MODULE 1 FUNCTIONS OF SEVERAL VARIABLES

UNIT 1 SOME BASIC CONCEPTS

CONTENTS

1.0 INTRODUCTION

In scientific problems, often times one discovers that a factor depends upon several other related factors. For instance, the area of rectangle depends on its length and breath, hence can say that area is the function of two variables i.e. its length and breadth. Potential energy of a body depends on gravity, density and height of the body, hence, we can also say that potential energy is a function of three variables i.e gravity, density and height etc. The strength of a material depends upon temperature, density, isotropy softness etc., here we can say that the strength of material is a function of many variables i.e. temperature, density, isotropy softness etc.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use Jacobian change variables in multiple integral;
- determine whether two or more functions are linearly dependent or independent; and
- identify the functions of two or more variables.

3.0 MAIN CONTENT

3.1 Functions of Several Variables

A function is composed of a domain set, a range set and a rule of correspondence that assigns exactly one element of the range to each element of the domain u, is called a function of two variables x and y if u has one definite value for every pair of variables of x and y. Symbolically, it is written as

$u = f(x, y)$.

The variables x and y are called independent variables while u is called the dependent variable.

Similarly, we can define u as a function of more than two variables.

In summary, we have that

$u(x) \Rightarrow$ a function of a single variable
$u(x_1, x_2) \Rightarrow$ a function of two variables
$u(x_1, x_2, x_3 \cdots\cdots x_n) \Rightarrow$ a function of several variables.

Example 1

If $f(x, y) = x^2 - 3xy + 6y$, find : (a) f(-1,1) and f(2,3).

(a) $f(x, y) = x^2 - 3xy + 6y$

$$f(-1,1) = (-1)^2 - 3(-1)(1) + 6(1)$$
$$f(-1,1) = 1 + 3 + 6 = 10$$

(b) $f(2,3) = 2^2 - 3(2)(3) + 6(3)$

$$f(2,3) = 4 - 18 + 18 = 4$$

3.2 Jacobian

Jacobian is a functional determinant (whose elements are functions) which is very useful in transformation of variables from Cartesian to polar, cylindrical and spherical coordinates in multiple integras. Let u(x,y) and v(x,y) be two given functions of two independent variables x and y.

The Jacobian of u and v with respect to x,y denoted by $J\begin{pmatrix} u & \mathrm{v} \\ x & \mathrm{y} \end{pmatrix} or \frac{\partial(u,v)}{\partial(x,y)}$ is a second order functional determinant defined as

$$J\begin{pmatrix} u & \mathrm{v} \\ x & \mathrm{y} \end{pmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Properties of Jacobians

If u and v are the functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} x \frac{\partial(x,y)}{\partial(u,v)} = 1$$

If u,v are the functions of r,s where r,s are functions of x, y, then, $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} x \frac{\partial(r,s)}{\partial(x,y)}$

If functions u, v, w of three independent variables x,y,z are not independent, then, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Example 2

Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ in each of the following:

(i) $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$

(ii) $u = x^2 + y^2$, $v = 2xy$

Solution.

(i) $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$, using $J\begin{pmatrix} u & \text{v} \\ x & \text{y} \end{pmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$J\begin{pmatrix} u & \text{v} \\ x & \text{y} \end{pmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} (1-\frac{y^2}{x^2}) & (\frac{2y}{x}) \\ (-\frac{y^2}{x^2}) & (\frac{2y}{x}) \end{vmatrix}$$

$$= \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

Solution

(ii) $u = x^2 + y^2$, $v = 2xy$, using $J\begin{pmatrix} u & \text{v} \\ x & \text{y} \end{pmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$J = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

= (2x)(2x)-(2y)(-2y)

$= 4x^2 + 4y^2$

$= 4(x^2 + y^2)$

Example 3

If u=xyz, $v = x^2 + y^2 + z^2$, w=x+y+z ,find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution

Since u, v,w are explicitly given, so , first we evaluate

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

= yz(2y-2z)-zx(2x-2z)+xy(2x-2y)

=2[yz(y-z)-zx(x-z)+xy(x-y)]

$= 2[x^2y - x^2z - xy^2 + xz^2 + y^2z - yz^2]$

$= 2[x^2(y-z) - x(y^2+z^2) + yz(y-z)]$

$= 2(y-z)[x^2 - x(y+z) + yz]$

$= 2(y-z)[y(z-x) - x(z-x)]$

$= 2(y-z)(z-x)(y-x)$

$= -2(x-y)(y-z)(z-x)$ The idea can be easily extended to three or several variables thus:

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example 4

Jacobian can be applied to polar coordinate r and θ, thus, $x = rCos\theta$ and $y = rSin\theta$.

Then,

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \qquad (1)$$

But $\frac{\partial x}{\partial r} = Cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$ (2)

$\frac{\partial y}{\partial r} = Sin\theta$ and $\frac{\partial y}{\partial \theta} = rCos\theta$

Substituting equation (2) into (1) gives

$$J = \begin{vmatrix} Cos\theta & -rSin\theta \\ Sin\theta & rCos\theta \end{vmatrix}$$

$$= rCos^2\theta - (-rSin^2\theta)$$

$$= r\left[Coc^2\theta + Sin^2\theta\right] = r$$

Since $Cos^2\theta + Sin^2\theta = 1$

$$\therefore \quad J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

3.3 Function Dependence and Independence

Two functions $u(x)$ and $v(x)$ defined on an interval $0 < x < 1$ are said to be functionally (linearly) dependent on $0 < x < 1$ if there exist '$\exists$' two constants k_1 and k_2 where not both zero, such that'$\exists$'

$k_1 u(x) + k_2 v(x) = 0$ for x, $\forall x$. (i)

However, the two functions $u(x)$ and $v(x)$ defined on interval $0 < x < 1$ are said to be functionally (linearly) independent on $0 < x < 1$, if the only constants k_1 and k_2 such that'$\exists$' for all x in the interval where both constants k_1 and k_2 are zeros i.e, when u or v can not be expressed as proportional to the other. Otherwise, u and v are linearly dependent if (i) holds for some k_1 and k_2 not both zero.

Example 5

Show that the functions $v(x) = e^{ax}$ and $u(x) = e^{bx}$ are linearly dependent on the interval. $0 < x < 1$.

