MODULE 4 INTEGRAL TRANSFORMS

UNIT 1 INTEGRAL TRANSFORMS

CONTENTS

1.0 INTRODUCTION

The integral transform method is one of the best methods used in handling problems involving mechanical vibrations. The integral transform method is given by

$$F(p) = \int_a^b f(x)k(x,\rho)dx$$

With the inverse,

$$f(x) = \sum_{p=a}^{b} F(p)H(x,\rho)$$

$F(\rho)$ is the integral transform of $f(x)$ and $k(x,\rho)$ is called the kernel of the transformation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state various form of integral transform;
- state Fourier Sine series and Fourier Cosine series;

- apply Fourier transform to solve some fourth, third and second order differential equations; and
- develop techniques and methods through transformation or along with transform to be able to solve physical and mechanical problems (vibrations).

3.0 MAIN CONTENT

3.1 Finite Fourier Transform

Let $f(x)$ be a function defined in the interval $a \le x \le b$ i.e. $f(x)$ is defined on x-space. Let $k(x,\rho)$ be a function x of and some parameter ρ.

Then the integral transform method is given by,

$$F(p) = \int_a^b f(x)k(x,\rho)dx \qquad (1)$$

$F(\rho)$ is called an integral transform of $f(x)$ and $k(x,\rho)$ is called the kernel of the transform

Symbolically,

$$F = Tf \qquad (2)$$

where T is an integral operator which means multiply what follows T by $k(x,p)$ and integrate the product with respect to x between the limit of $'a'$ and $'b'$. The new function $F(\rho)$ can be regarded as the image of $f(x)$ produced by T.

$F(p)$ is defined on p-space/image-space.

For integral transform to be a useful concept, it is necessary that there should exist an inverse operator T^{-1} which yields a unique $F(t)$ from a given $F(\rho)$. From equation (2) you have that:

$$f = T^{-1}(F) \qquad (3)$$

Finding the operator T^{-1} is equivalent to solving equation (1) regardless an integral equation for $f(t)$

$$f(t) = \int_\alpha^\beta F(\rho)H(\rho,x)d\rho \qquad (4)$$

i.e. $F(t)$ is an integral transform of $F(\rho)$ with kernel $H(\rho,x)$.
A specification of the T^{-1} operator as in equation (4) is known as **Inversion Theorem**.

3.1 Finite Fourier Transforms

3.1.1 Half Range Fourier Sine Series

$$f(x) = \sum_{\rho=1}^{\infty} b_\rho Sin \frac{\rho \pi x}{L} \qquad 0 \le x \le L$$

Where

$$b\rho = \int_0^L f(x)\left\{\frac{2}{L}\sin\frac{\rho\pi x}{L}\right\}dx$$

$$k(x,\rho) = \frac{2}{L}\frac{\rho\pi x}{L}.$$

The image space is given by all the positive integral values of ρ. Hence $b\rho$ rather than $b(\rho)$.

3.1.2 Half Range Fourier Sine Series $0 \le x \le L$

$$f(x) = \frac{1}{2}a_0 \times \sum_{\rho=1}^{\infty} \cos\frac{\rho\pi x}{L}$$

Where

$$a\rho = \int_0^L f(x)\left\{\frac{2}{L}\cos\frac{\rho\pi x}{L}\right\}dx$$

3.1.3 Ordinary Fourier Series

$$f(x) = \sum_{\rho=-\infty}^{\infty} C_\rho e^{\left(i\frac{\rho\pi x}{L}\right)}$$

$$= \sum_{\rho=-\infty}^{\infty} C_\rho \exp\left(\frac{\rho\pi x}{L}\right)$$

Where $-L \le x \le L$

$$C\rho = \int_{-L}^{L} f(x)\left\{\frac{1}{2L}\exp\left(-i\frac{\rho\pi x}{L}\right)\right\}dx$$

3.2 The Fourier Transform

3.2.1 Fourier Sine Transforms

$$F_S(\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(x)\sin\rho(x)dx \qquad (5)$$

$0 \le x \le \infty$

With inversion

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty F_S(\rho)\sin\rho(x)d\rho$$

$0 \le \rho \le \infty$

Since kernel for operator and its inversion.

3.2.2 Fourier Cosine Transforms

$$F_c(\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(x)\cos\rho(x)dx \qquad (7)$$

With the inversion

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} F_c(\rho)\cos\rho(x)dx \qquad (8)$$

Same kernel $\cos\rho(x)$ for operator and its inversion.

3.2.3 Ordinary Fourier Transforms

$$F(\rho) = (2\pi)^{-1/2} \int_{-\infty}^\infty f(x)e^{i\rho(x)}dx \qquad (9)$$

The kernel $k(x,\rho) = e^{i\rho x}$

With inversion is

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^\infty F(\rho)e^{-i\rho(x)}d\rho$$

Then $H = (\rho, x)e^{-i\rho(x)}$ (10)

have $k \ne H(\rho, x)$ (11)

If $f(x)$ is even then $f(-x) = f(x)$

and $F(\rho) = F_c(\rho)$ (12)

But if $f(x)$ is odd then $f(-x) = -f(x)$ and

Thus $F(\rho) = iF_c(\rho)$ (13)

From equation (9) above, you can deduce that;

$$(2\pi)^{-\frac{1}{2}} F(\rho) = \int_{-\infty}^{\infty} f(x)e^{-i\rho(x)}dx$$
$$= \int_{-\infty}^{0} f(x)e^{-i\rho(x)}dx + \int_{0}^{\infty} f(x)e^{-i\rho(x)}dx \quad (14)$$

But if

$$x = -t$$
$$\Rightarrow x = 0 \Rightarrow t = 0$$
$$x = -\infty \Rightarrow t = 0$$
$$\therefore \quad dx = -dt$$

Thus, you have

$$\int_{-\infty}^{0} f(x)e^{-i\rho(x)}dx = \int_{0}^{\infty} f(-t)e^{-i\rho(x)}dt \quad (15)$$

$$(2\pi)^{-\frac{1}{2}} F(\rho) = \int_{-\infty}^{0} f(x)e^{-i\rho(x)}dx + \int_{0}^{\infty} f(-x)e^{-i\rho(x)}dx \quad (16)$$

If $f(x)$ is even then $f(-x) = f(x)$

$\therefore$ Equation (16) becomes

$$\int_{0}^{\infty} f(x)\left[e^{i\rho(x)} + e^{-i\rho(x)}\right]dx \text{ for even } f(x)$$
$$2\int_{0}^{\infty} f(x)\cos\rho(x)dx = (2\pi)^{\frac{1}{2}} F(\rho) \quad (17)$$

But, for odd $f(x)$

$$\int_{0}^{\infty} f(x)\left[e^{i\rho(x)} - e^{-i\rho(x)}\right]dx$$
$$= 2i\int_{0}^{\infty} f(x)\sin\rho(x)dx \quad (18)$$

