MODULE 4 PARTIAL DIFFERENTIABILITY AND TOTAL DIFFERENTIABILITY OF FUNCTION OF SEVERAL VARIABLE

UNIT 1 PARTIAL DIFFERENTIABILITY OF FUNCTION OF SEVERAL VARIABLE

CONTENTS

1.0 INTRODUCTION

Differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x. This rate of change is called the **derivative** of *y* with respect to *x*. In more precise language, the dependence of y upon x means that *y* is a function of *x*. This functional relationship is often denoted $y = f(x)$, where f denotes the function. If x and *y* are real numbers, and if the graph of y is plotted against *x*, the derivative measures the slope of this graph at each point.

The simplest case is when y is a linear function of x, meaning that the graph of y against x is a straight line. In this case, $y = f(x) = m x + b$, for real numbers m and b, and the slope m is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

where the symbol Δ (the uppercase form of the Greek letter Delta) is an abbreviation for "change in." This formula is true because

$y + \Delta y = f(x + \Delta x) = m (x + \Delta x) + b = m x + b + m \Delta x = y + m \Delta x.$

It follows that $\Delta y = m \Delta x$.

This gives an exact value for the slope of a straight line. If the function f is not linear (i.e. its graph is not a straight line), however, then the change in y divided by the change in x varies: differentiation is a method to find an exact value for this rate of change at any given value of x.

Rate of change as a limiting value

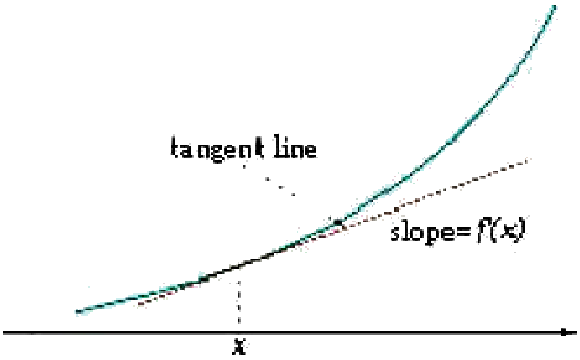


Figure 1: The tangent line at (x, f(x))

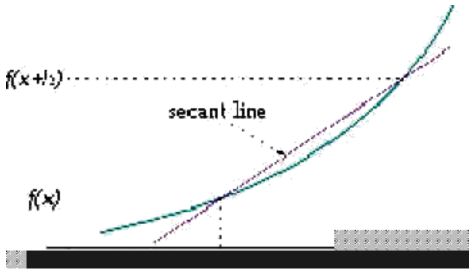


Figure 2: The secant to curve y= f(x) determined by points (x, f(x)) and (x+h, f(x+h))

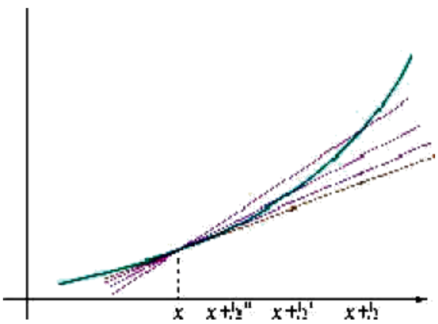


Figure 3: The tangent line as limit of secants

The idea, illustrated by Figure s 1-3, is to compute the rate of change as the limiting value of the ratio of the differences Δy / Δx as Δx becomes infinitely small.

In Leibniz's notation, such an infinitesimal change in x is denoted by dx, the derivative of y with respect to x is written

$$\frac{dy}{dx}$$

Suggesting the ratio of two infinitesimal quantities. (The above expression is read as "the derivative of *y* with respect to *x*", "d y by d x", or "d y over d x". The oral form "d y d x" is often used conversationally, although it may lead to confusion.)