Solution

Suppose $k_1 e^{ax} + k_2 e^{bx} = 0 \quad \forall x$ in $0 < x < 1$ (1)

Multiplying equation (1) by e^{-ax}, we obtain

$k_1 e^{ax} e^{-ax} + k_2 e^{bx} e^{-ax} = 0$ (2)

$k_1 + k_2 e^{(b-a)x} = 0$ (3)

differentiating equation (3) we obtain

$(b-a) k_2 e^{(b-a)x} = 0$ (4)

$(b-a) e^{(b-a)x} \neq 0$ since $b - a \neq 0$ then it implies that

$b = 0$ (5)

Substituting (5) into (1), and differentiating w.r.t.x, we obtain

$k_1 a e^{ax} = 0$ (6)

$\Rightarrow \quad a = 0$, since $e^{ax} \neq 0$.

Example 6

Show that the functions $v(x) = e^{ax}$ and $u(x) = e^{bx}$ are linearly independent on the interval. $0 < x < 1$.

Solution: If

$$k_1 e^{ax} + k_2 x e^{ax} = 0 \qquad (1)$$

$$(k_1 + k_2 x)e^{ax} = 0 \qquad (2)$$

Since $e^{ax} \neq 0$, $\Rightarrow \quad k_1 + k_2 x = 0 \qquad (3)$

Differentiating equation (3) we obtain

$$k_1 = 0 \qquad (4)$$

Substituting (5) into (1), however

$$k_1 e^{ax} = 0 \quad \Rightarrow \quad k_1 = 0 .\text{Since } e^{ax} \neq 0 \qquad (5)$$

3.3.1 Testing For Linear Dependence or Otherwise

A method called Wronskian of the function could also be used to test for linear dependence or otherwise. Thus, consider the functions $u(x)$ and $v(x)$ and the first derivatives $u'(x)$ and $v'(x)$, therefore we can define the Wronski determinant or Wroskian:

$$\text{Wronskian } = W\big(v(x).u(x)\big) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix}$$

$$= v(x)u'(x) - u(x)v'(x)$$

Results:
v(x), u(x) are linearly independent if $W \neq 0$
Otherwise linearly dependent when W=0.

Example 7

Determine whether the following functions v(x) and u(x) are linearly dependent or independent.

v(x) =cosbx, u(x) = sinbx with $b \neq 0$
v(x) = e^{ax}, $u(x) = e -^{ax}$ with $a \neq 0$.
Solution

$v(x) = \cos bx$, $v'(x) = -b\sin bx$, $u(x) = \sin bx$ and $u'(x) = \cos bx$.

(a) $$W(v(x).u(x)) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix} = \begin{vmatrix} \cos bx & \sin bx \\ -\sin b & b\cos bx \end{vmatrix}$$

$$= b(\cos^2 bx + \sin^2 bx)$$
$$= b \neq 0$$

So v(x) and u(x) are linearly independent.

$v(x) = e^{ax}$, $v'(x) = -ae^{ax}$, $u(x) = e^{-ax}$ and $u'(x) = -ae^{-ax}$.

(b) $$W(v(x).u(x)) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix} = \begin{vmatrix} e^{ax} & e^{-ax} \\ ae^{ax} & -ae^{-ax} \end{vmatrix}$$
$$= -ae^0 - ae^0$$
$$= -a(e^0 + e^0)$$

With $a \neq 0$. So v(x) and u(x) are linearly dependent.

SELF ASSESSMENT EXERCISE

Determine whether the following pair of functions are linearly dependent as the case may be

i. (a) $u(x) = x, \quad v(x) = e^{2x}$
(b) $u(x) = 2Sinhx, \quad v(x) = Cosx$
(c) $u(x) = x^3, \quad v(x) = 3x^3$

ii. (a) Show that the function $u(x)$ and $v(x)$ defined by $u(x) = x^2, \quad v(x) = x|x|$ are linearly Independent for the interval $0 < x < 1$. Compute the Wronskian of these functions.

iii. If $f(x,y) = x^4 - 2xy + 4y^2$,
Find (a) $f(1,-1)$, (b) $f(0,-3)$ and
(c) $\dfrac{f(x,y+k) - f(x,y)}{12}$

iv. If $f(x,y) = \dfrac{4x+2y}{2-2xy}$,
Find (a) $f(1,-3)$, (b) $\dfrac{f(2+h,3) - f(2,3)}{h}$

v. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.

Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta.$

vi. If $u = x^2, v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$

3.4 Multiple Integral

3.4.1 Double Integral

Definition: In this case the integrand is a function $f(x, y)$ that is given for all (x, y) in a closed bounded region R of the $x-y$ plane.

Let $f(x, y)$ be a single valued continuous function within a region R bounded by a close curve C. Then the region R is called

The region of integration. However, double integral can be defined thus:

$$\int_c^d \int_a^b f(x,y)dxdy \text{ or } \iint_r f(x,y)dA \qquad (1)$$

3.4.4.1 Evaluation of Double Integrals

Consider $a \le x \le b$ and $g(x) \le y \le h(x)$ so that $y = g(x)$ and $y = h(x)$ represents the boundary of R. Then

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y) \right] dx \qquad (2)$$

Similarly, if R can be described thus

$$c \le y \le d, \; v(y) \le x \le u(y)$$

So that $x = v(y)$ and $x = u(y)$. Then

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_{v(y)}^{u(y)} f(x,y)dx \right] dy \qquad (3)$$

In this case, one first calculates the integral within the square brackets. Then further integration is then performed.

Properties of Double Integrals

1. $\iint_D af(x,y)ds = a\iint_D f(x,y)ds$, a =constant

2. $$\iint_D [f(x,y)+g(x,y)]ds = \iint_D f(x,y)ds + \iint_D g(x,y)ds$$

3. $$\iint_D f(x,y)ds = \iint_{D1} f(x,y)ds + \iint_{D2} f(x,y)ds$$

Were D is the union of disjointed domains D1 and D2

Example 5

Evaluate the integrals

(i) $\int_0^1 \int_0^1 (x^2 + y^2)dydx$

(ii) $\int_1^2 \int_0^1 (x^2 + y^2)dydx$

Solution

(i) $$\int_0^1 \left[\int_0^1 (x^2 + y^2)dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \tfrac{1}{3} y^3 \right]_0^1 dx$$

$$= \int_0^1 \left[(x^2 + \tfrac{1}{3}) - 0 \right] dx = \int_0^1 (x^2 + \tfrac{1}{3})dx$$

$$= \frac{1}{3}x^3 + \frac{1}{3}x \Big|_0^1 = \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

