MODULE 3

UNIT 1 RESIDUE INTEGRATION METHOD

CONTENTS

1.0 INTRODUCTION

Since there are various methods of determining the coefficients of a Laurent series, without using the integral formulas. We intend (may) use the formula for b_1 for evaluating complex integrals in a very elegant and simple fashion. b_1 will be called the residue or $f(z)$ at $z = z_0$. The powerful method may also be applied for evaluation certain real integrals, as we shall see in section 3.3 and 3.4 of module 3 and unit 1.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine and explain Residue;
- use Residue to evaluate integrals; and
- show that the Residue integration method can be extended to the case of several singular points of $f(z)$ inside C.

3.0 MAIN CONTENT

3.1 Residues

Let us first explain what a residue is and how it can be used for evaluating Integrals

$$\oint_C f(z)dz.$$

There will be counter integral taken around a simple closed path C.
If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem and we are done.

If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots\cdots$$

That converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z = z_0| < R$. Now comes the key idea. The coefficient b_1 of the first negative power $\frac{1}{(z = z_0)}$ of this Laurent series is given by the integral formula, with n=1, that is,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz,$$

Since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients, we can find b_1 by one of these methods and then use the formula for b_1 for evaluating the integral:

1. $$\boxed{\oint_C f(z)dz = 2\pi i b_1.}$$

   Here we integrate in the counterclockwise sense around the simple closed path that contains $z = z_0$ in its interior.

   The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we shall denote it by

2. $$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)}$$

Example1

Evaluation of an Integral by Means of a Residue

Integrate the function $f(z) = z^{-4}$ around the unit circle C in the counterclockwise sense.

Solution

We obtain the Laurent series thus:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Which converges for $[z[> 0$ (that is for all $z \neq 0).)$ This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue of $f(z)$ at $z = 0$ is $b_1 = \frac{1}{3}!$.

From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Example 2

Use Laurent Series to Integrate $f(z) = 1/(z^3 - z^4)$ around the circle C: $|z| = 1/2$ in the clockwise sense.

Solution

$z^3 - z^4 = z^3(1-z)$ Shows $f(z)$ that $z = 0$ and $z = 1$. Now $z = 1$ lies outside C.

Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$ that

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad 0 < |z| < 1$$

We see it from this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i$$

Caution! Had we use the wrong series (II) say:

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots\cdots \qquad (|z| < 1),$$

We would have obtained the wrong answer 0. Explain!

3.1.1 Two Formulas for Residues at Simple Poles

Before we continue the integration, we ask the following question: To get a residue, a single coefficient of a Laurent series, must we divide the whole series or is there a more economical way? For poles, there is. We shall derive, once and for all, some formulas for residues at poles, so that in this case we no longer need the whole series.

Let $f(z)$ have a simple pole at $z = z_0$

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad 0 < |z - z_0| < R$$

Here $b_1 = 0$ (why?) Multiply both sides by $z - z_0$ we have

$$(z - z_0) f(z) = b_1 + (z - z_0)\left[a^0 + a_1(z - z_0) + \cdots\cdots\right]$$

We now let $z \to z_0$. The right hand side approaches b_1. This gives

$$\operatorname*{Re s}_{z=z_0} f(z) = b_1 = \lim_{z=z_0}(z - z_0) f(z) \tag{3}$$

Example 3

Residue at a Simple Pole

$$\operatorname*{Re s}_{z=z_0} \frac{9i+1}{z(z^2+1)} = \lim_{z=i}(z-i)\frac{9i+1}{z(z+i)} = \left[\frac{9z+1}{z(z+i)}\right]_{z=1}$$
$$= \frac{10i}{-2} = -5i$$

Another, sometimes simpler formula for the residue at a simple pole is obtained by starting from

$$f(z) = \frac{p(z)}{q(z)}$$

with analytic $p(z)$ and $q(z)$ where we assume that $p(z_0) \neq 0$ and $q(z)$ has a simple zero at $z - z_0$ (so that $f(z)$ has a simple pole at $z - z_0$ ad wanted. By the definition of a simple zero, $q(z)$ has a Taylor series of the form

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots$$

This we substitute into $f = p/q$ and then f into (3), finding

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z=z_0}(z - i)\frac{p(z)}{q(z)} = \lim_{z=z_0}\left[\frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''(z_0)/2 + \cdots]}\right]_{z=1}$$

We now see that on the right, a factor $z - z_0$ is cancelled and resulting denominator has the limit $q'(z_0)$. Hence our second formula for the residue at a pole is

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q(z_0)}. \qquad (4)$$

Example 4

Residue at a Simple Pole Calculated by Formula (4)

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} == \left[\frac{9z + i}{3z^2 + 1)}\right]_{z=i} = \frac{10i}{-2} = -5i$$

Example 5

Another Application of Formula (4)

$$f(z) = \frac{\cos \pi z}{z^4 - 1}.$$

Solution

$p(z) = \cos \pi z$ is entire, and $q(z) = z^4 - 1$ has a simple zero at1,i,-1,-i. Hence $f(z)$ has a simple pole at these points (and no further poles).