3.3 Fourier Integral Formular

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\cos\rho(x-t)dt \qquad (19)$$

Note that from (9) and (10) you have that:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)} d\rho \int_{-\infty}^{\infty} f(t)e^{i\rho(x)}dt \qquad (20)$$

You have now prove that equations (19) equals (20)
Consider equation (19)

$\int_{-\infty}^{\infty} f(t)\cos\rho(x-t)dx$ is a an even function of ρ

So that (19) can be re-written in the form

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\cos\rho(x-t)dt \qquad (21)$$

Since $\int_{0}^{\infty} g(\rho)d\rho = \frac{1}{2}\int_{-\infty}^{\infty} g(\rho)d\rho$

$g(\rho)$ is even

Hence $$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\sin\rho(x-t)dt \qquad (22)$$

In other to arrive at equation (19), you have equation (21) equals (22) because

$$\cos\theta = i\sin\theta = e^{-i\theta}$$

$$\therefore \quad F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} f(t)e^{-i\rho(x-t)}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)} d\rho \int_{-\infty}^{\infty} f(t)e^{i\rho(x)}dt$$

Which is equal to (20).

3.4 Transforms of Derivatives

$$F(\rho) = F(y(x)) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y(x)e^{i\rho(x)}dx \qquad (23)$$

You shall now transform $y'(x) = F(y'(x))$

$$\therefore F(y'(x)) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y'(x)e^{i\rho(x)}dx \qquad (24)$$

Using integration by parts, you have

$$\left(\frac{1}{2\pi}\right)^{-1/2}\left\{t(x)e^{i\rho(x)}\Big|_{-\infty}^{\infty} - i\rho\int_{-\infty}^{\infty} y(x)e^{i\rho(x)}dx\right\} \qquad (25)$$

suppose $y(x) \to 0$ as $x \to \pm\infty$

$$\therefore \quad \left(\frac{1}{2\pi}\right)^{-1/2}\int_{-\infty}^{\infty} y'(x)e^{i\rho(x)}dx = i\rho\left(\frac{1}{2\pi}\right)^{-1/2}\int_{-\infty}^{\infty} y(x)e^{i\rho(x)}dx$$

$$= i\rho(Y(\rho))$$

$$\therefore \qquad F(y'(x)) = i\rho(Y(\rho)). \qquad (26)$$

$$y''(x) = \left(\frac{1}{2\pi}\right)^{-1/2}\int_{-\infty}^{\infty} y'(x)e^{i\rho(x)}dx \qquad (27)$$

Integration by parts,

$$\left(\frac{1}{2\pi}\right)^{-1/2}\left\{y'(x)e^{i\rho(x)}\Big|_{-\infty}^{\infty} - i\rho\int_{-\infty}^{\infty} y'(x)e^{i\rho(x)}dx\right\} \qquad (28)$$

suppose $y'(x) \to 0$

Then you have

$$-i\rho\int_{-\infty}^{\infty} y'(x)e^{i\rho(x)}dx.$$

Which $-\rho\left[F(y'(x))\right] = i\rho\left(-i\rho(Y(\rho))\right)$

$$= -i\rho^2\left[Y(\rho)\right]$$

$$= -\rho^2\left(y(x)\right) \qquad (29)$$

Suppose you have

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = f(x)$$

$$y \to 0, \qquad y' \to 0 \text{ as } \quad x \to \pm\infty \qquad (30)$$

In other to arrive at equation (19), you equation (21)
Because $\cos\theta = i\sin\theta = e^{-i\theta}$.

$$\therefore \; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho\int_{-\infty}^{\infty} f(t)e^{-i\rho(x-t)}dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)} d\rho \int_{-\infty}^{\infty} f(t) e^{i\rho(x)} dt.$$

Which is equal to (20).

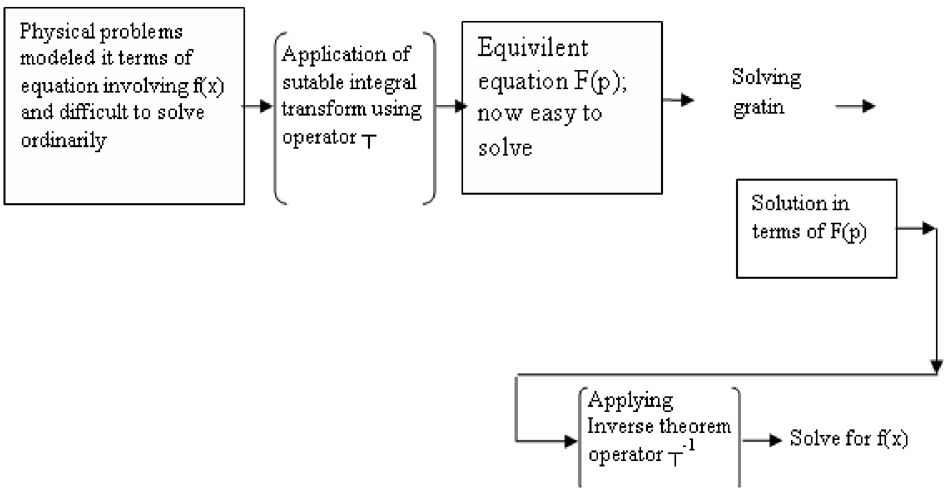
$$F(y'' + y' + y) = G(\rho)$$
$$\therefore -\rho^2 Y(\rho) - iY(\rho) + Y(\rho) = G(\rho)$$
$$Y(\rho)\left[-\rho^2 - i\rho + 1\right]$$
$$Y(\rho) = \left[\frac{G(\rho)}{-(\rho^2 + i\rho - 1)}\right] \quad (31)$$

4.0 CONCLUSION

In this unit, you treated the various forms of integral transform. The Fourier sine and cosine series representation were discussed. The inverse theorem was also considered.

5.0 SUMMARY

The general scheme of solving problem by integral transform is summarized below;



This is the diagrammatic expression of the summary.

6.0 TUTOR-MARKED ASSIGNMENT

i. State the method of integral transforms and its inverse. State also the Kernels of the method and its inverse
ii. Discuss briefly the inverse theorem.
iii. State the three theorems of finite Fourier transforms.
iv. If $F(\rho) = F(y(x)) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y(x) e^{i\rho(x)} dx$
use the transformation $y'(x) = F(y''(x))$, proof that $F(y'(x)) = i\rho[Y(\rho)]$.

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 2 FOURIER SERIES APPLICATION

CONTENTS

1.0 INTRODUCTION

Fourier series arises from the task of representing a given periodic function $f(x)$ by trigonometric series. The Fourier series coefficients are determined from $f(x)$ by Euler formula.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine Fourier coefficients;
- find the convergence and sum of Fourier series; and
- use Euler formula for the Fourier coefficients.