(ii) $$\int_1^2 \left[\int_0^1 (x^2 + y^2)dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \tfrac{1}{3} y^3 \right]_0^1 dx$$

$$= \int_1^2 \left[(x^2 + \tfrac{1}{3}) - 0 \right] dx = \int_1^2 (x^2 + \tfrac{1}{3})dx$$

$$= \frac{1}{3}x^3 + \frac{1}{3}x \Big|_1^2 = \frac{10}{3} - \frac{2}{3}$$

$$= \frac{8}{3}$$

3.4.4.2 Double Integral in Polar Coordinates

This is defined by

$$\int_{\theta_2}^{\theta_1} \int_{r_2}^{r_1} f(r,\theta)drd\theta$$

Example 6

Evaluate the integrals

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2Cos\theta} r^2 dr d\theta.$$

Solution

$$\int_{-\pi/2}^{\pi/2}\left[\int_0^{2Cos\theta} r^2 dr\right]d\theta = I \qquad (1)$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2Cos\theta} d\theta \qquad (2)$$

$$= \int_{-\pi/2}^{\pi/2}\frac{(2Cos\theta)^3}{3}d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{8}{3}Cos^3\theta d\theta \qquad (3)$$

Using trigonometric identity to simplify $Cos^3\theta$

Thus $Cos3\theta = Cos(2\theta+\theta) = Cos2\theta Cos\theta - Sin2\theta Sin\theta$

$$= (Cos^2\theta - Sin^2\theta)Cos\theta - (2Sin\theta Cos\theta)Sin\theta$$

$$= Cos^3\theta - Sin^2\theta Cos\theta - 2Sin^2\theta Cos\theta$$

$$= Cos^3\theta - 3Sin^2\theta Cos\theta$$

$$= Cos^3\theta - 3[1 - Cos^2\theta]Cos\theta$$

$$= Cos^3\theta - 3Cos\theta + Cos^3\theta$$

$$= 4Cos^3\theta - 3Cos\theta$$

$$\therefore \quad Cos^3\theta = \frac{1}{4}Cos3\theta + \frac{3}{4}Cos\theta \qquad (4)$$

Hence, substituting (4) into (3) we obtain

$$I = \frac{8}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{4}Cos3\theta + \frac{3}{4}Cos\theta\right)d\theta$$

$$I = -\frac{2}{3}\left[\frac{1}{3}Sin3\theta + 3Sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{2}{3}\left[\left(\frac{1}{3}Sin\tfrac{3}{2}\pi + 3Sin\tfrac{\pi}{2}\right) - \left(\frac{1}{3}Sin(-\tfrac{3}{2}\pi) + 3Sin(-\tfrac{\pi}{2})\right)\right] \qquad (5)$$

But $Sin\,{}^3\!/_2\pi = -1, \quad Sin\,{}^\pi\!/_2 = 1$

Similarly, $Sin - {}^3\!/_2\pi = -1$ and $Sin - {}^\pi\!/_2 = -1$ (6)

Substituting (6) into (5)

$$I = -\frac{2}{3}\left[\left(\frac{1}{3}(-1)+3\right)-\left(\frac{1}{3}(1)+3(-1)\right)\right]$$
$$= -\frac{2}{3}\left[\left(\frac{1}{3}+3\right)-\left(\frac{1}{3}+3\right)\right]$$
$$= -\frac{2}{3}\left[\frac{8}{3}+\frac{8}{3}\right] = -\frac{2}{3}\left(\frac{16}{3}\right) = -\frac{32}{9}$$
$$I = -3\frac{5}{9}$$

3.4.4.3 Triple Integral

Definition: A function of three variables is involved in triple integral. However, in triple integral, integration is carried out thrice. It is then define as:

$\iiint\limits_{v} f(x,y,z)dxdydz$ over the region v

$\int\limits_{v} f(x,y,z)dv$. This can also be used to find the volume of any shape.

Example 7

Evaluate

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz$$

Solution

$$\int_{-1}^{1}\left[\int_{0}^{z}\left(\int_{x-z}^{x+z}(x+y+z)dy\right)dx\right]dz$$
$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(xy+\tfrac{1}{2}y^2 zy\right)_{x-z}^{x+z}dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(\left[x(x+z)+\tfrac{1}{2}(x+z)^2+z(x+z)\right]-\left[x(x-z)+\tfrac{1}{2}(x-z)^2+z(x-z)\right]\right)dz\right]$$
$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(4xz+2z^2\right)dx\right]dz$$
$$= \int_{-1}^{1}\left[2x^2z+2xz^2\right]_{0}^{z}dz$$
$$= \int_{-1}^{1}4z^3dz = z^4\Big|_{-1}^{1} = 1-1 = 0$$

$= 0$

Example 8

Evaluate

$I = \iiint_v \left(3x^2 + 3y^2 + 3z^2\right)dv$ by changing to polar coordinate.

Thus $x = rSin\theta Cos\phi$, $y = rSin\theta Cos\phi$ and $z = rCos\theta$.

Solution

$$I = 24\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2 dr(rSin\theta d\phi)(rd\theta)$$
$$= \frac{24}{5}\int_0^{\pi/2}\int_0^{\pi/2} a^5 Sin\theta d\theta d\phi$$
$$= \frac{24}{5}a^5\int_0^{\pi/2}(-Cos\theta)_0^{\pi/2} d\phi$$
$$= \frac{24}{5}a^5 \cdot \frac{\pi}{2} = \frac{24}{5}a^5\pi.$$

4.0 CONCLUSION

In conclusion, the student should be able to use Jacobian method to change the variable in multiple integral and to determine whether two functions are linearly dependent or independent. Also to solve integral, multiple.

5.0 SUMMARY

The following are discussed in the unit:

Functions of variable defined thus, $u(x_1, x_2, x_3 \cdots\cdots x_n)$. Jacobian of (uv) was discussed and extend it to three or several variables, thus

$$J\begin{pmatrix} u & \text{v} \\ x & \text{y} \end{pmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{(\partial x, y, z)}$$

Jacobian was also applied to polar coordinate thus

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r\,.$$

The functional dependence of two functions $u(x)$ and $v(x)$ was discussed thus:

$k_1u(x) + k_2v(x) = 0$, $\forall x$ where k_1 and k_2 are constants and are not zero. While the functional independence of two functions $u(x)$ and $v(x)$ was also discussed thus:

$$k_1u(x) + k_2v(x) = 0 \quad \forall x, \text{ where } k_1 = k_2 = 0.$$

Testing for linear (independence) dependent was discussed using Wronskian method which involves the determinant thus

$$W\big(v(x), u(x)\big) = v(x)u'(x) - u(x)v'(x) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix}$$

Lastly, multiple integral was discussed.

