MODULE 5 COMPOSITE DIFFERENTIATION, EULER'S THEOREM, IMPLICIT DIFFERENTIATION

UNIT 1 COMPOSITE DIFFERENTIATION

CONTENTS

1.0 INTRODUCTION

In calculus, the chain rule is a formula for computing the derivative of the composition of two or more functions. That is, if f is a function and g is a function, then the chain rule expresses the derivative of the composite function $f \circ g$ in terms of the derivatives of f and g.

Calculate the derivatives of each function. Write in fraction form, if needed, so that all exponents are positive in your final answer. Use the "modified power rule" for each.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use chain rule to solve mathematical problems;
- solve composites of more than two functions;
- use the quotient rule to solve composite functions;
- identify problems in composite function which could be solve by the use of higher derivative;
- proof the chain rule; and
- explain the rule in higher dimension.

3.0 MAIN CONTENT

Statement of the Rule

The simplest form of the chain rule is for real-valued functions of one re l variable. It says that if g is a function that is differentiable at a point c (i.e. the derivative g′(c) exists) and f is a function that is differentiable at g(c), then the composite function f ∘ g is differentiable at c, and the derivative is

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

The rule is sometimes abbreviated as

$$(f \circ g)' = (f' \circ g) \cdot g'.$$

If $y = f(u)$ and $u = g(x)$, then this abbreviated form is written in Leibniz notation as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The points where the derivatives are evaluated may also be stated explicitly:

$$\left.\frac{dy}{dx}\right|_{x=c} = \left.\frac{dy}{du}\right|_{u=g(c)} \cdot \left.\frac{du}{dx}\right|_{x=c}.$$

Further examples

The chain rule in the absence of formulas

It may be possible to apply the chain rule even when there are no formulas for the functions which are being differentiated. This can happen when the derivatives are measured directly. Suppose that a car is driving up a tall mountain. The car's speedometer measures its speed directly. If the grade is known, then the rate of ascent can be calculated using trigonometry. Suppose that the car is ascending at 2.5 km/h. Standard models for the Earth's atmosphere implies that the temperature drops about 6.5 °C per kilometer ascended (see lapse rate). To find the temperature drop per hour, we apply the chain rule. Let the function $g(t)$ be the altitude of the car at time t, and let the function $f(h)$ be the temperature h kilometers above sea level. f and g are not known exactly: For example, the altitude where the car starts is not known and the temperature on the mountain is not known. However, their derivatives are known: f′ is −6.5 °C/km, and g′ is 2.5 km/h. The chain rule says that the derivative of the composite function is the product of the derivative of f and the derivative of g. This is − 6.5 °C/km · 2.5 km/h = −16.25 °C/h.

One of the reasons why this computation is possible is because f′ is a constant function. This is because the above model is very simple. A more accurate description of how the temperature near the car varies over time would require an accurate mod l of how the temperature varies at different altitudes. This model may not have a constant derivative. To compute the temperature change in such a model, it would be necessary to know g and not just g′, because without knowing g it is not possible to know where to evaluate f′.

Composites of more than t o functions

The chain rule can be applied to composites of more than two functions. To take the derivative of a composite of more than two functions, notice that the composite of f, g, and h (in that order) is the composite of f with $g \circ h$. The chain rule says that to compute the derivative of $f \circ g \circ h$, it is sufficient to compute the derivative of f and the derivative of $g \circ h$. The derivative of f can be calculated directly, and the derivative of $g \circ h$ can be calculated by applying the chain rule again.

For concreteness, consider the function

$$y = e^{\sin x^2}.$$

This can be decomposed as the composite of three functions:
$y = f(u) = e^u,$
$u = g(v) = \sin v,$
$v = h(x) = x^2,$

Their derivatives are:

$$\frac{dy}{du} = f'(u) = e^u,$$

$$\frac{du}{dv} = g'(v) = \cos v,$$

$$\frac{dv}{dx} = h'(x) = 2x.$$

The chain rule says that the derivative of their composite at the point x = a is:

$$(f \circ g \circ h)'(a) = f'((g \circ h)(a))(g \circ h)'(a) = f'((g \circ h)(a))g'(h(a))h'(a)$$

In Leibniz notation, this is:

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=g(h(a))} \cdot \left.\frac{du}{dv}\right|_{v=h(a)} \cdot \left.\frac{dv}{dx}\right|_{x=a},$$

or for short,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

The derivative function is therefore:

$$\frac{dy}{dx} = e^{\sin x^2} \cdot \cos x^2 \cdot 2x.$$

Another way of computing this derivative is to view the composite function $f \circ g \circ h$ as the composite of $f \circ g$ and h. Applying the chain rule to this situation gives:

$$(f \circ g \circ h)'(a) = (f \circ g)'(h(a))h'(a) = f'(g(h(a))g'(h(a))h'(a).$$

This is the same as what was computed above. This should be expected because $(f \circ g) \circ h = f \circ (g \circ h)$.

The quotient rule

The chain rule can be used to derive some well-known differentiation rules. For example, the quotient rule is a consequence of the chain rule and the product rule. To see this, write the function $f(x)/g(x)$ as the product $f(x) \cdot 1/g(x)$. First apply the product rule:

$$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) \\ &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{d}{dx}\left(\frac{1}{g(x)}\right). \end{aligned}$$

To compute the derivative of 1/g(x), notice that it is the composite of g with the reciprocal function, that is, the function that sends x to $1/x$. The derivative of the reciprocal function is $-1/x^2$. By applying the chain rule, the last expression becomes:

$$f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(-\frac{1}{g(x)^2} \cdot g'(x)\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

which is the usual formula for the quotient rule.

Derivatives of inverse functions

Suppose that y = g(x) has an inverse function. Call its inverse function f so that we have x = f(y). There is a formula for the derivative of f in terms of the derivative of g. To see this, note that f and g satisfy the formula

$f(g(x)) = x.$

Because the functions $f(g(x))$ and x are equal, their derivatives must be equal. The derivative of x is the constant function with value 1, and the derivative o $f(g(x))$ is determined by the chain rule. Therefore we have:

$f'(g(x))g'(x) = 1.$

To express f′ as a function of an independent variable y, we substitute f(y for x wherever it appears. Then we can solve for f′.

$$f'(g(f(y))g'(f(y)) = 1$$
$$f'(y)g'(f(y)) = 1$$
$$f'(y)\frac{1}{g'(f(y))}.$$

For example, consider the function $g(x) = e^x$. It has an inverse which is denoted $f(y) = \ln y$. Because $g`(x) = e^x$, the above formula says that

$$\frac{d}{dy}\ln y = \frac{1}{e^{\ln y}} = \frac{1}{y}.$$

This formula is true whenever g is differentiable and its inverse f is also differentiable. This formula can fail when one of these conditions is not true. For example, consider $g(x) = x^3$. Its inverse is $f(y) = y^{1/3}$ which is not differentiable at zero. If we attempt to use the above formula to compute the derivative of f at zero, then we must evaluate 1/g′(f(0)). f(0) = 0 and g′(0) = 0, so we must evaluate 1/0, which is undefined. Therefore the formula fails in this case. This is not surprising because f is not differentiable at zero.