3.0 MAIN CONTENT

3.1 Fourier Series

3.1.1 Euler Formula for the Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π that can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (1)$$

That is to say, you assume the convergence of the series and has $f(x)$ as its sum.

In any function $f(x)$ of such, you shall determine the coefficients a_n and b_n of the corresponding series.

(1) To determine a_0 ,you shall integrate both sides of the equation 1, from $-\pi \le x \le \pi$

Thus, you have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \left[\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]\right]dx$$

$$= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx$$

$$= a_0 x\Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx\Big|_{-\pi}^{\pi} - \sum_{n=1}^{\infty}\frac{b_n}{n}\cos nx\Big|_{-\pi}^{\pi}$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty}\frac{1}{n}\left[a_n(\sin n\pi - \sin(-n\pi)) - (\cos n\pi - \cos(-n\pi))\right]$$

$$= 2\pi a_0 \qquad (2)$$

Hence

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (3)$$

To determine $a_1, a_2, \cdots a_n$ using the same procedure. However, multiplying equation (1) by $\cos mx$, when m is any fixed real number, and integrate from $-\pi \le x \le \pi$

$$\therefore \int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi}\left[a_0 + \left[\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]\right]\cos mx dx \qquad (4)$$

$$= a_0\int_{-\pi}^{\pi}\cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx dx \qquad (5)$$

Evaluate (5) term by term, you have

$$a_0\int_{-\pi}^{\pi}\cos mx dx = a_0\left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi} = 0 \qquad (6)$$

Using trigonometric identities

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx \cos mx dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\left[\cos(n+m)x + \cos(n-m)x\right]x dx \qquad (7)$$

Similarly,

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx \cos nx dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\left[\sin(n+m)x + \sin(n-m)x\right]x dx \qquad (8)$$

From (7), you have,

$$\int_{-\pi}^{\pi} \cos(n+m)x dx = \frac{\sin(n+m)x}{n+m}\bigg|_{\pi}^{\pi} = 0 \tag{9}$$

and

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \frac{\sin(n-m)x}{n-m}\bigg|_{\pi}^{\pi} = 0 \tag{10}$$

for $n \neq m$

but if $n = m$ you have that

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \int_{-\pi}^{\pi} \cos(0)x dx = \int_{-\pi}^{\pi} dx.$$

because $\cos 0 = 1$

$$\therefore \quad \int_{-\pi}^{\pi} dx = x\bigg|_{-\pi}^{\pi} = 2\pi \tag{11}$$

From equation (8) you obtain thus

$$\int_{-\pi}^{\pi} \sin(n+m)x dx = -\frac{\cos(n+m)x}{n+m}\bigg|_{\pi}^{\pi} = 0 \tag{12}$$

and

$$\int_{-\pi}^{\pi} \sin(n-m)x dx = -\frac{\cos(n-m)x}{n-m}\bigg|_{\pi}^{\pi} = 0 \tag{13}$$

Substituting equations (9), (10), and (11) into (7), you have

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & \text{n} \neq \text{m} \\ \pi & \text{n} = \text{m} \end{cases} \tag{14}$$

and substituting equations (12), (13), and (14) into (8) gives

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx = \begin{cases} 0 & \text{n} \neq \text{m} \\ \pi & \text{n} = \text{m} \end{cases} \tag{15}$$

Then, in view of equations (14), (15) and (6), equation (5) becomes:

$$\int_{-\pi}^{\pi} f(x)\cos mxdx = a_n(0) + \sum_{n=m}^{\infty} a_n\pi + \sum_{n=1}^{\infty} b_n(0)$$
$$= a_m\pi \qquad (16)$$

$$\therefore \; a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx \qquad (17)$$

$b_1, b_2, \cdots b_n$ can also be obtained in the same manner, by multiplying equation (1) by $\sin mx$ and integrate from $-\pi \le x \le \pi$.

Using the trigonometric identities and manipulation, you have

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi}\left[a_0 + \left[\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]\right]\sin mxdx \qquad (18)$$

Integrating term by term, you see that the right hand side becomes

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi} a_n \sin mxdx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos nx \sin mxdx$$
$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \sin mxdx \qquad (19)$$

Using the same principle as before

$$\int_{-\pi}^{\pi} a_n \sin mxdx = 0 \qquad (20)$$

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mxdx = 0 \qquad (21)$$

$for\ n = 1,2,3,\cdots\cdots$

but

$$\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \sin mxdx = \frac{1}{2}\left[\int_{-\pi}^{\pi}\left[\cos(n-m)x - \cos(n+m)x\right]xdx\right]$$

$$\frac{1}{2}\frac{(-1)\sin(n-m)x}{(n-m)} - \frac{1}{2}\frac{(-1)\sin(n+m)x}{(n+m)}0 \qquad (22)$$

$n \neq m$

but for $n = m$

$$\frac{1}{2}\int_{-\pi}^{\pi}\cos(0)dx = \frac{1}{2}\int_{-\pi}^{\pi}dx = \pi$$

$$\therefore \int_{-\pi}^{\pi}\sin nx \cos mx dx = \begin{cases} 0 & \text{n} \neq \text{m} \\ \pi & \text{n} = \text{m} \end{cases} \qquad (23)$$

$\therefore$ substituting equation (23) into (19) you obtain thus

$$\int_{-\pi}^{\pi} f(x)\sin mx dx = b_n \pi$$

$$\Rightarrow \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \qquad (24)$$

For $m = 1, 2, \cdots$

Writing n in place of m in equation (17) and (24) respectively, you have

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx \\ &and \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \end{aligned}\right\} \qquad (25)$$

This is called the Euler formula.

These numbers given in equation (25) are called the Fourier coefficients of $f(x)$. However, the trigonometric series in equation (1) with coefficients given by (25) is called the Fourier series of $f(x)$.

Example 1

Find the Fourier coefficients of the periodic function $f(x)$ where

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < \text{x} < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases}$$

and $\qquad f(x + 2\pi) = f(x)$.

