MODULE 6 TAYLOR'S SERIES EXPANSION

UNIT 1 FUNCTIONS OF TWO VARIABLES

CONTENTS

1.0 INTRODUCTION

Functions of Two Variables

Definition of a function of two variables

Until now, we have only considered functions of a single variable.

However, many real-valued functions consist of two (or more) variables. E.g., the area of a rectangular shape depends on both its width and its height. And, the pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume. You define a function of two variables as follows:

A function f of two variables is a relation that assigns to every ordered pair of input values *x, y* in a set called the domain of a unique output value denoted by*, f (x, y).* The set of output values is called the *range*.

Since the domain consists of ordered pairs, you may consider the domain to be all (or part) of the x-y plane.

Unless otherwise stated, you will assume that the variables x and y and the output Value *f (x, y).*

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- solve problems on partial derivatives in calculus;
- solve problems on higher order partial derivative;
- state and apply clairauts theorem;
- solve problem on maxima and manima;
- identify Taylor series of function of two variable; and
- understand analytical function.

3.0 MAIN CONTENT

Partial Derivatives in Calculus

Let f(x,y) be a function with two variables. If we keep y constant and differentiate f (assuming f is differentiable) with respect to the variable *x*, we obtain what is called the partial derivative of f with respect to x which is denoted by

$$\frac{\partial f}{\partial x} \text{ or } f_x$$

You might also define partial derivatives of function f as follows:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

You now present several examples with detailed solution on how to calculate partial derivatives.

Example 1: Find the partial derivatives f_x and f_y if *f*(*x*, *y*) is given by

$f(x , y) = x^2 y + 2x + y$

Solution:

Assume y is constant and differentiate with respect to x to obtain

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2 y + 2x + y]$$
$$= \frac{\partial}{\partial x}[x^2 y] + \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial x}[y] = [2xy] + [2] + [0] = 2xy + 2$$

Now assume x is constant and differentiate with respect to y to obtain

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y + 2x + y]$$

$$= \frac{\partial}{\partial y}[x^2y] + \frac{\partial}{\partial y}[2x] + \frac{\partial}{\partial y}[y] = [x^2] + [0] + [1] = x^2 + 1$$

Example 2: Find fx and fy if f(x, y) is given by

f(x , y) = sin(x y) + cos x

Solution:

Differentiate with respect to x assuming y is constant

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin(xy) + \cos x] = y\cos(xy) - \sin x$$

Differentiate with respect to y assuming x is constant

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(xy) + \cos x] = x\cos(xy)$$

Example 3: Find f_x and f_y if f(x, y) is given by

f(x , y) = x $e^{x y}$

Solution:

Differentiate with respect to x assuming y is constant

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[xe^{xy}] = e^{xy} + xye^{xy} = (xy + 1)e^{xy}$$

Differentiate with respect to y

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[xe^{xy}] = x^2e^{xy}$$

Example 4: Find f_x and f_y if f(x, y) is given by

f(x , y) = ln (x^2 + 2y)

Solution

Differentiate with respect to x to obtain

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[In(x^2 + 2y)] = \frac{2x}{x^2 + 2y}$$

Differentiate with respect to y

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[In(x^2 + 2y)] = \frac{2}{x^2 + 2y}$$

Example 5: Find $f_x(2, 3)$ and $f_y(2, 3)$ if f(x , y) is given by

$f(x, y) = y x^2 + 2y$

Solution:

You first find f_x and f_y

$f_x(x,y) = 2x y$

$f_y(x,y) = x^2 + 2$

We now calculate $f_x(2, 3)$ and $f_y(2, 3)$ by substituting x and y by their given values

$f_x(2,3) = 2 (2)(3) = 12$

$f_y(2,3) = 2^2 + 2 = 6$

Exercise: Find partial derivatives f_x and f_y of the following functions

1. $f(x, y) = x e^{x + y}$
2. $f(x, y) = \ln(2x + yx)$
3. $f(x, y) = x \sin(x - y)$

Answer to Above Exercise:

1. $f_x = (x + 1)e^{x+y}$, $f_y = x e^{x+y}$

2. $f_x = 1/x$, $f_y = 1/(y + 2)$

3. $f_x = x \cos(x - y) + \sin(x -)$, $f_y = -x \cos(x - y)$

More on partial derivatives and multivariable functions. Multivariable Functions

Higher Order Partial Derivatives

Just as you had higher order derivatives with functions of one variable you will also have higher order derivatives of functions of more than one variable. However, this time you will have more options since you do have more than one variable. Consider the case of a function of two variables, $f(x,y)$ since both of the first order partial derivatives are also functions of x and y you could in turn differentiate each with respect to x or y. This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations that you'll use to denote them.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The second and third second order partial derivatives are often called mixed partial derivatives since you are taking derivatives with respect to more than one variable. Note as well that the order that you take the derivatives in is given by the notation for each these. If you are using the subscripting notation, e.g. f_{xy}, then you will differentiate from left to right. In other words, in this case, you will differentiate first with respect to x and then with respect to y. With the fractional notation e.g. $\frac{\partial^2 f}{\partial y \partial x}$, it is the opposite. In these cases we differentiate moving along the denominator from right to left. So, again, in this case you first differentiate with respect to x and then with respect to y.

Let's take a quick look at an example.

Example 1 Find all the second order derivatives for

$f(x,y) = \cos(2x) - x^2 e^{5y} + 3y^2$

Solution

You'll first need the first order derivatives so here they are.

$$f_x(x,y) = -2\sin(2x) - 2xe^{5y}$$

$$f_y(x,y) = -5x^2 e^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xy} = -4\cos(2x) - 2\,e^{5y}$$
$$f_{xy} = -10xe^{5y}$$
$$f_{yx} = -10xe^{5y}$$
$$f_{yy} = -25x^2e^{5y} + 6$$

Notice that you dropped the *(x,y)* from the derivatives. This is fairly standard and you will be doing it most of the time from this point on. You will also be dropping it for the first order derivatives in most cases.

Now let's also notice that, in this case, $f_{xy} = f_{yx}$. This is not by coincidence If the function is "nice enough" this will always be the case. So, what's "nice enough"? The following theorem tells you.

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Now, do not get too excited about the disk business and the fact that you have the theorem is for a specific point. In pretty much every example in this class if the two mixed second order partial derivatives are continuous then they will be equal.

Example 2 Verify Clairaut's Theorem for $f(x,y) = xe^{-x^2y^2}$.

Solution

You'll first need the two first order derivatives.

$$f_x(x,y) = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2}$$
$$f_y(x,y) = -2yx^3e^{-x^2y^2}$$

Now, compute the two fixed second order partial derivatives.

$$f_{xy}(x,y) = -2yx^2e^{-x^2y^2} - 4x^4y^3e^{-x^2y^2} = -6x^2ye^{-x^2y^2} + 4x^4y^3e^{-x^2y^2}$$

$$f_{yx}(x,y) = -6yx^2e^{-x^2y^2} + 4x^3y^4e^{-x^2y^2}$$

Sure enough they are the same.

So far you have only looked at second order derivatives. There are, of course, higher order derivatives as well. Here are a couple of the third order partial derivatives of function of two variables.

$$f_{xyx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$
$$f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

Notice as well that for both f these we differentiate once with respect to y and twice with respect to x. There is also another third order partial derivative in which you can do this, f_{xxy}. There is an extension to Clairaut's Theorem that says if all three of these are continuous then they should all be equal,

$$f_{yxy} = f_{xyx} = f_{yxx}$$

To this point you've only looked at functions of two variables, but everything that you've done to this point will work regardless of the number of variables that you've got in the function and there are natural extensions to Clairaut's theorem to all of these cases as well. For instance, $f_{xz}(x, y, z) = f_{xx}(x, y, z)$ provided both of the derivatives are continuous.

