MODULE 8 THE JACOBIANS

UNIT 1 JACOBIAN

CONTENTS

1.0 INTRODUCTION

Jacobian

The Jacobian of functions $f_i(x_1,x_2, ...,x_n),i$= 1, 2, ...,n, of real variables x_i is the determinant of the matrix whose *i*th row lists all the first-order partial derivatives of the function $f_i(x_1,x_2, ...,x_n)$. Also known as Jacobin an determinant.

(or functional determinant), a determinant $|a_{ik}|_I^n$ with elements $a_{ik} = \partial y_i/\partial x_k$ where $y_i = f_i(x_1, . .., x_n), 1 \leq i \leq n$, are functions that have continuous partial derivatives in some region Δ. It is denoted by

$$\frac{D(y_1, \dots, y_n)}{D(x_1, \dots, x_n)}$$

The Jacobian was introduced by K. Jacobi in 1833 and 1841. If, for example, n = 2, then the system of functions

(1) $y_l = f_1(x_1, x_2)\ y_2 = f_2(x_1, x_2)$

defines a mapping of a region Δ, which lies in the plane x_1x_2, onto a region of the plane y_1y_2. The role of the Jacobian for the mapping is largely analogous to that of the derivative for a function of a single variable. Thus, the absolute value of the Jacobian to some point M is equal to the local factor by which areas at the point are altered by the mapping; that is, it is equal to the limit of the ratio of the area of the image of the neighborhood of M to the area ofthe neighborhood as the dimensions of the neighborhood approach zero. The Jacobian at M is positive if mapping (1) does not change the orientation in the neighborhood of M, and negative otherwise.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- recognise the Jacobian rule; and
- how to use the Jacobian

3.0 MAIN CONTENT

If the Jacobian does not vanish in the region A and if $\varphi(y_1, y_2)$ is a function defined in the region Δ_1 (the image of Δ), then

$$\iint_{\Delta_1} \phi\,(y_1, y_2) dy_1 dy_2 = \iint_{\Delta_1} \phi\,[f_1(x_1, x_2), f_2(x_1, x_2)] \left|\frac{D(y_1, y_2)}{D(x_1, x_2)}\right| dx_1 dx_2$$

(the formula for change of variables in a double integral). An analogous formula obtains for multiple integrals. If the Jacobian of mapping (1) does not vanish in region A, then there exists the inverse mapping

$x_1 = \psi(y_1, y_2)$ $x_2 = \psi_2(y_1, y_2)$

and

$$\frac{D(x_1, x_2)}{D(y_1, y_2)} = 1 \colon \frac{D(y_1, y_2)}{D(x_1, x_2)}$$

(an analogue of the formula for differentiation of an inverse function). This assertion finds numerous applications in the theory of implicit functions.

In order for the explicit expression, in the neighborhood of the point $M\left(x_1^{(0)}, \dots, x_n^{(0)}, y_1, \dots, y_m\right)$ of the functions $y_1, \dots y_m$ that are implicitly defined by the equations

(2) $F_k(x_1. \dots, x_n, y_1. \dots, y_m) = 0 \quad i \leq k \leq m$

to be possible, it is sufficient that the coordinates of *M* satisfy equations (2), that the functions F_k have continuous partial derivatives, and that the Jacobian

$$\frac{D(F_1, \dots, F_m)}{D(y_1, \dots, y_m)}$$

be nonzero at *M*. The Jacobian is been classified into two:

The Jacobian matrix and the Jacobian determinant.

Examples1.Let F:$R^2 \rightarrow R^2$ be the mapping defined by

$$F(x,y) = \begin{pmatrix} X^2 + Y^2 \\ e^{xy} \end{pmatrix} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

Find the Jacobian matrix $J_f(p)$ for p = (1,1)

The Jacobian matrix at an arbitrary point (x,y) is

$$\begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dy}{dx} & \frac{dg}{dy} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ ye^{xy} & xe^{xy} \end{pmatrix}$$

Hence when x=1 ,y=1, you find $Jf(1, 1) = \begin{pmatrix} 2 & 2 \\ e & e \end{pmatrix}$

2. Let F :R^2 $R^3 \rightarrow$ be the mapping defined by

$$F(x,y) = \begin{pmatrix} xy \\ \sin x \\ x^2 y \end{pmatrix}$$

Find $J_F(P)$ at the point $P = \left(\pi, \ \frac{\pi}{2}\right)$.