Higher derivatives

Faà di Bruno's formula generalizes the chain rule to higher derivatives. The first few derivatives are

$$\frac{d(f\circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{d^2(f\circ g)}{dx^2} = \frac{d^2f}{dg^2}\left(\frac{dg}{dx}\right)^2 + \frac{df}{dg}\frac{d^2g}{dx^2}$$

$$\frac{d^3(f\circ g)}{dx^3} = \frac{d^3f}{dg^3}\left(\frac{dg}{dx}\right)^3 + 3\frac{d^2f}{dg^2}\frac{dg}{dx}\frac{d^2g}{dx^2} + \frac{df}{dg}\frac{d^3g}{dx^3}$$

$$\frac{d^4(f\circ g)}{dx^4} = \frac{d^4f}{dg^4}\left(\frac{dg}{dx}\right)^4 + 6\frac{d^3f}{dg^3}\left(\frac{dg}{dx}\right)^2\frac{d^2g}{dx^2} + \frac{d^2f}{dg^2}\left\{4\frac{dg}{dx}\frac{d^3g}{dx^3} + 3\left(\frac{d^2g}{dx^2}\right)\right.$$

Proofs of the chain rule

First proof

One proof of the chain rule begins with the definition of the derivative:

$$(f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Assume for the moment that g(x) does not equal g(a) for any x near a. Then the previous expression is equal to the product of two factors:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

When g oscillates near a, then it might happen that no matter how close one gets to a, there is always an even closer x such that g(x) equals g(a). For example, this happens for g(x) = x^2sin (1/x) near the point a = 0. Whenever this happens, the above expression is undefined because it involves division by zero. To work around this, introduce a function Q as follows:

$$Q(y) = \begin{cases} \frac{f(y) - f(g(a))}{y - g(a)}, y \neq g(a) \\ f'(g(a)), \qquad y = g(a). \end{cases}$$

We will show that the difference quotient for f 0 g is always equal to:

$$Q(g(x)). \frac{g(x) - g(a))}{x - a}$$

Whenever g(x) is not equal to g(a), this is clear because the factors of g(x) - g(a) cancel. When g(x) equals g(a), then the difference quotient for $f \circ g$ is zero because f(g(x)) equals f(g(a)), and the above product is zero because it equals f′(g(a)) times zero. So the above product is always equal to the difference quotient, and to show that the derivative of $f \circ g$ at a exists and to determine its value, we need only show that the limit as x goes to a of the above product exists and determine its value.

To do this, recall that the limit of a product exists if the limits of its factors exist. When this happens, the limit of the product of these two factors will equal the product of the limits of the factors. The two factors are Q(g(x)) and (g(x) - g(a))/(x - a). The latter is the difference quotient for g at, and because g is differentiable at a by assumption, its limit as x tends to a exists and equals g′(a).

It remains to study Q(g(x)). Q is defined wherever f is. Furthermore, because f is differentiable at g(a) by assumption, Q is continuous at g(a) g is continuous at a because it is differentiable at a, and therefore $Q \circ g$ is continuous at *a*. So its limit as *x* goes to *a* exists and equals Q(g(a)), which is *f(g(a))*

This shows that the limits of both factors exist and that they equal $f'(g(a))$ $g`(a)$, and respectively. Therefore the derivative of $f \circ g$ at $a$ exists and equals $f`(g(a))$ $(g(a))$

Second proof

Another way of proving the chain rule is to measure the error in the linear approximation determined by the derivative. This proof has the advantage that it generalizes to several variables. It relies on the following equivalent definition of differentiability at a point: A function g is differentiable at a if there exists a real number g′(a) and a function ε(h) that tends to zero as h tends to zero, and furthermore

$$g(a+h)-g(a)=g'(a)h+\varepsilon(h)h.$$

Here the left-hand side represents the true difference between the value of g at a and at a + h, whereas the right-hand side represents the approximation determined by the derivative plus an error term.

In the situation of the chain rule, such a function ε exists because g is assumed to be differentiable at a. Again by assumption, a similar function also exists for f at g(a). Calling this function η, we have

$$f(g(a)+k)-f(g(a))=f'(g(a))k+\eta(k)k.$$

The above definition imposes no constraints on η(0), even though it is assumed that η(k) tends to zero as k tends to zero. If you set η(0) = 0, then η is continuous at 0.

Proving the theorem requires studying the difference $f(g(a+h))-f(g(a))$ as h tends to zero. The first step is to substitute for $g(a+h)$ using the definition of differentiability of g at a:

$$f(g(a+h))-f(g(a))=f(g(a)+g'(a)h+\varepsilon(h)h)-f(g(a)).$$

The next step is to use the definition of differentiability of f at $g(a)$. This requires a term of the form $f(g(a)+k)$ for some k. In the above equation, the correct k varies with h. Set $k_h = g'(a)h+\varepsilon(h)h$ and the right and side becomes $f(g(a)+k_h)-f(g(a))$. Applying the definition of the derivative gives:

$$f(g(a)+k_h)-f(g(a))=f'(g(a))k_h+\eta(k_h)k_h.$$

To study the behavior of this expression as h tends to zero, expand k_h. After regrouping the terms, the right-hand side becomes:

$$f'(g(a)g'(a)h+\left[f'\big(g(a)\big)\varepsilon(h)+\eta(k_h)g'(a)+\eta(k_h)\varepsilon(h)\right]h.$$

Because ε(h) and $\eta(k_h)$ tend to zero as *h* tends to zero, the bracketed terms tend to zero as *h* tends to zero. Because the above expression is equal to the difference *f(g a + h)*) − *f*(*g*(*a*)), by the definition of the derivative $f \circ g$ is differentiable at *a* and its derivative is *f′(g(a))g′(a).*

The role of Q in the first proof is played by η in this proof. They are related by the equation:

$$Q(y) = f'(g(a)) + \eta(y - g(a)).$$

The need to define Q at g(a) is analogous to the need to define η at zero. However, the proofs are not exactly equivalent. The first proof relies on a theorem about products of limits to show that the derivative exists. The second proof does not need this because showing that the error term vanishes proves the existence of the limit directly.

The chain rule in higher dimensions

The simplest generalization of the chain rule to higher dimensions uses the total derivative. The total derivative is a linear transformation that captures how the function changes in all directions. Let $f : \mathbf{R}^m \to \mathbf{R}^k$ and $g: \mathbf{R}^n \to \mathbf{R}^m$ be differentiable functions, and let *D* be the total derivative operator. If **a** is a point in $\mathbf{R}^n$, then the higher dimensional chain rule says that:

$$D_{\mathbf{a}}(f \circ g) = D_{g(\mathbf{a})} f \circ D_{\mathbf{a}} g,$$

or for short,

$$D(f \circ g) = Df \circ Dg.$$

In terms of Jacobian matrices, the rule says

$$J_{\mathbf{a}}(f \circ g) = J_{g(\mathbf{a})}(f) J_{\mathbf{a}}(g),$$

That is, the Jacobian of the composite function is the product of the Jacobians of the composed functions. The higher-dimensional chain rule can be proved using a technique similar to the second proof given above.