6.0 TUTOR-MARKED ASSIGNMENT

i. Evaluate the double integrals

(a) $\int_{-\pi}^{\pi}\int_{-1}^{1} xydxdy$

(b) $\int_{1}^{2}\int_{-x}^{x} e^{y}Coshxdydx$

(c) $\int_{1}^{2}\int_{y}^{y^2+1} x^2 ydxdy$

ii. Evaluate the following triple integral

(a) $\int_{-\pi}^{\pi}\int_{0}^{2}\int_{x-z}^{x+z}(x+y+z)\,dxdydz$

(b) $\iiint \frac{dxdydz}{x^2+y^2+z^2}$ where $x^2+y^2+z^2=a$

(c) Compute the volume of the solid enclosed by

(i) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \quad x=0, y=0, z=0$

(ii) $x^2+y^2-2ax=0, \quad z=0, \ x^2+y^2=z^2$

iii. Determine whether the following pair of functions are linearly dependent or independent as the case may be.

(a) $u(x)=x, v(x)=e^{2x}$

(b) $u(x)=2Sinhx, v(x)=Cosx$

(c) $u(x)=x^3, v(x)=3x^3$

iv. (a) show that the functions $u(x)$ and $v(x)$ defined by $u(x)=x^2, u(x)=x|x.|$ are linearly independent for the interval $0\le x\le 1$.

(b) Compute the Wronskian of the function in 4(a)

v. Evalute $\iint_R (x+y)^2\,dxdy$, where R is a region bounded by the parallelogram x+y=0, x+y=2, 3x-2y=0, and 3x-2y=3.

vi. Evalute $\iint_R (x^2+y^2)\,dxdy$, where R is a region in the first quadrant bounded by $x^2-y^2=a$, $x^2-y^2=b$, 2xy=d, 0<a<b, 0<c<d

7.0 REFERENCES/FURTHER READING

Hernandez, V., J. E. Roman, and V. Vidal. "SLEPC: A Scalable and Flexible Toolkit for the Solution of Eigenvalue Problems." *ACM Trans. Math. Soft.* 31, no. 3 (2005).

McLachlan, R. I. "Families of High-Order Composition Methods." *Numerical Algorithms* 31 (2002).

McLachlan, R. I. and G. R. W. Quispel. "Splitting Methods." *Acta Numerica* 11 (2002).

Mazzia. F. and C. Magherini, "Test Set for Initial Value Problem Solvers-Release 2.4." *Dept. of Mathematics, University of Vari and INdAM, Research Unit of Bari.* 2008.

Moler, C. B. *Numerical Computing with MATLAB*. SIAM, (2004).

Murray, R. Spiegel *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc. (1974).

Nedialkov, N. and J. Pryce "Solving Differential Algebraic Equations by Taylor Series (I): Computing Taylor Coefficients." *BIT* 45, no.3 (2005)

Olsson, H. and G. Söderlind. "The Approximate Runge–Kutta Computational Process." *BIT* 40, no. 2 (2000).

Quarteroni, A., R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer-Verlag, (2000).

Ramana B.V *Higher Engineering Mathematics*. New Delhi: Tata McGraw-Hill Publishing Company Limited. (2008).

Rubinstein, B. "Numerical Solution of Linear Boundary Value Problems." *Mathematica MathSource* package, http://library.wolfram.com/database/MathSource/2127.

Shampine, L. F. "Solving in Matlab." *Journal of Numerical Mathematics* 10, no. 4 (2002).

Shampine, L. F. and S. Thompson. "Solving Delay Differential Equations with dde23." Available electronically from http://www.runet.edu/~thompson/webddes/tutorial.pdf.

Shampine, L. F. and S. Thompson. "Solving DDEs in MATLAB." *Appl. Numer. Math.* 37 (2001).

Shampine, L. F., I. Gladwell, and S. Thompson. *Solving ODEs with MATLAB*. Cambridge University Press, 2003.

Sofroniou, M. and G. Spaletta. "Increment Formulations for Rounding Error Reduction in the Numerical Solution of Structured Differential Systems." *Future Generation Computer Systems* 19, no. 3 (2003).

Sofroniou, M. and G. Spaletta. "Construction of Explicit Runge–Kutta Pairs with Stiffness Detection." *Mathematical and Computer Modelling*, special issue on The Numerical Analysis of Ordinary Differential Equations, 40, no. 11–12 (2004).

Sofroniou, M. and G. Spaletta. "Derivation of Symmetric Composition Constants for Symmetric Integrators." *Optimization Methods and Software* 20, no. 4–5 (2005).

Sofroniou, M. and G. Spaletta. "Hybrid Solvers for Splitting and Composition Methods." *J. Comp. Appl. Math.*, special issue from the International Workshop on the Technological Aspects of Mathematics, 185, no. 2 (2006).

Stephenor, G.. *Mathematical Methods for Science Students*. London: Longman, Group Limited. (1977)

Stroud, K.A... *Engineering Maths*. 5th Edition Palgraw. (1995)

Sprott, J. C. "A Simple Chaotic Delay Differential Equation." *Phys. Lett. A.* (2007).

Stewart, G. W. "A Krylov-Schur Algorithm for Large Eigenproblems." *SIAM J. Matrix Anal. Appl.* 23, no. 3, (2001).

Tang, X. H. and X. Zou. "Global Attractivity in a Predator-Prey System with Pure Delays." *Proc. Edinburgh Math. Soc.* (2008).

Unger, J., A. Kroner, and W. Marquardt. "Structural Analysis of Differential Algebraic Equation Systems—Theory and Applications." *Computers Chem. Engg.* 19, no. 8 (1995).

Verma, P.D.S.. *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd. (1995)

Verner, J. H. "Numerically Optimal Runge-Kutta Pairs with Interpolants." *Numerical. Algorithms* (2010).

Zennaro, M. "The Numerical Solution of Delay Differential Equations." Lecture notes, Dobbiaco Summer School on Delay Differential Equations and Applications, (2006).

UNIT 2 VECTOR FIELD THEORY

CONTENTS

1.0 INTRODUCTION

Vector function represents vector fields which have various physical and geometrical applications.

The basic concepts of differential calculus can be extended to vector function in a simple and natural fashion.

Vector functions are useful for representing and investigating curves and application in mechanics as path of moving bodies.

