MODULE 2

Unit 1 Functions of Complex Variables
Unit 2 Integration of Complex Plane

UNIT 1 FUNCTIONS OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

CONCEPTS OF SETS IN THE COMPLEX PLANE

Definition: The term set of points in the complex plane is the collection of finite or infinite points. Examples: the points on a line, the solution of quadratic equation and the points in the interior of a circle made up of sets respectively.

A set is called open if every point of S has a neighbourhood consisting entirely of points that belongs to S. that is the points in the interior of a circle or a square from an open set, and so do the points of the "right half – plane" $\operatorname{Re} z = 0 > 0$.

An open set S is to be connected if any two of its points can be joined by a broken line of finitely many straight line segments all of where points belong to S.

Likewise, an open connected set is called a domain. Thus, an open disk annulus is domain. An open square with a diagonal removed is not a domain since this set is not connected.

The complement of a set S in the complex plane is defined to be the set of all points of the complex plane that do not belong to S. A set is said to be closed if its complements is open. Example: the point on and inside the unit circle form a closed set.

A boundary point of a set S is a point every neighbourhood of which contains both points that belong to S and points that do not belong to S.

Example: if a set S is open, then no boundary point belongs to S, if S is closed, then every boundary point belongs to S.

A region is a set consisting of a domain plus, perhaps, some or all of its boundary points.

Next we shall consider functions of complex variables but before this we introduce complex functions first.

Complex functions

Definition: A real function F defined on a set S of real numbers is a rule that assigns to every X in S a real number f(x), called the value of f at x. Now in complex, S is a set of complex numbers and a function f defined on S is a rule that assigns to every Z in ρ a complex number w, called the value of f at z. we write $w = f(z)$

Here z varies in S and is called a Complex Variable. The set S is called the domain of definition of f.

Example 1

$w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the range of f. w is a complex, and we write $w = u + iv$, where u and v are the real and the imaginary parts, respectively. Now w is depends on $z = x + iy$. Hence, u becomes a real function of x and y. and so does v. we may thus write:

$w = f(z) = u(x, y) + iv(x, y)$.

This shows that a complex function f(z) is equivalent to a pair of real functions u(x, y) and v(x, y), each depending on the two real variables x and y.

Example 2

Function of a complex variable.

Let $w = z^2 + 3z$. Find u and v and calculate the values of f at z = 1 + 3i and
z = 2 – i.

Let the real part of w be defined thus $u = x^2 - y^2 + 3x$ and the imaginary part of w i.e. $v = 2xy + 3y$.

$$\therefore f(1+3i) = (1+3i)^2 + 3(1+3i) = -5 + 15i$$

Recall that $i^2 = -1$.

Let $w = z^2 + 3z$. Find u and v and calculate the values of f at z = 2 – i.

Let the real part of w be defined thus $u = x^2 - y^2 + 3x$ and the imaginary part of w i.e. $v = 2xy + 3y$.

$$\therefore f(2-i) = (2-i)^2 + 3(2-i) = 9 - 5i$$

Recall that $i^2 = -1$.

Example 3

Function of a complex variable.

Let $w = z^2 + 5z.$ Find u and v and calculate the values of f at z = 2 – i.

Let the real part of w be defined thus $u = x^2 - y^2 + 3x$ and the imaginary part of w i.e. $v = 2xy + 3y.$

$$\therefore f\,(2 - \mathrm{i}) = (2 - \mathrm{i})^2 + 5(2 - \mathrm{i}) = 13 - 7i$$

Recall that $i^2 = -1$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- complex numbers;
- complex analytical function;
- Cauchy – Riemann equation;
- Cauchy's theorem and inequality;
- integral transforms vis a vis: Fourier and Laplace transforms; and
- convolution theory and their applications.

3.0 MAIN CONTENT

3.1 Complex Numbers

It was observed early in history that there are equations which are not satisfied by any real number. Examples are:

$$x^2 = -3 \qquad\qquad or \qquad x^2 - 10x + 40 = 0$$

This led to the invention of complex numbers.

Definition

A complex number z is an ordered pair (x, y) of real numbers x, y and we write

$$z = (x, y).$$

We call x the real part of z and y the imaginary part of z and write

$$\mathrm{Re}\, z = x, \qquad\qquad lm\, z = y$$

Example 4

Re (4, -3) = 4 and lm (4, -3) = -3,

Example 5
Identify the real part and the imaginary part in the equation

a. $z = 4 - 3i$; b. $z = -5 + 3i$

a. Re (z) = 4 and lm (z) = -3,
b. Re (z) = - 5 and lm (z) = 3,

Furthermore, we defined two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ to be equal if and only if their real parts are equal and their imaginary parts are equal.

$z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Addition of complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

1. $$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Multiplication of complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

2. $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

We shall say more about these arithmetic operations and discuss examples below, but we first want to introduce a much more convenient form of writing them as points in the plane.

3.1.1 Representation in the Form z = x + iy

A complex number whose imaginary part is zero is of the form (x, 0). For such numbers we simply have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$
$$and$$
$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for the real numbers. This suggests that we identify (x, 0) with the real number x. hence the complex number system is an extension of the real number system.

The complex number (0,1) is denoted by i.

$$i = (o, 1)$$

and is called the imaginary unit. We show that it has the property.

3. $i^2 = -1$

Indeed, from (2) we have

$i^2 = (0.1)(0.1) = (-1,0) = -1$ *futheremore, for every real y we obtain from* (2)

$$iy = (0,1)(y,0) = (0, y)$$

Combining this with the above x = (x, 0) and using (1), that is,

$$(x, y) = (x, 0) + (0, y),$$

We see that we can write every complex number z = (x, y) in the form

$$z = x + iy$$

or z = x + yi. This is done in practice almost exclusively.

Example 6

Complex Numbers, their Real and Imaginary Parts

$z = (4, -3) = 4 - 3i,$ $\quad \operatorname{Re}(4 - 3i) = 4,$ $\quad \lim(4 - 3i) = -3$

$z = \left(\frac{-1}{2}, 0\right) = \left(\frac{-1}{2} + 0i\right),$ $\quad \operatorname{Re}\left(\frac{-1}{2}\right) = \frac{-1}{2},$ $\quad \lim\left(\frac{-1}{2}\right) = 0$

$z = (0, \pi) = 0 + \pi i,$ $\quad \operatorname{Re}(\pi i) = 0,$ $\quad \lim(\pi i) = \pi$

3.1.2 Complex Plane

This is a geometric representation of complex numbers as points in the plane. It is of great importance in applications. This idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal x – axis, called the real axis, and the vertical y – axis called the imaginary axis. On both axes we choose the same unit of length (Fig. 4). This is called a **Cartesian coordinate system.** We now plot z = (x, y) = x + iy as the point P with coordinates x, y. The xy – plane in which the complex numbers are represented in this way is called the **complex plane** or *Argand diagram.* Figure 5 shows an example.

Instead of staying ***"the point represented by z in the complex plane"*** we say briefly and simply "the point z in the complex plane" this will cause no misunderstandings.

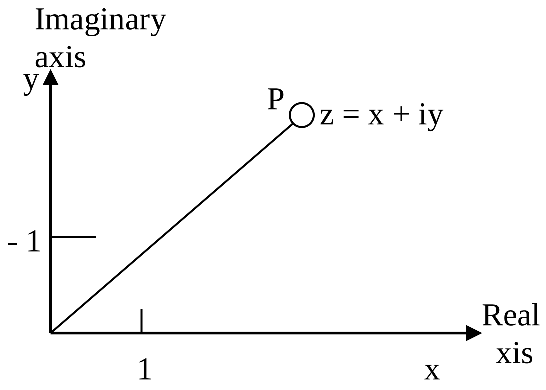


Fig.4: The Complex Plane

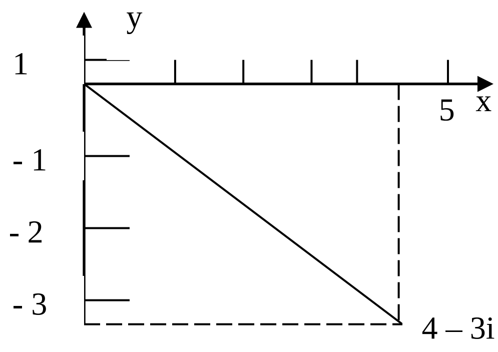


Fig. 5: The number 4 – 3i in the Complex Plane

3.1.3 Arithmetic Operations

We can make use of the notations z = x + iy and of the complex plane. Addition of the sum of $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ can now be written

4. $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2).$
$z_1 + z_2 = (x_1 + x_2) + (iy_1 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Example 7

a. $(5+i)+(1+3i) = (5+1)+(i+3i) = 6+4i.$
b. $(-3+i)+(3+2i) = (-3+3)+(i+2i) = 0+3i.$
c. $(4-i)+(-6-3i) = (4-6)+(-i-3i) = -2-4i.$

We see that addition of complex numbers is in accordance with the "parallelogram law" by which forces are added in mechanics.

Subtraction is defined to be the inverse operation of addition. That is the difference $z = z_1 - z_2.$

5. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$

Example 8

a. $(5+i)-(1+3i) = (5-1)+(i-3i) = 4-2i$
b. $(-3+i)-(3+2i) = (-3-3)-(i-2i) = -6+i$
c. $(4-i)-(-6-3i) = (4+6)-(-i+3i) = 10-2i$

Multiplication: The Product z_1z_2 in (2) can now be written

6. $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$
$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

This is easy to remember since it is obtained formally by the rules of arithmetic for real numbers and using (3), that is $i^2 = -1$

Example 9

a. $(5+i)(1+3i) = 5+15i + i + 3i^2 = 2+16i$
b. $(3+i)(2-3i) = 6-9i + 2i - 3i^2 = 9-7i$
c. $(-2-i)(1-5i) = -2+10i - i + 5i^2 = -7+9i$

Division is defined to be the inverse operation of multiplication. That is, the quotient z = z_1 / z_2 is the complex number z = x + iy for which

7. $z_1 = zz_2 = (x+iy)(x_2 + iy_2) \qquad (z_2 \neq 0).$

We show that for $z_2 \neq 0$ *the qutotient* $z = x + iy = z_1 / z_2$ is given by

8. $$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_1 + iy_2)(x_2 - iy_2)}$$

where $(x_2 - iy_2)$ is the conjugate of $(x_2 + iy_2)$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

Example 10

a. If $z_1 = 9 - 8i$ and $z_2 = 5 + 2i$, then

$$z = \frac{z_1}{z_2} = \frac{9 - 8i_1}{5 + 2i} = \frac{(9 - 8i)(5 - 2i)}{(5 + 2i)(5 - 2i)}$$

$$= \frac{45 - 18i - 40i - 16}{25 + 4} = \frac{29 - 58i}{29} = 1 - 2i. = z$$

The reader may check this result by showing that

$zz_2 = (1 - 2i)(5 + 2i) = 9 - 8i = z_1.$

b. If $z_1 = 3 - 2i$ and $z_2 = 5 + 2i$, then

$$z = \frac{z_1}{z_2} = \frac{3 - 2i}{5 + 2i} = \frac{(3 - 2i)(5 - 2i)}{(5 + 2i)(5 - 2i)}$$

$$= \frac{15 - 6i \ - 10i \ + 4}{25 \ + \ 4} = \frac{19 - 16i}{29} = \frac{19}{29} - \frac{16i}{29} = \text{z}$$

The reader may check this result by showing that

$$zz_2 = \left(\frac{19}{29} - \frac{16i}{29}\right)(5 + 2i) = 3 - 2i = z_1.$$

3.1.4 Properties of the Arithmetic Operations

From the familiar laws for real numbers we obtain for any complex numbers z_1, z_2, z_3, z the following laws (where z = x +iy):

$z_1 + z_2 = z_2 + z_1$……….commutative law of addition
$z_1 z_2 = z_2 z_1$ ……………… commutative law of multiplication

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$...associative law of addition
$(z_1 z_2) z_3 = z_1 (z_2 z_3)$……………………associative law of multiplication

9. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ ……distributive law

$$0 + z = z + 0 = z$$
$$z + (-z) = (-z) + z = +z - z = 0$$
$$z \,.\, 1 = z$$

3.1.5 Complex Conjugate Numbers

Let z = x + iy be any complex number. Then x – iy is called the conjugate of z and is denoted by $\bar{z}$, thus,

$$z = x + iy, \qquad \bar{z} = x - iy.$$

Example 11

The conjugate of $z = 5 + 2i$ is $\bar{z} = 5 - 2i$.