In general, you can extend Clairaut's theorem to any function and mixed partial derivatives. The only requirement is that in each derivative you differentiate with respect to each variable the same number of times. In other words, provided you meet the continuity condition, the following will be equal

$$f_{ssrtsrr} = f_{trsrssr}$$

because in each case you differentiate with respect to t once, s three times and r three times.

Let's do a couple of exam les with higher (well higher order than two anyway) order derivatives and functions of more than two variables.

Example 3 Find the indicated derivative for each of the following functions.

(a) Find f_{xxyzz} for $f(x,y,z) = z^3y^2$ In (x)

(b) Find $\frac{\partial^3 f}{\partial y \partial x^2}$ for $f(x, y) = e^{xy}$

Solution

(a) Find f_{xxyzz} for $f(x, y, z) = z^3y^2\text{In}(x)$

In this case remember that you differentiate from left to right. Here are the derivatives for this part.

$$f_x = \frac{z^3y^2}{x}$$
$$f_{xx} = \frac{z^3y^2}{x^2}$$
$$f_{xxy} = \frac{2z^3y}{x^2}$$
$$f_{xxyx} = \frac{6z^3y}{x^2}$$
$$f_{xxyxx} = \frac{12z^3y}{x^2}$$

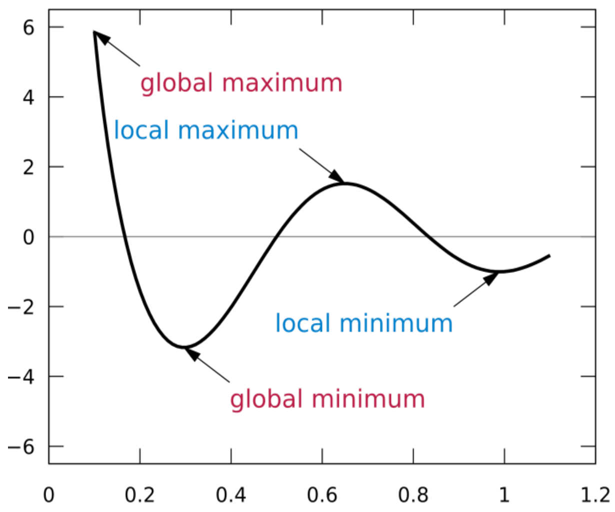
(b) Find for $\frac{\partial^3 f}{\partial y \partial x^2}$ for $f(x, y) = e^{xy}$

Here we differentiate from right to left. Here are the derivatives for this function.

$$\frac{\partial f}{\partial x} = ye^{xy}$$
$$\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}$$
$$\frac{\partial^3 f}{\partial y \partial x^2} = 2ye^{xy} + xy^2e^{xy}$$

Maxima and minima

For other uses, see Maxima (disambiguation) and Maximum (disambiguation). For use in statistics, see Maximum (statistics).



Local and global maxima and minima for cos $3\pi x/x, 0.1 \leq x \leq 1.1$

In mathematics, the **maximum** and **minimum** (plural: maxima and minima) of a function, known collectively as **extrema** (singular: extremum), are the largest and smallest value that the function takes at a point either within a given neighborhood (local or relative extremum) or on the function domain in its entirety (global or absolute extremum). More generally, the maximum and minimum of a set (as defined in set theory) are the greatest and least element in the set. Unbounded infinite sets such as the set of real numbers have no minimum and maximum.

To locate extreme values is the basic objective of optimization

real-valued function f define on a real line is said to have **a local (or relative) maximum point** at the point x^*, if there exists some $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ when $|x - x^*| < \varepsilon$. The value of the function at this point is called **maximum** of the function. Similarly, a function has a **local minimum point** at x*, if $f(x^*) \leq f(x)$ when $|x - x^*| < \varepsilon$. The value of the function at this point is called minimum of the function. A function has a **global (or absolute) maximum point** at x* if $f(x^*) \geq f(x)$ for all x. Similarly, a function has a **global (or absolute) minimum point** at x^* if $f(x^*) \leq f(x)$ for all x. The global maximum and **global minimum points** are also known as the arg max and arg min: the argument (input) at which the maximum (respectively, minimum) occurs.

Restricted domains: There may be maxima and minima for a function whose domain does not include all real numbers. A r al-valued function, whose domain is any set, can have a global maximum and minimum. There may also be local maxima and local mini ma points, but only at points of the domain set where the concept of neighborhood is define d. A neighborhood plays the role of the set of x such that $|x - x^*| < \varepsilon$.

A continuous (real-valued) function on a compact set always takes maxi um and minimum values on that set. An important example is a function whose domain is a closed (and bounded) interval of real numbers (see the graph above). The neighborhood requirement precludes a local maximum or minimum at an endpoint of an interval. However, an endpoint may still be a global maxim m or minimum. Thus it is *not always true*, for finite domains, that a global maximum (mini um) must also be a local maximum (minim m).

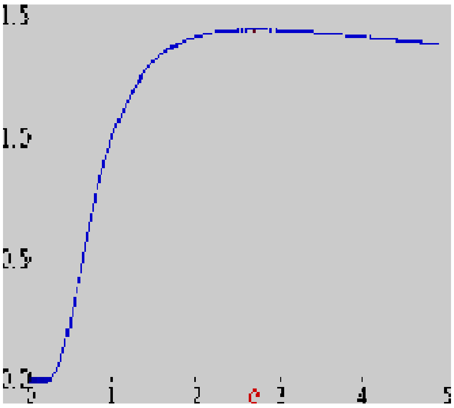
Finding functional maxima and minima

Finding global maxima and minima is the goal of mathematical optimization. If a function is continuous on a closed interval, then by the extreme value theorem global maxima and minima exist. Furthermore, a global maximum (or minimum) either must be a local maximum (or minimum) in the interior of the domain, or must lie on the boundary of the domain. So a method of finding a global maximum (or minimum) is to look at all the local maxima (or minima) in the interior, and also look at the maxima (or mini a) of the points on the boundary; and take the biggest (or smallest) one.

Local extrema can be found by Fermat's theorem, which states that they must occur at critical points. One can distinguish whether a critical point is a local maximum or local minimum by using the first derivative test and second derivative test.

For any function that is defined piecewise, you finds a maxima (or minima) by finding the maximum (or minimum) of each piece separately; and then seeing which one is biggest (or smallest).

Examples



The global maximum of $\sqrt[x]{x}$ occurs at $x = e$.

- The function x^2 has a unique global minimum at x = 0.
- The function x^3 has no global minima or maxima. Although the first derivative ($3x^2$) is 0 at x = 0, this is an inflexion point.
- The function $\sqrt[x]{x}$ has a unique global maximum at x = e. (See figure a right)
- The function x^{-x} has a unique global maximum over the positive real numbers at x = 1/e.
- The function $x^3/3 - x$ has first derivative $x^2 - 1$ and second derivative 2x. Setting the first derivative to 0 and solving for x gives stationary points at −1 and +1. From the sign of the second derivative we can see that −1 is a local maximum and +1 is a local minimum. Note that this function has no global maximum or minimum.
- The function $|x|$ has a global minimum at $x = 0$ that cannot be found by taking derivatives, because the derivative do s not exist at $x = 0$.
- The function $\cos(x)$ has infinitely many global maxima at 0, ±2π, ±4, ..., and infinitely many global minima at ± , ±3π, ...
- The function 2 $\cos(x)$ has infinitely many local maxima and minima, but no global maximum or minimum.
- The function $\cos(3\pi x)/x$ with $0.1 \le x \le 1.1$ has a global maximum at $x = 0.1$ (a boundary) , a global minimum near $x = 0.3$, a local maximum near $x = 0.6$, and a local minimum near $x = 1.0$. (See figure above.)
- The function $x^3 + 3x^2 - 2x + 1$ defined over the closed interval (segment) [−4,2] has two extrema: one local maximum at $x = -1 - \frac{\sqrt{15}}{3}$, one local minimum at $x = -1 + \frac{\sqrt{15}}{3}$, a global maximum at x = 2 and a global minimum at x = −4.