The higher-dimensional chain rule is a generalization of the one-dimensional chain rule. If *k*, m, and n are 1, so that f : R → R and g : R → R, then the Jacobian marices of *f* and g are 1 × 1. Specifically, they are:

$$J_a(g) = \left(g'(a)\right),$$
$$J_{g(a)}(f) = \left(f'(g(a))\right).$$

The Jacobian of $f \circ g$ is the product of these 1×1 matrices, so it is *f`*(*g*(*a*)) *g`*(*a*), as expected from the one-dimensional chain rule. In the language of linear transformations, $Da(g)$ is the function which scales a vector by a factor of $g`(a)$ and $Dg(a)(f)$ is the function which scales a vector by a factor of $f`(g(a))$. The chain rule says that the composite of these two linear transformations is the linear transformation $Da(f \circ g)$, and therefore it is the function that scales a vector by $f`(g(a))g`(a)$.

Another way of writing the chain rule is used when f and g are expressed in terms of their components as $\mathbf{y} = f(\mathbf{u}) = (f_1(\mathbf{u}), ..., f_k(\mathbf{u}))$ and $\mathbf{u} = g(\mathbf{x}) = (g_1(\mathbf{x}), ..., g_m(\mathbf{x}))$. In this case, the above rule for Jacobian matrices is usually written as:

$$\frac{\partial(f_1, \ldots, f_k)}{\partial(x_1, \ldots, x_n)} = \frac{\partial(f_1, \ldots, f_k)}{\partial(u_1, \ldots, u_m)} \frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_n)}.$$

The chain rule for total derivatives implies a chain rule for partial derivatives. Recall that when the total derivative exists, the partial derivative in the ith coordinate direction is found by multiplying the Jacobian matrix by the ith basis vector. By doing this to the formula above, we find:

$$\frac{\partial(f_1, \ldots, f_k)}{\partial x_i} = \frac{\partial(f_1, \ldots, f_k)}{\partial(u_1, \ldots, u_m)} \frac{\partial(g_1, \ldots, g_m)}{\partial x_i}.$$

Since the entries of the Jacobian matrix are partial derivatives, we may simplify the above formula to get:

$$\frac{\partial(f_1, \ldots, f_k)}{\partial x_i} = \sum_{\ell=1}^{m} \frac{\partial(f_1, \ldots, f_k)}{\partial u_\ell} \frac{\partial g_\ell}{\partial x_i}.$$

More conceptually, this rule expresses the fact that a change in the x_i direction may change all of g_l through g_k, and any of these changes may affect f.

In the special case where $k = 1$, so that f is a real-valued function, then this formula simplifies even further:

$$\frac{\partial f}{\partial x_i} = \sum_{\ell=1}^{m} \frac{\partial f}{\partial u_\ell} \frac{\partial g_\ell}{\partial x_i}$$

Example

Given $u = x^2 + 2y$ where $x = rsin(t)$ and $y = sin^2(t)$, determine the value of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial t}$ using the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (2x)(\sin(t)) + (2)(0) = 2r\sin^2(t)$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = (2x)\,(r\cos(t)) + (2)\,(2\sin(t)\cos(t))$$

$$= 2\,(r\sin(t))\,r\cos(t) + 4\sin(t)\cos(t) = 2\left(r^2+2\right)\sin(t)\cos(t).$$

Higher derivatives of multivariable functions

Faà di Bruno's formula for higher-order derivatives of single-variable functions generalizes to the multivariable case. If is a function of u = g(x) as above, then the second derivative of $f \circ g$ is:

$$\frac{\partial^2(f \circ g)}{\partial x_i \partial x_j} = \sum_k \frac{\partial f}{\partial x_k}\frac{\partial^2 g_k}{\partial x_i \partial x_j} + \sum_{k,\ell} \frac{\partial^2 f}{\partial u_k \partial u_\ell}\frac{\partial g_k}{\partial x_i}\frac{\partial g_\ell}{\partial x_j}$$

The composite function chain rule notation can also be adjusted for the multivariate case:

Given $z = f(u)$ and $u = g(x,y)$

such that $z = f\left[g(x,y)\right]$

Then the partial derivatives of z with respect to its two independent variables are defined as:

$$\frac{\partial z}{\partial x} = \frac{dy}{du}\cdot\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dy}{du}\cdot\frac{\partial u}{\partial y}$$

Let's do the same example as above, this time using the composite function notation where functions within the z function are renamed. Note that either rule could be used for this problem, so when is it necessary to go to the trouble of presenting the more formal composite function notation? As problems become more complicated, renaming parts of a composite function is a better way to keep track of all parts of the problem. It is slightly more time consuming, but mistakes within the problem are less likely.

Given $z = (2\pi + y^2)^3$

Let $z = f(u) = u^3$ and $u = g(x,y) = 2\pi + y^2$

Then $\frac{\partial z}{\partial x} = \frac{dy}{du}.\frac{\partial u}{\partial x} = (3u^2)(2) = 6u^2$

and $\frac{\partial z}{\partial y} = \frac{dy}{du}.\frac{\partial u}{\partial y} = (3u^2)(2y) = (6y)u^2$

The final step is the same, rep lace u with function g:

$$\frac{\partial z}{\partial x} = 6u^2 = 6(2x + y^2)^2$$

$$\frac{\partial z}{\partial y} = 6y(u)^2 = (6y)(2x + y^2)^2$$

Multivariate function

The rule for differentiating multivariate natural logarithmic functions, with appropriate notation changes is as follows:
Given
$z = f(u) = \text{In}(u) and\ u = g(x, y)$

Such that z = In g(x, y)
Then the partial derivatives of z with respect to its independent variables are defined as:

$$\frac{\partial z}{\partial x} = \frac{1}{u} \cdot \frac{\partial g}{\partial x} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{u} \cdot \frac{\partial g}{\partial y} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial y}$$

Let's do an example. Find the partial derivatives of the following function:

$$z = \ln(2x^2 + 4y^2)$$

$$\frac{\partial z}{\partial x} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial x} = \frac{1}{(2x^2 + 4y^2)} \cdot (4x) = \frac{4x}{(2x^2 + 4y^2)}$$

$$\frac{\partial z}{\partial y} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial y} = \frac{1}{(2x^2 + 4y^2)} \cdot (8y) = \frac{4y}{(x^2 + 2y^2)}$$

The rule for taking partials of exponential functions can be written as:

Given $z = f(u) = e^u$ $u = g(x, y)$
Such that $z = e^{g(x,y)}$

Then the partial derivatives of z with respect to its independent variables are defined as:

$$\frac{\partial z}{\partial x} = e^{g(x,y)} . \frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = e^{g(x,y)} . \frac{\partial g}{\partial y}$$

One last time, we look for partial derivatives of the following using the exponential function rule:

$$z = e^{g(x,y)}$$

$$\frac{\partial z}{\partial x} = e^{g(x,y)}.\frac{\partial g}{\partial x} = e^{(3xy)^2}.(3y^2) = (3y^2)e^{(3xy^2)}$$

$$\frac{\partial z}{\partial y} = e^{g(x,y)}.\frac{\partial g}{\partial y} = e^{(3xy)^2}.(6xy) = (6xy)e^{(3xy^2)}$$

Higher order partial and cross partial derivatives

The story becomes more complicated when we take higher order derivatives of multivariate functions. The interpretation of the first derivative remains the same, but there are now two second order derivatives to consider.