Integral theorems will be considered in the later path of this unit's i.e Line Integral, Gauss, Stokes and Greens theorems.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- appreciate vector field and vector function;

- understand the vector field theory, using vector function to investigate curves and their applications in mechanics; and
- use integral theorem to solve some physical problems. Study of Line Integral, Gauss, Stokes and Greens theorems and their applications.

3.0 MAIN CONTENT

3.1 Vector Field Theory

A scalar function is a function that is defined at each point of a certain set of points in space and whose values are real numbers depending only on the points in real space but not on the particular choice of the coordinate system.

Furthermore, the distance of $f(x, y, z)$ of any point p from a fixed point p_0 in space is a scalar function whose domain of definition D is the whole space. $f(x, y, z)$ defines a scalar field in space. Introducing a Cartesian coordinate x_0, y_0, z_0. Then the distance

$$f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

3.2 Relations between Vector Field and Functions

A vector $v(p)$ is a function that is defined on some point set D in space i.e. the set of points of a curve, a surface or a three dimensional region and associates with each point p in D a vector $v(p)$.

While a vector field is given in D. We introduce Cartesian coordinates x, y, z then we may write our vector function in terms of compound function.

$$v(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$$

or using i, j, k,. Thus

$$v(x, y, z) = v_1(x, y, z)i + v_2(x, y, z)j + v_3(x, y, z)k$$

But we should keep in mind that v depends only on that points of its domain of definition, and at the point defines the same vector for every choice of the coordinate system. The velocity of a moving fluid, gravitational force are the examples of vector point function.

Our notation in simple scalar and vector quantities in the pre-requisite course mathematical methods I and II are the same with that under discussion. The only difference is that the components v_1, v_2, v_3 of v now becomes functions of x, y, z since v is a function of x, y, z.

3.2.1 Example of Vector Field (Velocity Field)

At any instant, the velocity vectors $v(p)$ of a rotating body B constitute a vector field, the so called velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotations then

$$v(x, y, z) = w \times [z, y, z] = w \times (xi + yj + zk)$$

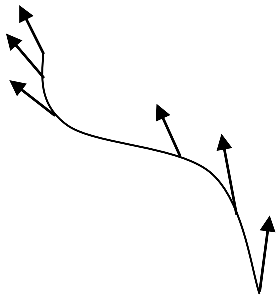


Fig. 1 Field of Tangent Vectors of a Curve

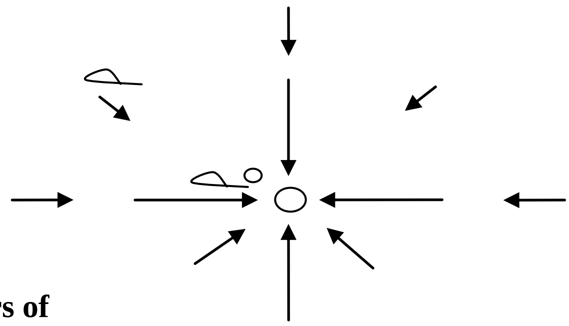


Fig. 2: Gravitational Field

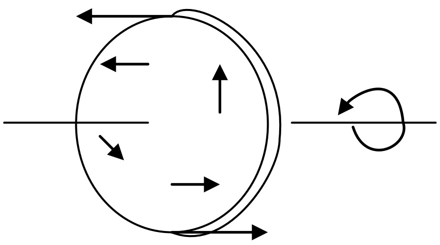


Fig. 3: A Rotating Body and the Corresponding Velocity Field

where x, y, z are the coordinates of any point p of B at the instant under consideration. If the coordinates are such that the z-axis of rotation and w points in the positive direction, then w = wk and

$$v = \begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = w(-yi + xj) = w[-y, x, 0]$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 3.

Example of Vector Field (Field of Force)

a. If the velocity at any point (x,y,z) within a moving fluid is known at a certain time, then a vector field is defined.

b. $v(x, y, z) = xyi - yz^2kj + x^2zk$ defines a vector field. A vector field which is independent of time is called a stationary steady-state vector field.

c. Let a particle A of mass M be fixed at a point p_0 and let a particle B of mass M to be free to take up various positions p in space. Then A attracts B. According to Newton's Law of gravitation, the corresponding gravitational force p is directed from p to p_0, and its magnitude is proportional to $1/r^2$ where r is the distance between p and p_0 say.

d. $$|p| = \frac{GM_A M_B}{r^2}$$

where G is the gravitational constant.

Hence p defines a vector field in space. If we introduce Cartesian coordinate such that p_0 has the coordinates x_0, y_0, z_0 and p has the coordinates x, y, z, then by Pythagoras theorem.

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \quad (2)$$

Introducing the vector assuming $r > 0$ then

$$r = (x - x_0)i + (y - y_0)j + (z - z_0)k \quad (3)$$

we have $|r| = r$ and $\left(-1/r\right)$ is a unit vector in the direction of p; the minus sign indicates that p is directed from $p + p_0$ .Fig. 2.
Hence substituting (1) into (3) we obtain

$$p = |p|\left(-\frac{1}{r}r\right) = -\frac{GM_A M_B}{r^3}r$$

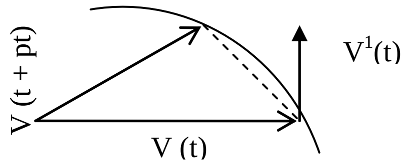
$$= -\frac{GM_A M_B}{r^3}\left[(x-x_0)i + (y-y_0)j + (z-z_0)k\right] \quad (4)$$

Hence, this vector function describes the gravitational force acting on B.

Derivative of a Vector Function

A vector function $v(t)$ is said to be differentiable at a point t if the limit exists. The vector is called the derivative of $v(t)$.

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$



Partial Derivatives of a Vector Function

The way of introducing partial derivation to vector analysis is obvious. Indeed, let the components of a vector function.

