivative of V with respect to r and h are respectively

$$\frac{dV}{dr} = \overbrace{\frac{2\pi rh}{3}}^{\frac{\partial V}{\partial r}} + \overbrace{\frac{\pi r^2}{3}}^{\frac{\partial V}{\partial h}}\frac{dh}{dr}$$

and

$$\frac{\partial V}{\partial h} = \overbrace{\frac{\pi r^2}{3}}^{\frac{\partial V}{\partial h}} + \overbrace{\frac{2\pi rh}{3}}^{\frac{\partial V}{\partial r}}\frac{dr}{dh}$$

The difference between the total and partial derivative is the elimination of indirect dependencies between variables in partial derivatives.

If (for some arbitrary reason) the cone's proportions have to stay the same, and the height and radius are in a fixed ratio *k*,

$$k = \frac{h}{r} = \frac{\mathrm{d}\,h}{\mathrm{d}\,r}.$$

This gives the total derivative with respect to *r*:

$$\frac{\mathrm{d}\,V}{\mathrm{d}\,r} = \frac{2\pi r h}{3} + k\frac{\pi r^2}{3}$$

Equations involving an unknown function's partial derivatives are called partial differential equations and are common in physics, engineering, and other sciences and applied disciplines.

Notation

For the following examples, let f be a function in x, y and z.

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = f_x = \partial_x f$$

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \partial_{xx} f$$

Second-order mixed derivatives:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = \partial_{yx} f$$

Higher-order partial and mixed derivatives:

$$\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} = f^{(i,j,k)}$$

When dealing with functions of multiple variables, some of these variables may be related to each other, and it may be necessary to specify explicitly which variables are being held constant.

In fields such as statistical mechanics, the partial derivative of f with respect to x, holding y and z constant, is often expressed as

$$\left(\frac{\partial f}{\partial x}\right)_{y,z}$$

Anti-derivative analogue

There is a concept for partial derivatives that is analogous to anti derivatives for regular derivatives. Given a partial derivative, it allows for the partial recovery of the original function.

Consider the example of $\frac{\partial z}{\partial x} = 2x + y.$ The "partial" integral can be taken with respect to x (treating y as constant, in a similar manner to partial derivation):

$$z = \int \frac{\partial z}{\partial x} dx = x^2 + xy + g(y)$$

Here, the "constant" of integration is no longer a constant, but instead a function of all the variables of the original function except x. The reason for this is that all the other variables are treated as constant when taking the partial derivative, so any function which does not involve x will disappear when taking the partial derivative, and we have to account for this when we take the ant derivative. The most general way to represent this is to have the "constant" represent an unknown function of all the other variables. Thus the set of functions

$x^2+xy+ g(y)$, where g is any one-argument function, represents the entire set of functions invariables x,y that could have produced the x-partial derivative $2x+y$.

If all the partial derivatives of a function are known (for example, with the gradient), then the antiderivatives can be matched via the above process to reconstruct the original function up to a constant

Example 3

For the function

$$f(x, y) = x^2 + x^3y^2 + y^4$$

find the partial derivatives of f with respect to x and y and compute the rates of change of the function in the x and y directions at the point (-1,2).

Initially we will not specify the values of x and y when we take the derivatives; we will just remember which one we are going to hold constant while taking the derivative. First, hold y fixed and find the partial derivative of f with respect to x:

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = 2x + 3x^2y^2$$

Second, hold *x* fixed and find the partial derivative of *f* with respect to *y*:

Now, plug in the values x = -1 and y = 2 into the equations. We obtain f_x (-1,2) = 10 and f_y (-1,2) = 28.

Partial Derivatives for Functions of Several Variables

We can of course take partial derivatives of functions of more than two variables. If f is a function of n variables x_1, x_2, ...,x_n, then to take the partial derivative of f with respect to x_i we hold all variables besides x_i constant and take the derivative.

Example 4

To find the partial derivative of f with respect to t for the function

$$f(x, y, z, t) = x^2 + y^2 + z^2 + t^2 + xyzt^{-3}$$

we hold x, y, and z constant and take the derivative with respect to the remaining variable t. The result is

$$\frac{\partial f}{\partial t}(x, y, z, t) = 0 + 0 + 0 + 2t - 3xyzt^{-4}$$

Interpretation

$\frac{\partial f}{\partial x}$ is the rate at which f changes as x changes, for a fixed (constant) y.

$\frac{\partial f}{\partial y}$ is the rate at which f changes as y changes, for a fixed (constant) x.

Higher Order Partial Derivatives

If f is a function of x, y, and possibly other variables, then

$\frac{\partial^2 f}{\partial x^2}$ is defined to be $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$

Similarly,

$\frac{\partial^2 f}{\partial y^2}$ is defined to be $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$

$\frac{\partial^2 f}{\partial y \partial x}$ is defined to be $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$

$\frac{\partial^2 f}{\partial x \partial y}$ is defined to be $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

The above second order partial derivatives can also be denoted by f_{xx}, f_{yy}, f_{xy}, and f_{yx} respectively.

The last two are called **mixed derivatives** and will always be equal to each other when all the first order partial derivatives are continuous.

Some examples of partial derivatives of functions of several variables are shown below, variable as we did in Calculus I.

Example 1: Find all of the first order partial derivatives for the following functions.

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

(b) $w = x^2y - 10y^2z^3 + 43x - t\tan(4y)$

(c) $h(s,t) = t^7\ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

(d) $f(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$

Solution

(a) $\boldsymbol{f(x,y) = x^4 + 6\sqrt{y} - 10}$

Let's first take the derivative with respect to *x* and remember that as you do so all the *y*'s will be treated as constants. The partial derivative with respect to *x* is,

$f_x(x,y) = 4x^3$

Notice that the second and the third term differentiate to zero in this case. It should be clear why the third term differentiated to zero. It's a constant and you know that constants always differentiate to zero. This is also the reason that the second term differentiated to zero. Remember that since you are differentiating with respect to *x* here you are going to treat all *y*'s as constants. That means that terms that only involve *y*'s will be treated as constants and hence will differentiate to zero.

Now, let's take the derivative with respect to *y*. In this case you treat all *x*'s as constants and so the first term involves only *x*'s and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to *y*.

$$f_x(x, y) = \frac{3}{\sqrt{y}}$$

(b) $\boldsymbol{w = x^2y - 10y^2z^3 + 43x - t\tan(4y)}$

With this function you've got three first order derivatives to compute. Let's do the partial derivative with respect to *x* first. Since you are differentiating with respect to *x* you will treat all *y*'s and all *z*'s as constants. This means that the second and fourth terms will differentiate to zero since they only involve *y*'s and *z*'s.

This first term contains both *x*'s and *y*'s and so when you differentiate with respect to *x* the *y* will be thought of as a multiplicative constant and so the first term will be differentiated just as the third term will be differentiated.