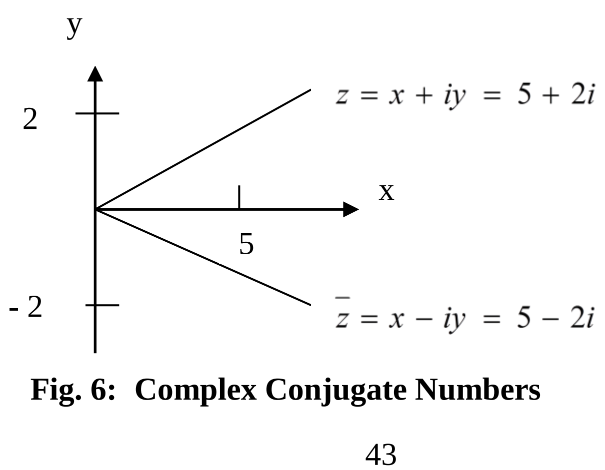


Fig. 6: Complex Conjugate Numbers

Conjugates are useful since $z\bar{z} = x^2 + y^2$ is real, a property we have used in the above division. Moreover, addition and subtraction yields $z + \bar{z} = 2x, \quad z - \bar{z} = 2iy,$ so that we can express the real part and the imaginary part of z by the important formulas.

10. $\quad \text{Re}\, z = x = \frac{1}{2}\left(z + \bar{z}\right), \qquad lm\, z = y = \frac{1}{2i}\left(z - \bar{z}\right)$

Example 12

If z = 6 – 5i, then we have $\bar{z}$ = 6 + 5i and from (10) we obtain

$$x = \frac{1}{2}\left(6 - 5i + 6 + 5i\right) = 6 \ \ and$$

$$y = \frac{1}{2i}(6 - 5i - 6 - 5i) = \frac{1}{2i}(0 - 10i)$$

$$= \frac{-10i}{2i} = -5$$

z is real if and only if y = 0, hence $\bar{z}$ = z by (10).
z is said to be pure imaginary if and only if x = 0, hence $\bar{z}$ = -z. Then working with conjugates is easy, since we have

11. $$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \quad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

In this section we were mainly concerned with complex numbers, their arithmetic operations and their representation as points in the complex plane. The next section we shall discuss the use of polar coordinates in the complex plane and situations in which **polar coordinates** are advantageous.

3.2 Polar Form of Complex Number Powers and Roots

It is often practical to express complex numbers z = x + iy in terms of polar coordinates r, θ, these are defined by:

1. $\quad x = r\cos\theta, \ y = r\sin\theta$

By substituting this we obtain the polar form of z,

2. $\quad z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

r is called the absolute value or modulus of z and is denoted by $|z|$. Hence

3. $\quad |z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 7).

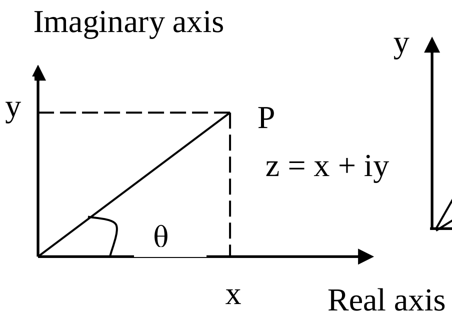
Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 301).

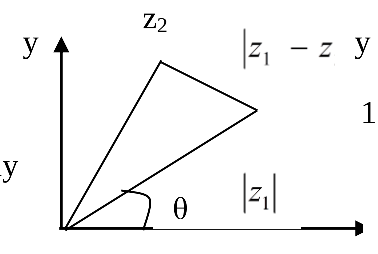
θ is called the **argument** of z and is denoted by arg z. thus (Fig. 7).

4. $\theta = \arg\ z = arc\tan\dfrac{y}{x}$ $(z \neq 0)$.

Geometrically, θ is the directed angle from the positive x – axis to OP in fig. 7. Here, as in calculus, all angles are measured in radians and positive in the counterclockwise series.

$|z| = r$





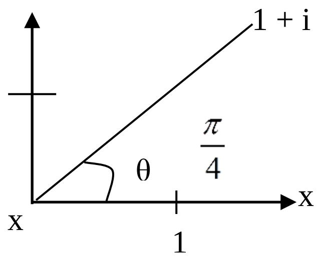


Fig. 7: Complex Plane, Polar Form of a Complex Number

Fig. 8: Distance between two points Complex Number

Fig. 9: Example 1

For z = 0 this angle θ is undefined. (Why?) For given z ≠ 0 it is determined only up to integer multiples of 2π. The value of θ that lies in the interval – π < θ ≤ π is called the principal value of the argument of z (≠ 0) and is denoted by Arg. z. Thus θ = Arg z satisfies by definition.

$-\pi$ < Arg z ≤ π.

Polar Form of Complex Numbers Principal Value

Example 11

Let z = 1 + I (cf. Fig. 9). Then

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),\ |z| = \sqrt{2},\ \ \arg z = \frac{\pi}{4} \pm 2n\pi \quad (n = 0, 1, \ldots\ldots..0.)$$

The principal value of the argument is arg z = π/4, other values are -7π/4, 9π/4, etc.

Example 12

Let z = $3 + 3\sqrt{i}$, *then* $z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, the absolute value of z is $|z| = 6$, and the principal value of arg z is Arg z = π/3.

Caution! In using (4), we must pay attention to the quadrant in which z lies, since tan θ has period π, so that the arguments of z and –z have the same tangent. Example: for θ_1 = arg (1+i) and θ_2 = arg (-1 – i) we have tan θ_1 = tan θ_2 = 1.

Triangle Inequality

For any complex numbers we have the importance **triangle inequality**

5. $|z_1 + z_2| \leq |z_1| + |z_2|$ (*Fig*. 303)

Which we shall use quite frequently, this inequality follows by nothing that

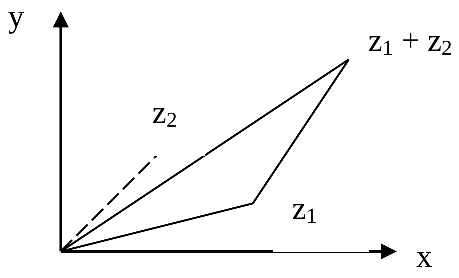


Fig 10: Triangle Inequality

The three points 0, z_1 and $z_1 + z_2$ are the vertices of a triangle (fig. 10) with sides $|z_1|, |z_2|$ *and* $|z_1 + z_2|$, and the side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob.45).

Example 13

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 \angle \sqrt{2} + \sqrt{13} = 5.020.$$

By induction the triangle inequality can be extended to arbitrary sums:

Example 14

If $z_1 = 5 + 3i$ and $z_2 = -2 + 3i$, then

$$|z_1 + z_2| = |3 + 6i| = \sqrt{27} = 5.196 \angle 4 + \sqrt{4} = 5.020.$$

By induction the triangle inequality can be extended to arbitrary sums:

6. $|z_1 + z_2 + ...+ z_n| \leq |z_1| + |z_2| + ...+ |z_n|;$

That is, the absolute value of a sum cannot exceed the sum of the absolute values of the terms.

3.2.1 Multiplication and Division in Polar Form

This will give us a better understanding of multiplication and division. Let:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \textit{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then, by (6), sec. 12.1, the product is at first

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta 2)].$$

The addition rules for the sine and cosine (6) in appendix 3.1) now yield

7. $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Taking absolute values and arguments on both sides, we thus obtain the important rules

8. $|z_1 z_2| = |z_1|\,|z_2|$

and

9. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ $\quad$ (*up to multiples of* 2π).

We now turn to division. The quotient $z = \frac{z_1}{z_2}$ is the number z satisfying $zz_2 = z_1$. Hence $|zz_2| = |z|\,|z_2| = |z_1|, \arg(zz_2) = \arg z + \arg z_2 = \arg z_1$.

This yield

10. $\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$ $\quad$ $(z_2 \neq 0)$

and

11. $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$ $\quad$ (*up to multiples of* 2π).

By combining these two formulas (10) and (11) we also have

12. $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

Example 15

Illustration of Formulas (8) – (11)

Let $z_1 = -2 + 2i$ *and* $z_2 = 3i$. *Then* $z_1 z_2$
$= -6 - 6i,\ z_1/z_2 = 2/3 + (2i/3)$
and for the arguments we obtain Arg $z_1 = 3\pi/4$, Arg $z_2 = \pi/2$.

$$Arg\ z_1 z_2 = \frac{-3\pi}{4} = Arg\ z_1 + Arg\ z_2 - 2\pi$$

$$Arg\ (z_1/z_2) = \frac{\pi}{4} = Arg\ z_1 - Arg\ z_2$$

Integer power of z

From (7) and (12) we have

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta),$$

$$z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)]$$

and more generally, for any integer n,

13. $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Example 16

Formula of De Moivre

For $|z|$ = r = 1, formula (3) yields the so – called formula of De Moivre

(13*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

This formula is useful for expressing cos nθ in terms of cos θ and sin θ. For instance when n = 2 and we take the real and imaginary parts on both sides of (13*), we get the familiar formulas.

$$\cos 2\theta = \cos^2\theta - \sin^2, \quad \sin 2\theta = 2\cos\theta\sin\theta.$$

3.2.2 Roots

If $z = w^n\ (n = 1, 2, ...)$, then to each value of w there corresponds one value of z, we shall immediately see that to a given z ≠ 0 there correspond precisely *n* distinct values of w. each of these values is called an nth root of z, and we write:

14. $w = \sqrt[n]{z}.$

Hence this symbol is multivalued, namely, n – valued, in contrast to the usual conventions made in real calculus. The n value of $\sqrt[n]{z}$ can easily be determined as follows. In terms of polar forms for z and

$$w = R(\cos\phi + i\sin\phi),$$

The equation $w^n = z$ becomes

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\phi + i\sin\theta)$$

By equating the absolute values on both sides we have

$$R^n = r, \text{ thus } R = \sqrt[n]{r}$$

Where the root is real positive and thus uniquely determined. By equating the arguments we obtain

$$n\phi = \theta + 2k\pi, \quad \text{thus } \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

Where k is an integer. For k = 0,1, …, n – 1 we get n distinct values of w. further integers of k would give values already obtained. For instance, k = n gives 2kπ/n = 2π, hence the w corresponding to k = 0, etc. consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

15. $$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) k = 0,1,\ldots n-1.$$

These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides.