Since $q'(z) = 4z^3$, we see from (4) that the residue equal the value for $\left(\frac{\cosh \pi i}{4z^3}\right)$ at those points, that is,

$$\frac{\cosh \pi}{4} \approx 2.8980, \quad \frac{\cosh \pi i}{4i^3} = \frac{\cos \pi}{-4i} = -\frac{i}{4}, \quad -\frac{\cosh \pi}{4}, \quad \frac{\cosh(-\pi i)}{4(-i)^3} = \frac{i}{4}.$$

3.1.2 Two Formulas for Residues at Simple Poles

Let $f(z)$ be analytic function that has pole of any order m>1 at a point $z = z_0$.Then, by the definition of such pole, the Laurent series of $f(z)$ converging near $z = z_0$ (except $z = z_0$) is

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

where $b_m \neq 0$. Multiplying both sides by $(z-z_0)^m$, we have

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1}$$
$$+ a_0(z-z_0)^m + +a_1(z-z_0)^{m+1} + \cdots.$$

We see that the residue b_1 of $f(z)$ at $z = z_0$ is now the coefficient of the power $(z-z_0)^{m-1}$ in the Taylor series of the function

$$g(s) = (z-z_0)^m f(z)$$

On the left, with center $z = z_0$.Thus by Taylor's theorem,

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$$

Hence if $f(z)$ has a pole of mth order at $z = z_0$, the residue is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)} \lim_{z=z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}. \qquad (5)$$

In particular, for a second-order pole (m=2),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z=z_0} \left\{ \left[(z-z_0)2 f(z)' \right] \right\}.$$

Example 6

Residue at a Pole of Higher Order

The function

$$f(z) = \frac{50z}{(z+4)(z-1^2)}$$

has a pole pole of second order at $z = 1$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z) \right] = \lim_{z\to 1} \frac{d}{dz}\left(\frac{50z}{z+4} \right) = 8$$

Example 7

Residue from a Partial Fraction

If $f(z)$ is rational, we can also determine its residue from partial fractions. In Example 6,

$$f(z) = \frac{50z}{(z+4)(z-1)^2} = \frac{-8}{z+4} + \frac{8}{z-1} + \frac{10}{(z-1)^2}.$$

This shows that the residue t $z = 1$ is 8 (as before),and at z=-4 (simple pole) it is -8. Why is this so? Consider $z = 1$. There the Laurent has two fractions as its principal part and the first fraction as the sum of the other part. This first fraction is analytic at $z = 1$, so that it has a Taylor series with centre $z = 1$, as it should be. Similarly, at z=4 the first fraction is the principal part of the Laurent series.

Example 8

Integration around a Second-order Pole

Counterclockwise integration around any simple closed path C such that $z = 1$ is inside C and z=4 outside C yields

$$\oint_C \frac{z}{(z+4)(z-1)^2} dz = \operatorname*{Res}_{z=1} 2\pi i \frac{z}{(z+4)(z-1)^2} = 2\pi i \frac{8}{50} \approx 1.0053i$$

So far we can evaluate integrals of analytic functions $f(z)$ over closed curve C when $f(z)$ has only *one* singular point inside C**.** In the next section we show that the residue integration method can be readily extended to the case of several singular points of $f(z)$ inside C**.**

3.2 Residue Theorem

So far we are in a position to evaluate contour integrals whose integrands have only a single isolated singularity inside the contour of integration. We shall now see that our simple method may be extended to the case when the integrand has several isolated singularity inside the contour. This extension is surprisingly simple, as follows

Residue Theorem

Let $f(z)$ be a function that is analytic inside a simple closed path C and on C, except for finitely many singular point $z_1, z_2, \cdots, z_k$ inside C. Then

$$\oint_C f(z) = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=1} f(z), \qquad (1)$$

The integral being taken in the clockwise sense around the path C

Proof: We enclose each of the singular points z_j in a circle C_j with radius small enough that k circles and C are all separated (fig. 43). Then

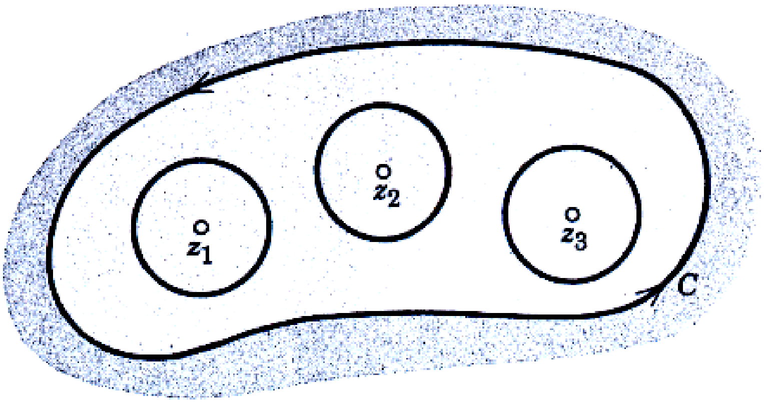

Fig. 43: Residue Theorem

$f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1 \cdots\cdots C_n$ and on the entire boundary of D. From the Cauchy's integral theorem we have