The Jacobian Matrix at an arbitrary point (x, y)

$$J_F(x,y) = \begin{pmatrix} y & x \\ \cos x & 0 \\ 2xy & x^2 \end{pmatrix}$$

Hence, $J_F\left(\pi, \ \frac{\pi}{2}\right) = \begin{pmatrix} \frac{\pi}{2} & \pi \\ -1 & 0 \\ \pi^2 & \pi^2 \end{pmatrix}$

4.0 CONCLUSION

In this unit, you have been able to recognise the Jacobian rule and how to use the formular.

5.0 SUMMARY

In this unit, you have studied the basic concept of Jacobian with the identification of the formula below as:

$$\frac{D(F_1, \dots, F_m)}{D(y_1, \dots, y_m)}$$

be nonzero at M.

6.0 TUTOR – MARKED ASSIGNMENT

1. Define the Jacobian matrix and the Jacobian determinant.
2. Compute the Jacobian matrix of the following cases below:
   a. $F(x,y) = (x+y, x^2y)$
   b. F(x,y) = (sinx, cosxy)
   c. $F(x,y,z) = (xyz, x^2z)$

7.0 REFERENCES/FURTHER READING

D.K. Arrowsmith and C. M. Place, Dynamical Systems, Section 3.3, Chap an & Hall, London, 1992. ISBN 0-412-39080-9.

Taken from http://www.sjcrothers.plasmaresources.com/schwarzschild.pd - On the Gravitational Field of a Mass Point according to Einstein's Theory by K. S chwarzschild - arXiv:physics/9905030 v1 (text of the original paper, in Wikisource).

UNIT 2 JACOBIAN DET RMINANT

CONTENTS

1.0 INTRODUCTION

The Jacobian of functions$f_i(x_1, x_2, ..., x_n)$, $i = 1, 2, ..., n$, of real variables xi is the determinant of the matrix whose ith row lists all the first-order partial derivatives of the function $f_i(x_1, x_2, ..., x_n)$.Also known as Jacobian determinant.

In vector calculus, the **Jacobian matrix**: is the matrix of all first-order partial derivatives of a vector- or scalar-valued function with respect to another vector. Suppose F: $R_n \rightarrow R_m$ is a function from Euclidean n-space to Euclidean m-space. Such a function is given by m real-valued component functions, $y_1(x_1,...,x_n)$, ..., $y_m(x_1,...,x_n)$. The partial derivatives of all these functions (if they exist) can be organized in an *m*-by-*n* matrix, the Jacobi n matrix *J* of *F*, as follows:

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

$$\frac{\partial(y_1, \ldots, y_m)}{\partial(x_1, \ldots, x_n)} J_F(x_1, \ldots, x_n)$$

This matrix is also denoted by $J_F(x_1, \ldots, x_n)$ and $\frac{\partial(y_1,\ldots, y_m)}{\partial(x_1,\ldots x_n)}$. If $(x_1, \ldots, x_n)$ usual orthogonal Cartesian coordinates, the *i*th row ($i = 1, \ldots, n$) of this mthe gradient of the i^{th} component function y_i:∇y_i. Note that some books define the Jacobian as the transpose of t e matrix given above.

The **Jacobian determinant** (often simply called the **Jacobian)** is the determinant of the Jacobian matrix (if $m = n$).

These concepts are named after the mathematician Carl Gustav Jacob Jacobi.

2.0 OBJECTIVE

At the end of this unit, you should be able to:

- apply the Jacobian concept;
- explain the Jacobian matrix;
- apply the inverse transformation; and
- solve problems on Jacobian determinant.

3.0 MAIN CONTENT

Jacobian matrix

The Jacobian of a function describes the orientation of a tangent plane to the function at a given point. In this way, the Jacobian generalizes the gradient of a scalar valued function of multiple variables which its lf generalizes the derivative of a scalar-valued function of a scalar. Likewise, the Jacobian can also be thought of as describing the amount of "stretching" that a transformation imposes. For example, if $(x_2,y_2) = f(x_1,y_1)$ is used to transform an image, the Jacobian of *f*, $J(x_1,y_1)$ describes how much the image in the neighborhood of (x_1,y_1) isstretched in the *x* and *y* directions.