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -dx + \int_{0}^{\pi} dx\right]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} - dx = \frac{1}{2\pi}(-x)\Big|_{-\pi}^{0} = \frac{1}{2\pi}\left[-0-(-\pi)\right]$$
$$= -\frac{1}{2}$$

and

$$\frac{1}{2\pi}\int_{0}^{\pi} dx = \frac{1}{2\pi}(x)\Big|_{0}^{\pi} = \frac{1}{2\pi}\left[\pi - 0\right]$$
$$= \frac{1}{2}$$
$$\therefore \ = -\frac{1}{2} + \frac{1}{2} = 0.$$

From equation (25) i.e.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx dx$$
$$= \frac{1}{\pi}\left[\frac{-\sin nx}{n}\Big|_{-\pi}^{0} + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$
$$\therefore \qquad a_n = 0$$

Similarly for

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx dx + \int_{0}^{\pi}\sin nx dx\right]$$
$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$
$$= \frac{1}{n\pi}\left[\cos 0 - \cos(-nx) - \cos nx + \cos 0\right]$$
$$= \frac{1}{n\pi}\left[2 - 2\cos(nx)\right]$$

N.B $\cos(-n\pi) = \cos(nx)$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi)\right]$$
$$= \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

N.B $\cos nx = (-1)^n$

$$b_n = \frac{2}{n\pi}[1+1] = \frac{4}{n\pi}$$

for $n = 1,3,5,\cdots$

$$b_n = \frac{2}{n\pi}[0] = 0$$

for $n = 2,4,6,\cdots$

$\therefore$ $b_1 = \frac{4}{\pi}$, $b_3 = \frac{4}{3\pi}$, $b_5 = \frac{4}{5\pi}$, etc

$b_2 = b_4 = b_6 = 0$

3.2 Even and Odd Numbers

Fourier coefficients of a function can be avoided if the function is odd or even. You say a function $y = g(x)$ is said to be even if

$$g(-x) = g(x) \quad \textit{for all } x. \tag{26}$$

While a function $h(x)$ is said to be odd if

$$h(-x) = -h(x) \quad \textit{for all } x. \tag{27}$$

However, it worth mentioning here that the function $\cos nx$ is even, while the function $\sin nx$ is odd.

If $g(x)$ is an even function, then

$$\int_{-L}^{L} g(xdx) = 2\int_0^L g(x)dx. \tag{28}$$

If $h(x)$ is an odd function, then

$$\int_{-L}^{L} h(x)dx = 0 \tag{29}$$

The product of both odd and even function is odd

$\therefore$ let $q(x) = g(x)h(x)$

and $q(-x) = g(-x)h(-x) = g(x)[-h(x)] = -q(x)$

3.2.1 Theorem 1 (Fourier Series of Even and Odd Functions)

The Fourier series of an even function $f(x)$ of periodic $2L$ is a "Fourier cosine series"

$$f(x) = a_0 + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \qquad (30)$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$n = 1,2,\cdots$

Also the Fourier series of an odd function $f(x)$ of period $2L$ is a "Fourier sine series"

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (31)$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (32)$$

In particular, this theorem implies that the Fourier series of an even function $f(x)$ of period $2L = 2\pi$ Fourier cosine series.

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots\cdots$$

with coefficients (33)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

n 2, 1, 2 ………. (34)

Similarly, the Fourier series of an odd function $f(x)$ of period 2π is a Fourier sine series.

$$f(x) = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots\cdots$$

with coefficients (35)

$$b_n = \frac{2}{\pi}\int_0^L f(x)\sin nx dx \qquad (36)$$

3.2.2 Theorem 2 (Sum of Functions)

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_2 f_1 and f_2.

The Fourier coefficients of a cf are c times the corresponding Fourier coefficients of f.

Example 2

The function $f^*(x)$ is the sum of the function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$ as in example 1 and the constant 1.

Hence from example 1 and theorem 2, above, you conclude that

$$f^*(x) = 1 + \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{6}\sin 6x + \cdots\cdots\right)$$

Example 3

Find the Fourier series of the function

$$f(x) = x + \pi \qquad \text{if } -\pi < x < \pi \text{ and}$$
$$f(x + 2\pi) = f(x)$$

Solution

Let $f = f_1 + f_2$ where $f_1 = x$ and $f_2 = \pi$.

The Fourier coefficients of f_2 are zero, except for the one (the constant term), which is π.

Hence, by theorem 2, the Fourier coefficients a_n, b_n are those f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$$

Integrating by parts you obtain

$$b_n = \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi}\right] + \frac{1}{\pi}\left[\int_0^{\pi}\cos ndx\right]$$

$$= \frac{2}{n} \cos n\pi$$

$$= \frac{2}{n}(-1)^n = \frac{2}{n} \text{ for odd } n$$

$$= -\frac{2}{n} \quad \text{for even } n$$

Hence, $b_1 = 2, b_2 = -1, b_3 = \frac{2}{3}, b_4 = -\frac{1}{2} \cdots\cdots$

Therefore the Fourier series of $f(x)$ is given thus;

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x \right)$$

4.0 CONCLUSION

The conclusion of this unit is embedded in the summary as discussed below.

5.0 SUMMARY

A Fourier series of a given function $f(x)$ of period 2π is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

With coefficients given as in equation (25).

Theorem 1 given conditions that is sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find the Fourier coefficients of the periodic function $f(x)$ where

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$$

and $f(x + 2\pi) = f(x)$

ii. Explain the term odd and even function of a Fourier series

iii. Find the Fourier series of the function

$f(x) = x + \pi$ if $0 < x < \pi$ and

$f(x + 2\pi) = f(x)$

iv Find the smallest positive period p of the following function

(a) $\cos x, \sin x, \cos 2x, \sin 2x$

v. If $f(x)$ and $g(x)$ have period p, show that

$h = af + bg(a, b, cons\tan t)$ has the period p.

Thus all functions of period p from a vector space.

vi. Evaluate the following integrals when

$n = 0,1,2,\cdots$

(a) $\int_0^{\pi/2} \cos nxdx$ (b) $\int_{\pi/2}^{\pi/2} x\cos nxdx$

(c) $\int_0^{\pi/2} e^x \cos nxdx$ (d) $\int_0^2 x^2 \cos nxdx$

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995). *Engineering Maths*. 5^{th} Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 3 LAPLACE TRANSFORMS AND APPLICATION

CONTENTS

1.0 INTRODUCTION

The Laplace transform is a method for solving differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem.

The Laplace transform is the most important method used in solving engineering mathematics.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- undergo the three main steps of solving initial and boundary value problem.

3.0 MAIN CONTENT

3.1 The Classical Laplace Transform

Let f be a function of the real variable t which is defined for all $t \geq 0$ and which is either continuous or at least sectionally continuous. The classical Laplace Transform † of f is the function $F_0(s)$ defined by the formula

$$F_0(s) \equiv \ell\,\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt. \qquad (1)$$

This definition of $F_0(s)$ clearly makes sense only for those values of s for which the infinite integral is convergent. For many applications it is enough to regard s as a real parameter, but in general it should be taken as complex, say $s = \sigma + i\omega$. Thus $F_0(s)$ is really a function of a complex variable defined over a certain region of the complex plane; the region of definition comprises just those values of s for which the infinite integral exists.