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots\cdots + \oint_{C_k} f(z)dz = 0 \qquad (2)$$

the integral along C being taken in the counterclockwise sense and the other integrals in the clockwise sense. We now reverse the sense of integration along $C_1 \cdots\cdots C_n$. Then the signs of the values of these integrals change, and we obtain from (2)

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots\cdots + \oint_{C_k} f(z)dz \qquad (3)$$

All these integrals are now taken in the clockwise sense. By (1) in the previous section

$$\oint_{C_j} f(z)dz = \operatorname*{Res}_{z=z_j} f(z),$$

So that (3) yields (1), and the theorem is proved.

This important theorem has various applications with complex and real integrals. We shall first consider some complex integrals.

Example 9

Integration by the Residue Theorem

Evaluate the following integral counterclockwise around any simple close path such that:

a. 0 and 1 are inside C
b. 0 is inside, 1outside,
c. 1 is inside, 0 outside,
d. 0and1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}$$

Solution

The integrand has simple poles at 0 and 1, with residues

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

Confirm this by (4) Ans.(a). ($2\pi i(-4+1) = -6\pi i,$) (b). $-8\pi i$ (c). $2\pi i$ (d). 0

Example 10

Integration by the Residue Theorem

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + z^{e^{\pi/2}}\right) dz$$

Solution

Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues (note: $e^{2\pi i} = 1$)

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16}, \quad \operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16},$$

and simple poles at ± 2 which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1+\frac{\pi}{z}+\frac{\pi^2}{2!z^2}+\frac{\pi^3}{3!z^3}+\cdots\right) = z+\pi+\frac{\pi^2}{z}\cdot\frac{1}{z}+\cdots.$$

Ans. $2\pi i(-6/-1/6+\pi^2/2) = \pi(\pi^2-1/4)i = 30.221i.$ by the residue theorem.

Example 10

Confirmation of an Earlier Result

Integrate $\dfrac{1}{(z-z_0)^m}$ (m a positive integer) in the clockwise sense around and simple close path C enclosing point $z = z_0$.

Solution

$\dfrac{1}{(z-z_0)^m}$ in its own Laurent series with centre $z = z_0$ consisting of this one- term principal path, and

$$\operatorname*{Res}_{z=z_0} \frac{1}{z-z_0} = 1, \qquad \operatorname*{Res}_{z=z_0} \frac{1}{(z-z_0)^m} = 0 \qquad (\mathrm{m} = 2,3\cdots\cdot).$$

In agreement with Example (2), we thus obtain

$$\oint_C \frac{dz}{(z-z_0)^m} = \begin{cases} 2\pi i & \text{if m} = 1 \\ 0 & \text{if m} = 2,3,\cdots \end{cases}$$

It should be very surprising to hear that our present *complex* integration method can be used for evaluating **real integrals** (incidentally, some of them difficult to evaluate by other methods). In the next section we discuss two methods for accomplishing this goal.

3.3 Evaluation of Real Integral

We want to show that residue theorem also yields a very elegant and simple method for evaluating certain classes of complicated real integrals.

Integrals of Rational fractions of $Cos\theta$ and $Sin\theta$

We first consider integrals of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \qquad (1)$$

where $F(\cos\theta, \sin\theta)$ is a real rational fraction of $Cos\theta$ and $Sin\theta$ [for example, $(\sin^2\theta)/(5-4\cos\theta)$ and is finite on the interval of integration. Setting $e^{i\theta} = z,$ we obtain

$$(2) \qquad \boxed{\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{2}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{2}\right) \end{aligned}}$$

and we see that the integrand becomes a rational function of z, say, $f(z)$.

As θ ranges from0 to 2π, the variable z ranges once around the unit circle $|z| = 1$in the counterclockwise sense. Since we have $d\theta = dz/iz$, and the given integral takes the form

$$I = \oint_C f(z)\frac{dz}{iz}, \qquad (3)$$

The integration being taken counterclockwise around the unit circle.