Functions of more than one variable

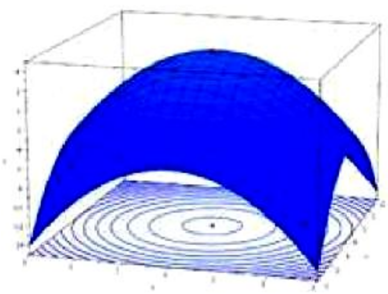
Second partial derivative test

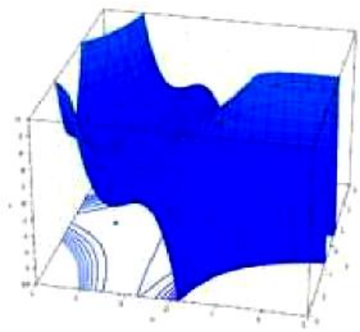
For functions of more than one variable, similar conditions apply. For example, in the (enlargeable) figure at the right, the necessary conditions for a local maximum are similar to those of a function with only one variable. The first partial derivatives as to z (the variable to be maximized) are zero at the maximum (the glowing dot on top in the figure). The second partial derivatives are negative. These are only necessary, not sufficient, conditions for a local maximum because of the possibility of a saddle point. For use of these conditions to solve for a maximum, the function z must also be differentiable throughout. The second partial derivative test can help classify the point as a relative maximum or relative minimum.

In contrast, there are substantial differences between functions of one variable and functions of more than one variable in the identification of global extrema. For example, if a bounded differentiable function *f* defined on a closed interval in the real line has a s ingle critical point, which is a local minimum, then it is also a global minimum (use the intermediate value theorem and Rolle's theorem to prove this by reduction and absurdum). In two and more dimensions, this argument fails, as the function shows:

$$f(x, y) = x^2 + y^2(1 - x)^3, \qquad x, y \in \mathbb{R}$$

Its only critical point s at (0,0), which is a local minimum with *f* 0,0) = 0. However, it cannot be a global one, because $f(4,1) = -11$.





The global maximum is the point at the top Counterexample

In relation to sets

Maxima and minima are more generally defined for sets. In general, if an ordered set *S* has a greatest element *m*, *m* is a maximal element. Furthermore, if *S* is a subset of an ordered set *T* and *m* is the greatest element of *S* with respect to order induced by *T*, *m* is a least upper bound of *S* in *T*. The similar result holds for least element, minimal element and greatest lower bound.

In the case of a general partial order, the **least element** (smaller than all other) should not be confused with a **minimal element** (nothing is smaller). Likewise, a greatest element of a partially ordered set (poset) is an upper bound of the set which is contained within the set, whereas a **maximal element** *m* of a poset *A* is an element of *A* such that if $m \leq b$ (for any *b* in *A*) then $m = b$. Any least element or greatest element of a poset is unique, but a poset can have several minimal or maximal elements. If a poset has more than one maximal element, then these elements will not be mutually comparable.

In a totally ordered set, or *chain*, all elements are mutually comparable, so such a set can have at most one minimal element and at most one maximal element. Then, due to mutual comparability, the minimal element will also be the least element and the maximal element will also be the greatest element. Thus in a totally ordered set we can simply use the terms ***minimum*** and ***maximum***. If a chain is finite then it will always have a maximum and a minimum. If a chain is infinite then it need not ha e a maximum or a minimum. For example, the set of natural numbers has no maximum, though it has a minimum. If an infinite chain S is bounded, then the closure Cl(S) of he set occasionally has a minimum and a maximum, in such case they are called the **greatest lower bound** and the **least upper bound** of the set *S*, respectively.

TAYLOR SERIES

The Maclaurin series for any polynomial is the polynomial itself.

The Maclaurin series for $(1 - x)^{-1}$ for |x| < 1 is the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

so the Taylor series for x− 1 at a = 1 is

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots.$$

By integrating the above Maclaurin series you find the Maclaurin series for log(1 − x), where log denotes the natural logarithm:

$$-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

and the corresponding Taylor series for log(x) at a = 1 is

$$(x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \cdots.$$

The Taylor series for the exponential function ex at a = 0 is

$$1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The above expansion holds because the derivative of e^x with respect to x is also e^x and e^o equals 1. This leaves the terms $(x-0)^n$ in the numerator and n! in the denominator for each term in the infinite sum.

History

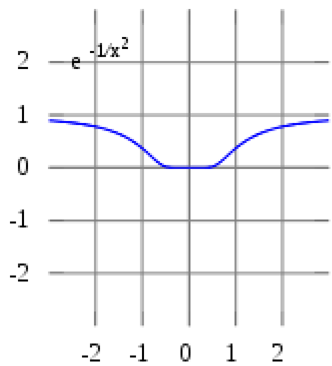
The Greek philosopher Zeno considered the problem of summing an infinite series to achieve a finite result, but rejected it as an impossibility: the result was Zeno's paradox. Later, Aristotle proposed a philosophical resolution of the paradox, but the mathematical content was apparently unresolved until taken up by Democritus and then Archimedes. It was through Archimedes's method of exhaustion that an infinite number of progressive subdivisions could be performed to achieve a finite result. Liu Hui independently employed a similar method a few centuries later

In the 14th century, the earliest examples of the use of Taylor series and closely related methods were given by Madhava of Sangamagrama though no record of his work survives; writings of later Indian mathematicians suggest that he found a number of special cases of the Taylor series, including those for the trigonometric functions of sine, cosine, tangent, and arctangent. The Kerala School of astronomy and mathematics further expanded his works with various series expansions and rational approximations until the 16th century.

In the 17th century, James Gregory also worked in this area and published several Maclaurin series. It was not until 1715 however that a general method for constructing these series for all functions for which they exist was finally provided by Brook Taylor, after whom the series are now named.

The Maclaurin series was named after Colin Maclaurin, a professor in Edinburgh, who published the special case of the Taylor result in the 18th century.

Analytic functions



The function $e^{\frac{-1}{x^2}}$ is not analytic at $x = 0$: the Taylor's series is identically 0, although the function is not.

If f(x) is given by a convergent power series in an open disc (or interval in the real line) centered at be, it is said to b analytic in this disc. Thus for x in this disc, f is given by a convergent power series

$$f(x)\sum_{n=0}^{\infty} a_n(x-b)^n$$

Differentiating by *x* the above formula *n* times, then setting *x*=*b* gives:

$$\frac{f^{(n)}(b)}{n!} = a_n$$

and so the power series expansion agrees with the Taylor's series. Thus a function is analytic in an open disc centered at b if and only if its Taylor's series converges to the value of the function at each point of the disc.

If *f*(*x*) is equal to its Taylor's series everywhere it is called entire. The polynomials and the exponential function e^x and the trigonometric functions sine and cosine are examples of entire functions. Examples of functions that are not entire include the logarithm, the trigonometric function tangent, and its inverse arctan. For these functions the Taylor's series do not converge if x is far from a. Taylor's series can be used to calculate the value of a entire function in every point, if the value of the function, and of all of its derivatives, are known at a single point.

4.0 CONCLUSION

In this unit, you have been introduced to partial derivative in calculus and some higher order partial derivative. Clairauts theorem was stated and applied. You have been introduced to Maxima and minima, functions of more than one variable and the relation of maxima and minima to set.