3.1.1 Elementary Applications of the Laplace Transform Depend Essentially on Three Basic Properties

i. **Linearity.** If the Laplace Transforms of f and g are $F_0(s)$ and $G_0(s)$ respectively, and if a_1 and a_2 are any (real) constants, then the Laplace Transform of the function h defined by

is

$$h(t) = a_1 f(t) + a_2 g(t)$$

$$H_0(s) = a_1 F_0(s) + a_2 G_0(s). \qquad (2)$$

The proof is trivial.

ii. **Transform of a Derivative**. If f is differentiable (and therefore continuous) for $f \geq 0$, then

$$= sF_0(s) - f(0). \qquad (3)$$

Proof

Using integration by parts you have

$$\ell\{f'(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \left[e^{-st} f(t)\right]_0^{\infty} se^{-st} f(t)dt$$

$$= -f(0) + s \int_0^{\infty} e^{-st} f(t)dt$$

Since $\lim e^{-st}f(t) = 0$

Corollary. If f is n-times differentiable for $t \geq 0$, then

$$\ell\{f^{(n)}(t)\} = s^n F_0(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots - f^{(n-1)}(0).$$

iii. **The Convelution Theorem**. Let f and g have Laplace Transforms $F_0(s)$ and $G_0(s)$ respectively, and define h as follows:

$$H(t) = \int_0^{t} f(\tau)g(t-\tau)d\tau, \quad t \geq 0.$$

Then,

$$\ell\{h(t)\} = F_0(s)G_0(s). \qquad (4)$$

(Recall that h, as defined here, is the convolution of the functions u(t)f(t) and u(t)g(t). If f and g happen to be functions which vanish identically for all negative values of t then the above result can be expressed in the form:

The Laplace Transform of the convolution of f and g is the product of the individual Laplace Transform.

Proof

The Laplace Transform of h is given by

$$H_0(s) = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt.$$

Now,

$$\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau$$

because u(t - τ) = 1 for all τ such that τ < t
and u(t - τ) = 0 for all τ such that τ > t.

Hence

$$H_0(s) = \int_0^\infty e^{-st}\left[\int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau\right]dt.$$

Again,

$$\int_0^\infty g(t-\tau)u(t-\tau)e^{-st}dt = \int_\tau^\infty g(t-\tau)e^{-st}d\tau$$

because u(t - τ) = 1 for all t such that t > τ,
and u(t - τ) = 0 for all t such that t < τ.

Thus,

$$H_0(s) = \int_0^\infty f(\tau)\left[\int_\tau^\infty g(t-\tau)e^{-st}dt\right]d\tau.$$

And so putting T = t - τ, you get

$$H_0(s) = \int_0^\infty f(\tau)\left[\int_0^\infty g(T)e^{-s(T+\tau)}dT\right]d\tau.$$

Since T = 0 when t = τ.
That is,

$$H_0(s) = \int_0^\infty f(\tau)e^{-st}d\tau\int_0^\infty g(T)e^{-sT}dT = F_0(s)D_0(s).$$

Remark

The change in the order of integration in the proof given above is justified by the absolute convergence of the integrals concerned.

3.1.2 Applications of Laplace

The most immediate application of these properties is in the solution of ordinary differential equations with constants. Consider the case of the general second-order equation

$$a\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + cy = f(t) \qquad (5)$$

Where y(0) = α and d'(0) = β. If ℓ[y(t)] = Y_0(s) then

$$\ell\left\{\frac{dy}{dt}\right\} = sY_0(s) - \alpha\text{, and } \ell\left\{\frac{d^2y}{dt^2}\right\} = 2Y0(s) - \alpha s - \beta.$$

Taking Laplace Transforms of both sides of (5.5) therefore gives

$$a[s^2Y_0(s) - \alpha s - \beta] + 2b\,[sY_0(s) - \alpha] + cY_0(s) = F_0(s).$$

That is,

$$Y_0(s) = \frac{F_0(s)}{as^2 + 2bs + c} + \frac{a\alpha s + (a\beta + 2b\alpha)}{as^2 + 2bs + c} \qquad (6)$$

Y_0(s) is thus given explicitly as a function of s, and what remains is an **inversion problem**; that is to say you need to determine a function y(t) whose Laplace Transform is Y_0(s). The question of uniqueness which naturally arises at this point is not, in practice, a serious problem. In brief, if y_1 and y_2 are any two functions which have the same Laplace Transform Y_0(s), then they can differ in value only on a set of points which is (in a sense which can be made precise) a negligibly small set. In fact, you have the following situation:

if $\ell[y_1(t)] = \ell[y_2(t)]$ then $\int_0^{\infty} |y_1(t) - y_2(t)|\, dt = 0.$

With this proviso in mind, you admit the slight abuse of notation involved, and write:

$$y(t) \equiv \ell^{-1}[Y_0(s)] = \ell^{-1}\left\{\frac{F_0(s)}{as^2 + 2bs + c}\right\} + \ell^{-1}\left\{\frac{a\alpha s + (a\beta + 2b\alpha}{as^2 + 2bs + c}\right\} \qquad (7)$$

where y is defined for all t > 0.

A more serious problem from the practical point of view is that of implementing the required inversion; that is, of division effective procedures which allow us to recover a function f(t) given its Laplace Transform F_0(s). In a large number of commonly occurring cases this can be done by expressing F_0(s) as a combination of standard functions of s whose inverse transforms are known .

Note that with zero initial conditions, (y(0) = y'(0) = 0), the differential equation (5) can be regarded as representing a linear time-invariant system which transforms a given input signal f into a corresponding output y. This output function y is the **particular integral** associated with f and, using the Convolution Theorem, it can be expressed in terms of the appropriate impulse response function characterizing the system:

$$Y(t) = \int_0^t f(\tau)h_1(t-\tau)d\tau = \ell^{-1}[F_0(s)H_0$$

Where

$$H_0(s) = \int_0^\infty e^{-st}h(t)\,dt = \frac{1}{as^2 + 2bs + c}$$

Non-zero initial conditions correspond to the presence of stored energy in the system at time t = 0. The response of the system to this stored energy is independent of the particular input f and is given by the **complementary function.** The complete solution (valid for all t > 0) of the equation (5) can be written in the form.

$$Y(t) = \ell^{-1}[F_0(s)H_0(s)] + \ell^{-1}[a\alpha s + (a\beta + 2b\alpha)]H_0(s)]. \qquad (8)$$

In applying the classical Laplace transform technique to (5) you are tacitly assuming that the system which it is being taken to represent is **unforced** for t < 0; that is, that the response which you compute from (5) is actually the response to the excitation f(t)u(t). This is sometimes expressed by saying that the input is **suddenly applied** at time t = 0.