Example 11

An Integral of the Type (1)

Show by the present method that

$$0\int^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

Solution

We use $\cos\theta = (z + 1/z)$ and $d\theta = \dfrac{dz}{iz}$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2}+\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_C \frac{dz}{c+\frac{i}{2}(z^2+2\sqrt{2z}+1)} = \frac{2}{i}\oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

We see that the integrand has two simple poles, one at $z_1 = \sqrt{2}+1,$ which lies outside the unit circle. C: $|z| = 1$ and is thus of no interest, and the other at $z_2 = \sqrt{2-1}$ inside C, where the residue is

$$\operatorname*{Re s}_{z=z2} \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = \left[\frac{1}{z-\sqrt{2}-1}\right]_{z=\sqrt{2}-1} = -\frac{1}{2}.$$

Together with the factor $-2/i$ in front of the integral this yields the desired result $2\pi i(-2/i)(-1/2) = 2\pi$

3.3.1 Improper Integrals of Rational Function

We now consider the real integral of the type

$$\int_{-\infty}^{\infty} f(x)dx \qquad (4)$$

Such an integral, for which the interval of integration is not finite, is called an **improper integral**, and it has the meaning

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to-\infty}\int_0^b f(x)dx. \qquad (5a)$$

If both limit exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \qquad (5b)$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of denominator. Then the limit in (5a) exists, and we may start from (5b). We may consider the corresponding contour integral

$$\oint_C f(z)dz \qquad (5c)$$

Around a path C on the diagram below. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper-half plane, and if we choose R large enough, then

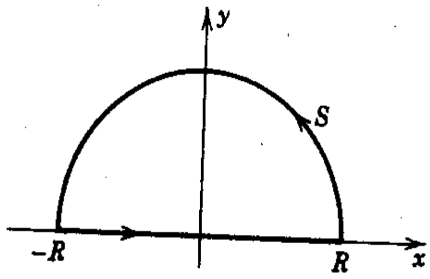

Fig. 44: Path C of the Contour Integral in (5*)

C encloses all these poles .By the residue theorem we then obtain

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

When the sun consists of all the residues, of $f(z)$ at the point in the upper half-plane at which $f(z)$ has a pole. From this we have

(6) $$\int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

We prove that $R \to \infty$,the value of the integral over the semicircle S approaches zero. If we set $z = R\,e^{i\theta}$, then S is represented by $R = const,$ and as z ranges along S, the variable θ ranges from0 to π.Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0. By the ML-inequality

$$\left|\int_S f(z)dz\right| < \frac{k}{R^2}\pi R = \frac{k\pi}{R} \qquad (R > R_0)$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

(7) $$\boxed{\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)}$$

the sum being extended over the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper half-plane.

Example 12

An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Solution

Indeed, $f(z) = \dfrac{1}{(1+z^4)}$ has four simple poles at the points

$$z_1 e^{\pi i/4}, \quad z_2 e^{3\pi i/4}, \qquad z_3 e^{-3\pi i/4}, \qquad z_4 e^{-\pi i/4}$$

The first two of these poles lie in the upper-half plane. We find

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4} e - ^{3\pi i/4},$$
$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4} e - ^{9\pi i/4}$$

By (1) and (7), in the current section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{4}(-e^{\pi i/4} + e^{-\pi i/4}) = \pi \sin\frac{\pi}{4} = \frac{\pi}{2\sqrt{2}}.$$

Since $1/(1 + x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

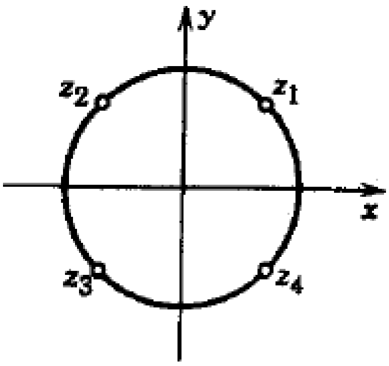

Fig. 45: Example 2

Example 13

Another Improper Integral

Using (7) show that

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{x^4 + 5x^2 + 4} dx = \frac{\pi}{6}.$$

Solution

The degree of denominator is two units higher than that of the numerator, so that our method again applies. Now

$$f(z) = \frac{p(z)}{q(z)} = \frac{z^2 - 1}{z^4 + 5z^2 + 4} = \frac{z^2 - 1}{(z^2 + 4)(z^2 + 1)}$$

has simple poles at $2i$ and i in the upper-plane (and at $-2i$ and $-i$ in the lower half-plane, which are of no interest here). We calculate the residues from (4), noting that $q'(z) = 4z^3 + 10z$,

$$\operatorname*{Res}_{z=2i} f(z) = \left[\frac{z^2 - 1}{4z^3 + 10z}\right]_{z=2i} = \frac{5}{12i}, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{z^2 - 1}{4z^3 + 10z}\right]_{z=i} = \frac{-2}{6i}$$

$Ans. 2\pi i(5/12i - 1/3i) = \frac{\pi}{6}$, as asserted.

Looking back, we realise that the key ideas of our present methods were these. In the first method we mapped the interval of integration on the real axis onto a closed curved in the complex plane (the unit circle). In the second method we attached to an interval on the real axis a semi circle such that we got a closed curve in the complex plane, which we then "blew up." This second method can be applied to further types of integrals, as we show in the next section, the last in the chapter.