First, there is the direct second-order derivative. In this case, the multivariate function is differentiated once, with respect to an independent variable, holding all other variables constant. Then the result is differentiated a second time, again with respect to the same independent variable. In a function such as the following:

$$z = f(x,y)$$

There are 2 direct second-order partial derivatives, as indicated by the following examples of notation:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = Z_{xx}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = Z_{yy}$$

These second derivatives can be interpreted as the rates of change of the slopes of the function z.

Now the story gets a little more complicated. The cross-partials, f_{xy} and f_{yx} are defined in the following way. First, take the partial derivative of z with respect to x. Then take the derivative again, but this time, take it with respect to y, and hold the x constant. Specially, think of the cross partial as a measure of how the slope (change in z with respect to x) changes, when the y variable changes. The following are examples of notation for cross-partials:

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

You'll discuss economic meaning further in the next section, but for now, we'll just show an example, and note that in a function where the cross-partials are continuous, they will be identical. For the following function:

$$z = 2x^3 + 3xy + 2y^2$$

Take the first and second partial derivatives.

$$z = 2x^3 + 3xy + 2y^2$$

$$z_x = 6x^2 + 3y \qquad z_y = 3x + 4y$$

$$z_{xx} = 12x \qquad z_{yy} = 4$$

Now, starting with the first partials, find the cross partial derivatives:

$$z_x = 6x^2 + 3y \qquad z_y = 3x + 4y$$

$$z_{xy} = 3 \qquad z_{yx} = 3$$

4.0 CONCLUSION

In this unit, you have been introduced to the composite differentiation also called the chain rule. You have known the Composites of more than two functions. You have also known the quotient rule. You have solved problems on higher derivative with the use of composite differentiation. You have proof the chain rule and known the rule in higher dimension.

5.0 SUMMARY

In this unit, you have studied:

- The chain rule
- Composites of more than two functions
- The quotient rule
- Higher derivative
- Proof of the chain rule
- The rule in higher dimension

6.0 TUTOR-MARKED ASSIGNMENT

1. What are the second – order derivatives of the function
   $F(x,y)= xy^2 + x^3 y^5$
2. Express x- and y- derivatives of $W(x^3 y^3)$ in terms of x,y.

3. What are the second - order derivatives of the function
   $F(x,y) = x^4 y^6$
4. What are the second – order derivatives of the function
   K(x,y) = ln (2x-3y).
5. What are the second – order derivatives of the function
   $R(x,y) = x^{\frac{1}{2}} y^{\frac{1}{3}}$
6. What are the second – order derivatives of the function
   $N(x,y) = \tan^{-1}(x, y)$.

7.0 REFERENCES/FURTHER READING

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UNIT 2 EULER'S THEOREM

CONTENT

1.0 INTRODUCTION

In number theory, **Euler's theorem** (also known as the **Fermat–Euler theorem** or **Euler's totient theorem**) states that if n and a are coprime positive integers, then

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

where φ(n) is Euler's totient function and "... ≡ ... (mod n)" denotes congruence modulo n.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state and prove the Euler's theorem.

3.0 MAIN CONTENT

The converse of Euler's theorem is also true: if the above congruence holds for positive integers a and n, then a and n are coprimes.

The theorem is a generalization of Fermat's little theorem, and is further generalized by Carmichael's theorem.

The theorem may be used to easily reduce large powers modulo *n*. For example, consider finding the ones place decimal digit of 7^{222}, i.e. 7^{222} (mod 10). Note that 7 and 10 are coprime, and $\varphi(10) = 4$. So Euler's theorem yields 74 ≡ 1 (mod 10), and we get $7^{222} \equiv 7^{4\times55+2} \equiv (7^4)^{55}\times7^2 \equiv 1^{55}\times7^2 \equiv 49 \equiv 9$ (mod 10).

In general, when reducing a power of a modulo n (where a and n are coprime), one needs to work modulo φ(n) in the exponent of a:
if x ≡ y (mod φ(n)), then ax ≡ ay (mod n).

Euler's theorem also forms the basis of the RSA encryption system: encryption and decryption in this system together amount to exponentiating the original text by k(p(n)+1 for some positive integer k, so Euler's theorem shows that the decrypted result is the same as the original.

Proofs

1. Leonhard Euler published a proof in 1789. Using modern terminology, one may prove the theorem as follows: the numbers b which are relatively prime to n form a group under multiplication mod n, the group G of (multiplicative) units of the ring Z/*n*Z. This group has $\varphi(n)$ elements. The element a: = a (mod n) is a member of the group G, and the order o(a) of a (the least k > 0 such that $a^k = 1$) must have a multiple equal to the size of G. (The order of a is the size of the subgroup of G generated by a, and Lagrange's theorem states that the size of any subgroup of G divides the size of G.)

   Thus for some integer M > 0, $M \cdot o(a) = \varphi(n)$. Therefore $a^{\varphi(n)} = a^{o(a) \cdot M} = (a^{o(a)})^M = 1^M = 1$. This means that $a^{\varphi}(n) = 1 \pmod{n}$.

2. Another direct proof: if a is coprime to n, then multiplication by a permutes the residue classes mod n that are coprime to n; in other words (writing R for the set consisting of the φ(n) different such classes) the sets { *x* : *x* in *R* } and { *ax* : *x* in *R* } are equal; therefore, the two products over all of the elements in each set are equal. Hence, $P \equiv a^{\varphi}(n)P \pmod{n}$ where P is the product over all of the elements in the first set. Since P is coprime to n, it follows that $a^{\varphi}(n) \equiv 1$ (mod

4.0 CONCLUSION

In this unit, you have stated and proved the Euler’s theorem

5.0 SUMMARY

In this unit, you have known the statement of Euler’s theorem and proved Euler’s theorem.

6.0 TUTOR-MARKED ASSIGNMENT

State and prove Euler’s theorem.

7.0 REFERENCES/FURTHER READING

Hernandez Rodriguez and Lopez Fernandez, A Semiotic Reflection on the Didactics of the Chain Rule, The Montana Mathematics Enthusiast, ISSN 1551-3440, Vol. 7, nos.2&3, pp.321–332.

Apostol, Tom (1974). Mathematical analysis (2nd ed. ed.). Addison Wesley. Theorem 5.5.

UNIT 3 IMPLICIT DIFFERENTIATION

CONTENTS

1.0 INTRODUCTION

Most of our math work thus far has always allowed us to solve an equation for y in terms of *x*. When an equation can be solved for y you call it an *explicit* function. But not all equations can be solved for y. An example is:
$x^3 + y^3 = 6xy$

This equation cannot be solved for y. When an equation cannot be solved for y, you call it an implicit function. The good news is that you can still differentiate such a function. The technique is called *implicit differentiation.*

When you implicitly differentiate, you must treat y as a composite function and therefore you must use the chain rule with y terms. The reason for this can be seen in Leibnitz notation: $\frac{d}{dx}$. This notation tells you that you are differentiating with respect to *x*. Because *y* is not native to what are differentiating with respect to, you need to regard it as a composite function. As you know, when you differentiate composite function you must use the chain rule.