3.2 Laplace Transforms of Generalised Functions

If a is any positive number then there is no specialty in extending the definition of the classical, one-sided, Laplace Transform to apply to the case of a delta function located at t = a, or to any of its derivatives located there; for a direct application of the appropriate sampling property gives immediately

$$\ell\{\delta_a(t)\} = \ell\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa} \qquad (9)$$

$$\ell\{\delta'(t-a)\} = \int_0^\infty e^{-st}\delta'(t-a)dt = -\left[\frac{d}{dt}(e^{-st})\right]_{t=a} = se^{-sa} \qquad (10)$$

and so on

Now take the case of a function f defined by a relation of the form

$$f(t) = \phi_1(t)u(a-t) + \phi_2(t)u(t-a) \qquad (11)$$

where a > 0, and ϕ_1 and ϕ_2 are continuously differentiable functions. Using the notation

$$f'(t) = \phi_1'(t)u(a-t) + \phi_2'(t)u(t-a) \quad \text{(for all } t \neq a\text{)}$$

and

$$Df(t) = \phi_1'(t)u(a-t) + \phi_2'(t)u(a-t) + [\phi_2(a) - \phi_1(a)]\delta(t-a)$$
$$\equiv f'(t) + [f(a+) - f(a-)]\delta(t-a). \qquad (12)$$

Using integration by parts to evaluate the Laplace integral you have

$$\int_0^{\infty} e^{-st} f'(t)dt = \int_0^{a} \phi_1'(t)e^{-st}dt + \int_0^{\infty} \phi_2'(t)e^{-st}dt$$

$$= \Big[e^{-st}\phi_1(t)\Big]_a^{\infty} + s\int_0^{a} \phi_1(t)e^{-st}dt + \Big[e^{-st}\phi_2(t)\Big]_a^{\infty} \int_a^{\infty} \phi_2(t)e^{-st}dt$$

$$= s\left[\int_0^{a}\phi_1(t)e^{-st}dt + \int_a^{\infty}\phi_2(t)e^{-st}dt\right] - e^{-as}\left[\phi_2(a) - \phi_1(a)\right] - \phi_1(0)$$

$$\equiv sF_o(s) - f(0) - e^{-as}\left[f(a+) - f(a-)\right] \qquad (13)$$

so that a modification of the derivative rule is required when you adhere to the classical meaning of the term "derivative" in the case of discontinuous functions.

On the other hand, from (12) you get

$$\int_0^{\infty} e^{-st}[Df(t)]dt = \int_0^{\infty} e^{-st} f'(t)dt + [f(a+) - f(a-)]\, e^{-as}$$

$$= sF_0(s) - f(0) \qquad (14)$$

and the usual form of the derivative rule continues to apply.

The result (13) makes sense even when you allow a to tend to zero, for then you get

$$\ell\,[f'(t)] = \int_0^{\infty} \phi_2'(t)e^{-st}dt = s\int_0^{\infty} \phi_2(t)e^{-st}dt - \phi_2(0)$$

$$= sF_0(s) - f(0+). \qquad (15)$$

However, a complication arises with regard to $\ell\,[Df(t)]$ when a = 0. If you have

Then $f(t) = \phi_1(t)u(-t) + \phi_2(t)u(t)$

$Df(t) = \phi_1'(t)u(-t) + [\phi_2(0) - \phi_1(0)]\,\delta(t)$

and so,

$$\ell\,[Df(t)] = \ell\,[\phi_2'(t)] + [\phi_2(0) - \phi_1(0)]\ \ell\,[\delta(t)]$$

$$= s\,\ell\,[\phi_2(t)] - \phi_2(0) + [\phi_2(0) - \phi_1(0)]\ \Delta(s)$$

$$\equiv sF_0(s) - f(0+) + [f(0+) - f(0-)]\ \Delta(s). \qquad (16)$$

The difficulty is that, as remarked in Sec. 4.5, the Laplace Transform of the delta function (which you have denoted by $\Delta(s)$) is not defined by the Laplace integral

$$\int_0^{\infty} e^{-st}\delta(t)dt = \int_{-\infty}^{+\infty} e^{-st}u(t)\ \delta(t)dt.$$

The role of the delta function as a (generalized) impulse response function suggests that you should have $\Delta(s) = 1$ for all s, and this is the definition most usually adopted. However the discussion on the significance of the formal product $u(t)\delta(t)$ shows that there are grounds for taking $\Delta(s) = \frac{1}{2}$, for all s; other values for $\Delta(s)$ have also at the issue cannot be resolved simply by an appeal to the definition of δ as a limit, nor by means of the formulation as a (Riemann) Stieltjes integral. In the latter case, for example, you have for an arbitrary continuous integrand f

$$\int_0^{\infty} f(t)du_c(t) = (1 - c)f(0) \qquad (17)$$

You could therefore obtain $\Delta(s) \equiv 1$ by choosing c = 0 or, equally well, $\Delta(s) \equiv \frac{1}{2}$ by choosing $c = \frac{1}{2}$. Whatever value you choose for $\Delta(s)$ the relation (16) is bond to be consistent with the behaviour of δ as the derivative of the unit step function u. for, since

$$\ell\,[u(t)] = \int_0^{\infty} e^{-st}dt = 1/s,$$

You have

$$\ell\,[u'(t)] = [s\left(\frac{1}{s}\right) - u(0+)] + \Delta(s)[u(0+) - u(0-)]$$
$$= (1 - 1) + \Delta(s)\,(1 - 0) = \Delta(s).$$

On the other hand care must be taken to ensure that the correct form of (16) is used when a specific definition of $\Delta(s)$ has been decided on. Thus, for $\Delta(s) = 1$ you get

$$\ell\,[Df(t)] = sF_0(s) - f(0-)$$
$$= sF_0(s) \qquad (18)$$

Whenever f(t) = 0 for all t < 0.

But for $\Delta(s) = \frac{1}{2}$ the result becomes

$$\ell\,[Df(t)] = sF_0(s) - \frac{1}{2}[f(0+) + f(0-)].$$

In what follows, you shall adopt the majority view and define $\Delta(s)$ to be 1 for all values of s. Similarly, you shall take the Laplace Transform of δ' to be s; the analogue of (19) then becomes

$$\ell\,[D^2f(t)] = s^2F_0(s) - sf(0-) - f'(0-) \qquad (19)$$
$$= s^2F_0(s)$$

whenever f(t) = 0 for all t < 0. The convenience of these definitions is readily illustrated by the following derivation of the Laplace Transform of a **periodic function:**