5.0 SUMMARY

In this unit you have studied:

- Partial derivatives in calculus
- Higher order partial derivative
- Clairauts theorem
- Maxima and manima
- Taylor series of function of two variable
- Analytical function

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READING

Stewart, James (2008). *Calculus*: *Early Transcendentals* (6th ed.). Brooks/Cole. ISBN 0-495-01166-5.

Larson, Ron; Edwards, Bruce H. (2009). *Calculus* (9th ed.). Brooks/Cole. ISBN 0547-16702-4.

Thomas, George B.; Weir, Maurice D.; Hass, Joel (2010). Thomas' Calculus: Early Transcendentals (12th ed.). Addison-Wesley. ISBN 0-321-58876-2.
Maxima and Minima From MathWorld--A Wolfram Web Resource.

Thomas Simpson's work on Maxima and Minima at Convergence

Apostol, Tom (1967), *Calculus*, Jon Wiley & Sons, Inc., ISBN 0-471-00005-1.

Bartle; Sherbert (2000), *Introduction to Real Analysis* (3rd ed.), John Wiley & Sons, Inc., ISBN 0-471-32148-6.

Hörmander, L. (1976), *Linear Partial Differential Operators, Volume 1,* Springer-Verlag, ISBN 978-3540006626.

Klein, Morris (1998), *Calculus: An Intuitive and Physical Approach*, Dover, ISBN 0-486-40453-6.

Pedrick, George (1994), *A First Course in Analysis,* Springer-Verlag, ISBN 0-387-94108-8.

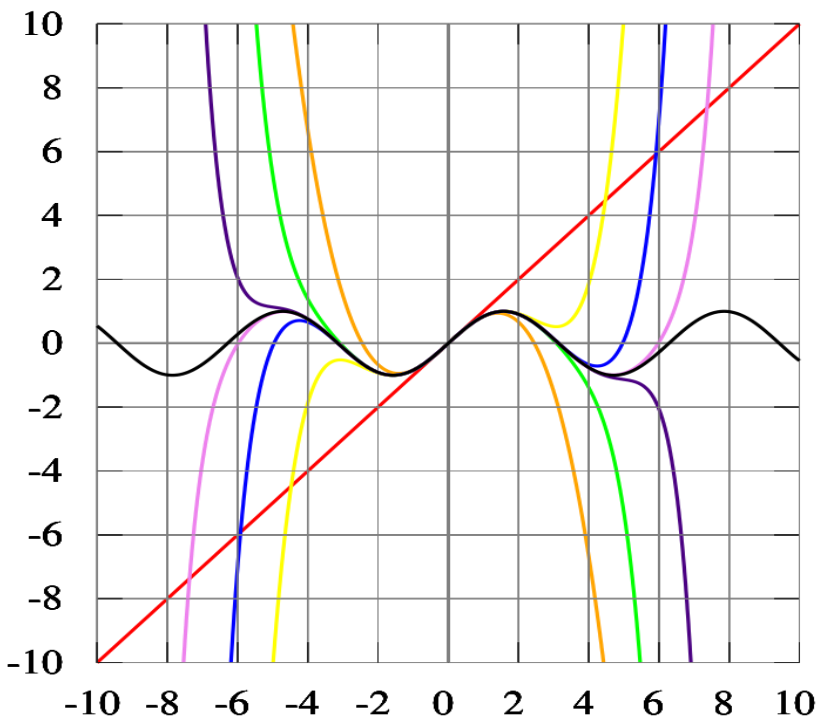
Stromberg, Karl (1981), Introduction to classical real analysis, Wadsworth, Inc., ISBN 978-0534980122.

Rudin, Walter (1987), *Real and complex analysis,* 3rd ed., McGraw-Hill Book Company, ISBN 0-07-054234

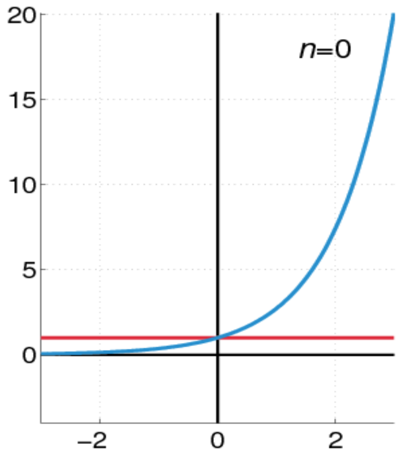
UNIT 2 TAYLOR'S SERIES OF EXPANSION FOR FUNCTIONS OF TWO VARIABLES

CONTENTS

1.0 INTRODUCTION



As the degree of the Taylor's polynomial rises, it approaches the correct function. This image shows sin(x) and its Taylor's approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13.



The exponential function e^x (in blue), and the sum of the first n+1 terms of its Taylor's series at 0 (in red).

In mathematics, a **Taylor's series** is a representation of a function as an in finite sum of terms that are calculated from the values of the function's derivatives at a single point.

The concept of a Taylor's series was formally introduced by the English mathematician Brook Taylor's in 1715. If the Taylor's series is centered at zero, then that series is also called a Maclaurin's series, named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor's series in the 18th century.

It is common practice to approximate a function by using a finite number of terms of its Taylor's series. Taylor's theorem gives quantitative estimates on the error in this approximation. Any finite number of initial terms of the Taylor's series of a function is called a Taylor's polynomial. The Taylor's series of a function is the limit of that function's Taylor's polynomials, provided that the limit exists. A function may not be equal of its Taylor's series, even if its Taylor's series converges at every point. A function that is equal to its Taylor's series in an open interval (or a disc in the complex plane) is known as an analytic function.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- definition Taylor's series of functions of two variables;
- solve problems on analytical problem;
- use the Taylor's series to solve analytic function;
- solve problems that involve approximation and convergence;
- state Maclaurine's series of some common functions;
- calculation of Taylor's series; use Taylor's series in calculations;
- explain Taylors's series in several variables; and
- explain Fractional Taylor's series.

3.0 MAIN CONTENT

Definition

The Taylor series of a real or complex function f(x) that is infinitely differentiable in a neighborhood of a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

which can be written in the more compact sigma notation as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-b)^n$$

where n! denotes the factorial of n and $f^{(n)}$ (a) denotes the nth derivative o f f evaluated at the point a. The zeroth derivative of f is defined to be f itself and $(x - a)^0$ and 0! are both defined to be 1. In the case that a = 0, the series is also called a Maclaurin's series.

Examples

The Maclaurin's series for any polynomial is the polynomial itself.

The Maclaurin's series for $(1 - x)^{-1}$ for |x| < 1 is the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

so the Taylor series for x−1 at a = 1 is

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

By integrating the above Maclaurin's series we find the Maclaurin's series for log(1 − x), where log denotes the natural logarithm:

$$-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

and the corresponding Taylor's series for log(x) at a = 1 is

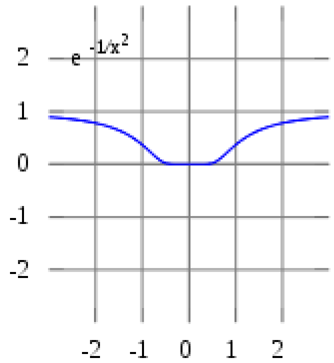
$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots.$$

The Taylor's series for the exponential function e^x at a = 0 is

$$1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The above expansion holds because the derivative of e^x with respect to x is also e^x and e^0 equals 1. This leaves the terms $(x-0)^n$ in the numerator and $n!$ in the denominator for each term in the infinite sum.

Analytic functions



The function e^{-1/x^2} is not analytic at $x = 0$: the Taylor's series is identically 0, although the function is not.