Let's now try to differentiate the implicit function, $x^3 + y^3 = 6xy$

$$x^3 + y^3 = 6xy$$

This is a "folium of Descartes" curve. This would be very difficult to solve for y, so you will need to use implicit differentiation.

$$\frac{d}{dx}(x^3 + y^3 = 6xy)$$

Here you show with Leibnitz notation that you are implicitly differentiating both sides of the equation.

$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}$

On the left side you need to individually take the derivative of each term. On the right side you have to use the product rule (6x.y)

$$3x^2 + 3y^2y' - 6xy' + 6y$$

Here you take the individual derivatives. Note: Where did the *y'* come from? Because you are differentiating with respect to *x*, you need to use the chain rule on the *y*. Notice that you did use the product rule on the right side.

$$3y^2y' - 6xy' = 6y - 3x^2$$

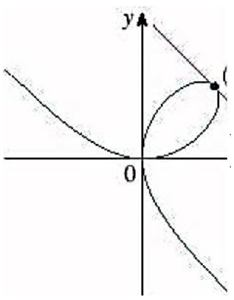
Now you get the *y'* terms on the same side of the equation.

$$y'(3y^2 - 6x = 6y - 3x^2$$

Now you factor *y'* out of the expression on the left side.

$$y' \frac{6y-3x^2}{3y^2-6x} = \frac{3(2y-x^2)}{3(y^2-2_2)}$$

Now you divide both sides by the$3y^2 - 6x$ factor and simplify.



You can see in a plot of the implicit function that the slope of the tan gent line at the point (3,3) does appear to be - 1.

Another example: Differentiate: $x^2 - 2xy + y^3 = c$

Given implicit function

$$x^2 - 2xy + y^3 = c$$

Doing implicit differentiation on the function.

$$2x - (2xy' + 2y) + 3y^2y' = 0$$

Note the use of the product rule on the second term

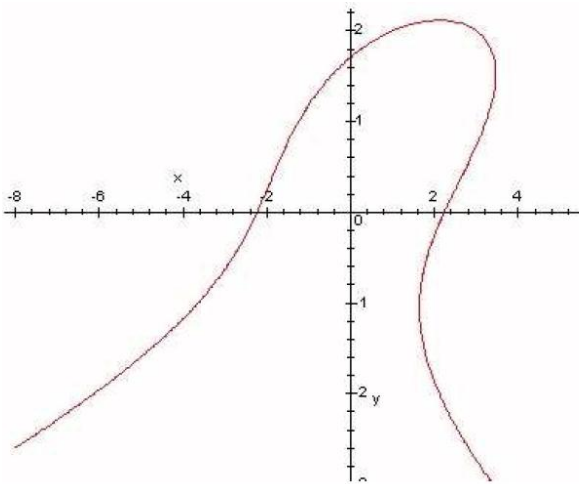
$$2x - 2xy' - 2y + 3y^2y' = 0$$

$$-2xy' + 3y^2y' = 2y - 2x$$

$$y'(3y^2 - 2x) = 2y - 2x$$

You do the algebra to solve for y'.

$$y' = \frac{2y - 2x}{3y^2 - 2x}$$



Here you see a portion of plot of the implicit equation with *c* set equal to 5.. When does it appear that the slope of the tangent line will be zero? It appears to be at about (2.2,2.2).

You take our derivative, set it equal to zero, and solve.

$$0 = \frac{2y - 2x}{3y^2 - 2x}$$
$$0 = 2y - 2x$$
$$y = x$$

Now putting *x* = original implicit equation, you find that...

$$x^2 - 2xy + y^3 = 5$$
$$x^2 - 2x^2 + y^3 = 5$$
$$x^3 - x^2 - 5 = 0$$

You still must use a computer algebra system to solve this cubic equation. The one real answer is shown:

X = y = 2.116343299

This answer does seem consistent with your visual estimate.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recall the derivatives of Inverse Trigonometric Functions;
- define and identify Implicit differentiation;
- recall formula for two variables;
- appreciate applications in economics; and
- solve Implicit differentiation problems.

3.0 MAIN CONTENT

Links to other explanations of Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

Thanks to implicit differentiation, you can develop important derivatives that you could not have developed otherwise. The inverse trigonometric functions fall under this category. You will develop and remember the derivatives of the inverse sine and inverse tangent.

$y = \sin^{-1} x$	Inverse sine function.
$\sin y = x$	This is what inverse sine means.
$\cos y \frac{dy}{dx} = 1$	You implicitly differentiate both sides of the equation with respect to x. Because we are differentiating with respect to x, you need to use the chain rule on the left side.
$\frac{dy}{dx} = \frac{1}{\cos y}$	You solve the equation for $\frac{dy}{dx}$
$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$	This is because of the trigonometric identity, $\sin^2 y + \cos^2 y = 1$
$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	Refer back to the equation in step two above. You have our derivative

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x \frac{1}{1+x^2}$$

Implicit differentiation

In

$y = \tan^{-1} x$	The inverse tangent function
$\tan y = x$	This is what inverse tangent means

$\sec^2 y \frac{dy}{dx} = 1$ You implicitly differentiate both sides of the equation with respect to x. Because you are differentiating with respect to x, you need to use the equation for the chain rule on the left side. $\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\frac{dy}{dx} = \frac{1}{1+\tan^2 x}$ This is because of the trigonometric identity, $\tan^2 y + 1 = \sec^2 y$.

$\frac{dy}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ Refer back to the equation in step two above. You have your derivative.

In calculus, a method called imp licit differentiation makes use of the chain rule to differentiate implicitly defined functions.

As explained in the introduction, y can be given as a function of x implicitly rather than explicitly. When you have an equation *R(x, y)* = 0, you may be able to solve it for y and then differentiate. However, sometimes it is simpler to differentiate R(x, y) with respect to *x* and *y* and then solve for *dy/dx.*

Examples

1. Consider for example

y + x + 5 = 0

This function normally can be manipulated by using algebra to change this equation to one expressing y in terms of an explicit function:

y = -x – 5,
where the right side is the explicit function whose output value is y. Differentiation then gives $\frac{dy}{dx} = -1$

Alternatively, one can totally differentiate the original equation:

$$\frac{dy}{dx} + \frac{dx}{dx} + \frac{d}{dx}(5) = 0;$$

$$\frac{dy}{dx} + 1 = 0.$$

Solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = 1 ,$$

the same answer as obtained previously.

2. An example of an implicit function, for which implicit differentiation might be easier than attempting to use explicit differentiation, is

$$x^4 + 2y^2 = 8$$

In order to differentiate this explicitly with respect to x, one would have to obtain (via algebra)

$$y = f(x) = \pm\sqrt{\frac{8 - x^4}{2}},$$

and then differentiate this function. This creates two derivatives: one for $y > 0$ and another for $y < 0$.