Let f be a function which vanishes identically outside the finite interval (0,T). The periodic extension of f, of period T, is the function obtained by summing the translates, f(t – kT), for k = 0, ± 1, ± 2,…., (see fig. 49)

$$f_T(t) = \sum_{k=-\infty}^{+\infty} f(t-kT) \qquad (20)$$

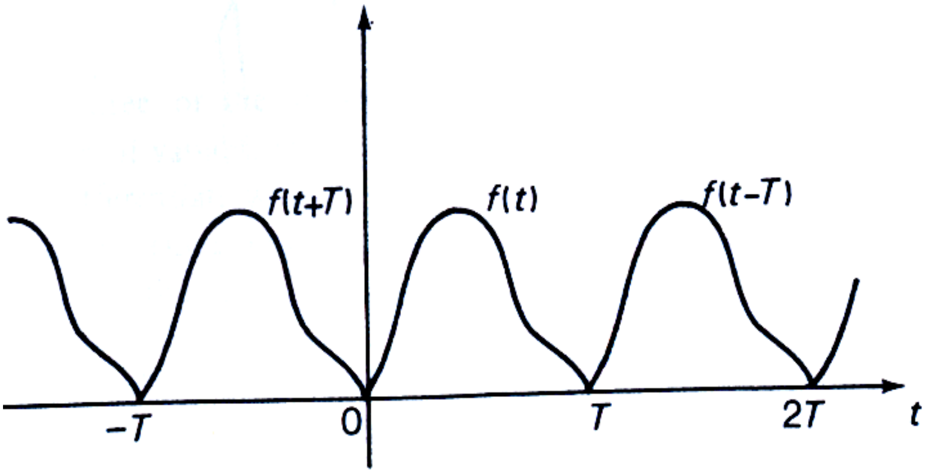


Fig. 49

You can write f_T as a convolution:

$$f_T(t) = \sum_{k=-\infty}^{+\infty}[f(t)*\delta(t-kT)] = f(t)*\sum_{k=-\infty}^{+\infty}\delta(t-kT). \qquad (21)$$

further, using the above definition of Δ(s), you obtain

$$\ell\left[\sum_{k=-\infty}^{+\infty}\delta(t-kT).\right] = \ell\left[\sum_{k=\infty}^{\infty}\delta(t-kT).\right]$$

$$= 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots = \frac{1}{1-e^{-sT}} \qquad (22)$$

The summation being valid provided that

$$|e^{-sT}| = |e^{-(}e^{-\alpha+i\omega)T} \,|\, e^{-\alpha T} < 1,$$

That is, for all s such that Re(s) > 0. Hence, appealing to the Conclusion

Theorem for the Laplace transform, (21) and (22) together yield

$$\ell\left[\sum_{k=-\infty}^{+\infty}\delta(t-kT).\right] = \frac{F_0(s)}{1-e^{-sT}} \qquad (23)$$

3.3 Computation of Laplace Transforms

If f is an ordinary function whose Laplace Transform exists (for some values of s) then you should be able to find that transform, in principle at least, by evaluating directly the integral which defines $F_0(s)$. It is usually simpler in practice to make use of certain appropriate properties of the Laplace integral and to derive specific transforms from them. The following results are easy to establish and are particularly useful in this respect:

(L.T.I) The first Translation Property. If $\ell\,[f(t)] = F_0(s)$, and if a is any real constant, then

$$\ell\,[e^{at}f(t)] = F_0(s - a).$$

(L.T.2) The Second Translation Property. If $\ell\,[f(t)] = F_0(s)$, and if a is any positive constant, then

$$\ell\,[u(t - a)f(t - a)] = e^{-as}F_0(s).$$

(L.T.3) Change of Scale. If $\ell\,[f(t)] = F_0(s)$, and if a is any positive constant, then

$$\ell\,[f(at)] = \frac{1}{a}F_0\left(\frac{s}{a}\right).$$

(L.T.4) Multiplication t. If $\ell\,[f(t)] = F_0(s)$, then

$$\ell\,[tf(t)] = -\frac{d}{ds}F_0(s) \equiv -F_0'(s).$$

(L.T.5) Transform of an Integral. If $\ell\,[f(t)] = F_0(s)$, and if the function g is defined by

$$g(t) = \int_0^t f(\tau)d\tau$$

then

$$\ell\,[g(t)] = \frac{1}{s}F_0(s).$$

The first three of the above properties follow immediately on making suitable changes of variable in the Laplace integrals concerned. For (L.T.4) you have only to differentiate with respect to s under the integral sign, while in the case of (L.T.5) it is enough to note that $g'(t) = f(t)$ and that $g(0) = 0$; the result then follows from the rule for finding the Laplace Transform of a derivative. Using these properties, an elementary basic table of standard transforms can be constructed without difficulty (Table 1). This list can be extended by using various special techniques. In particular, the results for the transforms of delta functions derived in the preceding section are of considerable value in this connection.

Table 1: Basic Table of Standard Transforms

fu(t)(t)	$F_0(s)$	Region of (absolute) convergence
u(t)	1/s	Re(s) > 0
t	$1/s^2$	Re(s) > 0
$t^n(n>1)$	$n!/s^{n+1}$	Re(s) > 0
e^{at}	$\frac{1}{s-a}$	Re(s) > a
e^{-at}	$\frac{1}{s+a}$	Re(s) > - a
sinh at	$\frac{a}{s^2-a^2}$	Re(s) > | a |
cosh at	$\frac{s}{s^2-a^2}$	Re(s) > | a |
sin at	$\frac{a}{s^2+a^2}$	Re(s) > 0
cos at	$\frac{s}{s^2+a^2}$	Re(s) > 0

Example 1

Find the Laplace transform of the triangular waveform show in fig. 50.

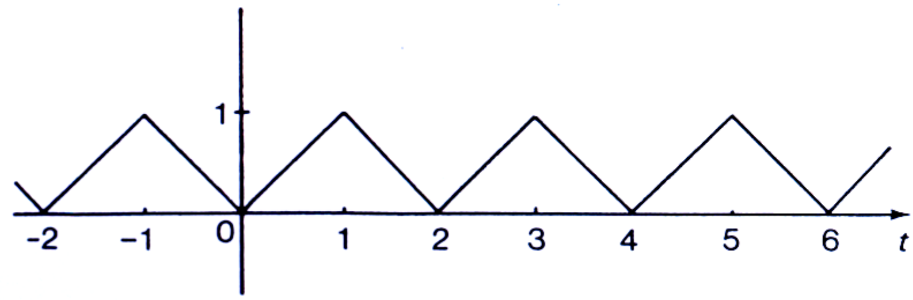
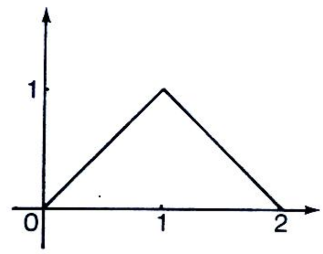


Fig. 50: Laplace Transform of the Triangle Waveform

You shall obviously expect to use the formula (23) for the Laplace Transform of the periodic extension of a function f, but the first need is to establish the transform of this function f itself. In fig. 51 there is shown a decomposition of the required function into a combination of ramp functions:

$$f(t) = tu(t) – 2(t – 1)u(t – 1) + (t – 2)u(t – 2)$$

Fig. 5.4 (a).



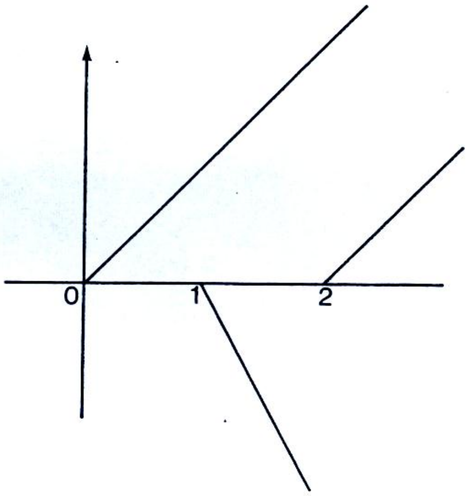


Fig. 5.4 (b).