If f(x) is given by a convergent power series in an open disc (or interval in the real line) centered at b, it is said to b analytic in this disc. Thus for x in this disc, f is given by a convergent power series

$$f(x)\sum_{n=0}^{\infty} a_n(x-b)^n.$$

Differentiating by *x* the above formula *n* times, then setting *x*=*b* gives:

$$\frac{f^{(n)}(b)}{n!} = a_n$$

and so the power series expansion agrees with the Taylor's series. Thus a function is analytic in an open disc centered at b if and only if its Taylor's series converges to the value of the function at each point of the disc.

If *f*(*x*) is equal to its Taylor's series everywhere it is called entire. The polynomials and the exponential function expand the trigonometric functions sine and cosine are examples of entire functions. Examples of functions that are not entire include the logarithm, the trigonometric function tangent, and its inverse arctan. For these functions the Taylor's series do not converge if x is far from a. Taylor's series can be used to calculate the value of a entire function in every point, if the value of the function, and of all of its derivatives, are known at a single point.

Uses of the Taylor's series for analytic functions include:

The partial sums (the Taylor's polynomials) of the series can be used as approximations of the entire function. These approximations are good if sufficiently many terms are included.

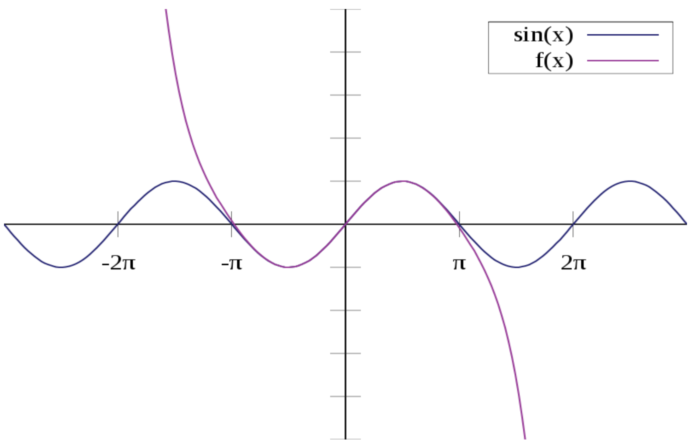
Differentiation and integration of power series can be performed term by term and is hence particularly easy.

An analytic function is uniquely extended to a holomorphic function on an open disk in the complex plane. This makes the machinery of complex analysis available.

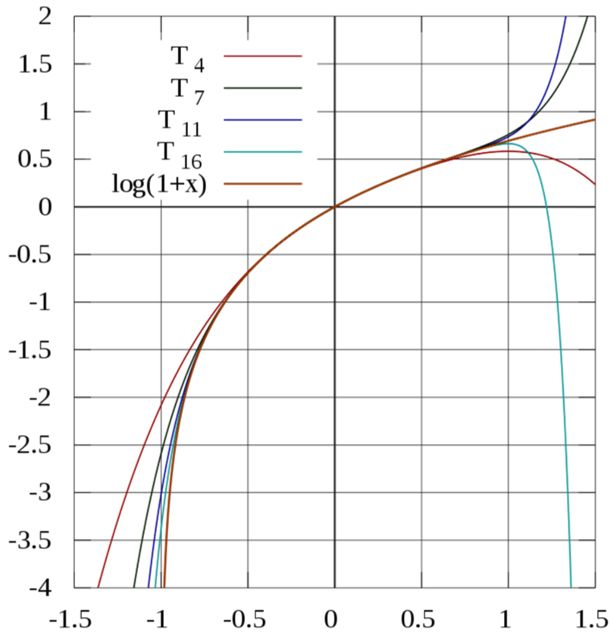
The (truncated) series can be used to compute function values numerically, (often by recasting the polynomial into the Chebyshev form and evaluating it with the Clenshaw algorithm).

Algebraic operations can be one readily on the power series representation; for instance the Euler's formula follows from Taylor's series expansions for trigonometric and exponential functions. This result is of fundamental importance in such fields as harmonic analysis.

Approximation and convergence



The sine function (blue) is closely approximated by its Taylor's polynomial of degree 7 (pink) for a full period centered at the origin.



The Taylor's polynomials for log(1 + x) only provide accurate approximations in the range $-1 < x \leq 1$. Note that, for $x > 1$, the Taylor's polynomials of higher degree are ***worse*** approximations.

Pictured on the right is an accurate approximation of sin(x) around the point x = 0. The pink curve is a polynomial of degree seven:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

The error in this approximation is no more than $|x|^9/9!$. In particular, for $-1 < x < 1$, the error is less than 0.000003.

In contrast, also shown is a picture of the natural logarithm function log(1+ x) and some of its Taylor's polynomials around $a = 0$. These approximations converge to the function only in the region $-1 < x \leq 1$; outside of this region the higher-degree Taylor's polynomials are worse approximations for the function. This is similar to Runge's phenomenon.

The **error** incurred in approximating a function by its *n*th-degree Taylor's polynomial is called the **remainder** or *residual* and is denoted by the function $R_n(x)$. Taylor's theorem can be used to obtain a bound on the size f the remainder.

In general, Taylor's series need not be convergent at all. And in fact the set of functions with a convergent Taylor's series is a meager set in the Fréchet space of smooth functions. Even if the Taylor's series of a function *f* does converge, its limit need not in general be equal to the value of the function *f*(*x*). For example, the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at x = 0, and has all derivatives zero there. Consequently, the Taylor's series of f(x) about x = 0 is identically zero. However, f(x) is not equal to the zero function, and so it is not equal to its Taylor's series around the origin.

In real analysis, this example shows that there are infinitely differentiable functions *f*(*x*) whose Taylor's series are not equal to *f(x)* even if they converge. By contrast in complex analysis there are no holomorphic functions *f(z)* whose Taylor's series converges to a value different from f(z). The complex function e^{-z-2} does not approach 0 as z approaches 0 along the imaginary axis and its Taylor's series is thus not defined there.

More generally, every sequence of real or complex numbers can appear a coefficients in the Taylor's series of an infinitely differentiable function defined on the real line, a consequence of Borel's lemma (see also Non -analytic smooth function and application to Taylor's series). As a result, the radius of convergence of a Taylor's series can be zero. There are even infinitely differentiable functions defined on the real line whose Taylor's series have a radius of convergence 0 everywhere.

Some functions cannot be written as Taylor's series because they have a singularity; in these cases, one can often still achieve a series expansion if one allows also negative powers of the variable *x*; see Laurent's series. For example, $f(x) = e^{-x-2}$ can be written as a Laurent's series.

There is, however, a generalization of the Taylor's series that does converge to the value of the function itself for any bonded continuous function on (0,∞), using the calculus of finite differences. Specifically, one has the following theorem, due to Einar Hille, that for any t > 0,

$$\lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n f(a)}{h^n} = f(a+t).$$

Here Δ_h^n is the *n*-th finite difference operator with step size h. The series is precisely the Taylor's series, except that divided differences appear in place of differentiation: the series is formally similar to the Newton series. When the function f is analytic at a, the terms in the series converge to the terms of the Taylor's series, and in this sense generalizes the usual Taylor's series.

In general, for any infinite sequence *ai*, the following power series identity holds:

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \Delta^n a_i = e^{-u} \sum_{j=0}^{\infty} \frac{u^j}{j!} a_{i+j}.$$

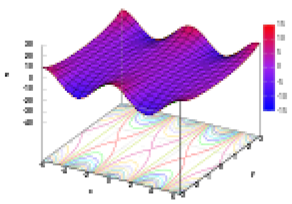
So in particular,

$$f(a+t) = \lim_{h\to 0+} e^{-t/h} \sum_{j=0}^{\infty} f(a+jh)\frac{(t/h)^j}{j!}.$$

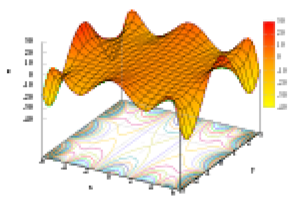
The series on the right is the expectation value of $f(a + X)$, where X is a Poisson distributed random variable that takes the value jh with probability $e^{-t/h}(t/h)^j/j!$. Hence

$$f(a+t) = \lim_{h\to 0+} \int_{-\infty}^{\infty} f(a+x)dP_{t/h,h}(x).$$

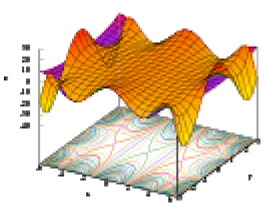
The law of large numbers implies that the identity holds.
List of Maclaurin's series of some common functions



The real part of the co sine function in the complex plane.