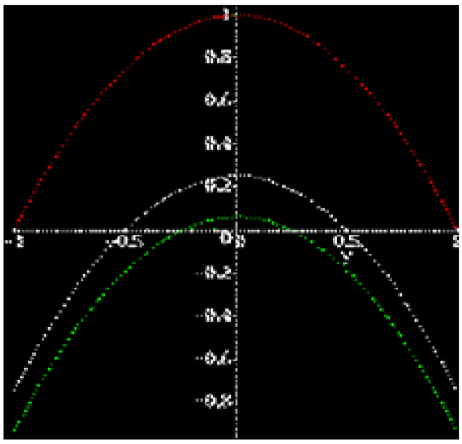
The most common approach to turn this intuitive idea into a precise definition uses limits, but there are other methods, such as non-standard analysis.

Derivatives

Bound as we humans are to three special dimensions, multi-variable functions can be very difficult to get a good feel for (Try picturing a function in the 17th dimension and see how far you get!) You can at least make three-dimensional models of two-variable functions, but even then at a stretch to our intuition. What is needed is a way to cheat and look at multi-variable functions as if they were one-variable functions.

You can do this by using **partial functions**. A partial function is a one-variable function obtained from a function of several variables by assigning constant values to all but one of the independent variables. What we are doing is taking two-dimensional "slices" of the surface represented by the equation.

For Example: $z = x^2-y^2$ can be modeled in three dimensional space, but personally you find it difficult to sketch! In the section on critical points a picture of a plot of this function can be found as an example o f a saddle point. But by alternately setting x =1 (red), x = 0.5 (white), and x = 0.25 (green), we can take slices of $z = x^2-y^2$ (each one a plane parallel to the z-y plane) and see different partial functions. We can get a further idea of the behavior of the function by considering that the same curves are obtained for x=-1, -0.5 and -0.25.



Food For Thought: How do partial functions compare to level curves and level surfaces? If the function *f* is a continuous function, does the level set or surface have to be continuous? What about partial functions?

All of this helps us to get to our main topic, that is, partial differentiation. We know how to take the derivative of a single function? What about the derivative of a multi-variable function? What does that even mean? Partial Derivatives are the beginning of an answer to that question.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

1. Identify and solve partial derivatives; and
2. Solve second partial derivatives.

3.0 MAIN CONTENT

A **partial derivative** is the rate of change of a multi-variable function when we allow only one of the variables to change. Specifically, we differentiate with respect to only one variable, regarding all others as constants (now we see the relation to partial functions!). Which essentially means if you know how to take a derivative, you know how to take a partial derivative.

A partial derivative of a function f with respect to a variable x, say $z = f(x, y_1, ,y_2,...y_n)$ (where the y_i's are other independent variables) is commonly denoted in the following ways:

$\frac{\partial z}{\partial x}$ (referred to as ``partial z, partial x")

$\frac{\partial f}{\partial x}$ (referred to as ``partial f, partial x")

Note that this is not the usual derivative "d". The funny ``d" symbol in the notation is called ``roundback d", ``curly d" or ` `del d" (to distinguish from ``delta d"; the symbol is actually a ``lowercase Greek `delta' ").

The next set of notations for partial derivatives is much more compact and especially used when you are writing down something that uses lots of partial derivatives, especially if they are all different kinds:

z_x (referred to as ``partial z, partial x")

f_x (referred to as ``partial f, partial x")

$f_x(x, y)$ (referred to as ``partial f, partial x")

Any of the above is equivalent to the limit

$$f_x = \lim_{x \to \Delta h} \frac{f(x + \Delta h, y) - f(x, y)}{\Delta x}$$

To get an intuitive grasp of partial derivatives, suppose you were an ant crawling over some rugged terrain (a two-variable function) where the x-axis is north-south with positive x to the north, the y-axis is east-west and the z-axis is up-down. You stop at a point P = (x_0, y_0, z_0) on a hill and wonder what sort of slope you will encounter if you walk in a straight line north. Since our longitude won't be hanging as we go north, the y in our function is constant. The slope to the north is the value of $f_x(x_0, y_0)$.

The actual calculations of partial derivatives for most functions are very easy! Treat every independent variable except the one we are interested in as if it were a cons ant and apply the familiar rules!

Example:

Let's find f_x and f_y of the function $z = f = x^2 - 3x^2y + y^3$. To find f_x, we will treat y as a constant and differentiate. So, $f_x = 2x - 6xy$. By treating x as a constant, we find $f_y = -3x^2 + 3y^2$.