Fig. 51 (b)

A straightforward application of the second translation property (L.T.2) immediately gives

$$F_0(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{e^{-2s}}{s^2} = \left[\frac{1-e^{-s}}{s}\right]^2 = \frac{4}{s^2}e^{-s}\sinh^2\frac{s}{2}.$$

Hence, applying (5.23)

$$\ell\,[f_T(t)] = \left[\frac{4}{s^2}e^{-s}\sinh^2\frac{s}{2}\right]\left[\frac{1}{1-e^{-2s}}\right] = \frac{2\sinh^2 s/2}{s^2\sinh s} = \frac{\tanh s/2}{s^2}.$$

4.0 CONCLUSION

In this unit you considered the Laplace transform atum from practical point of view and illustrate its use by important engineering problems, many of them related to ordinary differential equations.

5.0 SUMMARY

The main purpose of the Laplace transformation is the solution of differential equations and systems of such equations, as well as corresponding initial value problems.

The Laplace transform f(s) = ℓ (f) of a function f(t) depend by.

F(s) ℓ (f) $\int_0^{\delta} e^{-st} f(t)dt$

Further, more discussion, the Laplace of the derivation such that.

ℓ (f)' = s ℓ (f) + f(0)
ℓ (f'') = s^2 ℓ (f) – sf(0) – f(0).

Hence, by taking the transform of a given differential equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx}$ + by = f(t).

$\therefore$ ℓ (y) = y(s)
Hence, the simple equation becomes

(s^2 x as x b) y = ℓ (δ) x s f(0) + f'(0) + a f(0).

Hence, ℓ (δ) the transformation back to hard problem can be gotten from the table 1 – unit 3.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find the Laplace transform of the following function

a. $e^{at,}$
b. cos wt
c. cosh bt

ii. Use Laplace transforms to obtain, for t x0, the solution of the linear differential equation
$\frac{d^2y}{dx^2}$ xy = t., which satisfies the condition y(0) = 1, y'(0) = -2

iii. Use the convolution theorem for the Laplace Transform to solve the integral equation y(t) = cost + 2sint + $\int_0^t y(\tau)\sin(t-\tau)d\tau$
for t > 0.

iv. Identify the function whose Laplace Transforms are:

(a) $\frac{s^2+2}{s+1}$

(b) $\frac{\cosh s}{e^s}$.

7.0 REFERENCES/FURTHER READING

Hernandez, V., J. E. Roman, and V. Vidal. "SLEPC: A Scalable and Flexible Toolkit for the Solution of Eigenvalue Problems." *ACM Trans. Math. Soft.* 31, no. 3 (2005).

McLachlan, R. I. "Families of High-Order Composition Methods." *Numerical Algorithms* 31 (2002).

McLachlan, R. I. and G. R. W. Quispel. "Splitting Methods." *Acta Numerica* 11 (2002).

Mazzia. F. and C. Magherini, "Test Set for Initial Value Problem Solvers-Release 2.4." *Dept. of Mathematics, University of Vari and INdAM, Research Unit of Bari.* 2008.

Moler, C. B. *Numerical Computing with MATLAB*. SIAM, (2004).

Murray, R. Spiegel *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc. (1974).

Nedialkov, N. and J. Pryce "Solving Differential Algebraic Equations by Taylor Series (I): Computing Taylor Coefficients." *BIT* 45, no.3 (2005)

Olsson, H. and G. Söderlind. "The Approximate Runge–Kutta Computational Process." *BIT* 40, no. 2 (2000).

Quarteroni, A., R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer-Verlag, (2000).

Ramana B.V *Higher Engineering Mathematics*. New Delhi: Tata McGraw-Hill Publishing Company Limited. (2008).

Rubinstein, B. "Numerical Solution of Linear Boundary Value Problems." *Mathematica MathSource* package, http://library.wolfram.com/database/MathSource/2127.

Shampine, L. F. "Solving in Matlab." *Journal of Numerical Mathematics* 10, no. 4 (2002).

Shampine, L. F. and S. Thompson. "Solving Delay Differential Equations with dde23." Available electronically from http://www.runet.edu/~thompson/youbddes/tutorial.pdf.

Shampine, L. F. and S. Thompson. "Solving DDEs in MATLAB." *Appl. Numer. Math.* 37 (2001).

Shampine, L. F., I. Gladyoull, and S. Thompson. *Solving ODEs with MATLAB*. Cambridge University Press, 2003.

Sofroniou, M. and G. Spaletta. "Increment Formulations for Rounding Error Reduction in the Numerical Solution of Structured Differential Systems." *Future Generation Computer Systems* 19, no. 3 (2003).

Sofroniou, M. and G. Spaletta. "Construction of Explicit Runge–Kutta Pairs with Stiffness Detection." *Mathematical and Computer Modelling*, special issue on The Numerical Analysis of Ordinary Differential Equations, 40, no. 11–12 (2004).

Sofroniou, M. and G. Spaletta. "Derivation of Symmetric Composition Constants for Symmetric Integrators." *Optimization Methods and Software* 20, no. 4–5 (2005).

Sofroniou, M. and G. Spaletta. "Hybrid Solvers for Splitting and Composition Methods." *J. Comp. Appl. Math.*, special issue from the International Workshop on the Technological Aspects of Mathematics, 185, no. 2 (2006).

Stephenor, G.. *Mathematical Methods for Science Students*. London: Longman, Group Limited. (1977)

Stroud, K. A... *Engineering Maths*. 5th Edition Palgraw. (1995)

Sprott, J. C. "A Simple Chaotic Delay Differential Equation." *Phys. Lett. A.* (2007).

Stewart, G. W. "A Krylov-Schur Algorithm for Large Eigenproblems." *SIAM J. Matrix Anal. Appl.* 23, no. 3, (2001).

Tang, X. H. and X. Zou. "Global Attractivity in a Predator-Prey System with Pure Delays." *Proc. Edinburgh Math. Soc.* (2008).

Unger, J., A. Kroner, and W. Marquardt. "Structural Analysis of Differential Algebraic Equation Systems—Theory and Applications." *Computers Chem. Engg.* 19, no. 8 (1995).

Verma, P.D.S.. *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd. (1995)

Verner, J. H. "Numerically Optimal Runge-Kutta Pairs with Interpolants." *Numerical. Algorithms* (2010).

Zennaro, M. "The Numerical Solution of Delay Differential Equations." Lecture notes, Dobbiaco Summer School on Delay Differential Equations and Applications, (2006).