If a function is differentiable at a point, its derivative is given in coordinates by the Jacobian, but a function doesn't need to be differentiable for the Jacobian to be defined, since only the partial derivatives are required to exist.

The importance of the Jacobi n lies in the fact that it represents the best linear approximation to a differentiable function near a given point. In this sense, the Jacobian ***i*** the derivative of a multivariate function.

If **p** is a point in $\mathbf{R}^n$ and F is differentiable at **p**, then its derivative is given by $J_F(\mathbf{p})$. In this case, the linear map describe by $J_F(\mathbf{p})$ is the best linear approximation of F near the point p, in the sense that

$$F(\mathbf{x}) = F(\mathbf{p}) + J_F(\mathbf{p})(\mathbf{x} - \mathbf{p}) + o(\|\mathbf{x} - \mathbf{p}\|)$$

For **x** close to **p** and where *o* is the little o-notation (for $x \to p$) and || x – p|| is the distance between **x** and **p**.

In a sense, both the gradient and Jacobian are "first derivatives" — the former the first derivative of a scalar function of several variables, the latter the first derivative of a vector function of several variables. In general, the gradient can be regarded as a special version of the Jacobian: it is the Jacobian of a scalar function of several variables.

The Jacobian of the gradient has a special name: the Hessian matrix, which in a sense is the "second derivative" of the scalar function of several variables in question.

Inverse

According to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function. That is, for some function F: R^n - R^n and a point p in R^n,

$$J_{F^{-1}}(F(p)) = [J_F(p)]^{-1}.$$

It follows that the (scalar) inverse of the Jacobian determinant of a transformation is the Jacobian determinant of the inverse transformation.

Examples

Example 1: The transformation from spherical coordinates (r, θ, φ ,) to C rtesian coordinates (x_1, x_2, x_3) , is given by the function F : $R^+ \times [0,\pi] \times [0,2\pi) \to R^3$ with components:

$$x_1 = r \sin\theta \cos\phi$$
$$x_2 = r \sin\theta \sin\phi$$
$$x_3 = r \cos\theta.$$

The Jacobian matrix for this coordinate change is

$$J_F(r,\theta,\phi) = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \phi} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial \phi} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

The determinant is $r^2 \sin \theta$. As an example, since $dV = dx_1\, dx_2\, dx_3$ this determinant implies that the differential volume element $dV = r^2 \sin \theta dr d\theta$. Nevertheless this determinant varies with coordinates. To avoid any variation the new coordinates can be defined as

$w_1 = \frac{r^3}{3},\ w_2 = -\cos\theta,\ w_3 = \phi.$ Now the determinant equals to 1 and volume element becomes $r^2 dr\ \sin\theta\ d\theta\ d\phi = dw_1 dw_2 dw_3$

Example 2: The Jacobian matrix of the function F: $\mathbf{R}^3 \rightarrow \mathbf{R}^4$ with components

$y_1 = x_1$

$$y_2 = 5x_3$$
$$y_3 = 4x_2^2 - 2x_3$$
$$y_4 = x_3 \sin(x_1)$$

is

$$J_F(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} & \frac{\partial y_4}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3\cos(x_1) & 0 & \sin(x_1) \end{bmatrix}.$$

This example shows that the Jacobian need not be a square matrix.

Example 3:

$$x = r\ \cos\ \phi;$$
$$y = r\ \sin\ \phi.$$

$$J(r,\phi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial (r\cos\phi)}{\partial r} & \frac{\partial (r\cos\phi)}{\partial \phi} \\ \frac{\partial (r\sin\phi)}{\partial r} & \frac{\partial (r\sin\phi)}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix}$$