3.4 Further Types of Real Integrals

There are further classes of integrals that can be evaluated by applying the residue theorem to suitable complex integrals. In application such integral may arise in connection with integral transformations or representation of special functions. In the present section we shall consider two such classes of integrals. One of them is important in the problems involving the Fourier integral representation. The other class consists of real integral whose integrand is finite at some point in the interval of integration.

3.4.1 Fourier Integral

Real integral of the form

1. $\int_{-\infty}^{\infty} f(x) = \cos sxdx$ and $\int_{-\infty}^{\infty} f(x) = \sin sxdx$ (s real)

occur in connection with the Fourier integral.

If $f(x)$ is a rational function satisfying the assumptions on the degree stated in connection with (4), then the integral (1) may be evaluated in a similar to that used for the integral in (4) of the previous section. In fact, we may then consider the corresponding integral

$$\oint_C f(z)e^{isz}dz \qquad (s \text{ real and positive})$$

Over the contour C in sec 3.3 instead of (7), sec. 3.3, we get

$$\int_{-\infty}^{\infty} f(z)e^{isz}dz = 2\pi i \sum \operatorname{Re} s\left[f(z)e^{isz}\right] \qquad (s > 0) \qquad (2)$$

where the sum consists of the residue of $f(z)e^{isz}$ as its pole in the upper half-plane. Equating the and imaginary parts on both sides of (2), we have

$$\boxed{\begin{aligned}&\int_{-\infty}^{\infty} f(x)\cos sxdx = -2\pi i \sum \operatorname{Im}\operatorname{Re} s\left[f(z)e^{isz}\right],\\ &\int_{-\infty}^{\infty} f(x)\sin sxdx = 2\pi i \sum \operatorname{Re}\operatorname{Re} s\left[f(z)e^{isz}\right]\end{aligned}} \quad (s > 0)(3)$$

We remember that (7), was established by proving that the value of the integral over the semicircle S in fig. approaches zero as $R \to \infty$.

To establish (2) we should now prove the same fact for our present contour integral. This can be done as follows, Since S lies in the upper half-plane $y \geq 0$ and $s > 0$, we see that

$$\left|e^{isz}\right| = \left|e^{isx}\right|\left|e^{-isy}\right| = e^{-sy} \leq 1 \qquad (s > 0, \quad y \geq 0)$$

From this obtain the inequality

$$\left|f(z)e^{isz}\right| =) = \left|f(z\right|\left|e^{isz}\right| \leq \left|f(z\right| \qquad (s > 0, \quad y \geq 0)$$

which reduces our present problem to that in previous section.

Continuing as before, we see that the value of the integral under consideration approaches zero as R approaches infinity. This establishes (2), which implies (3).

Example 14

An Application of (3)

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2} dx = \frac{\pi}{k} e^{ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2} dx = 0 \qquad (s > 0, \ \mathrm{k} > 0)$$

Solution

In fact, $\dfrac{e^{isz}}{k^2 + x^2}$ has only one pole in the upper plane, namely, a simple pole at $z = ik$, and from (4) we obtain

$$\operatorname*{Re s}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \left[\frac{e^{-ks}}{2ik}\right].$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{isz}}{k^2 + z^2} dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{ks}.$$

Since $e^{isx} = \cos sx + i \sin sx,$ this yields the above results

3.4.2 Types of Real Improper Integrals

Another kind of improper integral is a definite integral

$$\int_A^B f(x) dx \tag{4}$$

whose integral becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty$$

Then the integral (4) means

$$\int_A^B f(x) dx = \lim_{\tau \to a} \int_A^{a-\tau} f(x) dx + \lim_{\eta \to 0} \int_{a+\eta}^B f(x) dx \tag{5}$$

where τ and η approaches zero independently and through positive values. It may happen that neither of these limits exists, if $\tau, \eta \to 0$ independently,

but

$$\lim_{\tau \to 0}\left[\int_{A}^{a-\tau} f(x)dx + \int_{a+\eta}^{B} f(x)dx\right] \tag{6}$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pv.v.} \int_A^B f(x)dx.$$

For example,

$$\text{pv.v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\tau \to 0}\left[\int_{-1}^{-\tau} \frac{dx}{x^3} + \int_{\tau}^{1} \frac{dx}{x^3}\right] = 0$$

the principal value exists although the integral itself has no meaning. The whole situation is quite similar to that discussed in the second part of the previous section.

To evaluate improper integral whose integrands have poles on the real axis, we use a part that avoids these singularities by following small semi-circles at the singular points; the procedure may be illustrated by the following example.