Second Partial Derivatives

Observe carefully that the expression f_{xy} implies that the function f is differentiated first with respect to x and then with respect to y, which is a natural inference since f_{xy} is really $(f_x)_y$.

For the same reasons, in the case of the expression,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

it is implied that we differentiate first with respect to y and then with respect to x.

Below are examples of **pure second partial derivatives:**

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Example:

Let’s find f_{xy} and f_{yx} of $f = e^{xy} + y(\sin x)$.

1. $f_x = ye^{xy} + y\cos x$
2. $f_{xy} = xye^{xy} + \cos x$
3. $f_y = xe^{xy} + \sin x$
4. $f_{yx} = xye^{xy} + \cos x$

In this example $f_{xy} = f_{yx}$. Is this true in general? Most of the time and in most will probably ever see, yes. More precisely, if

- both f_{xy} and f_{yx} exist for all points near (x_0,y_0)
- and are continuous at (x_0,y_0),

then $f_{xy} = f_{yx}$.

Partial Derivatives of higher order are defined in the obvious way. And as long as suitable continuity exists, it is immaterial in what order a sequence of partial differentiation is carried out.

Total differential (Definition)

There is the generalisation of the theorem in the parent entry concerning the real functions of several variables; here we formulate it for three variables:

Theorem. Suppose that S is a ball in $\boldsymbol{R}^3$, the function f:S $\rightarrow \boldsymbol{R}$ is continuous and has partial derivatives fx, fy, fz, in S and the partial derivatives are continuous in a point (x ,y ,z) of S . Then the increment

$$\Delta f = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z),$$

which f gets when one moves from(x ,y ,z) to another point $(x+\Delta x, y+\Delta y, z+\Delta z)$ of S , can be

$$\Delta f = [f'_x(x,y,z)\Delta x + f'_y(x,y,z)\Delta y + f'_z(x,y,z)\Delta z] + \langle\varrho\rangle\varrho.$$

Here, $\varphi = \sqrt{}$ Δx2+ Δy2+ Δz2 and (φ) is a quantity tending to 0 along with φ.

The former part of Δx is called the (total) differential or the exact differential of the function f in the point (x ,y ,z) and it is denoted by df(x ,y ,z) of briefly df . In the special case f(x ,y ,z) ≡ x , we see that df = Δx and thus Δx = dx ; similarly Δy = dy and Δz = dz. Accordingly, we obtain for the general case the more consistent notation

$$df = f'_x(x,y,z)dx + f'_y(x,y,z)dy + f'_z(x,y,z)dz,$$

where dx ,dy ,dz may be thought as independent variables.

We now assume conversely t at the increment of a function f in $\boldsymbol{R}_3$ can be split into two parts as follows:

$$f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) = |A\Delta x + C\Delta z| + (\varrho), (3)$$

where the coefficients A, B, C are independent on the quantities Δx, Δy, Δz and ϱ, (ϱ) are as in the above theorem. Then one can infer that the partial derivatives fx, fy, fz, exist in the point (x, y, z) and have the values A, B, C, respectively. In fact, if we choose $\Delta y - \Delta z - 0$ then ϱ = |Δx| whence (3) attains the form

$$f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z) = A\Delta x + (\Delta x)\Delta x$$

and therefore

A = lim Δx→ 0 Δxf(x+ Δx, y+Δy, z+ Δz) − f(x, y, z) = fx, (x, y, z).

Similarly we see the values of fy, and fz,.

Definition A function f in $\boldsymbol{R}^3$, satisfying the conditions of the above theorem is said to be differentiable in the point (x, y, z).

4.0 CONCLUSION

In this unit, you have identified and solved problem on partial differential of function of several variables. You have also used partial differential of function of several variables to solve problems on second partial derivatives.