The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and k = 0 in (15) is called the principal value of $w = \sqrt[n]{z}$

Example 17

Square Root

From (15) it follows that $w = \sqrt{z}$ has the two values

16a $$w_1 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and

16b. $$w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1$$

Which lie symmetric with respect to the origin. For instance, the square root of 4i has the values $\sqrt{4i} \quad \pm 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm\left(\sqrt{2} + i\sqrt{2}\right)$.

From (16) we can obtain the much more practical formula

17. $$\sqrt{z} = \pm\left[\left(\sqrt{\frac{1}{2}(|z|) + x}\right) + (sign\, y)i\sqrt{\frac{1}{2}(|z| - x)}\right]$$

Where sign y = 1 if y ≥0, sign y = - 1 if y < 0, and all square toots of positive numbers are taken with the positive sign. This follows from (16) if we use the trigonometric identities.

$$\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1+\cos\theta)n} \quad \sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1-\cos\theta)}.$$

Multiply them by $\sqrt{r}$.

$$\sqrt{r}\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(r\, r\cos\theta)}, \quad \sqrt{r}\sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(r - r\cos\theta)},$$

Use r cos θ = x, and finally choose the sign of 1m $\sqrt{z}$ so that sign

$$\left[(\mathrm{Re}\sqrt{z})(1m\sqrt{z})\right] = \text{sign y (why?).}$$

Example 18

Complex Quadratic Equation

Solve $z^2 - (5+i)z + 8 - i = 0$

Solution

$$z = \frac{1}{2}(5+i) \pm \sqrt{\frac{1}{4}(5+i)^2 - 8 - i} = \frac{1}{2}(5+i) \pm \sqrt{-2 + \frac{3}{2}i}$$

$$= \frac{1}{2}(5+i) \pm \left|\sqrt{\frac{1}{2}\left(\frac{5}{2} + (-2)\right)} + i\sqrt{\frac{1}{2}\left(\frac{5}{2} - (-2)\right)}\right|$$

$$= \frac{1}{2}(5+i) \pm \left[\frac{1}{2} + \frac{3}{2}i\right]$$

$$= \begin{cases} 3+2i \\ 2-i \end{cases}$$

Example 19

Cube Root of a Positive Real Number

If z is positive real, then $w = \sqrt[3]{z}$ has the real value $\sqrt[3]{r}$ and the complex values

$$\sqrt[3]{r}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt[3]{r}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$and \quad \sqrt[3]{r}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \sqrt[3]{r}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right).$$

For instance $\sqrt[3]{1} = 1, \frac{-1}{2} \pm \frac{1}{2}\sqrt{3i}$ $(fig.304)$. These are the roots of the equation $w^3 = 1$.

Example 20

***n*th Root of Unity**

Solve the equation $z^n = 1$.

Solution

From (15) we obtain

18. $$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n} \qquad k = 0,1,....,n-1.$$

If w denotes the value corresponding to k = 1, then the n values of $\sqrt[n]{1}$ can be written as 1, w, w^2, …, w^{n-1}. These values are the vertices of a regular polygon of n sides inscribed in the unit circle, with one vertex at the point 1. Each of these n values is called an nth root of unity. For instance, $\sqrt[4]{1}$ has the values 1, i, -1 and –i (Fig. 12 shows $\sqrt[5]{1}$). If w_1 is any nth root of an arbitrary complex number z, then the n values of $\sqrt[n]{z}$ are w_1, w_1w, w_1w^2, …. w_1w^{n-1}

Multiplying w_1 by w^k corresponds to increasing the argument of w_1 by $2k\pi/n$.

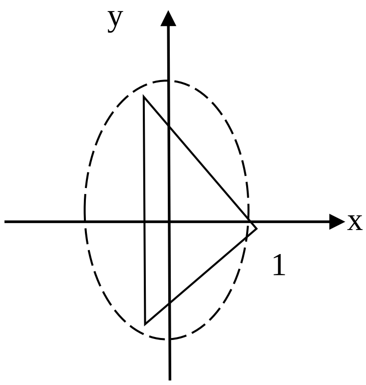


Fig 11. $\sqrt[3]{1}$

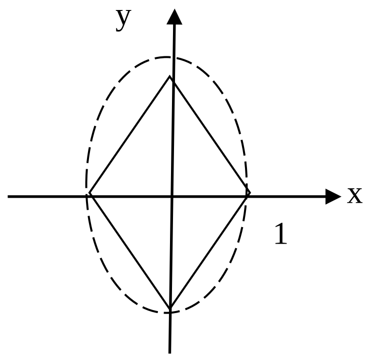


Fig 12. $\sqrt[4]{1}$

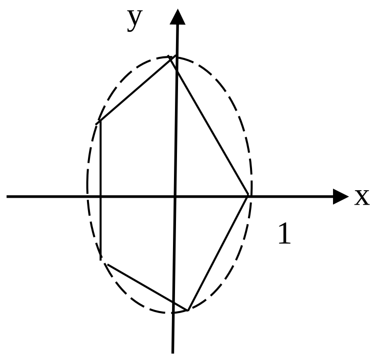


Fig 13. $\sqrt[5]{1}$

The student should be familiar with the problems related to the polar representation with particular care, since we shall need this representation quite often in our work. In the next section, we discuss some curves and regions in the complex plane which we shall also need in the chapters on complex analysis.

3.3 Curves on Regions in the Complex Plane

In this section we consider some important curves and regions and some related concepts we shall frequently need. This will also help us to become more familiar with the complex plane.

The distance between two points z and a is $|z-a|$. Hence a circle C of radius ρ and center at a (fig. 14) can be represented by;

1. $|z-a| = \rho.$

In particular, the so-called unit, that is the circle of radius 1 and center at the origin a = 0 (fig. 308), is given by;

$|z| = 1.$

Furthermore, the inequality

2. $|z-a| < \rho$

holds for every point z inside C: that is, (2) represents the interior of C. Such a region is called a circular disk or, more precisely, an open circular disk, in contrast to the closed circular disk.

$|z-a| \leq \rho.$

This consists of the interior of C and C itself. The open disk (2) is also called a neighborhood of the point a. Obviously, a has infinitely many such neighborhoods, each of which corresponds to a certain value of p

(> 0); and a belong to each of these neighborhoods, that is a, is a point of each of them.

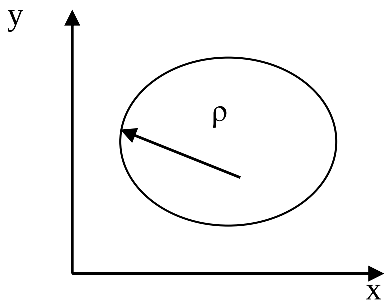


Fig 14. Circle in the Complex Plane

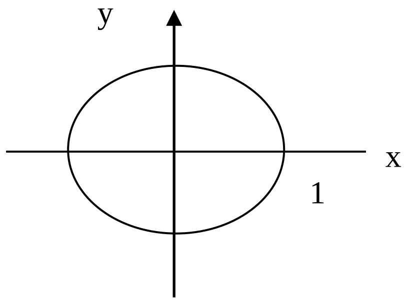


Fig 15. Unit Circle

Similarly, the inequality

$$|z-a| \quad \rho.$$

represents the exterior of the circle C. Furthermore, the region between two concentric circles of radii ρ_1 and ρ_2 (> ρ_1) can be represented in the form

3. $\rho_1 < |z-a| < \rho.$

Where a is the center of the circles. Such a region is called an open circular ring or open annulus (Fig. 16).

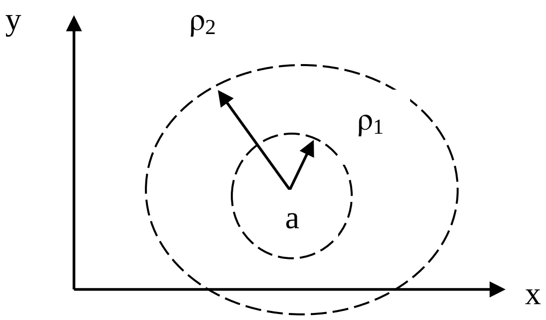


Fig 16. **Annulus in the Complex Plane**

Example 20

Circular Disk

Determine the region in the complex plane given by $|z-3+i| \leq 4$.

Solution: the inequality is valid precisely for all z whose distance from a = 3 – i does not exceed 4. Hence this is a closed circular disk of radius 4 with center at 3 – i.

Example 21

Unit Circle and Unit Disk

Determine each of the regions

(a) $|z| < 1$ (b) $|z| \leq 1$ (c) $|z| > 1$.

Solution

(a) The interior of the unit circle. This called the open unit disk.
(b) The unit circle and its interior. This is called the closed ad disk.
(c) The exterior of the unit circle.

By the (open) upper half we mean the set of all points z = x + iy such that $y > 0$. Similarly, the condition y < 0 defines the lower half – plane, $x > 0$ the right half – plane and x < 0 the left half – plane.

3.3.1 Some Concepts Related to Sets in the Complex Plane

We finally list a few concepts that are of general interest and will be used in our further work.

The term set of points in the complex plane means any sort of collection of a quadratic equation. The points on a line and the points in the interior of a circle are sets.

A set S is called open, if every point of S has a neighborhood consisting entirely of points that belong to S. for example, the neighborhood consisting entirely of points that belong to S. For example, the points in the interior of a circle or a square form an open set, and so do the points of the "right half – plane" Re z = x > 0.

An open set S is said to be connected if any two of its points can be joined by a broken line of finitely many straight line segments all of whose points belong to S. an open connected set is called a domain. Thus an open disk (2) and an open annulus (3) are domains. An open square with a diagonal removed is not a domain since this set is not connected. (Why?).

The complement of a set S in the complex plane is defined to be the set of all points of the complex plane that do not belong to S. A set is called closed if its complement is open. For example, the points on and inside the unit circle form a closed set ("closed unit disk" cf. example 2) since its complement $|z|$ > is open.

A boundary point of a set S is points every neighbourhood of which contains both points that belong to S and points that do not belong to S. For example; the boundary points of an annulus are the points on the two bounding circles.

Clearly, if a set S is open, then no boundary point belongs to S; is closed, and then every boundary point belongs to S.

A region is a set of a domain plus, perhaps, some or all of its boundary points. (The reader is warned that some authors use the term "region" for what we call a domain (following the modern standard terminology) and others make no distinction between the two terms.)

So far, we have been concerned with complex numbers and the complex plane (just as at the beginning of calculus, one talks about real numbers and the real line). In the next section, we start doing complex calculus: we introduce complex functions and derivatives. This will generalise familiar concepts of calculus.