Example 15

An Application

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(This is the limit of sine integral Si(x) as $x \to \infty$)

Solution

a. We do not consider $\frac{(\sin z)}{z}$ because this function does not behave suitably at infinity. We consider $\frac{e^{iz}}{z}$, which has a simple pole at z=0, and integrate around the contour in figure below. Since $\frac{e^{iz}}{z}$ is analytic inside and on C Cauchy's integral theorem gives

$$\oint_C \frac{e^{iz}}{z} dz = 0 \tag{7}$$

b. We prove that the value of the integral over the large semicircle C_1 approaches R as approaches infinity. Setting $z = R\,e^{i\theta}.dz = iR\,e^{i\theta}d\theta, \frac{dz}{z = id\theta}$ and therefore

$$\left|\int_C \frac{e^{iz}}{z} dz\right| = \left|\int_0^{\pi} e^{iz} id\theta\right| \le \int_0^{\pi} \left|e^{iz}\right| d\theta \qquad (z = R\,e^{i\theta})$$

In the integrant on the right,

$$\left|e^{iz}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right| = e^{-R\sin\theta}.$$

We insert this, $\sin(\pi - \theta) = \sin\theta$ to get an integral from 0 to $\pi/2$, and then $\varpi \ge 2\theta/\pi$ (when $0 \le \theta \le \pi/2$); to get an integral that we can evaluate:

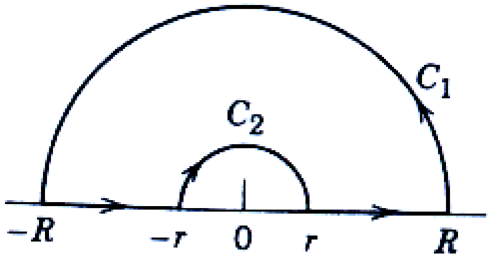

Fig. 46: Contour in Example 2

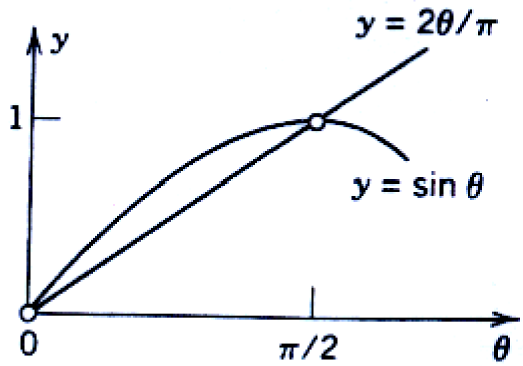

Fig. 47: Inequality in Example 2

$$\int_0^{\pi} \left|e^{iz}\right| d\theta = \int_0^{\pi} e^{-R\sin\theta} d\theta = \int_0^{\pi/2} e^{-R\sin\theta d\theta} d\theta$$

$$< 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R}(1 - e^{-R}) \to 0 \qquad \text{as} \quad \mathrm{R} \to \infty C_1$$

Hence the value of the integral over C_1 approaches as $\mathrm{R} \to \infty$

c. For the integral over small semicircle C_2 in figure above , we have

$$\int_{C_2} \frac{e^{iz}}{z} dz = \int_{C_2} \frac{dz}{z} + \int_{C_2} \frac{e^{iz} - 1}{z} dz$$

The first integral on the right equals $-\pi i$. The integral of the second integral is analytic and thus bounded, say, less than some constant M in absolute value for all z on C_2 and between C_2 and the x-axis. Hence by the $ML-$inequality, the absolute value of this integral cannot exceed $M\pi r$. This approaches $r \to 0$. Because of part (b), from (7) we thus obtain

$$\int_{C_2} \frac{e^{iz}}{z} dz = \mathrm{pv.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + \lim_{r \to 0} \int_{C_2} \frac{e^{iz}}{z} dz$$
$$= \mathrm{pv.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i = 0$$

Hence this principal value equals πi; its real part is 0 and its imaginary part is

$$\mathrm{pv.v.}\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \qquad (8)$$

d. Now the integrand in (8) is not singular at $x = 0$. Furthermore, Since for positive x the function $1/x$ decreases, the area under the curve of the integrand between two consecutive positive zeros decreases in a monotone fashion, that is, the absolute value of the integrals

$$I_n = \int_{n\pi}^{n\pi+\pi} \frac{\sin x}{x} dx \qquad \mathrm{n} = 0, 1, \cdots$$

From a monotone decreasing sequence, $|I_1|, |I_2|, \cdots$ and $I_n \to 0$ as $n \to \infty$. Since these integrals have alternating sign (why?), it follows from the Leibniz test that the infinite series $I_0 + I_1 + I_2 + \cdots$ converges. Clearly, the sum of the series is the integral

$$\int_0^{\infty} \frac{\sin x}{x} dx = \lim_{b \to \infty} \int_0^{b} \frac{\sin x}{x} dx$$

which therefore exists. Similarly the integral from 0 to $-\infty$ exists. Hence we need not take the principal value in (8), and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Since the integrand is an even function, the desired result follows.
In part (c) of example 2 we avoided the simple pole by integrating along a small semicircle C_2, and then we let C_2 shrink to a point. This process suggests the following.