One might find it substantially easier to implicitly differentiate the original function:

$$4x^3 + 4y\frac{dy}{dx} = 0,$$

giving,

$$\frac{dy}{dx} = \frac{-4x^3}{4y} = \frac{-x^3}{y}$$

3. Sometimes standard explicit differentiation cannot be used and, in order to obtain the derivative, implicit differentiation must be employed. An example of such a case is the equation $y - y = x$. It is impossible to express y explicitly as a function of x and therefore *dy/dx* cannot be found by explicit differentiation. Using the implicit method, *dy/dx* can be expressed:

$$5y^4\frac{dy}{dx} - \frac{dy}{dx} = \frac{dy}{dx}$$

Where $\frac{dy}{dx} = 1$ factoring out dx shows that

$$\frac{dy}{dx}(5y^4 - 1) = 1$$

which yields the final answer

$$\frac{dy}{dx} = \frac{1}{5y^4 - 1},$$

which is defined for $y \neq \pm\frac{1}{\sqrt[4]{5}}$.

Formula for two variables

"The Implicit Function Theorem states that if *F* is defined on an open disk containing (*a*,*b*), where *F(a,b)* = 0, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation F(*x*,*y*) = 0 defines y as a function of x near the point (a,b) and the function is given by..."[1]:§ 11.5

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y},$$

where F_x and F_y indicate the derivatives of *F* with respect to *x* and *y*.

The above formula comes from using the generalized chain rule to obtain the total derivative — with respect to x — of both sides of *F(x, y)* = 0:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

and hence

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Implicit function theorem

It can be shown that if R(x,y) is given by a smooth submanifold *M* in R^2, nd (*a*,*b*) is a point of this submanifold such that the tangent space there is not vertical (that is $\frac{\partial R}{\partial y} \neq 0$), then M in some small enough neighborhood of (a,b) is given by a parameterization *(x,f(x))* where f is a smooth function. In less technical language, implicit functions exist and can be differentiated, unless the tangent to the supposed graph would be vertical. In the standard case where we are given an equation

R(x,y) = 0

the condition on R can be checked by means of partial derivatives .

Applications in economics

Marginal rate of substitution

In economics, when the level set *R(x,y)* = 0 is an indifference curve for the quantities x and y consumed of two goods, the absolute value of the implicit derivative is interpreted as the marginal rate of substitution of the two goods: how much more of y one must receive in order to be indifferent to a loss of 1 unit of x.

IMPLICIT DIFFERENTIATION PROBLEMS

The following problems require the use of implicit differentiation. Implicit differentiation is nothing more than a special c se of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions y written EXPLICITLY as functions of x. For example, f

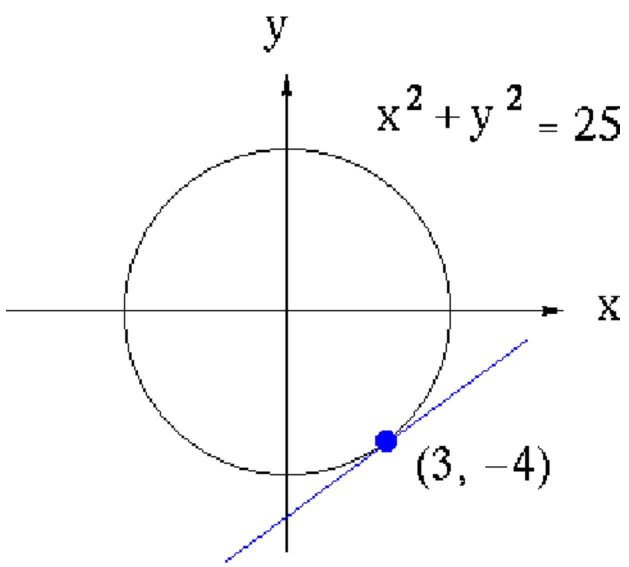
$y = 3x^2 - \sin(7x + 5)$,

then the derivative of y is *6x – 7cos (7x+5)*

However, some functions *y* are written implicitly as functions of *x*. A familiar example of this is the equation

$x^2 + y^2 = 25$,

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point (3, -4).



How could we find the derivative of y in this instance? One way is to first write *y* explicitly as a function of x. Thus,

$x^2 + y^2 = 25$,

$y^2 = 25 - x^2$, and

$y = \pm\sqrt{25 - x^2}$,

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point (3, -4) lies on the bottom semi-circle given by

$y = -\sqrt{25 - x^2}$,

the derivative of y is

$$y' = -(1/2)\left(25 - x^2\right)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}},$$

i.e.,

$$y' = \frac{x}{\sqrt{25 - x^2}}.$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{(3)}{\sqrt{25 - (3)^2}} = \frac{3}{4}.$$

Unfortunately, not every equation involving *x* and *y* can be solved explicitly for *y*. For the sake of illustration you will find the derivative of y without writing *y* explicitly as a function of *x*. Recall that the derivative (D) of a function of *x* squared, $(f(x))^2$, can be found using the chain rule:

$$D\{(f(x))^2\} = 2f(x)D\{f(x)\} = 2f(x)f'(x)$$

Since *y* symbolically represents a function of *x*, the derivative of y^2 can be found in the same fashion:

$$D\{y^2\} = 2y\ D\{y\} = 2yy'$$

Now begin with

$x^2 + y^2 = 25$.

Differentiate both sides of the equation, getting

$D\ (x^2 + y^2) = D\ (25)$,

$D\ (x^2) + D\ (y^2) = D\ (25)$, and

$2x + 2\ y\ y' = 0$,

so that

$2\ y\ y' = -\ 2x$,   and

$$y' = \frac{-2x}{2y} = \frac{-x}{y},$$

i.e.,

$$y' = \frac{-x}{y}.$$

Thus, the slope of the line tan gent to the graph at the point (3, -4) is

$$m = y' = \frac{-(3)}{(-4)} = \frac{3}{4}.$$

This second method illustrates the process of implicit differentiation. It is important to note that the derivative expression for explicit differentiation involves *x* only, while the derivative expression for implicit differentiation may involve both *x* and *y*.

The following problems range in difficulty from average to challenging.

PROBLEM 1: Assume that y is a function of x. Find y' = dy/dx for x^3 *+ y3 = 4.*

SOLUTION 1: Begin with $x^3 + y^3 = 4$. Differentiate both sides of the equation, getting

$D (x^3 + y^3) = D (4)$,

$D (x^3) + D (y^3) = D (4)$,

(Remember to use the chain rule on $D (y^3)$.)

$3x^2 + 3y^2 y' = 0$,

so that (Now solve for y' .)

$3y^2 y' = - 3x^2$, and

$$y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

PROBLEM 2: Assume that y is a function of x. Find $y^1 = {}^{dy}\!/_{dx}$ *for* $(x-y)^2 = x+y^{-1}$

SOLUTION 2: Begin with $(x-y)^2 = x + y - 1$. Differentiate both sides of the equation, getting

$D (x-y)^2 = D (x + y - 1)$,

$D (x-y)^2 = D (x) + D (y) – D (1)$

(Remember to use the chain rule on $D (x-y)^2$.)

$$2(x-y)\ D(x-y) = 1 + y' - 0,$$

$2 (x-y) (1- y') = 1 + y'$,

so that (Now solve for y' .)

$2 (x-y) - 2 (x-y) y' = 1 + y'$,

$- 2 (x-y) y' - y' = 1 - 2 (x-y)$,

(Factor out y' .)