An 8th degree approximation of the cosine function in the complex plane.



The two above curves put together.

Several important Maclaurin's series expansions follow. All these expansions are valid for complex arguments x.

Exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

Natural logarithm:

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } |x| < 1$$

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } |x| < 1$$

Finite geometric series:

$$\frac{1 - x^{m+1}}{1 - x} = \sum_{n=0}^{m} x^n \quad \text{for } x \neq 1 \text{ and } m \in \mathbb{N}_0$$

Infinite geometric series:

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Variants of the infinite geometric series:

$$\frac{x^m}{1 - x} = \sum_{n=m}^{\infty} x^n \quad \text{for } |x| < 1 \text{ and } m \in \mathbb{N}_0$$

$$\frac{1}{(1 - x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for } |x| < 1$$

$$\frac{x}{(1 - x)^2} = \sum_{n=1}^{\infty} n x^n \quad \text{for } |x| < 1$$

Square root:

$$\sqrt{1 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1 - 2n)(n!)^2 (4^n)} x^n = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \ldots \quad \text{for } |x| \leq$$

Binomial series (includes the square root for α = 1/2 and the infinite geometric series for α = −1):

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{for all } |x| < 1 \text{ and all complex } \alpha$$

with generalized binomial coefficients

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

Where the B_s are Bernoulli's numbers.

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| \le 1$$

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } |x| \le 1$$

Hyperbolic functions:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{for all } x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x$$

$$\tanh x = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\operatorname{arsinh}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| \le 1$$

$$\operatorname{artanh}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1$$

Lambert's W function:

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \text{ for } |x| < \frac{1}{e}$$

The numbers B_k appearing in the *summation* expansions of tan(x) and tanh(x) are the Bernoulli's numbers. The E_k in the expansion of sec(x) are Euler numbers.

Calculation of Taylor's series

Several methods exist for the calculation of Taylor's series of a large number of functions. One can attempt to use the Taylor's series as-is and generalize the form of the coefficients, or one can use manipulations such as substitution, multiplication or division, addition or subtraction of standard Taylor's series to construct the Taylor's series of a function, by virtue of Taylor's series being power series. In some cases, one can also derive the Taylor's series by repeatedly applying integration by parts. Particularly convenient is the use of computer algebra systems to calculate Taylor's series.

First example

Compute the 7th degree Maclaurin polynomial for the function

$$f(x) = \log \cos x, \quad x \in (-\pi/2, \pi/2)$$

First, rewrite the function as .

$$f(x) = \log(1 + (\cos x - 1))$$

You have for the natural logarithm (by using the big O notation)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

and for the cosine function

$$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)$$

The latter series expansion has a zero constant term, which enables us to substitute the second series into the first one and to easily omit terms of higher order than the 7th degree by using the big O notation

$$\begin{aligned} f(x) &= \log(1 + (\cos x - 1)) \\ &= (\cos x - 1) - \frac{1}{2}(\cos x - 1)^2 + \frac{1}{3}(\cos x - 1)^3 + O\left((\cos x - 1)^4\right) \\ &= \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2} + O(x^4)\right)^3 + O(x^8) \\ &= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^6}{24} + O(x^8) \\ &= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + O(x^8). \end{aligned}$$

Since the cosine is an even function, the coefficients for all the odd powers x, x^3, x^5, x^7, ... have to be zero.

Second example

Suppose you want the Taylor series at 0 of the function

$$g(x) = \frac{e^x}{\cos x}$$

You have for the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and, as in the first example,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Assume the power series is

$$\frac{e^x}{\cos x} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

Then multiplication with the denominator and substitution of the series of the cosine yields

$$\begin{aligned} e^x &= \left(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots\right)\cos x \\ &= \left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\ &= c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{4!}x^4 + c_1x - \frac{c_1}{2}x^3 + \frac{c_1}{4!}x^5 + c_2x^2 - \frac{c_2}{2}x^4 + \frac{c_2}{4!}x^6 + c_3x^3 - \frac{c_3}{2}x^5 + \frac{c_3}{4!}x^7 + \cdots \end{aligned}$$

Collecting the terms up to fourth order yields

$$= c_0 + c_1x + \left(c_2 - \frac{c_0}{2}\right)x^2 + \left(c_3 - \frac{c_1}{2}\right)x^3 + \left(c_4 + \frac{c_0}{4!} - \frac{c_2}{2}\right)x^4 + \cdots$$

Comparing coefficients with the above series of the exponential function yield the desired Taylor series

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots$$

Comparing coefficients with the above series of the exponential function yields the desired Taylor series

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots$$

Third example

Here we use a method called "Indirect Expansion" to expand the given function. This method uses the known function of Taylor’s series for expansion.

Q: Expand the following function as a power series of x

$(1 + x)e^x$.

You know the Taylor's series of function e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, -\infty < x < +\infty$$

Thus,

$$\begin{aligned}(1+x)e^x = e^x + xe^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{(n-1)!}\right) x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{n+1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n.\end{aligned}$$

Taylor's series in several variables

The Taylor's series may also be generalized to functions of more than one variable with

$$T(x_1, \ldots, x_d) =$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1+\cdots+n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}}\right)(a_1, \ldots, a_d).$$

For example, for a function that depends on two variables, x and y, t e Taylor's series to second order about the point (a, b) is:

$$\begin{aligned}f(x,y) &\approx f(a,b) + (x-a)\,f_x(a,b) + (y-b)\,f_y(a,b) \\ &\quad + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right],\end{aligned}$$

where the subscripts denote t e respective partial derivatives.

A second-order Taylor's series expansion of a scalar-valued function of more than one variable can be written compactly as

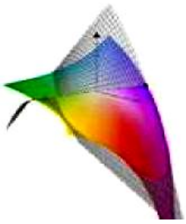
$$T(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} - \mathbf{a})^T \mathrm{D} f(\mathbf{a}) + \frac{1}{2!} (\mathbf{x} - \mathbf{a})^T \left\{\mathrm{D}^2 f(\mathbf{a})\right\} (\mathbf{x} - \mathbf{a}) + \cdots,$$

Ere Df (a) is the gradient of f evaluated at $x = a$ and D^2f(a) is the Hessian matrix. Applying the multi-index notation the Taylor's series for several variables becomes

$$T(\mathbf{x}) = \sum_{|\alpha|\geq 0} \frac{(\mathbf{x}-\mathbf{a})^{\alpha}}{\alpha!} (\partial^{\alpha} f)(\mathbf{a}),$$

which is to be understood as still more abbreviated multi-index version f the first equation of this paragraph, again in full analogy to the single variable case.

Example



Second-order Taylor's series approximation (in gray) of a function $f(x,y) = e^x\log(1 + y)$ around origin.