Here is the partial derivative with respect to *x*.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Let's now differentiate with respect to *y*. In this case all *x*'s and *z*'s will be treated as constants. This means the third term will differentiate to zero since it contains only *x*'s while the *x*'s in the first term and the *z*'s in the second term will be treated as multiplicative constants. Here is the derivative with respect to *y*.

$$\frac{\partial w}{\partial x} = x^2 - 20yz^3 - 28\sec^2(4y)$$

Finally, let's get the derivative with respect to *z*. Since only one of the terms involve*z*'s this will be the only non-zero term in the derivative. Also, the *y*'s in that term will be treated as multiplicative constants. Here is the derivative with respect to *z*.

$$\frac{\partial w}{\partial x} = -30y^2z^2$$

(c) $\boldsymbol{h(s, t) = t^7\ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}}$

With this one you'll not put in the detail of the first two. Before taking the derivative let's rewrite the function a little to help us with the differentiation process.

$$h(s, t) = t^7\ln(s^2) + 9t^{-3} - s^{\frac{4}{7}}$$

Now, the fact that you're using *s* and *t* here instead of the "standard" *x* and *y* shouldn't be a problem. It will work the same way. Here are the two derivatives for this function.

$$h_s(s,t) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$

$$h_s(s,t) = \frac{\partial h}{\partial s} = 7t^6\ln(s^2) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{\partial}{\partial x}(\ln(x)) = \frac{g'(x)}{g(x)}$$

(d) $\boldsymbol{f(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}}$

Now, you can't forget the product rule with derivatives. The product rule will work the same way here as it does with functions of one variable. You will just need to be careful to remember which variable we are differentiating with respect to.

Let's start out by differentiating with respect to *x*. In this case both the cosine and the exponential contain *x*'s and so you've really got a product of two functions involving *x*'s and so you'll need to product rule this up. Here is the derivative with respect to *x*.

$$f_x(x,y) = -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)e^{x^2y-5y^2} + \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}(2xy)$$

$$= -\frac{4}{x^2}\sin\left(\frac{4}{x}\right)e^{x^2y-5y^3} + 2xy\cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$$

Do not forget the chain rule for functions of one variable. You will be looking at the chain rule for some more complicated expressions for multivariable functions in a latter section. However, at this point you're treating all the *y*'s as constants and so the chain rule will continue to work as it did back in Calculus I.

Also, don't forget how to differentiate exponential functions,

$$\frac{\partial}{\partial x}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

Now, let's differentiate with respect to *y*. In this case you don't have a product rule to worry about since the only place that the *y* shows up is in the exponential. Therefore, since *x*'s are considered to be constants for this derivative, the cosine in the front will also be thought of as a multiplicative constant. Here is the derivative with respect to *y*.

$$f_x(x,y) = (x^2 - 15y^2)\cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$$

Example 2: Find all of the first order partial derivatives for the following functions.

(a) $z = \frac{9u}{u^2+5v}$

(b) $g(x, y, z) = \frac{x\sin(y)}{z^2}$

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$

Solution

(a) $z = \frac{9u}{u^2+5v}$

You also can't forget about the quotient rule. Since there isn't too much to this one, you will simply give the derivatives.

$$z_u = \frac{9(u^2 + 5v) - 9u(2u)}{(u^2 + 5v)^2} = \frac{-9u^2 + 45v}{(u^2 + 5v)^2}$$

$$z_v = \frac{(0)(u^2 + 5v) - 9u(5)}{(u^2 + 5v)^2} = \frac{-45u}{(u^2 + 5v)^2}$$

In the case of the derivative with respect to v recall that u's are constant and so when you differentiate the numerator you will get zero!

Now, you do need to be careful however to not use the quotient rule when it doesn't need to be used. In this case you do have a quotient, however, since the x's and y's only appear in the numerator and the z's only appear in the denominator this really isn't a quotient rule problem.

(b) $g(x, y, z) = \frac{x\sin(y)}{z^2}$

Let's do the derivatives with respect to x and y first. In both these cases the z's are constants and so the denominator in this is a constant and so we don't really need to worry too much about it. Here are the derivatives for these two cases.

$$g_x(x, y, z) = \frac{\sin(y)}{z^2} \qquad g_y(x, y, z) = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to z we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to z.

$$g(x, y, z) = x \sin(y)\, z^{-2}$$

$$g(x, y, z) = -2x \sin(y)\, z^{-3} = \frac{2x \sin(y)}{z^3}$$

You went ahead and put the derivative back into the "original" form just so you could say that you did. In practice you probably don't really need to do that.

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$

In this last part we are just going to do a somewhat messy chain rule problem. However, if you had a good background in Calculus I chain rule this shouldn't be all that difficult of a problem. Here are the two derivatives,

$$z_x = \frac{1}{2}(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}} \frac{\partial}{\partial x}(x^2 + \ln(5x - 3y^2))$$

$$= \frac{1}{2}(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}} \left(2x + \frac{5}{5x - 3y^2}\right)$$

$$\left(x + \frac{5}{2(5x - 3y^2)}\right)(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}}$$

$$z_y = \frac{1}{2}(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}} \frac{\partial}{\partial y}(x^2 + \ln(5x - 3y^2))$$
$$= \frac{1}{2}(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}} \left(\frac{-6y}{5x - 3y^2}\right)$$

$$\left(\frac{3y}{5x - 3y^2}\right)(x^2 + \ln(5x - 3y^2))^{\frac{-1}{2}}$$

So, there are some examples of partial derivatives. Hopefully you will agree that as long as we can remember to treat the other variables as constants these work is exactly the same manner that derivatives of functions of one variable do. So, if you can do Calculus I derivative you shouldn't have too much difficulty in doing basic partial derivatives.

There is one final topic that you need to take a quick look at in this section, implicit differentiation. Before getting into implicit differentiation for multiple variable. Functions, let's first remember how implicit differentiation works for functions of one variable.

Example 3 find $\frac{dy}{dx}$ for $3y^4 + x^7 = 5x$

Remember that the key to this is to always think of y as a function of x, or $y = y(x)$ and so whenever you differentiate a term involving y's with respect to x you will really need to use the chain rule which will mean that you will add on a $\frac{dy}{dx}$ to that term.

The first step is to differentiate both sides with respect to x.

$$12y^3 \frac{dy}{dx} + 7x^6 = 5$$

The final step is to solve for $\frac{\partial y}{\partial x}$

$$\frac{\partial y}{\partial x} = \frac{5 - 7x^6}{12y^3}$$

Now, you did this problem because implicit differentiation works in exactly the same manner with functions of multiple variables. If you have a function in terms of three variables x, y, and z you will assume that z is in fact a function of x and y. In other words, $z = z(x, y)$. Then whenever you differentiate z's with respect to x you will use the chain rule and add on a $\frac{\partial z}{\partial x}$.

Let's take a quick look at a couple of implicit differentiation problems.
Example 4 find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions

(a) $x^3z^2 - 5xy^5z = x^2 + y^3$

(b) $x^2 \sin(2y - 5z) = 1 + y\cos(6xz)$

Let's start with finding $\frac{\partial z}{\partial x}$. You first differentiate both sides with respect to x and remember to add on a $\frac{\partial z}{\partial x}$ whenever you differentiate a z

$$3x^3z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

Remember that since you are assuming $z = z(x, y)$ then any product of x's and z's will be a product rule! Now, solve for

$$3x^3z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

$$(2x^3z - 5xy^5)\frac{\partial z}{\partial x} = 2x - 3x^2z^2 + 5y^5z$$

$$\frac{\partial z}{\partial x} = \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}$$

Now you'll do the same thing for $\frac{\partial z}{\partial y}$ except this time you'll need to remember to add on a $\frac{\partial z}{\partial y}$ whenever you differentiate a z

$$2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} = 3y^2$$

$$(2x^3z - 5xy^5)\frac{\partial z}{\partial x} = 3y^2 + 25xy^4z$$

$$\frac{\partial z}{\partial x} = \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}$$