$y' [- 2 (x-y) - 1] = 1 - 2 (x-y)$, and

$$y' = \frac{1-2(x-y)}{-2(x-y)-1} = \frac{2y-2x+1}{2y-2x-1}.$$

PROBLEM 3 Assume that y is a function of x. Find y^1 for y = 5 in (3x+4y)

SOLUTION 3: Begin with$y = \sin(3x+4y)$. Differentiate both sides of the equation, getting

$$D(y) = D(\sin(3x+4y)),$$

(Remember to use the chain rule on)

$$y' = \cos(3x+4y)\ D(3x+4y),$$

$$y' = \cos(3x+4y)(3+4y'),$$

so that (Now solve for y' .)

$$y' = 3\cos(3x+4y) + 4y'\cos(3x+4y),$$

$$y' - 4y'\cos(3x+4y) = 3\cos(3x+4y)$$

(Factor out y'.)

$y'[1-4\cos(3x+4y)] = 3\cos(3x+4y)$ and

$$y' = \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}.$$

PROBLEM 4:Assume that y is a function of x. Find y^1 for $y = x^2y^3+x^3y^2$

SOLUTION 4: Begin with $y = x^2\, y^3 + x^3\, y^2$. Differentiate both sides of the equation, getting

$D(y) = D(x^2\, y^3 + x^3\, y^2)$,

$D(y) = D(x^2\, y^3) + D(x^3\, y^2)$

(Use the product rule twice.)

$$y' = \{x^2 D(y^3) + D(x^2)y^3\} + \{x^3 D(y^2) + D(x^3)y^2\},$$

(Remember to use the chain rule on $D(y^3)$ and $D(y^2)$.)

$$y' = \{x^2(3y^2y') + (2x)y^3\} + \{x^3(2yy') + (3x^2)y^2\},$$

$y' = 3x^2 y^2 y' + 2x y^3 + 2x^3 y y' + 3x^2 y^2$,

so that (Now solve for y' .)

$y' - 3x^2 y^2 y' - 2x^3 y y' = 2x y^3 3x^2 y^2$,

(Factor out y' .)

$y' [1 - 3x^2 y^2 - 2x^3 y] = 2x y^3 + 3x^2 y^2$, and

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}.$$

PROBLEM 5: Find $y^1 = \frac{dy}{dx}$ for $e^{xy} = e^{4x} - e^{5y}$

SOLUTION 5: Begin with $e^{xy} = e^{4x} - e^{5y}$. Differentiate both sides of the equation, getting

$$D(e^{xy}) = D(e^{4x}) - D(e^{5y})$$

$$e^{xy} D(xy) = e^{4x} D(4x) - e^{5y} D(5y)$$

$$e^{xy}(xy' + (1)y) = e^{4x}(4) - e^{5y}(5y')$$

so that (Now solve for y^1)

$$xe^{xy}y' + ye^{xy} = 4e^{4x} - 5e^{5y}y'$$

$$xe^{xy}y' + 5e^{5y}y' = 4e^{4x} - ye^{xy}$$

(Factor out y^1)

$$y'[xe^{xy} + 5e^{5y}] = 4e^{4x} - ye^{xy}$$

and

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}}$$

.

PROBLEM 6: Solve for y^1 if $\cos^2 x + \cos^2 y = \cos(2x+2y)$

***SOLUTION 6**:* Begin with $\cos^2 x + \cos^2 y = \cos(2x + 2y)$. Differentiate both sides of the equation, getting

$$D(\cos^2 x + \cos^2 y) = D(\cos(2x + 2y))$$

$$D(\cos^2 x) + D(\cos^2 y) = D(\cos(2x + 2y))$$

$$(2\cos x)D(\cos x) + (2\cos y)D(\cos y) = -\sin(2x + 2y)D(2x + 2y)$$

$$2\cos x(\sin x) + 2\cos y(-\sin y)(y') = -\sin(2x + 2y)(2 + 2y')$$

So that (Now solve for y^1)

$$-2\cos x\sin x - 2y'\cos y\sin y = -2\sin(2x + 2y) - 2y'\sin(2x + 2y)$$

$$2y'\sin(2x + 2y) - 2y'\cos y\sin y = -2\sin(2x + 2y) + 2\cos x\sin x$$

(Factor out y^1)

$$y'[2\sin(2x + 2y) - 2\cos y\sin y] = 2\cos x\sin x - 2\sin(2x + 2y)$$

$$y' = \frac{2\cos x\sin x - 2\sin(2x + 2y)}{2\sin(2x + 2y) - 2\cos y\sin y}$$

$$y' = \frac{2[\cos x\sin x - \sin(2x + 2y)]}{2[\sin(2x + 2y) - \cos y\sin y]}$$

and

$$y' = \frac{\cos x\sin x - \sin(2x + 2y)}{\sin(2x + 2y) - \cos y\sin y}$$

.

PROBLEM 7: Assume that y is a function of x^1. Find y^1 for $= 3+\sqrt{x^2+y^2}$

SOLUTION 7: Begin with $x = 3+\sqrt{x^2+y^2}$. Differentiate both sides of the equation, getting

$$D(x) = D(3+\sqrt{x^2+y^2})$$

$1 = (1/2)(x^2 + y^2)^{-1/2} D(x^2 + y^2)$,

$1 = (1/2)(x^2 + y^2)^{-1/2} (2x + 2y')$, so that (Now solve for y'.)

$$1 = \frac{(1/2)(2)(x + yy')}{\sqrt{x^2 + y^2}}$$

$$1 = \frac{x + yy'}{\sqrt{x^2 + y^2}} ,$$

$$\sqrt{x^2 + y^2} = x + yy'$$

$$\sqrt{x^2 + y^2} - x = yy'$$

and

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y}$$

PROBLEM 8: Find y1 if $\frac{x - y^3}{y + x^2} = x + 2$

SOLUTION 8: Begin with $\frac{x-y^3}{y+x^2} = x + 2$. Clear the fraction by multiplying both sides of the equation by y + x^2, getting

$$\frac{x - y^3}{y + x^2}(y + x^2) = (x + 2)(y + x^2) ,$$

or

x - y^3 = xy + 2y + x^3 + $2x^2$.

Now differentiate both sides f the equation, getting

D (x - y^3) = D (xy + 2y + x^3 + $2x^2$) ,

D (x) - D (y^3) = D (xy) + D (2y) + D (x^3) + D ($2x^2$) ,

(Remember to use the chain rule on D (y^3) .)

1 - 3 y^2 y' = (xy' + (1)y) + 2 y' + $3x^2$ + 4x ,

so that (Now solve for y' .)

1 - y - $3x^2$ - 4x = 3 y^2 y' + xy' + 2 y' ,

(Factor out y'.)

$1 - y - 3x^2 - 4x = (3y^2 + x + 2)y'$

and

$$y' = \frac{1 - y - 3x^2 - 4x}{3y^2 + x + 2}$$

PROBLEM 9: Find dy/dx *for* $\frac{y}{x^3} + \frac{x}{y^3} = x^2 y^4$

SOLUTION 9: Begin with $\frac{y}{x^3} + \frac{x}{y^3} = x^2 y^4$. Clear the fractions by multiplying both sides of the equation by $x^3 y^3$ getting

$$\left\{ \frac{y}{x^3} + \frac{x}{y^3} \right\} (x^3 y^3) = x^2 y^4 (x^3 y^3)$$
,

$$\frac{y x^3 y^3}{x^3} + \frac{x x^3 y^3}{y^3} = x^2 x^3 y^4 y^3$$

$y^4 + x^4 = x^5 y^7$.