SELF ASSESSMENT EXERCISE 1

Determine and sketch the sets represented by

1. $|z-2i| = 2$
2. $1 \le |z+1-i| \le 3$
3. $\operatorname{Re}(z^2) \le 1$
4. $|\arg z| < \frac{\pi}{4}$
5. $-\pi < 1m\, z \le \pi$
6. $\left|\frac{1}{z}\right| < 1$
7. $\left|\frac{z+1}{z-1} = 1\right|$
8. $\left|\frac{z+3i}{z-i}\right| = 1$
9. $lm \frac{2z+1}{4z-4} \le 1$
10. $z\bar{z} + (1+2i)z + (1-2i)\bar{z} + 1 = 0.$

3.4 Limit, Derivative and Analytic Functions

The functions with which complex is concerned are complex functions that are differentiable. Hence, we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be quite similar to that in calculus.

3.4.1 Complex Function

Recall from the calculus that a real function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number f(x) called the value of f at x.

Now in complex, S is a set of complex numbers. And a function f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z. write

$$w = f(z)$$

Here z varies in S and is called a complex variable. The set S is called the domain of definition of f.

Example 21

$w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the range of f.

W is complex, and we write w = u + iv, where u and v are the real and imaginary parts, respectively. Now w depends on z = x + iy. Hence u becomes a real function; of x and y, and so does v. We may thus write:

$$w = f(z) = u(x, y) + iv(x, y).$$

This shows that a complex function f(z) is equivalent to a pair of real functions u(x,y) and depending on the two real variables x and y.

Example 22

Function of a Complex Variable

Let w = f(z) = z^2 +3z. Find u and v and z = 2 –i.

Solution

$u = \text{Re } f(z) = x^2 + y^2 + 3x \, and \, v = 2xy + 3y$, also,

$$f(1+3i) = (1+3i)^2 + 3(1+3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i$$

This shows that u(1,3) = -5 and v (1,3) = 15, similarly.

$$f(2-i) = (2-i)^2 + 3(2-i) = 4i + 6 - 3i = 9 - 7i.$$

Example 23

Function of a Complex Variable

Let $w = f(z) = 2iz + 6\bar{z}.$ Find u and v and the value for f at $z = \frac{1}{2} + 4i$

Solution $f(z) = 2i(x + iy) + 6(x - iy)$

gives

$u(x, y) = 6x - 2y$ and $v(x, y) = 2x - 6y.$

Also

$$f\left(\frac{1}{2} + 4i\right) = 2i\left(\frac{1}{2} + 4i\right) + 6\left(\overline{\frac{1}{2} - 4i}\right) = i - 8 + 3 - 24i = -5 - 23i.$$

Limit, Continuity

A function f(z) is said to be limit l as z approaches a point z_0, written

1. $\lim_{z \to z_0} (f(z) = l$

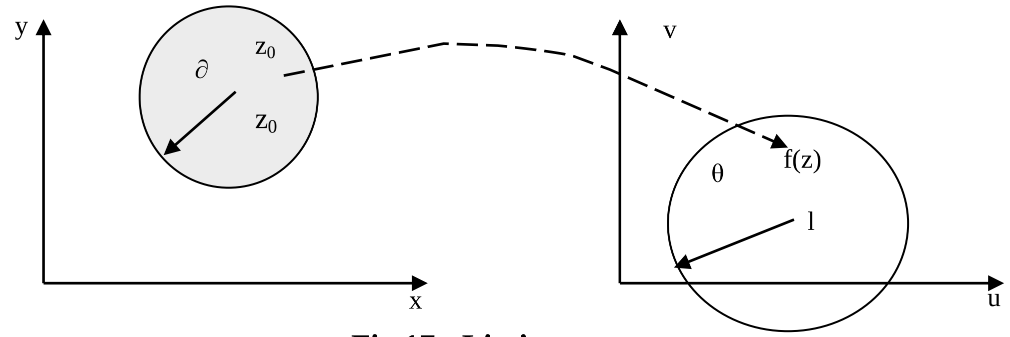


Fig 17: Limit

If f is defined in a neighborhood of z_0 (except itself) and if the values of f are "close" to l for all z "close" to z_0; that is, in precise terms, for every positive real $\in$ we can find a positive real ∂ such that for $z \neq z_0$ in the disk $|z - z_0| < \partial$ $(Fig.310)$ we have

2. $|f(z) - l| < \in;$

That is, for every $z \neq z_0$ in that the value of f lies in the disk (2).
Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real line, here, by definition, z may approach z_0 from any direction in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique. (Cf. Prob. 30)

A function f(z) is said to be continuous at z = z_0 if f(z_0) is defined and

3. $\lim_{z \to z_0} f(z) = f(z_0)$.

Note that by the definition of a limit this implies that f(z) is defined in some neighbourhood of z_0.

f(z) is said to be continuous in a domain if it is continuous at each point of this domain.

3.4.3 Derivative

The derivative of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

4. $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. Then f is said to be differentiable at z_0. if we write $\Delta z = z - z_0$ we also have

(4') $$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remember that this definition of a limit implies that f(z) is defined (at least) in a neighborhood of z_0. Also by that definition, z may approach z_0 from any direction. Hence differentially at z_0 means that, along whatever path z approaches z_0, the quotient in (4') always approaches a certain value and all these values are equal. This is important and should be kept in mind.

Example 24

Differentiability Derivatives

The function f(z) = z^2 is differentiate for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z.$$

The differentiation rules are the same as in real calculus, since their proofs are literally the same. Thus,

$$cf' = cf' \; (f+g)' = f' + g', \; (fg)' = f'g + fg', \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

As well as the chain rule and power rule $(z^n)' = nz^{n-1}$ (n integer) hold. Also, if f (z) is differentiable at z_0. It is continues s at z_0. (Cf. Prob. 34).

Example 25

$\bar{z}$ **not differentiable**

It is important to note that there are many simple functions that do not have a derivative at any point. For instance, $f(z) = \bar{z} = x - iy$ is such a function? Indeed, we write $\Delta z = \Delta x + i\Delta y, we\,have$

5. $$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\left(\overline{z+\Delta z}\right) - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y.}$$

but -1 along path II. Hence, by equation of (5) at $\Delta z \to 0$ does not exit at any z.

This example may be surprising, but it merely illustrates that differentiability of a complex function is a rather serve requirement. The idea of proof approach form different directions is based and will be discussed again in the next section.

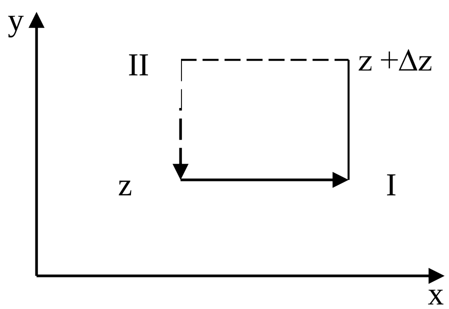


Fig. 18: Paths in (5)

3.4.1 Analytic Functions

These are the functions that are differentiable in some domain, so that we can do "calculus in complex." They are the main concern of complex analysis. Their introduction is our main goal in this section;

Definition (Analyticity)

A function f(z) is said to be analytical in a domain D if f(z) is defined and differentiable at all points of D. The function f(z) is said to be analytic at a point $z = z_0$ in D if f(z) is analytic in a neighbourhood (cf. sec. 12.3) of z_0.

Also, by analytical function we mean a function that is analytical in some domain.
Hence, analytical of f(z) at z_0 means that f (z) has a derivative at every point in some neighbourhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighbourhood). This concept is motivated by the fact that it is of no practical interest when a function is differentiable merely at a single point z_0 but not throughout some neighbourhood of z_0. Problem 28 gives an example.
An older term for analytical in D is regular in D, and a more modern term is holomorphic in D.

Example 26

Polynomids Rational Functions

The integer power 1, z, z^2, … and more generally, polynomials, that is function of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + ... c_n z^n$$

Where c_i, and i=1,2,3…. are complex constants, are analytical in the entire complex plane. The quotient of two polynomials g(z) and h(z).

$$f(z) = \frac{g(z)}{h(z)}.$$

is called a rational function. This f is analytic except at the points where *h*(*z*)= 0 here we assume that common factors of g and h have been cancelled partial fractions

$$\frac{c}{(z - z_0)^m} \qquad (c \neq 0)$$

(c and z_0 complex, m is a positive integer) are special rational functions, they are analytic except at z_0. It is in algebra that every rational function can be written as a sum of a polynomial (which may be 0) and finitely partial fractions.

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function. Indeed, complex analysis is concerned exclusively with analytic functions and although many will yield a branch of mathematics, that is most beautiful from the theoretical point of view and most useful for practical purposes.

Before we consider special analytic functions (exponential functions, cosine, sine etc.) let us give equations by means of which we can readily decide whether a function is analytic or not. These are the famous Cauchy–Riemann equation, which we shall discuss in the next section.

3.5 Cauchy – Riemann Equations

We shall now derive a very important criterion (a test) for the analyticity of a complex function.

$$w = f(z) = u(x, y) + i(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two equations

1. $u_2 = u_{y,}$ $\quad u_y = -v_2$.

Everywhere in D, here $u_x = \partial u / \partial x$, $u_y = \partial u / \partial y$ and similarly for u_x and u_y which are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorem 1 and 2 below. The equation (1) is called the Cauchy – Riemann equations. They are the most important equations in the whole unit.

Example 27

$f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z, and

$$u = x^2 - y^2 \; and \; v = 2xy$$

Satisfy (1), namely, $u_x = 2x = v_y \; and \; u_y = -2y = -v_x$ more examples will follow.

3.5.1 Theorem 1 (Cauchy Riemann Equations)

Let f(z) = u(x,y) + iv(x,y) be defined and continuous in some neighbourhood of a point z = x + iy and differentiable at z itself. Then at the point, the first – order partial derivatives of u and v exist and satisfy the Cauchy Riemann equations (1).

Hence if f(z) is analytic in a domain f' (z) at z exists. It is given by (1) at all points of D.

Proof

By assumption, the derivative f' (z) at z exists. It given by

2. $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

The idea of the proof is very simple, by the definition of a limit in complex (cf. sec. 12.4) we can let Δz approaches zero along any path in a neighbourhood of z. Thus, we may choose the two paths I and II in fig. 312 and equate the results. By comparing the real parts we shall obtain the first Cauchy Riemann equation and by comparing the imaginary parts we shall obtain the other equation in (1). The technical details are as follows.

We write Δz = Δx +iΔy. In terms of u and v, the derivative in (2) becomes

3. $$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

We first choose path I in fig. 312. Thus we let $\Delta y \to 0$ *first and then* $\Delta x \to 0$.

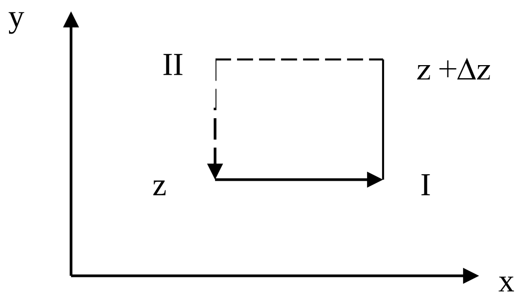


Fig. 19: Paths in (2)

After Δy becomes zero, Δz = Δx. then (3) becomes, if we first write the two u – terms and then two v-terms.