(b) $x^2\sin(2y - 5z) = 1 + y\cos(6xz)$

You'll do the same thing for this function as you did in the previous part. First let's find $\frac{\partial z}{\partial x}$

$$2x\sin(2y - 5z) + x^2\cos(2y - 5z)\left(-5\frac{\partial z}{\partial x}\right) = -y\sin(6xz)\left(6z + 6x\frac{\partial z}{\partial x}\right)$$

Don't forget to do the chain rule on each of the trig functions and when you are differentiating the inside function on the cosine you will need to also use the product rule. Now let's solve for $\frac{\partial z}{\partial x}$

$$2x\sin(2y - 5z) - 5\frac{\partial z}{\partial x}\cos(2y - 5z) = -6yz\sin(6xz) - 6xy\sin(6xz)\frac{\partial z}{\partial x}$$

$$2x\sin(2y - 5z) + 6yz\sin(6xz) = \left(5x^2\cos(2y - 5z) - 6xy\sin(6xz)\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = \frac{2x\sin(2y - 5z) + 6yz\sin(6xz)}{5x^2\cos(2y - 5z) - 6xy\sin(6xz)}$$

Now let's take care of $\frac{\partial z}{\partial y}$. This one will be slightly easier than the first one.

$$x^2 \cos(2y - 5z)\left(2 - 5\frac{\partial z}{\partial y}\right) = \cos(6xz) - y\sin(6xz)\left(6x\frac{\partial z}{\partial y}\right)$$

$$2x^2 \cos(2y - 5z) - 5x^2 \cos(2y - 5z)\frac{\partial z}{\partial y}\cos(6xz) - 6xy\sin(6xz)\frac{\partial z}{\partial y}$$

$$(6xy\sin(6xz) - 5x^2\cos(2y - 5z))\frac{\partial z}{\partial y} = \cos(6xz) - 2x^2\cos(2y - 5z)$$

$$\frac{\partial z}{\partial y} = \frac{\cos(6xz) - 2x^2\cos(2y - 5z)}{6xy\sin(6xz) - 5x^2\cos(2y - 5z)}$$

4.0 CONCLUSION

In this unit, you have defined a Partial derivative of a function of several variables. You have used the partial derivative of a function of several variables to know the geometric interpretation of a function and anti-derivative analogue has been identified. You have solved problems on partial derivative for function of several variables and identified higher order derivatives.

5.0 SUMMARY

In this unit, you have studied the following:

- the definition of Partial derivative of functions of several variable;
- the geometric interpretation of partial derivative of functions of several variables
- the identification of ant derivative analogue of partial derivative of functions of several variable
- solve problems on partial derivative for function of several variables
- the identification of higher order derivatives of functions of several variables

TUTOR MARKED ASSIGNMENT

1. Find the partial derivatives f_x and f_y if f(x , y) is given by

$$f(x, y) = x^2 y + 2x + y$$

2. Find f_x and f_y if f(x , y) is given by

$$f(x, y) = \sin(x y) + \cos x$$

3. Find f_x and f_y if f(x , y) is given by

$$f(x , y) = x e^{x y}$$

4. Find f_x and f_y if f(x , y) is given by

$$f(x , y) = \ln (x^2 + 2 y)$$

5. Find $f_x(2 , 3)$ and $f_y(2 , 3)$ if f(x , y) is given by

$$f(x , y) = y x^2 + 2 y$$

2. Find partial derivatives f_x and f_y of the following functions

A. $f(x , y) = x e^{x + y}$

B. $f(x , y) = \ln (2 x + y x)$

C. $f(x , y) = x \sin(x - y)$

7.0 REFERENCE

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UNIT 3 APPLICATION OF PARTIAL DERIVATIVE

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 3.6 Apply partial derivative of functions of several variable in Exterior derivative
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1.0 INTRODUCTION

The **partial derivative of f with respect to x** is the derivative of **f** with respect to x, treating all other variables as constant.

Similarly, the **partial derivative of f with respect to y** is the derivative of **f** with respect to y, treating all other variables as constant, and so on for other variables. The partial derivatives are written as $\partial f/\partial x$, $\partial f/\partial y$, and so on. The symbol "∂" is used (instead of "d") to remind us that there is more than one variable, and that we are holding the other variables fixed.

2.0 OBJECTIVES

In this Unit, you should be able to:

- apply partial derivative of functions of several variable in chain rule;
- apply partial derivative of functions of several variable in Curl (Mathematics);
- apply partial derivative of functions of several variable in derivatives;
- apply partial derivative of functions of several variable in D'Alamber operator;
- apply partial derivative of functions of several variable in double integral;

- apply partial derivative of functions of several variable in exterior derivative; and
- apply partial derivative of function of several variable in Jacobian matrix and determinant.

3.0 MAIN CONTENT

3.1 Applications of Partial Derivative of Functions in Several Variables.

Chain rule

Composites of more than two functions

The chain rule can be applied to composites of more than two functions. To take the derivative of a composite of more than two functions, notice that the composite of *f*, *g*, and h (in that order) is the composite of *f* with $g \circ h$. The chain rule says that to compute the derivative of $f \circ g \circ h$, it is sufficient to compute the derivative of *f* and the derivative of $g \circ h$. The derivative of *f* can be calculated directly, and the derivative of $g \circ h$ can be calculated by applying the chain rule again.

For concreteness, consider the function

$$y = e^{\sin x^2}$$

This can be decomposed as the composite of three functions:

$$y = f(u) = e^u$$

$$u = g(v) = \sin v$$

$$v = h(x) = x^2$$

Their derivatives are:

$$\frac{dy}{du} = f'(u) = e^u$$

$$\frac{du}{dv} = g'(v) = \cos v$$

$$\frac{dv}{dx} = h'(x) = 2x$$

The chain rule says that the derivative of their composite at the point *x* = *a* is:

$$(f \circ g \circ h)'(a) = f'\big((g \circ h)(a)\big)(g \circ h)'(a) = f'\big((g \circ h)(a)\big)g'(h(a))h'(a)$$

In Leibniz notation, this is:

$$\frac{dy}{dx} = \frac{dy}{du}\bigg|_{u = g(h(a))} \cdot \frac{du}{dv}\bigg|_{v = h(a)} \cdot \frac{dv}{dx}\bigg|_{x = a},$$

or for short,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

The derivative function is therefore:

$$\frac{dy}{dx} = e^{\sin x^2} . \cos x^2 . 2x$$

Another way of computing this derivative is to view the composite function $f \circ g \circ h$ as the composite of $f \circ g$ and h. Applying the chain rule to this situation gives:

$$(f \circ g \circ h)'(a) = (f \circ g)'(h(a))h'(a) = f'(g(h(a))g'(h(a))h'(a)$$

This is the same as what was computed above. This should be expected because $(f \circ g) \circ h = f \circ (g \circ h)$.

The quotient rule

The chain rule can be used to derive some well-known differentiation rules. For example, the quotient rule is a consequence of the chain rule and the product rule. To see this, write the function $f(x)/g(x)$ as the product $f(x) \cdot 1/g(x)$. First apply the product rule:

$$\frac{d}{dx} = \left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x).\frac{1}{g(x)}\right)$$

$$= f'(x).\frac{1}{g(x)} + f(x).\frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

To compute the derivative of $1/g(x)$, notice that it is the composite of g with the reciprocal function, that is, the function that sends x to $1/x$. The derivative of the reciprocal function is $-1/x^2$. By applying the chain rule, the last expression becomes:

$$f'(x).\frac{1}{g(x)} + f(x).\left(-\frac{1}{g(x)}.g'(x)\right) = \frac{f'(x)g'(x) - f(x)g'(x)}{g(x)^2}$$

which is the usual formula for the quotient rule.