$v = v_1 i + v_2 j + v_3 k$ be differentiable functions of n variables $t_1, t_2, t_3 \cdots\cdots t_n$. Then the partial derivative of v with respect to t is denoted by $\frac{\partial v}{\partial t}$ and is defined as the vector function.

$$\frac{\partial v}{\partial t} = \frac{\partial v_1}{\partial t} i + \frac{\partial v_2}{\partial t} j + \frac{\partial v_3}{\partial t} k$$

Example 1

Let $r(t_1, t_2) = aCot_1 i + aS\,\mathrm{int}_1\, j + 3t_2 k$

$$\frac{\partial r}{\partial t_1} = -aS\,\mathrm{int}_1 + aCot_1 i,$$

$$\frac{\partial r}{\partial t_2} = 3k$$

3.2.2 Line Integrals

Definition: Let $f(x)$ be a single real valued function in the interval $a \le x \le b$. Thus, we can define line integral as

$$\int_a^b f(x)dx$$

3.2.3 Evaluation of Line Integral

Evaluation of line integral $\int_a^b f(x)dx$ can be accomplished by two methods. Thus:

a. A line integral of a vector function $F(r)$ over a curve c is defined by

$$\int_c F(r)dr = \int_a^b F(r(t)).\frac{dr}{dt}dt \qquad (1)$$

b. In term of components, with $dr = dxi_i + dy_j + dz_k$

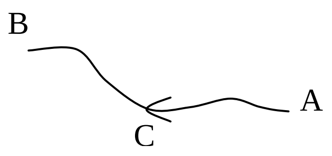
Then we obtain

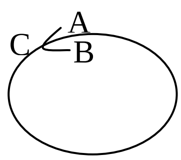
$$\int_c F(r)dr = \int_c (F_1dx + F_2dy + F_3dz)$$

$$= \int_c (F_1x' + F_2y' + F_3z')dt \qquad (2)$$

Where $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, $z' = \frac{dx}{dt}$ (3)

It is worth to mention that if the path of integration C in equation (1) above is a close curve that is





then.

Then instead of $\int_c$ we can also write $\oint_c$

3.2.4 General Properties of Line Integral

a. $\int_c kF.dr = k\int_c F.dr$ where k is a constant .

b. $\int_c (F+G)\cdot .dr = \int_c F\cdot .dr + \int_c G\cdot dr$

c. $\int_c F\cdot dr = \int_{c1} F\cdot .dr + \int_{c2} F\cdot .dr$

Where $c = c_1 + c_2$

3.2.5 Examples on Line Integrals

If $A = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int_c A..dr$ from (0,0,0) to (1,1,1) along the following parts C:

$x = t, y = t^2, z = t^3.$

The straight lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1).

The straight line joining (0,0,0) and (1,1,1).

Solution:

$$\int_c A..dr = \int_c [(3x^2 + 6y)i - 14yzj + 20xz^2k].(dxi + dyj + dzk)$$

$$= \int_c [(3x^2 + 6y)dx - 14yzdy + 20xz^2dz]$$

If $x = t, y = t^2, z = t^3$.,points (0,0,0) and (1,1,1) correspond to t=0 and t=1 respectively. Then

$$\int_c A..dr = \int_{t=0}^{t=1} (3t^2 + 6t^2)dt - 14(t^2)(t^3)d(t^2) + 20t(t^3)^2 d(t^3)$$

$$= \int_{t=0}^{t=1} (9t^2 - 28t^6 + 60t^9)dt$$

$$= [3t^3 - 4t^7 + 6t^{10}]_0^1 = 5$$

Along the straight line from (0,0,0) to (1,0,0) , y=0, z=0, dy=0 and dz=0 while x varies from 0 to 1. Then the integral over this point of the path is

$$\int_{x=0}^{t=1} (3x^2 + 6(0))dx - 14(0)(0)(0) + 20x(0)^2(0)$$

$$\int_{x=0}^{t=1} 3x^2 dx = [x^9]_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) , x=1, z=0, dx=0 while y varies from 0 to 1. Then the integral over this point of the path is

$$\int_{y=0}^{t=1} (3(1)^2 + 6(y))0 - 14y(0)dy + 20(1)(0)^2(0) = 0$$

Along the straight line from (1,1,0) to (1,1,1) , x=1, y=1, dx=0 , dy=0 while z varies from 0 to 1. Then the integral over this point of the path is

$$\int_{z=0}^{t=1}\left(3(1)^2+6(1)\right)0-14(1)z(0)dy+20(1)(z)^2\,dz$$

$$\int_{z=0}^{t=1}\left(20z^2\right)dz=\left[\frac{20t^3}{3}\right]_0^1=\frac{20}{3}$$

Adding $\int_c A..dr=1+0+\frac{20}{3}=\frac{23}{3}$

The straight line joining (0,0,0) and (1,1,1) is giving in parametric form by x=t, z=t. Then

$$\int_c A..dr=\int_{t=0}^{t=1}\left(3t^2+6t\right)dt-14(t)(t)d(t)+20(t)(t^2)dt$$

$$=\int_{t=0}^{t=1}\left(3t^2+6t-14t^2+20t^3\right)dt$$

$$=\int_{t=0}^{t=1}\left(6t-11t^2+20t^3\right)dt=\frac{13}{3}$$

3.3 Integral Theorem

3.3.1 Divergence Theorem of Gauss

For simplicity, divergence theorem of Gauss can be used to transform triple integral into surface integral over the boundary surface of a region in space. This is obvious because surface integral is simpler and easier to handle compared to triple integral.

Therefore, let T be closed bounded in a region space whose boundary is a piecewise smooth orient table surface S.

Let $f(x,y,z)$ be a vector function that is continuous and has continuous first partial derivative in some domain containing T. However, the transformation is done by the so called divergence theorem which involves the divergence of a vector function F.

Where divergence of F

$$\Rightarrow \qquad \operatorname{div} F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv = \iint_s F \cdot ndA \tag{2}$$

But
$$\iint_s F \cdot ndA = \iint_s \left(F_1 dydz + F_2 dxdz + F_3 dxdy\right) \tag{3}$$

Where $'n'$ is the outer unit normal vector of S.

but
$$F = F_1 i + F_2 j + F_3 k \tag{4}$$

and
$$n = Cos\alpha i + Cos\beta j + Cos\gamma k \tag{5}$$

where $\alpha, \beta,$ and γ are the angle between $'n'$ and the positive $x, y,$ and z axes respectively.

Next, we substitute equation (3) and (4) into (2) so we can obtain

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S \left(F_1 Cos\alpha i + F_2 Cos\beta j + F_3 Cos\gamma k\right) dA \tag{6}$$

But
$$Cos\alpha = dzdy, Cos\beta = dzdx, Cos\gamma = dxdy$$
$$\therefore \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) cdz = \iint_S F_1 dydz + F_2 dxdz + F_3 dydx \tag{7}$$

Example 2

Application of the Divergence Theorem

Harmonic Function

The theory of solution of Laplace gives thus:
$$\nabla^2 f = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 \tag{8}$$
and equation (1) is called **potential theory**.