5.0 SUMMARY

In this unit, you have studied:

1. Partial derivatives
2. Second partial derivatives

6.0 TUTOR – MARK ASSIGNMENTS

1. Find the first order derivative of the following function
   $F(x, y, z) = x^2 y z^4$
2. Find f_{xx} f_{yy} f_{zz} ,given that F(x, y, z) = sin(xyz)
3. Evaluate the second order derivative of
   $F_{xx} f_{yy} f_{zz} = x^3 y^4 + 2xy + z^4$
4. Evaluate the second order derivative of
   $F(x, y, z) = x^3 + y^2 z^3$

7.0 REFERENCE/FURTHER READING

Jacques, I. 1999. Mathematics for Economics and Business. 3rd Edition. Prentice Hall.

UNIT 2 TOTAL DIFFERENTIABILITY OF FUNCTION OF SEVERAL VARIABLES

CONTENTS

1.0 INTRODUCTION

In the case of a function of a single variable the differential of the function y = f(x) is the quantity

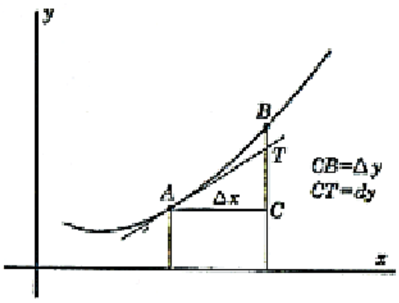


Fig. 2

dy = f '(x) Δx .
This quantity is used to compute the approximate change in the value of f(x) due to a change

Δx in x. As is shown in Fig. 2,

Δy = CB = f(x + Δx) - f(x)

while dy = CT = f '(x)Δx .

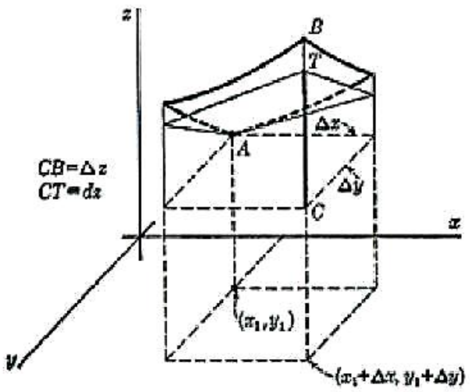


Fig. 3

When Δx is small the approximation is close. Line AT represents the tangent to the curve at point A.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

1. identify and solve problems on total differentials of functions of several variables

3.0 MAIN CONTENT

In the case of a function of two variables the situation is analogous. Let us start at point $\Delta(x_1, y_1, z_1)$ on the surface

$z = f(x, y)$

shown in Fig. 3 and let x and y change by small amounts Δx and Δy, respectively. The change produced in the value of the function z is

$\Delta z = CB = f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1)$.

An approximation to Δz is given by

$$CT = \left(\frac{\partial_z}{\partial_z}\right)_A \Delta x + \left(\frac{\partial_z}{\partial_y}\right)_A \Delta y$$

When Δx and Δy are small the approximation is close. Point T lies in that plane tangent to the surface at point A.

The quantity

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

is called the **total differential** of the function z = f(x, y). Because is customary to denote increments Δx and Δy by dx and dy, the total differential of a function z = f(x, y) is defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\ .$$

The total differential of three or more variables is defined similarly. For a function z = f(x, y, .. , u) the total differential is defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \cdots + \frac{\partial z}{\partial u}du\ .$$

Each of the terms represents a partial differential. For example, the term

$$\frac{\partial z}{\partial x}dx$$

is the partial differential of z with respect to x. The total differential is the sum of the partial differentials.

4.0 CONCLUSION

In this unit, you have identified and solved problems on total differentials of functions of several variables

5.0 SUMMARY

In this unit, you have studied total differentials of functions of several variables.

6.0 TUTOR – MARKED ASSIGNMENT

Find the total differentiability of the following:

a. F(x, y) = $x + 2xy + y^2$

b. F(x, y, z) = $x^4 + 2y^3 + z^2$

c. F(x, y, z) = $x^3 y^2 z^3$

d. F(x, y, z) = $4x^2 y^3 + z^2$

e. F(x, y, z) = $\sqrt{x^2y^2 - 2xyz}$

7.0 REFERENCES/FURTHER READING

James & James, Mathematics Dictionary

Middlemiss, Differential and Integral Calculus

Spiegel, Advanced Calculus

Taylor, Advanced Calculus