Compute a second-order Taylor's series expansion around point $(a,b) = (0,0)$ of a function

$$f(x,y) = e^x \log(1+y).$$

Firstly, we compute all partial derivatives we need

$$f_x(a,b) = e^x \log(1+y)\Big|_{(x,y)=(0,0)} = 0\,,$$

$$f_y(a,b) = \frac{e^x}{1+y}\Big|_{(x,y)=(0,0)} = 1\,,$$

$$f_{xx}(a,b) = e^x \log(1+y)\Big|_{(x,y)=(0,0)} = 0\,,$$

$$f_{xy}(a,b) = f_{yx}(a,b) = \frac{e^x}{1+y}\Big|_{(x,y)=(0,0)} = 1.$$

$$f_{yy}(a,b) = -\frac{e^x}{(1+y)^2}\Big|_{(x,y)=(0,0)} = -1\,,$$

The Taylor's series is

$$T(x,y) = f(a,b) + (x-a)\, f_x(a,b) + (y-b)\, f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots,$$

which in this case becomes

$$T(x,y) = 0 + 0(x-0) + 1(y-0) + \frac{1}{2}\left[0(x-0)^2 + 2(x-0)(y-0) + (-1)(y-0)^2\right] + \cdots$$
$$= y + xy - \frac{y^2}{2} + \cdots .$$

Since log(1 + *y*) is analytic in *y*| < 1, we have

$$e^x \log(1+y) = y + xy - \frac{y^2}{2} + \cdots$$

for |*y*| < 1.

Fractional Taylor series

With the emergence of fractional calculus, a natural question arises about what the Taylor's Series expansion would be. Odibat and Shawagfeh answered this in 2007. By using the Caputo fractional derivative, $0 < \alpha < 1$, and *x*| indicating the limit as we approach *x* from the right, the fractional Taylor's series can be written as

$$f(x+\Delta x) = f(x) + D_x^\alpha f(x+)\frac{(\Delta x)^\alpha}{\Gamma(\alpha+1)} + D_x^\alpha D_x^\alpha f(x+)\frac{(\Delta x)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.$$

4.0 CONCLUSION

In this unit, you have defined tailors series of function of two variables. You have studied analytical function and have used Taylors's series to solve problem s that involve analytical functions. You have studied approximation and convergence. You have also studied the list of Maclaurine's series of some common functions and have done some calculation of Taylor's series. You have also studied Taylors in several variables and the fractional Taylor's series.

5.0 SUMMARY

In this unit, you have studied the following:

- Definition Taylor's series of functions of two variables
- Solve problems on analytical problem
- Use the Taylor's series to solve analytic function
- Solve problems that involve approximation and convergence
- The list of Maclaurine's series of some common functions

- Calculation of Taylor's series
- Taylor's series in several variables
- Fractional Taylor's series

6.0 TUTOR – MARKED ASSIGNMENT

1. Use the Taylor's series to expand F(z) = $\frac{1}{z+1}$ about the point z = 1 ,and find the values of z for which the expansion is valid.
2. Use the Taylor's series to expand F(x) = $\frac{1}{x+2}$ about the point x = 1, and find the values of z for which the expansion is valid.
3. Use the Taylor's series to expand F(x) = $\frac{1}{x+2^2}$ about the point x = 2, and find the values of z for which the expansion is valid.
4. Use the Taylor's series to expand F(x) = $\frac{1}{x+4^2}$ about the point x = 2, and find the values of z for which the expansion is valid.

2. Use the Taylor's series to expand F(b) = $\frac{2}{b+2^3}$ about the point b = 1 ,and find the values of z for which the expansion is valid.

7.0 REFERENCES/FURTHER READING

Abramowitz, Milton; Stegun, Irene A. (1970), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications, Ninth printing

Thomas, George B. Jr.; Finney, Ross L. (1996), *Calculus and Analytic Geometry* (9th ed.), Addison Wesley, ISBN 0-201-53174-7

Greenberg, Michael (1998), *Advanced Engineering Mathematics* (2nd ed.), Prentice Hall, ISBN 0-13-321431

Abramowitz, M. and Stegun, I. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* 9th printing. New York: Dover, p. 880, 1972.

Arfken, G. "Taylor's Expansion." §5.6 in *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 303-313, 1985.

Askey, R. and Haimo, D. T. *"Similarities between Fourier and Power Series.*" Amer. Math. Monthly 103, 297-304, 1996.

Comtet, L. "Calcul pratique des coefficients de Taylor d'une fonction algébrique." Enseign. Math. 10, 267-270, 1964.

Morse, P. M. and Feshbach, H. *"Derivatives of Analytic Functions, Taylor and Laurent Series." §4.3 in Methods of Theoretical Physics,* Part I. New York: McGraw-Hill, pp. 374398, 1953.

Whittaker, E. T. and Watson, G. N. "Forms of the Remainder in Taylor's Series." §5.41 in *A Course in Modern Analysis*, 4th ed. Cambridge, England: Cambridge University Press, pp. 95-96, 1990.

UNIT 3 APPLICATIONS OF TAYLOR'S SERIES

CONTENT

1.0 INTRODUCTION

You started studying Taylor's Series because you said that polynomial functions are easy and that if you could find a way of representing complicated functions as series ("infinite polynomials") then maybe some properties of functions would be easy to study too. In this section, you'll show you a few ways in Taylor's series can make life easy.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- evaluate definite integrals with Taylor's series;
- understand the asymptotic behaviour with Taylor's series;
- understand the growth of functions with Taylor's series; and
- solve differential equations with Taylor's series.

3.0 MAIN CONTENT

Evaluating definite integrals

Remember that you've said that some functions have no anti derivative which can be expressed in terms of familiar functions. This makes evaluating definite integrals of these functions difficult because the Fundamental Theorem of Calculus cannot be used. However, if you have a series representation of a function, you can often times use that to evaluate a definite integral.

Here is an example. Suppose you want to evaluate the definite integral
$\int_0^1 \sin(x^2)\, dx$

The integrand has no anti derivative expressible in terms of familiar functions. However, you know how to find its Taylor's series: you know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

Now if you substitute $t = x^2$, you have

$$\mathrm{Sin}(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

In spite of the fact that you cannot anti differentiate the function; you can anti differentiate the Taylor's series:

$$\begin{aligned}\int_0^1 \sin(x^2)\, dx &= \int_0^1 (x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots)\, dx \\ &= (\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \ldots)\Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \ldots\end{aligned}$$

Notice that this is an alternating series so you know that it converges. If you add up the first four terms, the pattern becomes clear: the series converges to **0.31026.**

Understanding asymptotic behaviour

Sometimes, a Taylor's series can tell you useful information about how a function behaves in an important part of its domain. Here is an example which will demonstrate.

A famous fact from electricity and magnetism says that a charge **q** generates an electric field whose strength is inversely proportional to the square of the distance from the charge. That is, at a distance **r** away from the charge, the electric field is

$$E = \frac{kq}{r^2}$$

where k is some constant of proportionality.

Often times an electric charge is accompanied by an equal and opposite charge nearby. Such an object is called an electric dipole. To describe this, you will put a charge **q** at the point $x = d$ and a charge **-q** at $x = -d$.

Along the x axis, the strength of the electric fields is the sum of the electric fields from each of the two charges. In particular,

$$E = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2}$$

If you are interested in the electric field far away from the dipole, you can consider what happens for values of x much larger than d. You will use a Taylor's series to study the behaviour in this region.

$$E = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2} = \frac{kq}{x^2(1-\frac{d}{x})^2} - \frac{kq}{x^2(1+\frac{d}{x})^2}$$

Remember that the geometric series has the form

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \dots$$

If we differentiate this series, you obtain

$$\frac{1}{(1-u)^2} = 1 + 2u + 3u^2 + \ 4u^2 + \ \dots$$

Into this expression, you can substitute $u = -\frac{d}{x}$ to obtain

In the same way, if you substitute $u = -\frac{d}{x}$, we have

$$\frac{1}{(1+\frac{d}{x})^2} = 1 - \frac{2d}{x} + \frac{3d^2}{x^2} - \frac{4d^3}{x^3} + \dots$$

Now putting this together gives

$$\begin{aligned}
E &= \frac{kq}{x^2(1-\frac{d}{x})^2} - \frac{kq}{x^2(1+\frac{d}{x})^2} \\
&= \frac{kq}{x^2}[(1 + \frac{2d}{x} + \frac{3d^2}{x^2} + \frac{4d^3}{x^3} + \dots) - (1 - \frac{2d}{x} + \frac{3d^2}{x^2} - \frac{4d^3}{x^3} + \dots)] \\
&= \frac{kq}{x^2}[\frac{4d}{x} + \frac{8d^3}{x^3} + \dots] \\
&\approx \frac{4dq}{x^3}
\end{aligned}$$

In other words, far away from the dipole where *x* is very large, you see that the electric field strength is proportional to the inverse *cube* of the distance. The two charges partially cancel one another out to produce a weaker electric field at a distance.