Now, from the divergence theorem formula
$$\iiint_T divFdv = \iint_S f \cdot ndA \tag{9}$$

Where $F = \nabla f$ (10)

is gradient of scalar function.

$$divF = \nabla^2 f \quad (11)$$

and $F \cdot n = n \cdot \nabla f$

Hence,

$$\iiint_T \nabla^2 f dv = \iint_S \frac{\partial f}{\partial n} dA \quad (12)$$

Where

$$n \cdot gradf = \frac{\partial f}{\partial n} dA \quad (13)$$

we denote the directional derivative of f in the outer normal direction of S by $\frac{\partial f}{\partial n}$

However,

$$f \cdot n \equiv n \cdot \nabla f \equiv \frac{\partial f}{\partial n} \cdot dA \quad (14)$$

3.3.2 Green's Theorem

This theorem gives the relation between the integral over the boundary surface which encloses the volume. If F_1, F_2, F_3 are three functions of x, y, z and their derivatives $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ are continuous and single valued functions in a region V bounded by a closed surface S, then

$$\iint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \iint_S \left(PCos\alpha + QCos\beta + RCos\gamma \right) dA$$

As in (6) above

Where $Cos\alpha$, $Cos\beta$ and $Cos\gamma$ are the direction cosines normal to the surface S.

Example 3

Evaluate the surface integral

$$I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy \right)$$

where is the surface bounded by $z = 0, z = b, x^2 + y^2 = a^2$.

Solution

Using Green's theorem

$$I = \iiint_V \left(3x^2 + x^2 + x^2\right) dxdydz$$

$$= 4\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\int_0^b dz \right) dy \right] 5x^2 dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} (b) dy \right] 5x^2 dx$$

$$= 20b\int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

Substituting $x = aSin\theta$ or $x = aCos\theta$ we have $dx = aCos\theta d\theta$

$$= 20b\int_0^a \left(a^2 Sin^2\theta \sqrt{a^2 - a^2 Sin^2\theta}\right) Cos\theta d\theta$$

$$= 20a^4 b\int_0^a \left(Sin^2\theta \sqrt{1 - Sin^2\theta}\right) Cos\theta d\theta$$

but $\sqrt{1 - Sin^2\theta} = \sqrt{Cos^2\theta} = Cos\theta$

$$I = 20ba^4 \int_0^a \left(Sin^2\theta - Cos^2\theta\right) d\theta$$

$$= -20ba^4 \int_0^a \cos 2\theta d\theta$$

$$= 20ba^4 \left[\frac{\pi}{16}\right]$$

$$= \frac{5}{4}\pi a^4 b$$

3.3.3 Stoke's Theorem

This is the transformation between surface integrals and line integrals. Stoke's theorem involves the curl.

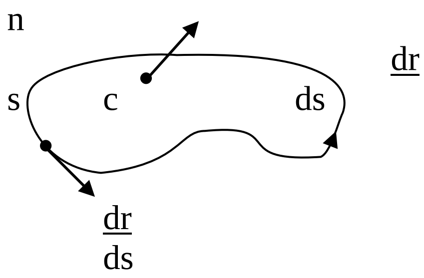
$$\text{Curl}\, F = \Delta x F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad (1)$$

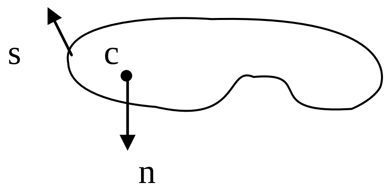
Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple close curve C.

Let $F(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\Delta x F)\cdot n dA = \oint_C F \cdot \frac{dr}{ds} \qquad (2)$$

where n is a unit normal vector of S and, also $\frac{dr}{ds}$ is the unit tangent vector and S the arc length of C.





$$\therefore \quad \iint_R \left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)N_1+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)N_2+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)N_3\right]dudv$$
$$=\oint_C \left(F_1dx+F_2dy+F_3dz\right) \qquad (3)$$

3.3.4 Green's Theorem in the Plane as a Special Case of Stoke's Theorem

Let $F=F_1i+F_2j+F_3k$ be a vector function that is continuously differentiable in a domain in the $x-y$ plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple close curve.

Then from equation (1)

$$\left(\Delta xF\right)\cdot n=\left(\Delta xF\right)\cdot k=\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}$$

Then the formula in Stoke's theorem now takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\right)dA=\oint_C \left(F_1dx+F_2dy\right)$$

Hence, Green's theorem in space is s special case of Stoke's theorem.

Example 4

Evaluation of line integral by Stoke's theorem.

Evaluate $\int_C \left(F\cdot\frac{dr}{ds}\right)ds,$ where C is the circle $x^2+y^2=4, \quad z=-3,$ oriented counterclockwise as seen by a person standing at the origin, and with respect to right-handed Cartesian coordinates $F=yi+xz^3j-zy^3k.$

Solution

As a surface S bounded by C we can take the plane circular disc $x^2 + y^2 = 4$ in the plane $z = -3$. Then n in Stoke's theorem points in the positive z-direction; thus n=k,. Hence $(\Delta xF) \cdot n$ is simply the component of curl (ΔxF) in the positive z-direction. Since F with $z = -3$ has the components $F_1 = y, F_2 = -27x$ and $F_3 = 3y^3$, we thus obtain

$$(\Delta xF) \cdot n = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} = 17 - 1 = 128$$

Hence, the integral over S in Stoke's theorem equals 128times the area 4π of the disk S.

$$\therefore \quad \begin{aligned} &[(\Delta xF) \cdot n]4\pi = -28 \cdot 4\pi = -112\pi \\ &= -352 \end{aligned}$$

4.0 CONCLUSION

In conclusion, the students must have understood vector field theory and also be able to relate vector field and vector function together respectively.

However, the Line Integral, Gauss's, Stoke's, and Green's theorem were discussed using the knowledge acquired from vector field theory.

5.0 SUMMARY

In summary, double integrals over a region in the plane can be transformed into line integrals over the boundary C of R by Green's theorem in the plane using

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

Also Triple integrals taken over a region T in space can be transformed into surface integrals over the boundary surface S of T by the divergence theorem of Gauss using,

$$\iiint_T divFdv = \iint_S f \cdot ndA$$

where n is the outer unit normal vector to S which implies Green's formulas.