3.4.3 Simple Poles on the Real Axis

If (z) *has a simple pole at* $z = a$ *on the real axis, then*

$$\lim_{r \to 0} \int_{C_2} f(z)dz = \pi i \operatorname{Re}_{z=a} s f(z).$$

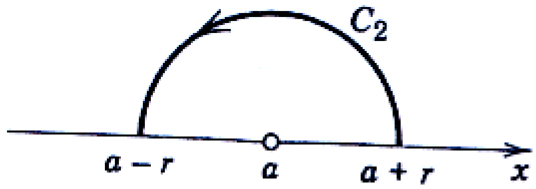

Fig. 48: Theorem 1

Proof

By the definition of a simple pole the integrand $f(z)$ has at $z = a$ the Laurent series

$$f(z) = \frac{b_1}{z-a} + g(z), \qquad b_1 = \operatorname{Re}_{z=a} s\, f(z)$$

where $g(z)$ is analytic on the semicircle of integration

$$C_2 : z = a + re^{i\theta}, \qquad 0 \le \theta = \pi$$

and for all z between C_2 and the x-axis. By integration,

$$\int_{C_2} f(z)dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} ire^{i\theta} d\theta + \int_{C_2} g(z)dz$$

The first integral on the right equals $-b_1\pi i$ .The second cannot exceed $M\pi r$ in absolute value, by the ML-inequality and $M\pi r \to 0$ as $r \to 0$.

We may combine this theorem with (7) or (3) in this section.

Thus,

$$\text{pv.v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Re} s f(z) + \pi i \sum \operatorname{Re} s f(z) \qquad (9)$$

(summation over all poles in the upper half-plane in the first sum, and on the x-axis in the second), valid for rational $f(x) = p(x)/q(x)$ with degree $q \geq$ degree $p+2$, having simple poles on the x-axis.

This is the end of unit 1, which added another powerful general integration method to the methods discussed in the chapter on integration. Remember that our present residue method is based on Laurent series, which we therefore had to discuss first.

In the next chapter we present a systemic discussion of mapping by analytic functions (**"conformal mapping"**) .Conformal mapping will then be applied to potential theory, our last chapter on complex analysis.

4.0 CONCLUSION

In this unit, we have seen that our simple method have been extended to the case when the integrand has several isolated singularities inside the contour. We also proved the residue theorem.

5.0 SUMMARY

The **residue** of an analytic function $f(z)$ at a point $z = z_0$ is the coefficient of $\frac{1}{z-z_0}$ the power in the Laurent series

$f(z) = a_0 + a_1(z-z_0) + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$ of $f(z)$ which converges near z_0 (except at z_0 itself). This residue is given by the integral 3.1

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)dz \qquad (1)$$

but can be obtained in various other ways, so that one can use (1) for evaluating integral over closed curves. More generally, the **residue theorem** (sec.3.2) states that if $f(z)$ is analytic in a domain D such except at finitely many points z_j and C is a simple close path in D such that no z_j lies on C and the full interior of C belongs to D, then

$$\oint_{C_j} f(z)dz = \frac{1}{2\pi i}\sum_j \operatorname*{Res}_{z=z_j} f(z) \qquad (2)$$

(summation only over those z_j that lie inside C).

This integration method is elegant and powerful. Formulas for the residue at **poles** are ($m =$ order of the pole)

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right), \qquad m = 1, 2, \cdots \qquad (3)$$

Hence for a simple pole ($m = 1$),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) \qquad (3^*)$$

Another formula for the case of a simple pole of $f(z) = p(z)/q(z)$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z)}{q'(z)} \qquad (3^{**})$$

Residue integration involves closed curves, but the real interval of integration $0 \le \theta \le 2\pi$ is transformed into the unit circle by setting $z = e^{i\theta}$, so that by residue integration we can integrate **real integrals** of the form (sec. 3.3)

$$\int_0^{2\pi} F(\cos\theta \sin\theta) d\theta$$

where F is a rational function of $\cos\theta$ and $\sin\theta$, such as, for instance, $\dfrac{\sin^2\theta}{5 - 4\cos\theta}$, etc.

Another method of integrating *real* integrals by residues is the use of a closed contour consisting of an interval $-R \le x \le R$ of the real axis and a semicircle $|z| = R$. From the residue theorem, if we let $R \to \infty$, we obtain for rational $f(x) = p(x)/q(x)$ (with $q(x) \ne 0$ and $q >$ degree $p + 2$)

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Re} s f(z) \qquad \text{(sec.3.3)}$$

$$\int_{-\infty}^{\infty} \cos sx dx = -2\pi \sum \operatorname{Im} \operatorname{Re} s\left[f(z)e^{isz}\right]$$

$$\int_{-\infty}^{\infty} \sin sx dx = 2\pi \sum \operatorname{Im} \operatorname{Re} s\left[f(z)e^{isz}\right] \qquad \text{(sec.3.4)}$$

(sum of all residues at poles in the upper-half plane). In sec.3.4, we also extend this method to real integrals whose integrands become infinite at some point in the interval of integration.