Understanding the growth of functions

This example is similar is spirit to the previous one. Several times in this course, you have used the fact that exponentials grow much more rapidly than polynomial. You recorded this by saying that

$$\lim_{n\to\infty} \frac{e^x}{x^n} = \infty$$

for any exponent n. Let's think about this for a minute because it is an important property of exponentials. The ratio $\frac{e^x}{x^n}$is measuring how large the exponential is compared to the polynomial. If this ratio was very small, you would conclude that the polynomial is larger than the exponential. But if the ratio is large, you would conclude that the exponential is much larger than the polynomial. The fact that this ratio becomes arbitrarily large means that the exponential becomes larger than the polynomial by a factor which is as large as you would like. This is what you mean when you say "an exponential grows faster than a polynomial."

To see why this relationship holds, you can write down the Taylor's series for e^x.

$$\frac{e^x}{x^n} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \frac{x^n + 1}{(n+1)!} + \cdots}{x^n}$$

$$= \frac{1}{x^n} + \frac{1}{x^{n-1}} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots$$

$$> \frac{x}{(n+1)!}$$

Notice that this last term becomes arbitrarily large as $x \to \infty$. That impl…………
are interested in does as well:

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$

Basically, the exponential e^x grows faster than any polynomial because it behaves like an infinite polynomial whose coefficients are all positive.

Solving differential equations

Some differential equations cannot be solved in terms of familiar functions (just as some functions do not have anti derivatives which can be expressed in terms of familiar functions).

However, Taylor's series can come to the rescue again. Here you will present two examples to give you the idea.

Example 1: You will solve the initial value problem

$$\frac{dy}{dx} = y$$
$$y(0) = 1$$

Of course, you know that the solution is $y(x) = e^x$, but you will see how to discover this in a different way. First, you will write out the solution in terms of its Taylor's series:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

Since this function satisfies the condition $y(0) = 1$, you must have $y(0) = a_0 = 1$.

You also have

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \ldots$$

Since the differential equation says that $\frac{dy}{dx} = y$, you can equate these two Taylor's series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

If we now equate the coefficients, you obtain:

$$a_0 = a_1 = 1, \quad a_1 = 1$$
$$a_1 = 2a_2, \quad a_2 = \frac{a_1}{2} = \frac{1}{2}$$

$$a_2 = 3a_3, \quad a_3 = \frac{a_2}{3} = \frac{1}{2\cdot 3}$$
$$a_3 = 4a_4, \quad a_4 = \frac{a_3}{4} = \frac{1}{2\cdot 3\cdot 4}$$

$$a_{n-1} = na_n, \quad a_n = \frac{a_{n-1}}{n} = \frac{1}{1\cdot 2\cdot 3 \ldots n} = \frac{1}{n!}$$
$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots = e^x$$

This means that as you expect.

Of course, this is an initial value problem you know how to solve. The method is in studying initial value problems that you do not know how to solve

Example 2: Here we will study *Airy's equation* with initial conditions:

$$y'' = xy$$
$$y(0) = 1$$
$$y'(0) = 0$$

This equation is important in optics. In fact, it explains why a rainbow appears the way in which it does! As before, you will write the solution as a series:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

Since you have the initial conditions, $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$.

Now you can write down the derivatives:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$$
$$y'' = 2a_2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + \ldots$$

The equation then gives

$$\begin{aligned} y'' &= xy \\ 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots &= x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) \\ 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots &= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots \end{aligned}$$

Again, you can equate the coefficients of x to obtain

$$\begin{array}{ll} 2a_2 = 0 & a_2 = 0 \\ 2 \cdot 3a_3 = a_0 & a_3 = \frac{1}{2 \cdot 3} \\ 3 \cdot 4a_4 = a_1 & a_4 = 0 \\ 4 \cdot 5a_5 = a_2 & a_5 = 0 \\ 5 \cdot 6a_6 = a_3 & a_6 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} \end{array}$$

This gives you the first few terms of the solution:

$$y = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \ldots$$

If you continue in this way, you can write down many terms of the series perhaps you see the pattern already?) And then draw a graph of the solution. This looks like this:

Notice that the solution oscillates to the left of the origin and grows like exponential to the right of the origin. Can you explain this by looking at the differential equation.

4.0 CONCLUSION

In this unit, you have been introduced to the application of Taylor's series and some basic ways of using Taylor's series such as the evaluating of definite integrals, understanding the asymptotic behaviour, understanding the growth of functions and solving differential equations. Some examples where used to illustrate the applications.

5.0 SUMMARY

Having gone through this unit, you now know that;

In this section, you show you ways in which Taylor's series can make life easy

- In evaluating definite integrals, you used series representation of evaluate some functions that have no anti derivative.

  Suppose you want to evaluate the definite integral

$$\int_0^1 \sin(x^2)dx$$

  The integrand has no anti derivative expressible in terms of familiar functions. However, you know how to find its Taylor's series: you know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

  Now if you substitute $t = x^2$, you have

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{10!} - \frac{x^{14}}{14!} + \ldots$$

  In spite of the fact that you cannot anti differentiate the function, you can anti differentiate the Taylor's series:

- We used Taylor's series to understand asymptotic behaviour of functions that behave in the important part of the domain. And some examples are shown to demonstrate,

- Taylor's series is used to understand the growth of functions. Because you know the fact that exponentials grow much more rapidly than polynomials. You recorded this by saying that

$$\lim_{n\to\infty} \frac{e^x}{x^n} = \infty$$

  for any exponent **n**.

- You used Taylor's series to solve problems which could not be solved ordinarily through differential equations.

6.0 TUTOR-MARKED ASSIGNMENT

1. Compute a second-order Taylor series expansion around point (a,b) = (0,0) of a function $F(x,y) = e^x \log(2+y)$

2. Show that the Taylor series expansion of $f(x,y) = e^{xy}$ about the point (2,3) .

7.0 REFERENCES/FURTHER READING

Arfken and Weber, Mathematical Methods for Physicists, 6th Edition, 352-354, Academic press 200

http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/series/appsTaylor.html September,29 2008

Broadhead MK (Broadhead, Michael K.) , geophysical prospecting 56, 5, 729-735 SEP 2008

Guyenne P (Guyenne, Philippe), Nicholls DP (Nicholls, David P.), Siam journal on scientific computing 30, 1, 81-101, 2007

Popa C (Popa, Cosmin), IEEE transactions on very large scale integration (VLSI) systems, 16, 3,318-321, MAR 2008

Janssen AJEM (Janssen, A. J. E. M.), Van Leeuwaarden JSH (Van Leeuwaarden, J. S. H.) stochastic processes and their applications 117, 12, 1928-1959, DEC 2007

Sahu S (Sahu, Sunil), Baker A J (Baker, A. J.) Source: international journal for numerical methods in fluids 55, 8, 737-783, NOV 20, 2007