Derivatives of inverse functions

Inverse functions and differentiation

Suppose that $y = g(x)$ has an inverse function. Call its inverse function f s o that we have $x = f(y)$. There is a formula for the derivative of f in terms of the derivative of g. To see this, note that f and g satisfy the formula

$$f(g(x)) = x.$$

Because the functions $f(g(x))$ and x are equal, their derivatives must be equal. The derivative of x is the constant function with value 1, and the derivative of $f(g(x))$ is determined by the chain rule. Therefore we have:

$$f'(g(x))g'(x) = 1.$$

To express f'as a function of an independent variable y, we substitute $f(\ y)$ for x wherever it appears. Then we can solve for f'.

$$f'\left(g\big(f(y)\big)g'\big(f(y)\big)\right) = 1$$

$$f'(y)g'(f(y)) = 1$$

$$f'(y) = \frac{1}{g'(f(y))}$$

For example, consider the function $g(x) = e^x$. It has an inverse which is denoted $f(y) = \ln y$. Because $g'(x) = e^x$, the above formula says that

$$\frac{d}{dy}\ln y = \frac{2}{e^{\ln y}} = \frac{1}{y}$$

This formula is true whenever g is differentiable and its inverse f is also differentiable. This formula can fail when one of these conditions is not true. For example, consider $g(x) = x^3$. Its inverse is $f(y) = y^{1/3}$, which is not differentiable at zero.

If we attempt to use the above formula to compute the derivative of f at zero, then we must evaluate $1/g'(f(0))$. $f(0) = 0$ and $g'(0) = 0$, so we must evaluate $1/0$, which is undefined. Therefore the formula fails in this case. This is not surprising because f is not differentiable at zero.

Higher derivatives

Faà di Bruno's formula generalizes the chain rule to higher derivatives. The first few derivatives are

$$\frac{d(f\circ g)}{dx}=\frac{df}{dg}\frac{dg}{dx}$$

$$\frac{d^2(f\circ g)}{dx^2}=\frac{d^2f}{dg^2}\left(\frac{dg}{dx}\right)^2+\frac{df}{dg}\frac{d^2f}{dx^2}$$

$$\frac{d^3(f\circ g)}{dx^3}=\frac{d^3f}{dg^3}\left(\frac{dg}{dx}\right)^3+3\frac{d^2f}{dg^2}\frac{dg}{dx}\frac{d^2g}{dx^2}+\frac{df}{dg}\frac{d^3f}{dx^3}$$

$$\frac{d^4(f\circ g)}{dx^4}=\frac{d^4f}{dg^4}\left(\frac{dg}{dx}\right)^4+6\frac{d^3f}{dg^3}\left(\frac{dg}{dx}\right)^2\frac{d^2g}{dx^2}+\frac{d^2f}{dg^2}\left\{4\frac{dg}{dx}\frac{d^3g}{dx^3}+3\left(\frac{d^2g}{dx^2}\right)^2\right\}+\frac{df}{dg}$$

Example

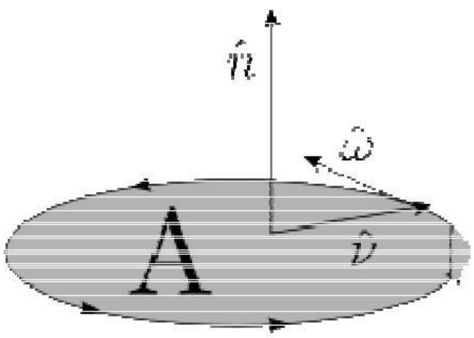
Given $u=x^2+2y$where$x=r\sin t$ and$y=\sin^2(t)$, determine the value of $\frac{du}{dr}$ and $\frac{du}{dt}$ using the chain rule

Curl (mathematics)

In vector calculus, the **curl** (or **rotor**) is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point.

The curl of a vector field **F**, denoted curl **F** or ∇× **F**, at a point is defined in terms of its projection onto various lines through the point. If $\hat{\mathbf{n}}$ is any unit vector, the projection of the curl of **F** onto $\hat{\mathbf{n}}$ is defined to be the limiting value of a closed line integral in a plane orthogonal to $\hat{\mathbf{n}}$ is the path used in the integral becomes infinitesimally close to the point, divided by the area enclosed.

As such, the curl operator maps C^1 functions from $\mathbf{R}^3$ to $\mathbf{R}^3$ to C^0 functions from $\mathbf{R}^3$ to $\mathbf{R}^3$.



Convention for vector orientation of the line integral

Implicitly, curl is defined by:

$$(\boldsymbol{\nabla} \times \mathbf{F}).\hat{\mathbf{n}} \overset{\text{def}}{=} \lim_{A \to 0} \frac{\oint_C \mathbf{F}.dr}{|A|}$$

The above formula means that the curl of a vector field is defined as th e infinitesimal area density of the *circulation* of that field. To this definition fit naturally (i) the Kelvin-Stokes theorem, as a global formula corresponding to the definition, and (ii) the following "easy to memorize" definition of the curl in orthogonal curvilinear coordinates , e.g. in Cartesian coordinates, spherical, or cylindrical, or even elliptical or parabolical coordinates:

$$(curl\ \boldsymbol{F})_3 = \frac{1}{a_1 a_2} \ . \ \left(\frac{\partial(a_2 F_2)}{\partial u_1} - \frac{\partial(a_1 F_1)}{\partial u_2} \right)$$

If (x_1,x_2,x_3) are the Cartesian coordinates and (u_1,u_2,u_3) are the curvilinear coordinates, then

$$a_i = \sqrt{\sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial u_i} \right)^2}$$

Usage

In practice, the above definition is rarely used because in virtually all case s, the curl operator can be applied using some set of curvilinear coordinates, for which simpler representations have been derived.

The notation $\nabla\times$ **F** has its origins in the similarities to the 3 dimensional cross product, and it is useful as a mnemonic in Cartesian coordinates if we take ∇ as a vector differential operator del. Such notation involving operators is common in physics and algebra. If certain coordinate systems are used, for instance, polar-toroidal coordinates (common in plasma physics) using the notation $\nabla\times$ **F** will yield an incorrect result.

Expanded in Cartesian coordinates (see: Del in cylindrical and spherical coordinates for spherical and cylindrical coordinate representations), $\nabla\times$ **F** is, for **F** composed of $[F_x, F_y, F_z]$:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ F_x & F_y & F_z \end{vmatrix}$$

Where **i**, **j**, and **k** are the unit vectors for the *x*, *y*, and *z*-axes, respectively. This expands as follows:[4]

$$\left(\frac{\partial F_x}{\partial_y} - \frac{\partial F_y}{\partial_z}\right)\boldsymbol{i} + \left(\frac{\partial F_x}{\partial_z} - \frac{\partial F_z}{\partial_x}\right)\boldsymbol{j} + \left(\frac{\partial F_y}{\partial_x} - \frac{\partial F_x}{\partial_y}\right)\boldsymbol{k}$$

Although expressed in terms of coordinates, the result is invariant under proper rotations of the coordinate axes but the result inverts under reflection.