Likewise, surface integrals over a surface with boundary curve c can be transformed into line integrals over C by Stokes's theorem.

$$\iint_S (\Delta nF) \cdot ndA = \oint_C \left(F \cdot \frac{dr}{ds} \right) ds$$

6.0 TUTOR-MARKED ASSIGNMENT

i. Compute $\int_c F(r) \cdot dr$ where
   (a) $F = y^2 i - x^4 j, \quad c : r = ti + t^{-1},$ for $1 \le t \le 3$
   (b) $F = x^2 i - y^2 j, \quad c : y = 1 - x^2,$ for $-1 \le x \le 1$

ii. Find the work done by the force $F = xi - zj + 2yk$ in the displacement;
   (a) Along the y axis from 0 to 1
   (a) Along the curve $z = y^4$, $x = 1$, from (1,0,1) to (1,1,1).

iii. Evaluate $\int_c (x^2 + y^2) \cdot ds$
   (a) Over the path $y = 2x$ from (0,0) to (1,2)
   (a) Over the path $y = -x$ from (1,-1) to (2,-2)

iv. Evaluate the relations between vector fields and vector functions.

v. State one example of a rotating body and the corresponding velocity field.

vi. Let the components of a vector function $r(t_1, t_2) = aCost_1 i + aS\text{int}_1\, j + 3t_2 k$ be differentiable functions on variables t_1 and t_2. Then find the partial derivatives of $r(t_1, t_2)$ with respect to t_1 and t_2 denoted by $\frac{\partial r}{\partial t_1}$ and $\frac{\partial r}{\partial t_2}$.

vii. Evaluate the surface integral
$$I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy \right)$$
where is the surface bounded by $z = 0, z = b, x^2 + y^2 = a^2$

viii. State and prove Stoke's theorem.

xiv. Evaluate $\int_C \left(F \cdot \frac{dr}{ds} \right) ds,$ where C is the circle $x^2 + y^2 = 4$, $z = -3,$ oriented counterclockwise as seen by a person standing at the origin, and with respect to right-handed Cartesian coordinates $F = yi + xz^3 j - zy^3 k.$

x. Show that vector function $F = (x^2 + yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational. Find the scalar potential

xi. Verify divergence theorem for the function
$F = 4xzi - y^3 j + yz$

over the unit cube $x=0, x=1, y=1$ and $z=0$ and $z=1$.

xii. Prove that div $(\underline{u}\times\underline{v})=\underline{v}\cdot Curl\underline{u}-uCurl\underline{v}$

xiii. Evaluate $\int_L \Phi\cdot dr$, where $\Phi = xyi+yzj+zxk$ and curve L

$$r = ti + t^2 j + t^3 k \text{ where } -1\le t\le 1.$$

7.0 REFERENCES/FURTHER READING

Hernandez, V., J. E. Roman, and V. Vidal. "SLEPC: A Scalable and Flexible Toolkit for the Solution of Eigenvalue Problems." *ACM Trans. Math. Soft.* 31, no. 3 (2005).

McLachlan, R. I. "Families of High-Order Composition Methods." *Numerical Algorithms* 31 (2002).

McLachlan, R. I. and G. R. W. Quispel. "Splitting Methods." *Acta Numerica* 11 (2002).

Mazzia. F. and C. Magherini, "Test Set for Initial Value Problem Solvers-Release 2.4." *Dept. of Mathematics, University of Vari and INdAM, Research Unit of Bari.* 2008.

Moler, C. B. *Numerical Computing with MATLAB*. SIAM, (2004).

Murray, R. Spiegel *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc. (1974).

Nedialkov, N. and J. Pryce "Solving Differential Algebraic Equations by Taylor Series (I): Computing Taylor Coefficients." *BIT* 45, no.3 (2005)

Olsson, H. and G. Söderlind. "The Approximate Runge–Kutta Computational Process." *BIT* 40, no. 2 (2000).

Quarteroni, A., R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer-Verlag, (2000).

Ramana B. V *Higher Engineering Mathematics*. New Delhi: Tata McGraw-Hill Publishing Company Limited. (2008).

Rubinstein, B. "Numerical Solution of Linear Boundary Value Problems." *Mathematica MathSource* package, http://library.wolfram.com/database/MathSource/2127.

Shampine, L. F. "Solving in Matlab." *Journal of Numerical Mathematics* 10, no. 4 (2002).

Shampine, L. F. and S. Thompson. "Solving Delay Differential Equations with dde23." Available electronically from http://www.runet.edu/~thompson/webddes/tutorial.pdf.

Shampine, L. F. and S. Thompson. "Solving DDEs in MATLAB." *Appl. Numer. Math.* 37 (2001).

Shampine, L. F., I. Gladwell, and S. Thompson. *Solving ODEs with MATLAB*. Cambridge University Press, 2003.

Sofroniou, M. and G. Spaletta. "Increment Formulations for Rounding Error Reduction in the Numerical Solution of Structured Differential Systems." *Future Generation Computer Systems* 19, no. 3 (2003).

Sofroniou, M. and G. Spaletta. "Construction of Explicit Runge–Kutta Pairs with Stiffness Detection." *Mathematical and Computer Modelling*, special issue on The Numerical Analysis of Ordinary Differential Equations, 40, no. 11–12 (2004).

Sofroniou, M. and G. Spaletta. "Derivation of Symmetric Composition Constants for Symmetric Integrators." *Optimization Methods and Software* 20, no. 4–5 (2005).

Sofroniou, M. and G. Spaletta. "Hybrid Solvers for Splitting and Composition Methods." *J. Comp. Appl. Math.*, special issue from the International Workshop on the Technological Aspects of Mathematics, 185, no. 2 (2006).

Stephenor, G.. *Mathematical Methods for Science Students*. London: Longman, Group Limited. (1977)

Stroud, K.A... *Engineering Maths*. 5th Edition Palgraw. (1995)

Sprott, J. C. "A Simple Chaotic Delay Differential Equation." *Phys. Lett. A.* (2007).

Stewart, G. W. "A Krylov-Schur Algorithm for Large Eigenproblems." *SIAM J. Matrix Anal. Appl.* 23, no. 3, (2001).

Tang, X. H. and X. Zou. "Global Attractivity in a Predator-Prey System with Pure Delays." *Proc. Edinburgh Math. Soc.* (2008).

Unger, J., A. Kroner, and W. Marquardt. "Structural Analysis of Differential Algebraic Equation Systems—Theory and Applications." *Computers Chem. Engg.* 19, no. 8 (1995).

Verma, P. D. S.. *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd. (1995)

Verner, J. H. "Numerically Optimal Runge-Kutta Pairs with Interpolants." *Numerical. Algorithms* (2010).

Zennaro, M. "The Numerical Solution of Delay Differential Equations." Lecture notes, Dobbiaco Summer School on Delay Differential Equations and Applications, (2006).