Now differentiate both sides of the equation, getting

$D (y^4 + x^4) = D (x^5 y^7)$,

$D (y^4) + D (x^4) = x^5 D (y^7) + D (x^5) y^7$,

(Remember to use the chain rule on $D (y^4)$ and $D (y^7)$.)

$4 y^3 y' + 4 x^3 = x^5 (7 y^6 y') + (5 x^4) y^7$,

so that (Now solve for y' .)

$4 y^3 y' - 7 x^5 y^6 y' = 5 x^4 y^7 - 4 x^3$,

(Factor out y' .)

$y' [4 y^3 - 7 x^5 y^6] = 5 x^4 y^7 - 4 x^3$,

and

$$y' = \frac{5x^4y^7 - 4x^3}{4y^3 - 7x^5y^6}.$$

PROBLEM 10: Assume that y is a function of x. Find y^1 for $(x^2+y^2)^3 = 8x^2y^2$

SOLUTION 10: Begin with $(x^2+y^2)^3 = 8x^2y^2$. Now differentiate both sides of the equation, getting

$D (x^2+y^2)^3 = D (8x^2y^2)$,

$3 (x^2+y^2)^2 D (x^2+y^2) = 8x^2 D (y^2) + D (8x^2) y^2$,

(Remember to use the chain rule on $D (y^2)$.)

$3 (x^2+y^2)^2 (2x + 2 y y') = 8x^2 (2 y y') + (16 x) y^2$,

so that (Now solve for y' .)

$6x (x^2+y^2)^2 + 6 y (x^2+y^2)^2 y' = 16 x^2 y y' + 16 x y^2$,

$6 y (x^2+y^2)^2 y' - 16 x^2 y y' = 16 x y^2 - 6x (x^2+y^2)^2$,

(Factor out y' .)

$y' [6 y (x^2+y^2)^2 - 16 x^2 y] = 1 x y^2 - 6x (x^2+y^2)^2$,

and

$$y' = \frac{16xy^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 16x^2y}.$$

Thus, the slope of the line tan gent to the graph at the point (-1, 1) is

$$m = y' = \frac{16(-1)(1)^2 - 6(-1)((-1)^2+(1)^2)^2}{6(1)((-1)^2+(1)^2)^2 - 16(-1)^2(1)} = \frac{8}{8} = 1,$$

and the equation of the tangent line is

$y - (1) = (1) (x - (-1))$

or

$y = x + 2$

PROBLEM 11: Find y^1 for $x^2+(y-x)^3 = 9$.

SOLUTION 11: Begin with $x^2 + (y-x)^3 = 9$. If x=1, then

$(1)^2 + (y-1)^3 = 9$

so that

$(y - 1)^3 = 8$,

y - 1 = 2,

y = 3,

and the tangent line passes through the point (1, 3) . Now differentiate both sides of the original equation, getting

$D (x^2 + (y-x)^3) = D (9)$,

$D (x^2) + D (y-x)^3 = D (9)$,

$2x + 3 (y-x)^2 D (y-x) = 0$,

$2x + 3 (y-x)^2 (y'-1) = 0$,

so that (Now solve for y' .)

$2x + 3 (y-x)^2 y'- 3 (y-x)^2 = 0$,

$3 (y-x)^2 y' = 3 (y-x)^2 - 2x$,

and

$$y' = \frac{3(y-x)^2 - 2x}{3(y-x)^2}.$$

Thus, the slope of the line tan gent to the graph at (1, 3) is

$$m = y' = \frac{3(3-1)^2 - 2(1)}{3(3-1)^2} = \frac{10}{12} = \frac{5}{6},$$

and the equation of the tangent line is

y - (3) = (5/6) (x - (1)) ,

or

y = (7/6) x + (13/6) .

PROBLEM 12: Assume that y is a function of x. Find $y^1 = {}^{dy}\!/_{dx}$ *for* $x^2y+y^4 = 4+2x$

SOLUTION 12: Begin with $x^2y + y^4 = 4 + 2x$. Now differentiate both side s of the original equation, getting

$D (x^2 y + y^4) = D (4 + 2x)$,

$D (x^2 y) + D (y^4) = D (4) + D (2x)$,

$(x^2 y' + (2x) y) + 4 y^3 y' = 0 + 2$,

so that (Now solve for y' .)

$x^2 y' + 4 y^3 y' = 2 - 2x y$,

(Factor out y' .)

$y' [x^2 + 4 y^3] = 2 - 2x y$,

and

(Equation 1)

$$y' = \frac{2-2xy}{x^2+4y^3}$$

Thus, the slope of the graph (the slope of the line tangent to the graph) at (-1, 1) is

$$y' = \frac{2-2(-1)(1)}{(-1)^2+4(1)^3} = \frac{4}{5}.$$

Since y'= 4/5, the slope of the graph is 4/5 and the graph is increasing at the point (-1, 1). Now determine the concavity of the graph at (-1, 1). Differentiate Equation 1, getting

$$y'' = \frac{(x^2+4y^3)D(2-2xy)-(2-2xy)D(x^2+4y^3)}{(x^2+4y^3)^2}$$

$$= \frac{(x^2+4y^3)((-2x)y'+(-2)y)-(2-2xy)(2x+12y^2y')}{(x^2+4y^3)^2}.$$

Now let x=-1, y=1 , and y'=4/5 so that the second derivative is

$$y'' = \frac{[(-1)^2 + 4(1)^3][(-2(-1))(4/5) + (-2)(1)] - [2 - 2(-1)(1)][2(-1) + 12(1)^2(4/5)]}{((-1)^2 + 4(1)^3)^2}$$

$$= \frac{(5)(8/5 - 2) - (4)(-2 + 48/5)}{25}$$

.

$$= \frac{-2 - (152/5)}{25}$$

$$= \frac{-162}{125}$$

Since y'' < 0, the graph is concave down at the point (-1, 1)

4.0 CONCLUSION

In this unit you have studied the derivative of inverse of trigonometric functions. You have known the definition of implicit differentiation and have identified problems on implicit differentiation. You have also studied the formula for two variables and implicit differentiation applications in economics. You have solved various examples on implicit differentiation.

5.0 SUMMARY

In this course you have studied

- The derivatives of Inverse Trigonometric Functions
- Definition and identification f Implicit differentiation
- The formula for two variables
- The applications in economic
- Implicit differentiation problems

6.0 TUTOR-MARKED ASSIGNMENT

Find the equation of the tangent line to the ellipse 25 $x^2 + y^2 = 109$

Find y' if $y^4 + 4y - 3x^3 \sin(y) = 2x + 1$.

Find y' if $xy^3 + x^2y^2 + 3x^2 - 6 = 1$.

Show that if a normal line to each point on an ellipse passes through the center of an ellipse, then the ellipse is a circle.

7.0 REFERENCES/FURTHER READING

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