In a general coordinate system, the curl is given by

$$(\boldsymbol{\nabla} \times \mathbf{F})^k = \epsilon^{klm}\partial_\ell F_m$$

Where ε denotes the Levi-Civita symbol, the metric tensor is used to lower the index on **F**, and the Einstein summation convention implies that repeated indices are summed over. Equivalently,

$$(\boldsymbol{\nabla} \times \mathbf{F})^k = \boldsymbol{e_k}\epsilon^{klm}\partial_\ell F_m$$

Where $\mathbf{e}_k$ are the coordinate vector fields. Equivalently, using the exterior derivative, the curl can be expressed as:

$$\boldsymbol{\nabla} \times \mathbf{F} = [* \, (dF^{\sigma})]^{\sharp}$$

Here and are the musical isomorphisms, and is the Hodge dual. This formula shows how to calculate the curl of F in any coordinate system, and how to extend the curl to any oriented three dimensional Riemannian manifold. Since this depends on a choice of orientation, curl is a chiral operation.

In other words, if the orientation is reversed, then the direction of the curl is also reversed.

Directional derivative

The directional derivative of a scalar function

$f(\vec{x}) = f(x_1, x_2, \dots x_n)$

along a unit vector
$\vec{u} = (u_1, \dots u_n)$

is the function defined by the limit

$$\boldsymbol{\nabla}_{\vec{u}} f(\vec{x}) = \lim_{h \to 0^+} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

(See other notations below.) If the function *f* is differentiable at $\vec{x}$, t hen the directional derivative exists along any unit vector $\vec{u}$ and one has

$$\boldsymbol{\nabla}_{\vec{u}} f(\vec{x}) = \boldsymbol{\nabla} f(\vec{x}).\vec{u}$$

where the $\boldsymbol{\nabla}$ on the right denotes the gradient and is the Euclidean inner product. At any point $\vec{x}$, the directional derivative of *f* intuitively represents the rate of change in *f* along $\vec{u}$at the point $\vec{x}$.

One sometimes permits non-unit vector, allowing the directional derivative to be taken in the direction $\vec{u}$, where $\vec{v}$ is any nonzero vector. In this case, one must modify the definitions to account for the fact that $\vec{v}$ may not be normalized, so one has

$$\boldsymbol{\nabla}_{\vec{v}} f(\vec{x}) = \lim_{h \to 0^+} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h|\vec{v}|},$$

or in case *f* is differentiable at $\vec{x}$,

$$\boldsymbol{\nabla}_{\vec{v}} f(\vec{x}) = \boldsymbol{\nabla} f(\vec{x}).\frac{\vec{v}}{|\vec{v}|}$$

Such notation for non-unit vectors (undefined for the zero vector), however, is incompatible with notation used elsewhere in mathematics, where the space of derivations in a derivation algebra is expected to be a vector space.

Notation

Directional derivatives can be also denoted by:

$$\boldsymbol{\nabla}_{\vec{u}} f(\vec{x}) \sim \frac{\partial f(\vec{x})}{\partial u} \sim f'_u(X) \sim D_u f(X) \sim \mathbf{u} \,.\, \boldsymbol{\nabla} f(X)$$

In the continuum mechanics of solids

Several important results in continuum mechanics require the derivatives of vectors with respect to vectors and of tensors with respect to vectors and tensors. The **directional derivative** provides a systematic way of finding these derivatives.

The definitions of directional derivatives for various situations are given below. It is assumed that the functions are sufficiently smooth that derivatives can be taken.

Derivatives of scalar valued functions of vectors

Let $f(\mathbf{v})$ be a real valued function of the vector **v**. Then the derivative of $f(\mathbf{v})$ with respect to **v** (or at **v**) in the direction **u** is the **vector** defined as

$$\frac{\partial f}{\partial \mathrm{v}} \,.\, \mathbf{u} = \boldsymbol{D} f(v)[\mathbf{u}] = \left[\frac{d}{d\alpha} f(\mathbf{v} + \alpha\mathbf{u})\right]_{\alpha=0}$$

For all vectors **u**

Properties:

1) If $f(v) = f_1(\mathbf{v}) + f_2(\mathbf{v})$ then $\frac{df}{d\mathrm{v}} \,.\, \mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}} + \frac{\partial f_2}{\partial \mathbf{v}}\right) .\, \mathbf{u}$

2) If $f(\mathbf{v}) = f_1(\mathbf{v}) f_2(\mathbf{v})$ then $\frac{\partial f}{\partial \mathrm{v}} \,.\, \mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}} \,.\, \mathbf{u}\right) f_2(\mathbf{v}) + f_1(\mathbf{v}) \left(\frac{\partial f_2}{\partial \mathbf{v}} \,.\, \mathbf{u}\right)$

3) $f(\mathbf{v}) = f_1\big(f_2(\mathbf{v})\big)$ then $\frac{\partial f}{\partial \mathrm{v}} .\, \mathbf{u} = \frac{\partial f_1}{\partial f_2} + \frac{\partial f_2}{\partial \mathbf{v}} \,.\, \mathbf{u}$

Derivatives of vector valued functions of vector

Let $\mathbf{f}(\mathbf{v})$ be a vector valued function of the vector **v**. Then the derivative of $f(\mathbf{v})$with respect to **v** (or at **v**) in the direction **u** is the **second order tensor** defined as

$$\frac{\boldsymbol{\partial} \mathbf{f}}{\boldsymbol{\partial} \mathbf{v}} \,.\, \mathbf{u} = \boldsymbol{D} f(v)[\mathbf{u}] = \left[\frac{d}{d\alpha} f(\mathbf{v} + \alpha\mathbf{u})\right]_{\boldsymbol{\alpha}=\mathbf{0}}$$

For all vector $\mathbf{u}$

Properties:

1) If $\mathbf{f}(\mathbf{v}) = \mathrm{f}_1(\mathbf{v}) + \mathrm{f}_2(\mathbf{v})$ then $\frac{df}{d\mathrm{v}}\ .\mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}} + \frac{\partial f_2}{\partial \mathbf{v}}\right)\ .\mathbf{u}$

2) If $\mathbf{f}(\mathbf{v}) = \mathrm{f}_1(\mathbf{v}) \times \mathrm{f}_2(\mathbf{v})$ then $\frac{\partial f}{\partial \mathrm{v}}\ .\mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}}\ .\mathbf{u}\right) \times f_2(\mathbf{v}) + f_1(\mathbf{v}) \times \left(\frac{\partial f_2}{\partial \mathbf{v}}\ .\mathbf{u}\right)$

3) $\mathbf{f}(\mathbf{v}) = \mathrm{f}_1(\mathrm{f}_2(\mathbf{v}))$ then $\frac{\partial f}{\partial \mathrm{v}}.\mathbf{u} = \frac{\partial f_1}{\partial f_2}\ .\left(\frac{\partial f_2}{\partial \mathbf{v}}\ .\mathbf{u}\right)$

Derivatives of vector valued functions of second-order tensors

Let $f(\mathbf{S})$ be a real valued function of the vector **S**. Then the derivative of $f(\mathbf{S})$with respect to **S** (or at **S**) in the direction **T** is the **second order tensor** defined as

1) If $\mathbf{f}(\mathbf{S}) = \mathrm{f}_1(\mathbf{S}) + \mathrm{f}_2(\mathbf{S})$ then $\frac{df}{d\mathrm{v}}\ .\mathbf{T} = \left(\frac{\partial f_1}{\partial \mathbf{S}} + \frac{\partial f_2}{\partial \mathbf{S}}\right)\ .\mathbf{T}$