6.0 TUTOR-MARKED ASSIGNMENT

i. Explain the term residues and how it can be used for evaluating integrals.

ii. Find the residues at the singular points of the following functions;

(a) $\dfrac{\cos 2z}{z^4}$ (b) $\tan z$ (c) $\dfrac{e^z}{(z+\pi i)^6}$

iii. Evaluate the following integrals where C is the unit circle (counterclockwise).

(a) $\oint_C \cot z dz$ (b) $\oint_C \frac{dz}{1-e^z}$ (c) $\oint_C \frac{z^2+1}{z^2-2z}$

iv. Show that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2-\cos\theta}} = 2\pi$$

7.0 REFERENCES/FURTHER READING

Hernandez, V., J. E. Roman, and V. Vidal. "SLEPC: A Scalable and Flexible Toolkit for the Solution of Eigenvalue Problems." *ACM Trans. Math. Soft.* 31, no. 3 (2005).

McLachlan, R. I. "Families of High-Order Composition Methods." *Numerical Algorithms* 31 (2002).

McLachlan, R. I. and G. R. W. Quispel. "Splitting Methods." *Acta Numerica* 11 (2002).

Mazzia. F. and C. Magherini, "Test Set for Initial Value Problem Solvers-Release 2.4." *Dept. of Mathematics, University of Vari and INdAM, Research Unit of Bari.* 2008.

Moler, C. B. *Numerical Computing with MATLAB*. SIAM, (2004).

Murray, R. Spiegel *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc. (1974).

Nedialkov, N. and J. Pryce "Solving Differential Algebraic Equations by Taylor Series (I): Computing Taylor Coefficients." *BIT* 45, no.3 (2005)

Olsson, H. and G. Söderlind. "The Approximate Runge–Kutta Computational Process." *BIT* 40, no. 2 (2000).

Quarteroni, A., R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer-Verlag, (2000).

Ramana B.V *Higher Engineering Mathematics*. New Delhi: Tata McGraw-Hill Publishing Company Limited. (2008).

Rubinstein, B. "Numerical Solution of Linear Boundary Value Problems." *Mathematica MathSource* package, http://library.wolfram.com/database/MathSource/2127.

Shampine, L. F. "Solving in Matlab." *Journal of Numerical Mathematics* 10, no. 4 (2002).

Shampine, L. F. and S. Thompson. "Solving Delay Differential Equations with dde23." Available electronically from http://www.runet.edu/~thompson/webddes/tutorial.pdf.

Shampine, L. F. and S. Thompson. "Solving DDEs in MATLAB." *Appl. Numer. Math.* 37 (2001).

Shampine, L. F., I. Gladwell, and S. Thompson. *Solving ODEs with MATLAB*. Cambridge University Press, 2003.

Sofroniou, M. and G. Spaletta. "Increment Formulations for Rounding Error Reduction in the Numerical Solution of Structured Differential Systems." *Future Generation Computer Systems* 19, no. 3 (2003).

Sofroniou, M. and G. Spaletta. "Construction of Explicit Runge–Kutta Pairs with Stiffness Detection." *Mathematical and Computer Modelling*, special issue on The Numerical Analysis of Ordinary Differential Equations, 40, no. 11–12 (2004).

Sofroniou, M. and G. Spaletta. "Derivation of Symmetric Composition Constants for Symmetric Integrators." *Optimization Methods and Software* 20, no. 4–5 (2005).

Sofroniou, M. and G. Spaletta. "Hybrid Solvers for Splitting and Composition Methods." *J. Comp. Appl. Math.*, special issue from the International Workshop on the Technological Aspects of Mathematics, 185, no. 2 (2006).

Stephenor, G.. *Mathematical Methods for Science Students*. London: Longman, Group Limited. (1977)

Stroud, K.A... *Engineering Maths*. 5th Edition Palgraw. (1995)

Sprott, J. C. "A Simple Chaotic Delay Differential Equation." *Phys. Lett. A.* (2007).

Stewart, G. W. "A Krylov-Schur Algorithm for Large Eigenproblems." *SIAM J. Matrix Anal. Appl.* 23, no. 3, (2001).

Tang, X. H. and X. Zou. "Global Attractivity in a Predator-Prey System with Pure Delays." *Proc. Edinburgh Math. Soc.* (2008).

Unger, J., A. Kroner, and W. Marquardt. "Structural Analysis of Differential Algebraic Equation Systems—Theory and Applications." *Computers Chem. Engg.* 19, no. 8 (1995).

Verma, P.D.S.. *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd. (1995)

Verner, J. H. "Numerically Optimal Runge-Kutta Pairs with Interpolants." *Numerical. Algorithms* (2010).

Zennaro, M. "The Numerical Solution of Delay Differential Equations." Lecture notes, Dobbiaco Summer School on Delay Differential Equations and Applications, (2006).