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

Since f'(z) exists, the two real limits on the right exist. By definition, they are the partial derivatives of u and v with respect to x. hence the derivative f' (z) of f(z) can be written

4. $$f'(z) = u_x + iv_x$$

Similarly, if we choose path II in fig 312, we let $\Delta x \to 0$ first and then $\Delta y \to 0$. After Δx becomes zero, Δz = iΔy, so that from (3) we now obtain

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x\,, y+\Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta\, y) - v(x, y)}{i\Delta y}$$

Since f'(z) exists, the limits on the right exist and yield partial derivatives with respect to y; noting that 1/i = -i, we obtain:

5. $f'(z) = -iu_y + v_y$

The existence of the derivatives f'(z) thus implies the existence of the four partial derivatives in (4) and (5). By equating the real parts u_x and v_y in (4) and (5) we obtain the first Cauchy – Riemann equation (1). Equating the imaginary part yields the other. This proves the first statements of the theorem and implies the second because of the definition of analyticity.

Formulas (4) and (5) are also quite practical for calculating derivatives f'(z), as we shall see.

Examples 28

Cauchy – Riemann Equations

$f(z) = z^2$ is analytic for all z. it follows that the Cauchy – Riemann equations must be satisfied (as we have verified above).

For $f(z) = \bar{z} = x - iy$ we have u = x, v = -y and see that the second Cauchy-Riemann equation is satisfied, u_y= - v_x = 0, but the first is not: $u_x = 1 \neq v_y = -1$. We conclude that $f(z) = \bar{z}$ is not analytic, confirming example 4 of sec. 12.4. Note the savings in calculation!

The Cauchy – Riemann equations are fundamental because they are not only necessary but also sufficient for a function to be analytic. More precisely, the following holds.

Theorem 2 (Cauchy – Riemann Equations)

If two real – valued continuous functions u(x,y) and v(x,y) of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy – Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

The proof of this theorem is more involved than the previous proof;
Theorems 1 and 2 are of great practical importance, since by using the Cauchy – Riemann equations we can now easily find out whether or not a given complex function is analytic.

Example 29

Cauchy – Riemann Equations

Is $f(z) = z^3$ analytic?

Solution

We find $u = x^3 – 3xy$ and $v = 3x^2y – y^3$. next we calculate
$u_x = 3x^2 – 3y^2$, $v_y = 3x^2 – 3y^2$
$u_y = -6xy$, $v_x = 6xy$

We see that the Cauchy – Riemann equations are satisfied for every z, hence $f(z) = z^3$ is analytic for every z, by theorem 2.

Example 30

Determination of an Analytic Function with given Real Part

We illustrate another class of practical; that can be solved by the Cauchy – Riemann equations.

Find the most general analytic function f(z) whose real part is

$$u = x^3 – y^2 – x.$$

Solution

We have $u_z = 2x – 1 = v_y$ by the first Cauchy – Riemann equation. This we integrate with respect to y;

$$v = 2xy - y + k(x).$$

As an important point, since we integrated a partial derivative with respect to y, the "constant" of integration k may depend on the other variable, x. (To understand this, calculate v_y from the v.) and the second Cauchy – Riemann equation.

$$u_y = -v_x = -2y + \frac{dk}{dx}$$

On the other hand, from the given $u = x^2 – y^2 – x$ we have $u_y = -2y$. By comparison, dk/dx = 0. Hence k = constant, which must be real. (Why?).

The result is

$$f(z) = u + iv = x^2 - y^2 - x + i(2xy - y + k).$$

We can express in terms of z, namely, f(z) = $z^2 - z + ik$.

Example 31

An Analytic Function of Constant Absolute Value is Constant

The Cauchy – Riemann equations also help to establish general properties of analytic functions.

For example, show that if f (z) is analytic in a domain D and $|f(z)| = k =$ constant in D, then f(z) = constant in D.

Solution

By assumption, $u^2 + v^2 = k^2$ by differentiation.

$$uu_x - vu_x = 0. \qquad uu_y + vv_y = 0.$$

From this and the Cauchy – Riemann equations.

6. (a) $uu_x - uu_y = 0.$ (b) $uu_y + uu_x = 0$

To get rid of u_y multiply (6a) by u and (6b) by v and add. Similarly to eliminate u_x, multiply (6a) by – v and (6b) by u and add. This yield.

$$\left(u^2 + v^2\right)u_x = 0. \qquad \left(u^2 + v^2\right)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then u = v, hence f = 0. if $k \neq 0$, *then* $u_x = u_y = 0$, hence by the Cauchy – Riemann equations, also $v_x = v_y = 0.$ together, u = constant and v = constant, hence f = constant.

If we use polar form z = r(cos θ + isin θ) and set f(z) = u(r, θ), then the Cauchy – Riemann equations are

7. $u_r = \frac{1}{r} v_\theta$ *and* $v_r = \frac{-1}{r} u_\theta$

The derivative can then be calculated from

8a. $f'(z) = (u_r + iv_r)(\cos\theta - i\sin\theta)$

or from

8b. $f'(z) = (v_\theta - iu_\theta)(\cos\theta - i\sin\theta)/r$.

Example 32 Cauchy – Riemann equations in polar form

let $f(z) = z^3 = r^3(\cos 3\theta + i\sin 3\theta)$.

$Then\, u = r^3\cos 3\theta, v = r^3\sin 3\theta$

By definition,

$$u_r = 3r^2\cos 3\theta, \qquad v_\theta = 3r^3\cos 3\theta,$$
$$v_r\, 3r^2\sin 3\theta, \qquad u_\theta = 3r^3\sin 3\theta$$

We see that (7) holds for all z $\neq 0$.this confirms that z^3 is analytic for all z $\neq 0$.(and we know that it is also analytic at (z = 0). From (8b) we obtain the derivative as expected.

$$f'(z) = 3r^2(\cos 3\theta + i\sin 3\theta)(\cos\theta - i\sin\theta) = 3z^2.$$

Laplace's Equation: Harmonic functions

One of the main reasons for the great practical importance of complex analysis in engineering mathematics results form the fact that the real part of an analytic function f = u + iv satisfies the so – called Laplace's equation.

9. $\nabla^2 u = u_{xx} + u_{yy} = 0.$

(∇^2 read "nabla squared") and the same holds fort the imaginary part

10. $\nabla^2 v = v_{xx} + v_{yy} = 0.$

Laplace's equation is one of the most equations in physics, occurring in gravitation, electrostatics, fluid flow, etc. (cf. chaps. 11, 17) let us discover why this equation arises in complex analysis.

Theorem 3 (Laplace's Equation)

If f(z) = u(x,y) + iv(x,y) is analytic in a domain d, then u and v satisfy Laplace's equation (9) and (10) in d and have continuous second partial derivatives in D.

Proof:

Differentiating $u_x = v_y$ with respect to x and $u_y = v_x$ with respect to y, we obtain

11. $u_{xx} = v_{yx}$ $u_{yy} = -v_{xy}$.

Now the derivative of an analytic function is itself analytic, as we shall prove later (in sec. 13.6). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal; $v_{yx} = v_{xy}$. By adding (11) we thus obtain (9). Similarly, (10), is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$.

Solutions of Laplace's equation having continuous second – order partial derivatives are called harmonic functions and their theory is called potential theory (cf. also sec. 11.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy – Riemann equations in a domain d, they are the real and imaginary parts of an analytic function f in d. Then v is said to be a conjugate harmonic function of u in d. (of course this use of the word "conjugate" has nothing to do with that employed in defining $\bar{z}$, the conjugate of a complex number z).

A conjugate of a given harmonic function can be obtained from the Cauchy – Riemann equations, as may be illustrated by the following example.

Example 33

Conjugate Harmonic Function

Verify that $u = x^2 - y^2 - y$ is harmonic in the complex plane and find a conjugate harmonic function of v of u.

Solution

$\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x \, and \, u_y = -2y - 1.$ hence a conjugate v of u must satisfy

$$v = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain.

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}$$

A comparison with the second shows that dh/dx = 1. This gives h(x) = x + c. hence $v = 2xy + x + c\,(c\, any\, real\, cons\tan t)$ is the most general conjugate harmonic of the given u.

The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$

The Cauchy – Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shown in chapter 17. In the remainder of this chapter we discuss elementary functions, one after the other, beginning with e^z in the next section. Without knowing these functions and their properties we would not be able to do any useful practical work. This is just as in calculus.

3.6 Exponential Function

The remaining sections of this chapter will be devoted to the most important elementary complex function, logarithm, trigonometric functions, etc we shall see that these complex functions can easily be defined in such a way that, for real values of the independent variable, the functions become identical with the familiar real functions. Some of the complex functions have interesting properties. Which do not show when the independent variable is restricted to real values. The student should follow the consideration with great care, because these elementary functions will be frequently needed in applications.

We begin with the complex exponential function also written as one of most important analytic functions. The definition of e^z in terms of the real functions e^x cos y and sin y is $e^z = e^x$(cos y + sin y). This definition is motivated by requirement that make e^z a natural extension of the real exponential function e^x, namely.

(a) e^z should reduce to the latter when z = x is real;

(b) e^z should be an entire function, that is analytic for all z, and resembling calculus, its derivative should be

2. $(e^z)^1 = e^z$

from (1) we see that (a) holds, since cos 0 = 1 and sin 0 = 0. that e^z is easily verified by the Cauchy-Riemann equations. Formula (2) then follows from (4) that

$(e^z)^1 = (e^z \cos y)_z + i(e^x \sin y)_x = e^z \cos y + ie^z \sin y = e^{z.}$

$e^{z.}$ has further interesting properties. Let us first show that, as in real, we have the functional relations

3. $e^{z_1+z_2} = e^{z_1 z_2}$

For any

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, indeed, by (1).
$= e^{x_1}\left(\cos y_1 + i\sin y_1\right) e^{x_2}\left(\cos y_2 + i\sin y_2\right).$

Since $e^{x_1}e^{x_2} = e^{x_1} + e^{x_2}$ for these real functions, by an application of the addition formulas for the cosine and sine functions (similar to that in sec. 12.2) we find that this equals

$e^{z_1 z_2} = e^{x_1}\left(\cos\left(y_1 + y_2\right) + i\sin\left(y_1 + y_2\right)\right) = e^{z_1 + z_2}$
As asserted. An

4. $\left|e^{iy}\right| = \left|\cos y + i\sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1.$

That is, for pure imaginary exponents the exponential function has absolute value one, a result the student should remember. From (7) and (1),

5. $\left|e^{z}\right| = e^{x}$. Hence arg $e^{z} = y + 2n\pi$ $(n = 0,1,2,\cdots)$

since $\left|e^{z}\right| = e^{x}$.shows that (1) is actually e^{x} in polar form.

Example 34

Illustration of Some Properties of the Exponential Function

Computation of values from (1) provides no problem. For instance, verify that

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i\sin 0.6) = 4.055(0.825 - 0.565i) = 3.347 - 2.290i,$$
$$\left|e^{1.4-0.6i}\right| = e^{1.4} = 4.055, \qquad \mathrm{Arg}\, e^{1.4-0.6i} = -0.6.$$

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we have from (5)

6. $e^{2\pi i} = 1$

Furthermore use (1), (5) or (6) to verify these important special values:

7. $e^{\pi i/2} = i$, $e^{\pi i} = -1$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$.

To illustrate (3), take the product of

$e^{2+i} = e^2(\cos i + i\sin 1) = e^{4-i}e^4(\cos 1 - i\sin 1)$
and verify that equals

$e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)-(4-i)}$.