2) If $\mathbf{f}(\mathbf{S}) = \mathrm{f}_1(\mathbf{S}) \times \mathrm{f}_2(\mathbf{S})$ then $\frac{\partial f}{\partial \mathbf{S}}\ .\mathbf{T} = \left(\frac{\partial f_1}{\partial \mathbf{S}}\ .\mathbf{T}\right) \times f_2(\mathbf{S}) + f_1(\mathbf{S}) \times \left(\frac{\partial f_2}{\partial \mathbf{v}}\ .\mathbf{T}\right)$

3) $\mathbf{f}(\mathbf{S}) = \mathrm{f}_1(\mathrm{f}_2(\mathbf{S}))$ then $\frac{\partial f}{\partial \mathrm{v}}.\mathbf{T} = \frac{\partial f_1}{\partial f_2}\ .\left(\frac{\partial f_2}{\partial \mathbf{S}}\ .\mathbf{T}\right)$

Derivatives of tensor valued functions of second-order tensors

Let $\mathbf{F}(\mathbf{S})$ be a real valued function of the vector **S**. Then the derivative of $\mathbf{F}(\mathbf{S})$with respect to **S** (or at **S**) in the direction **T** is the **fourth order tensor** defined as

$$\frac{\partial \mathbf{F}}{\partial \mathbf{S}}\ .\mathbf{T} = \boldsymbol{D}\mathbf{F}(S)[\mathbf{T}] = \left[\frac{d}{d\alpha}\mathbf{F}(\mathbf{S} + \alpha\mathbf{T})\right]_{\boldsymbol{\alpha=0}}$$

For all second order tensor **T**

Properties:

1) If $\mathbf{F}(\mathbf{S}) = \mathbf{F_1}(\mathbf{S}) + \mathbf{F_2}(\mathbf{S})$ then $\frac{dF}{dS}\ .\mathbf{T} = \left(\frac{\partial \boldsymbol{F_1}}{\partial \boldsymbol{S}} + \frac{\partial \boldsymbol{F_2}}{\partial \boldsymbol{S}}\right)\ .\mathbf{T}$

2) If $\mathbf{F}(\mathbf{S}) = \mathbf{F_1}(\mathbf{S})\ .\ \mathbf{F_2}(\mathbf{S})$ then $\frac{\partial F}{\partial \mathbf{S}}\ .\mathbf{T} = \left(\frac{\partial \boldsymbol{F_1}}{\partial \boldsymbol{S}}\ .\mathbf{T}\right).\ \boldsymbol{F_2}(\mathrm{S}) + \boldsymbol{F_1}(\mathrm{S})\ .\ \left(\frac{\partial F_2}{\partial \mathbf{vS}}\ .\mathbf{T}\right)$

3) $\mathbf{F}(\mathbf{S}) = \mathbf{F_1}(\mathbf{F_2}(\mathbf{S}))$ then $\frac{\partial \boldsymbol{F}}{\partial \mathbf{S}}.\mathbf{T} = \frac{\partial \boldsymbol{F}_1}{\partial \boldsymbol{F_2}}\ .\left(\frac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{S}}\ .\mathbf{T}\right)$

4) $f(\mathbf{S}) = f_1(f_2(\mathbf{S}))$ then $\frac{\partial f}{\partial \mathrm{v}}.\mathbf{T} = \frac{\partial f_1}{\partial \mathbf{F}_2}.\left(\frac{\partial \mathbf{F}_2}{\partial \mathbf{S}}.\mathbf{T}\right)$

Exterior derivative

The exterior derivative of a differential form of degree k is a differential form of degree $k + 1$. There are a variety of equivalent definitions of the exterior derivative.

Exterior derivative of a function

If f is a smooth function, then the exterior derivative of f is the differential of f. That is, $\mathrm{d}f$ is the unique one-form such that for every smooth vector field X, $\mathrm{d}f\,(X) = X\,f$, where Xf is the directional derivative of f in the direction of X. Thus the exterior derivative of a function (or 0-form) is a one-form.

Exterior derivative of a *k*-form

The exterior derivative is defined to be the unique **R**-linear mapping from k-forms to $(k+1)$-forms satisfying the following properties:

1. $\mathrm{d}f$ is the differential o ff for smooth functions f.
2. $\mathrm{d}(\mathrm{d}f) = 0$ for any smooth function f.
3. $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^p(\alpha\wedge\mathrm{d}\beta)$ where α is a p-form. That is to say, d is an anti-derivation of degree 1 on the exterior algebra of differential forms.

The second defining property holds in more generality: in fact, $\mathrm{d}(\mathrm{d}\alpha) = 0$ for any k-form α. This is part of the Poincarélemm. The third defining property implies as a special case that if f is a function and α a k-form, then $\mathrm{d}(f\alpha) = \mathrm{d}f\wedge\alpha + f\wedge d\alpha$ *because functions are forms of degree 0.*

Exterior derivative in local coordinates

Alternatively, one can work entirely in a local coordinate system ($x^1,...,x^n$). First, the coordinate differentials $\mathrm{d}x^1,...,\mathrm{d}x^n$ form a basic set of one-forms within th e coordinate chart. Given a multi-index $I = (i_1,...,i_k)$ with $1 \le i_p \le n$ for $1 \le p \le k$, the exterior derivative of a k-form

$$\omega = f_1\mathrm{d}x^I = f_{i1,i2,...,ik}\mathrm{d}x^{i1}\wedge\mathrm{d}x^{i2}\wedge \ldots \wedge\mathrm{d}x^{ik}$$

over $\mathbf{R}^n$ is defined as

$$\mathrm{d}w = \sum_{i=1}^{n}\frac{\mathrm{d}f_1}{\mathrm{d}x^i}\mathrm{d}x^i\wedge\mathrm{d}x^I$$

For general k-forms $\omega = \Sigma_I f_I dx_I$ (where the components of the multi-index I run over all the values in $\{1,...,n\}$, the definition of the exterior derivative is extended linearly.

Note that whenever i is one of the components of the multi-index I then $dx_i \wedge dx_I = 0$ (see wedge product).

The definition of the exterior derivative in local coordinates follows from the preceding definition. Indeed, if $\omega = f_I \, d_{i1} \wedge ... \wedge dx_{ik}$, then

$$d\omega = d\left(f_I dx^{i1} \wedge ... \wedge dx^{ik}\right)$$

$$= df_I \wedge \left(dx^{i1} \wedge ... \wedge dx^{ik}\right) + f_I d\left(dx^{i1} \wedge ... \wedge dx^{ik}\right)$$

$$= df_I \wedge dx^{i1} \wedge ... \wedge dx^{ik} + \sum_{p=1}^{k} (-1)^{(p-1)} f_I dx^{i1} \wedge ... \wedge dx^{ip-1} \wedge d^2 x^{ip} \wedge dx^{ip+1} \wedge ... \wedge dx^{ik}$$

$$= df_I \wedge dx^{i1} \wedge ... \wedge dx^{ik}$$

$$= \sum_{i=}^{k} \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^{i1} \wedge ... \wedge dx^{ik}$$

Here, we have here interpreted f_I as a zero-form, and then applied the properties of the exterior derivative.