The Jacobian determinant is equal to r. This shows how an integral in the Cartesian coordinate system is transformed into an integral in the polar coordinate system:

$$\iint_A dx\,dy = \iint_B r\,dr\,d\phi$$

Example 4: The Jacobian determinant of the function F: $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ with components

$$y_1 = 5x_2$$
$$y_2 = 4x_1^2 - 2\sin(x_2 x_3)$$
$$y_3 = x_2 x_3$$

is

$$\begin{vmatrix} 0 & 5 & 0 \\ 8x_1 & -2x_3\cos(x_2x_3) & -2x_2\cos(x_2x_3) \\ 0 & x_3 & x_2 \end{vmatrix} = -8x_1 \cdot \begin{vmatrix} 5 & 0 \\ x_3 & x_2 \end{vmatrix} = -40x_1x_2.$$

From this we see that F reverses orientation near those points where x_1 and x_2 have the same sign; the function is locally invertible everywhere except near points where $x_1 = 0$ or $x_2 = 0$. Intuitively, if you start with a tiny object around the point (1,1,1) and apply F to that object, you will get an object set with approximately 40 times the volume of the original one.

4.0 CONCLUSION

In this unit, you have studied the application of the Jacobian concept. You have known the Jacobian matrix and the application of the inverse transformation of Jacobian determinants. You have solved problems on Jacobian determinant.

5.0 SUMMARY

In this unit;

- you have studied application of the Jacobian concept
- you have known the Jacobi n matrix
- you have known the inverse transformation of Jacobian determinant
- you have solve problems o Jacobian determinant such as ;
- The Jacobian matrix of thefunction F : R3 → R4 with components

$$y_1 = x_1$$
$$y_2 = 5x_3$$
$$y_3 = 4x_2^2 - 2x_3$$
$$y_4 = x_3\sin(x_1)$$

is

$$J_F(x_1,x_2,x_3) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} & \frac{\partial y_4}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3\cos(x_1) & 0 & \sin(x_1) \end{bmatrix}.$$

This example shows that the Jacobian need not be a square matrix.

6.0 TUTOR-MARKED ASSIGNMENT

1. In each of the following cases, compute the Jacobian matrix of F, and evaluate at the following points;

   F(x,y) = (sinx,cosxy) at points (1,2)

   F(x,y,z) = (sinxyz,xz) at points (2,-1,-1)

   F(x,y,z) =(xz,xy,yz) at points (1,1,-1)

2. Transform the following from spherical coordinates (r, θ, ϕ) to Cartesian coordinate (x_1,x_2,x_3) by the function
   $F{:}R^{+}\times(0,\pi)\times(0,2\pi)\rightarrow R^3$ with components :

   r_1= r tan $\theta\cos\theta$

   r_2= r sinθ tan θ

   r = rsinθ_1

3. The Jacobian matrix of the function F: $R^3 \rightarrow R^4$ with components

   $y_1 = x_2$

   $y_2 = 4x_1$

   $y_3 = 5x_2^2 - 4x_3$

   $y_4 = x_1 \sin x_3$

4. The Jacobian matrix of the function F: $R^3 \rightarrow R^3$ with components

   $y_1 = 4x_1^2 - 3\sin x_2 x_3$

   $y_2 = 3x_2$

   $y_3 = x_2 3x_3$

The Jacobian matrix of the function F: $R^3 \rightarrow R^3$ with components

$x = r\tan\phi$

$y = r\cos\phi$

7.0 REFERENCES/FURTHER READING

Kudriavtsev, L. D. Matematicheskiianaliz, 2nd ed., vol. 2.Moscow, 1973.Il'in, V. A., and E. G. Pozniak.Osnovymatematicheskogoanaliza, 3rd ed..part I. Moscow, 1971.

The Great Soviet Encyclopedia, 3rd Edition (1970-1979). © 2010 The Gale Group, Inc. All rights reserved.

D.K. Arrowsmith and C.M. Place, Dynamical Systems, Section 3.3, Chapman & Hall, London, 1992.

UNIT 3 APPLICATION OF JACOBIAN

CONTENTS

1.0 INTRODUCTION

If $m = n$, then F is a function from m-space to n-space and the Jacobian matrix is a square matrix. You can then form its determinant, known as the **Jacobian determinant**. The Jacobian determinant is sometimes simply called "the Jacobian."