Finally, conclude from $\left|e^z\right| = e^x \neq 0$ in (8)that

8. $e^x \neq 0$ for all z

So here we have an entire function that never vanishes, in contrast to (non-constant) polynomials, which are also entire (Example 5 in Sec.2.4) but always have zero, as is proved in algebra. [Can you obtain (11) from (3) ?]

Periodicity of e^x with period $2\pi i$,

9. $e^{z+2\pi i} = e^z$ all z

is a basic property that follows from (1) and the periodicity of cosy and siny. It also follows from (3) and (9).] Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π .

10. $-\pi < y \leq \pi$

This infinite strip is called a **fundamental region** of e^x .

Example 35

Solution of an Equation

Find all solution of $e^x = 3 + 4i$

Solution

$\left|e^x\right| = e^x = 5, x = In\,5 = 1.609$ is a real part of all solutions. Furthermore, since $e^x = 5$,

$e^x \cos y = 3,\quad e^x \sin y = 4,\quad \cos y = 0.6,\quad \sin y = 0.8,\quad y = 0.927$.

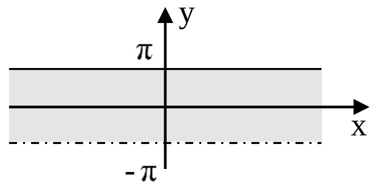


Fig. 20: Fundamental Region of the Exponential Function e^z I in the z-plane

Ans. $z = 1.609 + 0.927i \pm 2n\pi i \quad (n = 0,1,2,\cdots\cdot)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line x=1.609 at a distance 2π from their neighbours.

To summarise: many properties of $e^z = \exp z$ parallel to those of e^x; an exception is the periodicity of e^x with $2\pi i$, which suggested the concept of a fundamental region and causes the periodicity of cosz and sinz with the *real* period 2π, as we shall see in the next section. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

3.7 Trigonometric Functions, Hyperbolic Functions

Just as e^z extends e^x to complex, we want the complex trigonometric functions to extend the familiar real trigonometric functions. The idea of making the connection is the use of the Euler formulae.

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain

$$\cos x = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin x = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad x \; real$$

This suggests the following definitions for complex values $z = x + iy$

1. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$

Furthermore, in agreement with the definition from the real calculus we define

2. $\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$

and

3. $\sec z = \frac{1}{\cos z}, \qquad \operatorname{cosec} z = \frac{1}{\sin z}.$

Since e^z is entire, cosz and sinz are entire functions. Tanz and secz are not entire; they are analytic except at the point where cosz is zero; and

$\cot z$ and $\csc z$ are analytic except, where $\sin z = 0$. Formulas for the derivatives follows readily from $(e^z)' = e^z$ and (1)-(3); as in calculus,

4. $(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z,$

etc. Equation (1) also shows that **Euler's formula** *is valid in complex:*

5. $e^{iz} = \cos z + i \sin z$ for all *z*.

Real and imaginary parts of cosz and sinz are needed in computing values, and they also help in displaying properties of our functions. We illustrate this by typical example.

Example 36

Real and Imaginary Parts. Absolute Value. Periodicity

Show that

(a) $\cos z = \cos x \cosh y - i \sin x \sinh y$

6. (b) $\sin z = \sin x \cosh y + i \cos x \sinh y$

and

7. (a) $|\cosh z|^2 = \cos^2 x + \sinh^2 y$

(b) $|\sinh z|^2 = \sin^2 x + \sinh^2 y$

And give some application of these formulas.

Solution

From (1)

$$\cos z = \frac{1}{2}(e^{i()x+iy} + e^{-i(x+iy)})$$

$$= \frac{1}{2}e^{-y}(\cos x + i \sin y) + \frac{1}{2}e^{y}(\cos x - i \sin y)$$

$$= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2}i(e^{y} - e^{-y})\sin x.$$

This yields (6a) since, as is known from calculus,

8. $\cosh y = \frac{1}{2}(e^{y} + e^{-y}), \qquad \sin hy = \frac{1}{2}(e^{y} - e^{-y});$

(6b) is obtained similarly. From and $\cosh^2 y = 1 + \sinh^2 y$ we obtain
$|\cos|^2 = \cos^2 x(1+\sinh^2 y) + \sin^2 x + \sinh^2 y.$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2+3i\cos 2\cosh 3 - i\sin 2\sinh 3 = -4.190 - 9.109i$.

From (6) we see that cosz and sinz are *periodic with period* 2π , just as in real. Periodicity of tan z and cot z with period π now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine: whereas $|\cos x| \le 1$ and $|\sin x| \le 1,$ the complex cosine and sine functions are no longer bounded but approach infinity in absolute value as $y \to \infty,$ since $\sinh y \to \infty.$

Example 37

Solution of Equations. Zeros

Solve

(a) $\cos z = 5$ (which has no real solution),
(b) $\cos z = 0$
(c) $\sin z = 0$

Solution

(a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solution (3D-values)

$e^{iz} = e^{-y+ix} 5 \pm \sqrt{25-1} = 9.899$ and $0.101.$

Thus $e^{-y} = 9.899$ or $0.101, e^{ix} = 1, y = \pm 2.292, x = 2n\pi$

Ans. $z = \pm 2n\pi \pm 2.292i(n = 0,1,2,\cdots),$ can you obtain this by using (6a)?

(b) $\cos x = 0,\ \sinh y = 0,$ by (7a), $y = 0.$

Ans. $z = \pm\frac{1}{2}(2n+1)\pi \ \ (n = 0,1,2,\cdots).$

(c) $\sin sx = 0,\ \sinh y = 0,$ by (7b), $y = 0.$

Ans. $z = 2n\pi \ \ (n = 0,1,2,\cdots).$

Hence the only zeros of cos*z* and sin*z* are those of the real cosine and sine functions.

From the definition it follows immediately that all the familiar formulas for the real trigonometric functions continue to hold for complex values.

We mention in particular the addition rules

9. $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$
$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

and the formula

10. $\cos^2 z + \sin^2 z = 1.$

Some further useful formulas are inclined in the problem set.

HYPERBOLIC FUNCTIONS

The complex **hyperbolic cosine** and **sine** are defined by the formulas

11. $\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}).$

This suggested by the familiar definition for the real variable. These functions are shown below, with derivatives

12. $(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z,$

as in calculus. The other hyperbolic functions are defined by

$$\tan z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

13. $\sec hz = \dfrac{1}{\cosh z}, \qquad \csc hz = \dfrac{1}{\sin zh},$

Complex trigonometric and hyperbolic functions are related

If in (11), we replace z by *iz* and use (1), we obtain

14. $\cosh iz = \cos z, \qquad \sinh iz = i \sin z,$

From this, since cosh is even and sinh is odd, conversely

15. $\cos iz = \cosh z, \qquad \sin iz = i \sinh z,$

Apart from their practical importance, these formulas are remarkable in principle. Whereas in real calculus, the trigonometric and hyperbolic functions are of a different character, in complex these functions are intimately related. Moreover the Euler formula relates them to the exponential function. This situation illustrates that by working in complex, rather than in real, one can often gain a deeper understanding of **special functions.** This is one of the three main reasons of the practical importance of complex analysis, mentioned at the beginning of this chapter.

In the next section we discus the **complex logarithms,** which differ substantially from the real logarithm (which is simpler), and the student should work the next section with particular care.

4.0 CONCLUSION

To this end, we conclude by giving a summary of what we have covered.

5.0 SUMMARY

For arithmetic operations with **complex number**

1. $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta),$

   $r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x),$ and for their representation in the complex plane, see Sec 2.1 and 2.2

   A complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in domain *D* if it has a **derivative.**

2. $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

   Everywhere in *D*. Also, $f(z)$ is analytic at a point $z = z_0$ if it has a derivative in a neighbourhood of z_0 (not merely at z_0 itself).

   If $f(z)$ is analytic in *D*, then *u(x, y)* and $v(x, y)$ satisfy the (very important!) **Cauchy-Riemann** equations (Sec. 2.5).

3. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   everywhere in *D*. Then u and v also satisfy **Laplace's equation**

4. $$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

   everywhere in *D* . If *u(x, y)* and $v(x, y)$ are continuous and have continuous partial derivatives in *D* that satisfy (3) in *D*, then

$f(z) = u(x, y) + iv(x, y)$ is analytic in domain D .Sec. 2.5 the complex exponential function (Sec. 2.6)

5. $e^z = \exp z = e^z(\cos y + i\sin y)$
is periodic with $2\pi i$, reduces to e^z when $z = x(y = 0)$ and has the derivative e^z. The **trigonometric function**s are (Sec.2.7)
$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i\sin x \sinh y$
$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \sin x \cosh y - i\cos x \sinh y$
$\tan z = (\sin z)/\cos z, \cot z = 1/\tan z,$ etc.

6.0 TUTOR-MARKED ASSIGNMENT

i. Let $z_1 = 3 + 4i$ and $z_2 = 5 - 2i$
Find in the form $x + iy$
(a) $(z_1 - z_2)^2$ (b) $z_2/2z$

ii. Show that z is pure imaginary if and only if $\bar{z} = -z$.

iii. Find; (a) $|1 - i|^2$ (b) $\left|\frac{(3+4i)^4}{(3-4i)^3}\right|$

iv. Represent in polar form
(a) $i\sqrt{2}/3+3i$ (b) $4i$

v. Determine the principal value of the arguments of
(a) $-2 + 2i$ (b) $1 - i\sqrt{3}$

vi. Represent in form $x + iy$
(a) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\sqrt{50}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

vii. Determine and sketch the sets represented by
(a) $|z - 2i| = 2$ (b) $z\bar{z} + (1+2i)z + (1-2i) + 1 = 0$

viii. Find $f(2+i), f(-4+i)$ where $f(z)$ equals
(a) $3z^2 + z$ (b) $\frac{(z+1)}{(z-1)}$

ix. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

x. Prove the product rule $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$

xi. Are the following functions analytic?
(a) $f(z)z^4$ (b) $f(z)e^x(\cos y + i\sin y)$.

xii. Let v be a conjugate harmonic of u in some domain *D*. Show that then $h = u^2 - v^2$ is harmonic in *D*.

xiii. Derive the Cauchy-Riemann equations in polar form equation from equation 1.

xiv. Using the Cauchy-Riemann equations, show that e^x is analytic for all z.

xv. Compute e^z (in the form ($u + iv$) and $|e^z|$) when z equals

(a) $\pi - i/2$ (b) $-1 - \frac{7\pi i}{4}$

xvi. Show that $u = e^{xy}\cos\left(\frac{x^2}{2} - \frac{y^2}{2}\right)$ is harmonic and find a conjugate.

xvii. Prove that $\cos z, \sin z, \cosh z,$ and $\sinh z$ are entire functions.

xviii. What is the idea that led to the Cauchy-Riemann equations?

xix. State the Cauchy-Riemann equations from memory.

xx. What is an analytic function? Can a function be differentiable at a point z_0 without being analytic at z_0.

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UNIT 2 INTEGRATION OF COMPLEX PLANE

CONTENTS

1.0 INTRODUCTION

In this unit we defined and explained complex integrals. The most fundamental result in the whole unit is Cauchy's integral theorem. It implies, the importance of Cauchy integral formula.