Invariant formula

Alternatively, an explicit formula can be given for the exterior derivative o f a k-form ω, when paired with k+1 arbitrary smooth vector fields $V_1, V_2, ..., V_k$:

$$d\omega(V_1, ..., V_k) = \sum_i (-1)^{i-1} V_i \left(\omega(V_1, \ldots, \hat{V}_i, \ldots, V_k)\right)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([V_i, V_j], V_1, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k)$$

where $[V_i, V_j]$ denotes Lie bracket and the hat denotes the omission of that element:

$$\omega\left(V_1, \dots, \hat{V}_1, \dots, V_k\right) = \omega(V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_k)$$

In particular, for 1-forms we have: $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, where X and Y are vector fields.

Examples

1. Consider $\sigma = u\, dx^1 \wedge dx^2$ over a 1-form basis $dx^1,\ldots,dx^n$. The exterior derivative is:

$$d\sigma = d(u) \wedge dx^1 \wedge dx^2$$

$$= \left(\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx^i\right) \wedge dx^1 \wedge dx^2$$

$$= \sum_{i=3}^{n} \left(\frac{\partial u}{\partial x_i} dx^i \wedge dx^1 \wedge dx^2\right)$$

The last formula follows easily from the properties of the wedge product. Namely,$dx^i \wedge dx^i = 0$.

2. For a 1-form $\sigma = u\, dx + v dy$ defined over $\mathbf{R}^2$. We have, by applying the above formula to each term (consider $x^1 = x$ and $x^2 = y$) the following sum,

$$d\sigma = \left(\sum_{i=1}^{2} \frac{\partial u}{\partial x^i} dx^i \wedge dx\right) + \left(\sum_{i=1}^{2} \frac{\partial v}{\partial x^i} dx^i \wedge dy\right)$$

$$= \left(\frac{\partial u}{\partial x} dx \wedge dx + \frac{\partial u}{\partial y} dy \wedge dx\right) + \left(\frac{\partial v}{\partial x} dx \wedge dy + \frac{\partial v}{\partial y} dy \wedge dy\right)$$

$$= 0 - \frac{\partial u}{\partial y} dx \wedge dy + \frac{\partial v}{\partial x} dx \wedge dy + 0$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \wedge dy.$$

D'Alembert operator

In special relativity, electromagnetism and wave theory, the **d'Alembert operator** (represented by a box:$\sqcup$) also called the **d'Alembertian** or the **wave operator**, is the Laplace operator of Minkowski space. The operator is named for French mathematician and physicist Jean le Rondd'Alembert. In Minkowski space in standard coordinates (t, x, y, z) it has the form:

$$\sqsupset = \partial_\mu \partial^\mu = g_{\mu\nu} \partial^\nu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

Applications

The Klein–Gordon equation has the form
$$(\Box + m^2)\psi = 0.$$

The wave equation for the electromagnetic field in vacuum is

$$\Box A^{\mu} = 0$$

Where A^{μ} is the electromagnetic four-potential.

The wave equation for small vibrations is of the form

$$\Box_c u\,(x,t) \equiv u_{tt} - c^2 u_{xx} = 0,$$

Where $u(x,t)$ is the displacement.

Green's function

The Green's function $G(x - x')$.for the d'Alembertian is defined by the equation

$$\Box G(x - x') = \delta(x - x')$$

Where $\delta(x - x')$ is the Dirac delta function and $x$andx' are two points in Minkowski space.

Explicitly we have

$$G(t,x,y,z) = \frac{1}{2\pi} \ominus (t)\delta(t^2 - x^2 - y^2 - z^2)$$

Where $\ominus$is the Heaviside step function

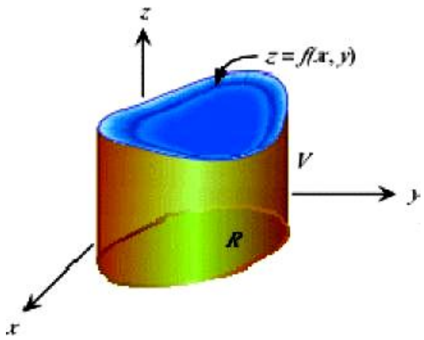
Double Integral

The **double integral of f(x, y) over the region R in the xy-plane** is define d as

$$\iint_R f(x,y)dx\,dy$$

= (volume above R and under the graph of *f*) - (volume below R and above the graph of *f*).

- The following figure illustrates this volume (in the case that the graph of f is above the region R).

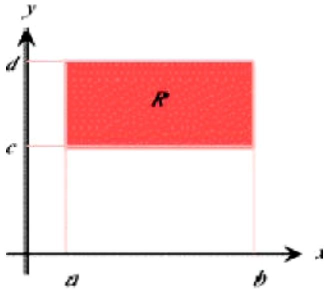


Computing Double Integrals

If R is the rectangle a ■ x■ b and c ■ y ■ d (see figure below) then

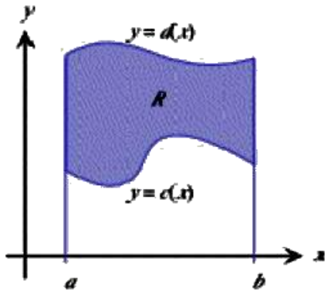
$$\iint_R f(x,y)dx\,dy = \int_c^d [\int_a^b f(x,y)dx]dy$$

$$= \int_a^b [\int_c^d f(x,y)dy]dx$$



If R is the region a■ x ■ b and c(x) ■ y ■ d(x) (see figure below) then we integrate over R according to the following equation.

$$\iint_R f(x,y)dx\,dy = \int_a^b [\int_{\mathrm{c}(x)}^{\mathrm{d}(x)} f(x,y)dy]dx$$



Jacobian Matrix

The Jacobian of a function describes the orientation of a tangent plane to the function the Jacobian generalizes the gradient of a scalar valued function of multiple variable derivative of a scalar-valued function of scalar. Likewise, the Jacobian can also be amount of "stretching" that a transformation imposes. For example, if $(x_2,y_2) = f(x_1,y_1)$ is used to transform an image, the Jacobian of f, $J(x_1,y_1)$ describes how much the image in the neighborhood of (x_1,y_1) directions.

If a function is differentiable at a point, its derivative is given in coordinates by the Jacobian, but a function doesn't need to be differentiable for the Jacobian to be defined, since only the partial derivatives are required to exist.

The importance of the Jacobian lies in the fact that it represents the best linear approximation to a differentiable function near a given point. In this sense, the Jacobian is the derivative of a multivariate function.

If **p** is a point in $\mathbf{R}^n$ and F is differentiable at **p**, then its derivative is given by $J_F(\mathbf{p})$. In this case, the linear map described by $J_F(\mathbf{p})$ is the best linear approximation of F near the point **p**, in the sense that

$$F(\mathbf{x}) = F(\mathbf{p}) + J_F(\mathbf{p})(\mathbf{x} - \mathbf{p}) + o(\|\mathbf{x} - \mathbf{p}\|)$$

for **x** close to **p** and where o is the little o-notation (for $x \to p$) and $\|\mathbf{x} - \mathbf{p}\|$ is the distance between **x** and **p**.

In a sense, both the gradient and Jacobian are "first derivatives" — the former the first derivative of a *scalar function* of several variables, the latter the first derivative of a *vector function* of several variables. In general, the gradient can be regarded as a special version of the Jacobian: it is the Jacobian of a scalar function of several variables.

The Jacobian of the gradient has a special name: the Hessian matrix, which in a sense is the "second derivative" of the scalar function of several variables in question.

Inverse

According to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function. That is, for some function F: $R^n \to R^n$ and a point p in R^n,

$$J_{F^{-1}}(F(p)) = [J_F(p)]^{-1}.$$

It follows that the (scalar) inverse of the Jacobian determinant of a transformation is the Jacobian determinant of the inverse transformation.