2.0 OBJECTIVE

3.0 MAIN CONTENT

Dynamical systems

Consider a dynamical system of the form x' = *F(x)*, where x' is the (component-wise) time derivative of x, and $F: R^n \to R^n$ is continuous and differentiable. If $F(x_0) = 0$, then x_0 is a stationary point (also called a fixed point). The behavior of the system near a stationary point is related to the eigenvalues of $J_F(x_0)$, the Jacobian of F at the stationary point. Specifically, if the eigenvalues all have a negative real part, then the system is stable in the operating point, if any eigenvalue has a positive real part, then the point is unstable.

The Jacobian determinant at a given point gives important information about the behavior of ***F*** near that point. For instance, the continuously differentiable function **F** is invertible near a point $\mathbf{p \in R^n}$ if the Jacobian determinant at **p** is non-zero. This is the inverse function theorem. Furthermore

If the Jacobian determinant at **p** is positive, then **F** preserves orientation near **p**; if it is negative, **F** reverses orientation. The absolute value of the Jacobian determinant at **p** gives us the factor by which the function ***F*** expands or shrinks volumes near **p**; this is why it occurs in the general substitution rule.

Uses

The Jacobian determinant is used when making a change of variables when evaluating a multiple integral of a function over a region within its domain. To accommodate for the change of coordinates the magnitude of the Jacobian determinant arises as a multiplicative factor within the integral. Normally it is required that the change of coordinates be done in a manner which maintains an injectivity between the coordinates that determine the domain. The Jacobian determinant, as a result, is usually well defined.

Newton's method

A system of coupled nonlinear equations can be solved iteratively by Newton's method. This method uses the Jacobian matrix of the system of equations

4.0 CONCLUSION

In this unit, you have known the application of Jacobian concept. You have studied the application of Jacobian matrix. You have used Jacobian in the application of inverse transformation and have also solved problems on Jacobian determinant.

5.0 SUMMARY

In this unit, you have studied the following:

- Application of the Jacobian concept
- Application of Jacobian on the Jacobian matrix
- Application of the Jacobian o the inverse concept
- Application of the Jacobian to solve problems on Jacobian determinant

6.0 TUTOR – MARK ASSIGNMENTS

1. Find the Jacobian determinant of the map below, and determine all the points where the Jacobian determinants is equal to zero(0).
   a. $F(x,y,z) = (xy,y,xz)$
   b. $F(x,y) = (e^{xy},x)$
   c. $F(x,y) = (xy, x^2)$

2. The transformation from spherical coordinates (r, θ, φ) to Cartesian coordinates (x_1, x_2, x_3), is given by the function
   F: $\mathbf{R}^+ \times [0,\pi] \times [0,2\pi) \rightarrow \mathbf{R}^3$ with components:

$$x_1 = r\ \sin\theta\ \cos\phi$$

$$x_2 = r\ \sin\theta\ \sin\phi$$

$$x_3 = r\ \cos\theta.$$

3. The Jacobian determinant of the function F: $\mathbf{R}^3 \to \mathbf{R}^4$ with components

$$y_1 = x_1$$
$$y_2 = 5x_3$$
$$y_3 = 4x_2^2 - 2x_3$$
$$y_4 = x_3 \sin(x_1)$$

4. The Jacobian determinant of the function F: $\mathbf{R}^3 \to \mathbf{R}^4$with components

$$x = r \cos \phi;$$
$$y = r \sin \phi.$$

5. The Jacobian determinant of the function F: $\mathbf{R}^3 \to \mathbf{R}^3$with components

$$y_1 = 5x_2$$
$$y_2 = 4x_1^2 - 2\sin(x_2 x_3)$$
$$y_3 = x_2 x_3$$

7.0 REFERENCES/FURTHER READING

Gradshteyn, I. S. and Ryzhik, I. M. "Jacobian Determinant." §14.313 in Ta bles of Integrals, Series, and Products, 6th ed. San Diego, CA: Academic Press, pp. 1068-1 69, 2000.

Kaplan, W. Advanced Calcul s, 3rd ed. Reading, MA: Addison-Wesley, pp. 98-99, 123, and 238-245, 1984.

Simon, C. P. and Blume, L. E .Mathematics for Economists. New York: W . W. Norton, 1994.

D.K. Arrowsmith and C.M. P lace, Dynamical Systems, Section 3.3, Chap an & Hall, London, 1992. ISBN 0-412-3 9080-9