We prove that if a function is analytic, it has derivatives of all orders. Hence, in this respect, complex analytic functions behave much more simply than real-valued functions of real variables. Interpretation by means of residues and applications to real integrals will be considered in Module 3.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- in applications there occur real integrals that can be evaluated by complex integration, whereas the usual methods of real integral calculus are not successful; and
- some basic properties of analytic function can be established by integration, but would be difficult to prove by other methods. The existence of higher derivatives of analytic functions is a striking property of this type.

3.0 MAIN CONTENT

3.1 Line Integral in the Complex Plane

As in real calculus, we distinguish between definite integrals, and indefinite integrals or ant derivatives. An **indefinite integral** is a function whose derivative equals a given analytic function in a region. By inverting known differentiation formulas we may find many types of indefinite integrals.

We shall now define *definite integrals*, or line integrals, of complex function $f(z)$, where $z = x + iy$ as follows;.

Path of Integration

In real calculus, a definite integral is taken over an interval (a segment) of the real line. In the case of a complex definite integral we integrate along a curve C in the complex plane, which will be called the *path of integration.*

Now a curve C in the complex plane can be represented in the form

$$z(t) = x(t) + iy(t) \qquad (a \le t \le b) \qquad (1)$$

where t is a real parameter. For example,

$$z(t) = t + 3it \qquad (0 \le t \le 2)$$

represent a portion of the line $y = 3x$ (sketch it!),

$$z(t) = 4\cos t + 4i\sin t \qquad (-\pi \le t \le \pi)$$

represent the circle $|z| = 4$, etc. (More example below)

C is called a smooth curve if C has a derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each of its points which is continuous and nowhere zero. Geometrically this means that C has a continuous turning tangent. This follow directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

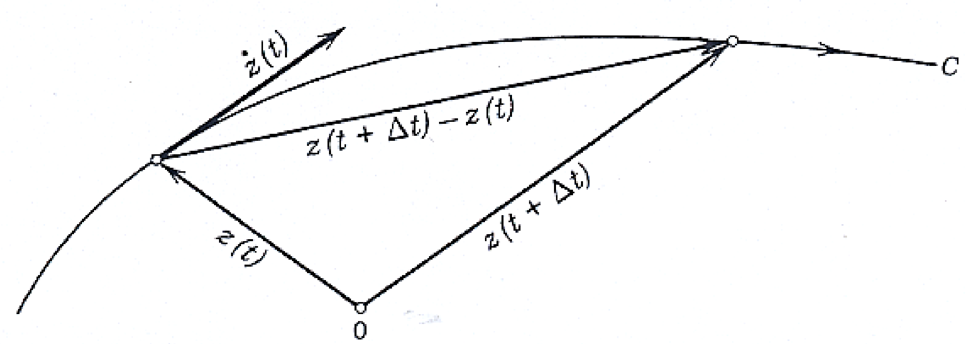


Fig. 21: Tangent vector z(t) of a curve C in the complex plane given by z(t). The arrow on the curve indicates the positive sense (sense of increasing t).

3.1.1 Definition of the Complex Line Integral

This will be similar to the method used in calculus. Let C be a smooth curve in the z-plane represented in the form (1). Let $f(z)$ be a continuous function defined (least) at each point of C. We subdivided ("partition") the interval $(a \le t \le b)$ in (1) by points of

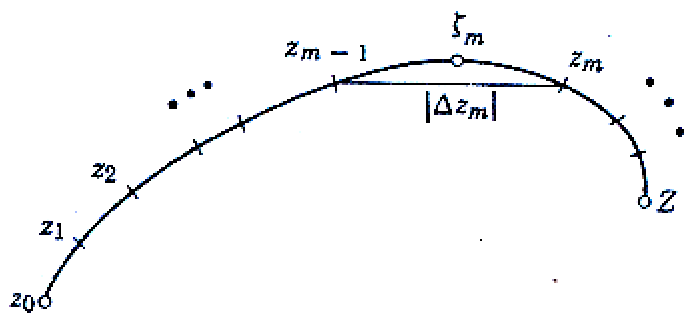


Fig. 22: Complex Line Integral

$$t_0(=a), t_1, \cdots . t_{n-1}, t_n(=b)$$

Where $t_0 < t_1 < \cdots t_n$. To do this subdivision there corresponds a subdivision of C by points

$$z_0, z_1, \cdots . z_{n-1}, z_n (= z),$$

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ξ_1 between z_0 and z_1 (that is, $\xi_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$, a point ξ_1 between z_0 and z_1 (that is, $\xi_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$, a point ξ_2 between z_1 and z_2 etc. Then we form the sum

$$S_n = \sum_{m=1}^{n} f(\xi_m) \qquad (2)$$

where

$$\Delta z_m = z_m - z_m - 1.$$

This we do for each $n = 1,2,3, \cdots$ in a completely independent manner, but in such a way that the greatest $|\Delta z_m|$ approaches zero as n approaches infinity. This gives a sequence of complex numbers $S_2, S_3, \cdots$. The limit of these sequence is called the **line integral** (or simply the integral) of $f(z)$, along the oriented curve C and is denoted by

$$\int_C f(z)dz \qquad (3)$$

The curve C is called t**he path of integration.** C is called a **closed path** if $z = z_0$, that is, if its terminal point coincides with its initial point.

(Example: a circle, a curve shaped like an 8, etc.) Then also writes

$$\oint_C \text{ instead of } \int_C$$

Examples follow in the next section.

General Assumption

All path of integration for complex line integral will be assumed to be **piecewise smooth,** *that is, to consist of finitely many smooth curves joined end to end.*

3.1.2 Existence of the Line Integral

From our assumption that $f(z)$ is continuous and C is piecewise smooth, the existence of the line integral (3) follows, as in the previous chapter let us write $f(z) = u(x, y) + iv(x, y)$. We also set

$$\xi_m = \xi_m + i\eta_m \text{ and } \Delta z_m = \Delta x_m + i\Delta y_m$$

then (2) may be written

$$S_n = \sum (u+iv)(\Delta x_m + i\Delta y_m) \qquad (4)$$

Where $u = u(\xi_m, \eta_m)$ and $v = v(\xi_m, \eta_m)$ we sum over m from1 to n. We may now split up S_n into four sums:

$$S_n = \sum u\Delta x_m - \sum v\Delta y_m + i\left[\sum u\Delta y_m + \sum v\Delta x_m\right]$$

These sums are real. Since f is continuous, u and v are continuous. Hence, if we let n approach infinity in the aforementioned way, then the greatest Δx_m and Δy_m will approach zero and each sum on the right becomes a real line integral:

$$\lim_{n\to\infty} S_n = \int_C f(z)dz = \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right] \qquad (5)$$

This shows that under our assumption (f continuous on C_1 and C_2 piecewise smooth) the line integral (3) exist and its value is independent of the choice of subdivisions and intermediate points ξ_m.

3.1.3 Three Basic Properties of Complex Line Integrals

We list three properties of complex line integrals that are quite similar to those of real definite integrals (and real line integrals) and follow immediately from the definition.

Integration is a linear operation, that is, a sum of two (or more) functions can be integrated term by term, and constant factors can be taken out from under the integral sign:

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1\int_C f_1(z)dz + k_2\int_C f_2(z)dz \qquad (6)$$

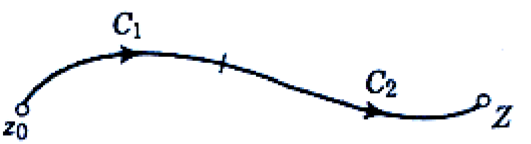


Fig. 23: Subdivision of Path (Formula (7)

Decomposing C into two portions C_1 and C_2 (Fig), we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad (7)$$

3. Reversing the sense of integration, we get the negative of the original value:

$$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz \qquad (8)$$

here the path C with endpoint z_0 and Z is the same; on the left we integrate from z_0 to Z , on the right from z_0 to Z .

Applications follow in the next section and problems at the end of it.

3.2 Two Integration Methods

Complex integration is rich in methods for evaluating integrals. We discuss first two of them, and others will follow later in this chapter.

3.2.1 First Method: Use of Representation of the Path

This method applies to any continuous complex function.

Theorem 1 (Integration by the use of the path)

Let C be a piecewise smooth path, represented by z = z(t), where $a \le t \le b$ *Let f(z) be a continuous function on C. Then*

$$\int_C f(z) = \int_a^b f[z(t)]\dot{z}(t)dt \qquad \left(i = \frac{dz}{dt}\right) \qquad (1)$$

Proof

The left-hand side of (1) is given by (5), Sec, 13.1, in terms of real integrals, and we show that the right-hand side of (1) also equals (5).

We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for $u[x(t), y(t)]$ and v for $v[x(t), y(t)]$.We also have $dx = \dot{x}dt$ and $dy = \dot{y}dt$. Consequently, in (1),

$$\int_a^b f[z(t)]\dot{z}(t)dt = \int_a^b (u + iv)(\dot{x} + i\dot{y})dt$$
$$= \int_C [udx - vdy + i(udy + vdx)],$$

Which is the right-hand side of (5), as claimed.

Steps in applying Theorem 1

Represent the path C in the form $z(t)\ a \le t \le b$
Calculate the derivative $\dot{z}(t) = dz/dt$
Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y)
Integrate $f[z(t)]\dot{z}(t)$ over t from a to b

Example 1

A Basic Result: Integral of $1/z$ around the unit circle

Show that

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i} \qquad (C \text{ the unit circle, clockwise}) \qquad (2)$$

The important result will be frequently needed.

Solution

We may represent the unit circle C in the form

$$\dot{z}(t) = \cos t + i \sin t \, \text{z} \qquad (0 \le t \le 2\pi).$$

So that the counterclockwise integration correspond to an increase of t from 0 to 2π. By differentiation,

$$\dot{z}(t) = -\sin t + i \cos t$$

Also $f[z(t)] = \dfrac{1}{z(t)}$. Formula (1) now yields the desired result

$$\begin{aligned}\oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{1}{\cos t + i \sin t}(-\sin t + i \cos t)dt \\ &= i\int_0^{2\pi} dt \\ &= 2\pi i\end{aligned}$$

The Euler formula helps us to save work by representing the unit circle simply in the form

$$z(t) = e^{it}$$

Then

$$\frac{1}{z(t)} e^{-it}, \quad dz = ie^{it}dt.$$

As before, we now get more quickly

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it} = i\int_0^{2\pi} dt$$
$$= 2\pi i \, .$$

Example 2

Integral of Integer Powers

Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 is a constant.
Integrate in the clockwise sense around the circle C of radius ρ with centre at z_0

Solution

We may represent the unit circle C in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \text{z} \qquad (0 \le t \le 2\pi).$$

Then we have
$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it} dt,$
and we obtain

$$\oint_C (z - z_0)^m \, dz = \int_0^{2\pi} \rho^m e^{imt} dt$$
$$= \int_0^{2\pi} e^{i(m+1)t} dt.$$

By the Euler formula (5), the right-hand side equals
$i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t + i\int_0^{2\pi} \sin(m+1)t\right].$

When $m = -1$, we have $\rho^{m+1} = 1, \cos 0 = 1, \sin 0 = 0$ and thus obtain $2\pi i$. For integer $m \neq 1$ each of the two integer is zero because we integrate over an interval of length $2\pi i$, equal to a period of sine and cosine. Hence the result is

$$\oint_C (z - z_0)^m \, dz = \begin{cases} 2\pi i & (\text{m} = \text{-1}) \\ 0 & (\text{m} \neq \text{-1 and integer}). \end{cases} \qquad (3)$$

Let us now illustrate the following important fact. If we integrate a function $f(z)$, from a point z_0 to a point z_1 along different path, we generally get the values of the integral. In other words, a complex line integral generally depends not only on the end point of the path but also on the geometric shape of the path.