Uses

Dynamical systems

Consider a dynamical system of the form $x' = F(x)$, where x' is the (component-wise) time derivative of x, and F: $R^n \to R^n$ is continuous and differentiable. If $F(x_0) = 0$, then x_0 is a stationary point (also called a fixed point). The behavior of the system near a stationary point is related to the eigenvalues of $J_F(x_0)$, the Jacobian of F at the stationary point. Specifically, if the eigenvalues all have a negative real part, then the system is stable in the operating point, if any eigenvalue has a positive real part, then the point is unstable.

Newton's method

A system of coupled nonlinear equations can be solved iteratively by Newton's method. This method uses the Jacobian matrix of the system of equations.

The following is the detail code in MATLAB

```
function s = Jacobian (f, x, tol) % f is a multivariable function handle, x is a starting point

ifnargin == 2
tol = 10^-5 ;
end

while 1
% if x and f(x) are row vectors, we need transpose operations here
y = x' - jacob(f, x)\f(x)';      % get the next point
if norm(f(y))<tol                % check error tolerate
s = y';
return;
end
x = y';
end

function j = jacob(f, x) % approximately calculate Jacobian matrix

k = length(x);
j = zeros(k, k);
for m = 1: k
        x2 = x;
        x2(m) =x(m)+0.001;
        j(m, :) = 1000*(f(x2)-f(x)); % partial derivatives in m-th row end
```

Jacobian determinant

If $m = n$, then F is a function from n-space to n-space and the Jacobian matrix is a square matrix. We can then form its determinant, known as the Jacobian determinant. The Jacobian determinant is sometimes simply called "the Jacobian."

The Jacobian determinant at a given point gives important information about the behavior of ***F*** near that point. For instance, the continuously differentiable function ***F*** is invertible near a point $\mathbf{p} \in \mathbf{R}^n$ if the Jacobian determinant at **p** is non-zero. This is the inverse function theorem. Furthermore, if the Jacobian determinant at p is positive, then ***F*** preserves orientation near **p**; if it is negative, ***F*** reverses orientation. The absolute value of the Jacobian determinant at *p* gives us the factor by which the function ***F*** expands or shrinks volumes near **p**; this is why it occurs in the general substitution rule.

Uses

The Jacobian determinant is used when making a change of variables when evaluating a multiple integral of a function over a region within its domain. To accommodate for the change of coordinates the magnitude of the Jacobian determinant arises as a multiplicative factor within the integral. Normally it is required that the change of coordinates be done in a manner which maintains an injectivity between the coordinates that determine the domain. The Jacobian determinant, as a result, is usually well defined.

Examples

Example 1: The transformation from spherical coordinates (r, θ, φ) to Cartesian coordinates (x^1, x^2, x^3), is given by the function $F: R^+ \to R^3$ with components:

$$x_1 = r\ \sin\theta\ \cos\phi$$

$$x_2 = r\ \sin\theta\ \sin\phi$$

$$x_3 = r\ \cos\theta.$$

The Jacobian matrix for this coordinate change is

$$J_F(r,\theta,\phi) = \begin{bmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_1}{\partial \theta} & \dfrac{\partial x_1}{\partial \phi} \\ \dfrac{\partial x_2}{\partial r} & \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial x_2}{\partial \phi} \\ \dfrac{\partial x_3}{\partial r} & \dfrac{\partial x_3}{\partial \theta} & \dfrac{\partial x_3}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

The determinant is $r^2 \sin\theta$. As an example, since $dV = dx_1\, dx_2\, dx_3$ this determinant implies that the differential volume element $dV = r^2 \sin\theta\, dr d\theta\, d\phi$. Nevertheless this determinant varies with coordinates. To avoid any variation the new coordinates can be defined as $w_1 = \frac{r^3}{3}, w_2 = -\cos\theta, w_3 = \phi$ Now the determinant equals to 1 and volume element becomes $r^2 dr \ \sin\theta \ d\theta \ d\phi = dw_1 dw_2 dw_3$

Example 2: The Jacobian matrix of the function F : $R^3 \to R^4$ with components

$$y_1 = x_1$$
$$y_2 = 5x_3$$
$$y_3 = 4x_2^2 - 2x_3$$
$$y_4 = x_3 \sin(x_1)$$

$$J_F(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} & \frac{\partial y_4}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3\cos(x_1) & 0 & \sin(x_1) \end{bmatrix}.$$

This example shows that the Jacobian need not be a square matrix.

Example 3

$$x = r \cos \phi;$$
$$y = r \sin \phi.$$

$$J(r, \phi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial(r\cos\phi)}{\partial r} & \frac{\partial(r\cos\phi)}{\partial \phi} \\ \frac{\partial(r\sin\phi)}{\partial r} & \frac{\partial(r\sin\phi)}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix}$$

The Jacobian determinant is equal to r. This shows how an integral in the Cartesian coordinate system is transformed into an integral in the polar coordinate system:

$$\iint_A dx\, dy = \iint_B r\, dr\, d\phi$$

Example 4. The Jacobian determinant of the function F: $\mathbf{R}^3 \to \mathbf{R}^3$ with components

$$y_1 = 5x_2$$
$$y_2 = 4x_1^2 - 2\sin(x_2x_3)$$
$$y_3 = x_2x_3$$

is

$$\begin{vmatrix} 0 & 5 & 0 \\ 8x_1 & -2x_3\cos(x_2x_3) & -2x_2\cos(x_2x_3) \\ 0 & x_3 & x_2 \end{vmatrix} = -8x_1 \cdot \begin{vmatrix} 5 & 0 \\ x_3 & x_2 \end{vmatrix} = -40x_1x_2.$$

From this we see that **F** reverses orientation near those points where x_1 and x_2 have the same sign; the function is locally invertible everywhere except near points where $x_1 = 0$ or $x_2 = 0$. Intuitively, if you start with a tiny object around the point (1,1,1) and apply F to that object, you will get an object set with approximately 40 times the volume of the original one

4.0 CONCLUSION

In this unit you have applied partial derivative of functions of several variables to solve chain rule and curl (mathematics). You have also applied partial derivative of functions of several variable solve derivatives and D'Alamber operator. You have applied partial derivative of functions of several variables in Double integral and exterior derivative. You also used partial derivative of function of several variables in Jacobian matrix and determinant.

5.0 SUMMARY

In this unit, you have studied the:

- Application of partial derivative of functions of several variable in Chain rule.
- Application of partial derivative of functions of several variable in Curl (Mathematics)
- Application of partial derivative of functions of several variable in Derivatives
- Application of partial derivative of functions of several variable in D' Alamber operator
- Application of partial derivative of functions of several variable in Double integral
- Application of partial derivative of functions of several variable in Exterior derivative
- Application of partial derivative of function of several variable in Jacobian matrix and determinant

6.0 TUTOR – MARKED ASSIGNMENT

1. Find the equation of the tangent plane to $z = \ln(2x + y)$ $z = \ln(2x + y)$ at (1, 3)
2. Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$
3. Find the absolute minimum and absolute maximum of $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by -1≤x≤1 and -1≤y≤1
4. Find the absolute minimum and absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$
5. Find the partial derivatives of the following in the second order:
   a. $F(x, y) = x^2 - 2xy + 6x - 2y + 1$
   b. $F(x, y) = \theta^{xy}$

7.0 REFERENCES/FURTHER READING

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