Example 3

Integral of Non-analytic Function

Integrate $f(z) = x$ from 0 to 1.
along C^* in fig. 325 below.
along C consisting of C_1 and C_2.

Solution

a. C^* can be represented by $z(t) = t + it \quad (0 \le t \le 1)$. Hence
$\dot{z}(t) = +i$ and $f[z(t)] = x(t) = 1$ (on C^*).

We now calculate

$$\int_C \operatorname{Re} z dz = \int_0^1 t(1+i)dt$$
$$= \frac{1}{2}(1+i).$$

b. C_1 can be represented by $z(t) = t \ (0 \le t \le 1)$. Hence
$\dot{z}(t) = 1$ and $f[z(t)] = x(t) = 1$ (on C_1).
C_2 can be represented by $z(t) = t + it \quad (0 \le t \le 1)$. Hence
$\dot{z}(t) = 1$ and $f[z(t)] = x(t) = 1$ (on C_2).
Using (7) , we calculate

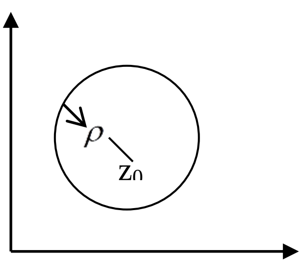


Fig. 24 Path in Example 2

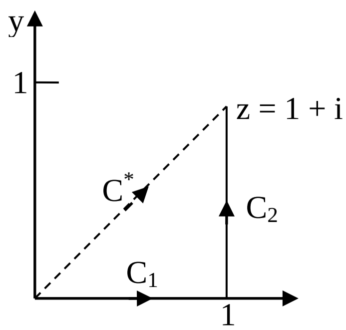


Fig. 25. Path in Example 3

$$\int_C \operatorname{Re} z dz = \int_{C_1} \operatorname{Re} z dz + \int_{C_2} \operatorname{Re} z dz = \int_0^1 t dt + \int_0^1 1 \cdot t dt$$
$$= \frac{1}{2} + i$$

Note that this result is differ from the result in (a).

3.2.2 Second Method: Indefinite Integration

In real calculus, if for given $f(x)$ we know an $F(x)$ such that $F'(x) = f(x)$,
then we can apply the formula

$$\int_a^b f(x)dx = F(b) - F(a)$$

This method extends to complex functions. We shall see that it is simpler than the previous method, but, of course, we have to find an $F(z)$ whose derivative $F'(z)$ equals the given function $f(z)$ that we want to integrate. Clearly, differentiation formulas will often helps us in finding such an $F(z)$, so that this method becomes of great practical importance.

Theorem 2 (Indefinite Integration of Analytic Functions)

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all path in D joining two point z_0 and z_1 in D we have

4. $$\boxed{\int_{z_0}^{z_1} f(x)dz = F(z_1) - F(z_0)} \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 *and* z_1 *instead of* C, *since we get the same value for all those* C *from* z_0 *and* z_1).

This theorem will be proved by using Cauchy's integral theorem which we discuss in the next section…

Example 4

$$\int_0^{1+i} z^3 dz = \frac{1}{3} z^3 \Big|_0^{1+i}$$

$$= \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3} i$$

Example 5

$$\int_{-\pi i}^{\pi i} \cos z dz = \sin z \Big|_{-\pi i}^{\pi i}$$

$$= 2\sin \pi i = 2i \sinh \pi = 23.097i$$

Example 6

$$\int_{8+3\pi i}^{8-3\pi i} e^{z/2}dz = 2e^{z/2}\Big|_{8+3\pi}^{8-3\pi}$$
$$= 2(e^{4-3\pi i/2} - e^{4+3\pi i/2})$$
$$= 0$$

Since e^z is periodic with period $2\pi i$.

3.2.3 Bound for Absolute Value of Integrals

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

6. $\left|\int_C f(z)dz\right| \leq ML$ **(ML -inequality)**;

here L is the length of C and M a constant such that $|f(z)| \leq M$ everywhere on C.

Proof:

We consider S_n as given by (2). By the generalized triangle inequality (6), we obtain

$$|S_n| = \left|\sum_{m=1}^{n} f(\xi_m)\Delta z_m\right| \leq \sum_{m=1}^{m} |f(\xi_m)||\Delta z_m|$$
$$\leq M\sum_{m=1}^{n} |\Delta z_m|.$$

Now Δz_m is the length of the chord whose end points are z_{m-1} and z_m. Hence the sum on the right represents the length L^* of the broken line of the chord whose endpoints are $z_0, z_1, \cdots, z_n (n = Z)$. If n approaches infinity in such a way that the greatest $|\Delta z_m|$ approaches zero, then L^* approaches the length L of the curve C, by the definition of the length of a curve. From this the inequality (6) follows.

We cannot see for (6) how close to the bound ML the actual absolute value of the integral is, but this will be no hardship in applying (6). For the time being we explain the practical use of (6) by a simple example.

Example 8

Find a upper bound for the absolute value of the integral

$$\int_C z^2 dz, \qquad C \text{ the straight-line segment from 0 to } 1+i$$

Solution

$L = \sqrt{2}$ and $|f(z)| = |z^2| \leq 2$ on C gives by (6)

$$\left|\int_C z^2 dz\right| \leq 2\sqrt{2} = 2.8284$$

The absolute value of the integral is

$$\left|-\frac{2}{3} + \frac{2}{3}i\right| = \frac{2}{3}\sqrt{2} = 0.9428$$

In the next section we discuss the most important theorem of the whole chapter, **Cauchy's integral theorem,** which is the basic in itself and has far reaching consequences which we shall explore, above all the existence of all higher derivatives of an analytic function, which are themselves analytic functions.

3.3 Cauchy's Integral Theorem

Cauchy's integral theorem is very important in complex analysis and has various theoretical and practical consequences. To state this theorem, we shall need the following concepts.

A closed path *C* is called a **simple close path** if *C* does not intersect or touch itself (see diagram below). For example a circle is simple, an eight- shaped curve is not.

A domain *D* in the complex plane is called a **simply connected domain** if every closed path in *D* encloses only points of *D*. A domain that is not simply connected is called *multiply connected.*

For instance, the interior of a circle ("circular disk"), ellipse or square is

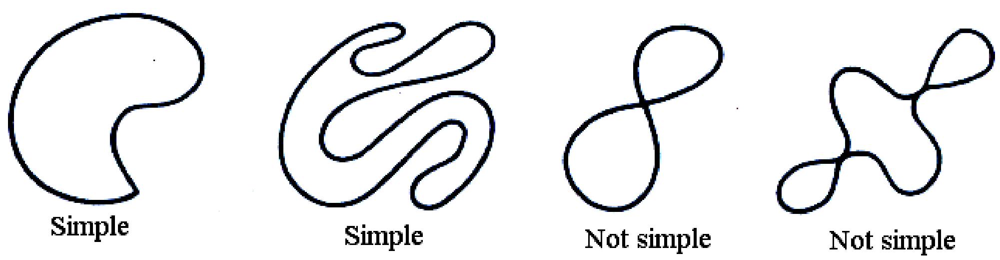


Fig. 326. Closed paths

simply connected. More generally, the interior of a simple closed curve is simply connected. A circular ring or annulus is multiply connected (more precisely: doubly connected). The figure below shows further examples.

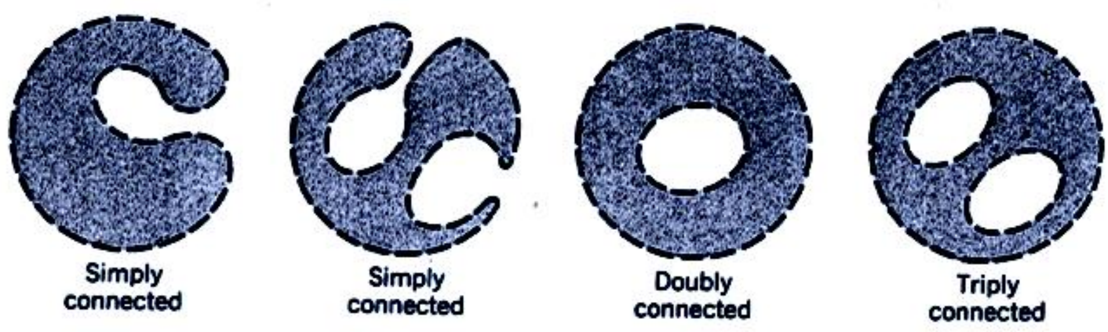


Fig. 27: Simply and Multiply Connected Domain

Recalling that, by definition, a function is a *single-valued* relation, we can now state Cauchy's integral theorem as follows. This theorem is sometimes also called the **Cauchy-Gaursat theorem.**

3.3.1 Cauchy's Integral Theorem

If $f(z)$ be analytic in a simply connected domain D, then for every simple close path C in D,

1. $$\int_C f(z)dz = 0$$

Proof

If we make assumption –as Cauchy did- that the derivative $f'(z)$ of $f(z)$ is continuous in D (existence of $f'(z)$ in D being a consequences of analyticity), then Cauchy's theorem follows from a basic theorem on real

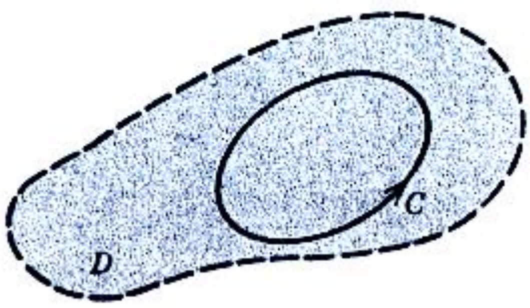


Fig. 28: Cauchy's Integral Theorem

line integrals (proof below). Goursat finally proved Cauchy's theorem without the assumption that $f'(z)$ is continuous (optional proof at the end of this chapter). Before we go into details, let us consider some example in order to really understand what is going on.

We mention that a closed path is sometimes called a contour and an integral over such a path a **contour integral.**

Example 9

$\int_C e^z dz = 0$, $\int_C \cos z dz = 0$ $\int_C z^n dz = 0$ $(n = 0,1,\cdots)$

For any closed path, since these functions are (analytic for all *z*).

Example 10

$\int_C \sec z dz = 0$, $\int_C \frac{dz}{z^2+4} = 0$

where C is the unit circle. $\sec z = \frac{1}{\cos z}$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C ; none lie on C. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2\pi i$ outside *C*.

Example 11

$\int_C \bar{z} dz = 2\pi i$

(C the unit circle, counterclockwise) does not contradict Cauchy's theorem, since $f(z) = \bar{z}$ is not analytic, so that the theorem does not apply. (Verify this result!)

Example 12

$\int_C \frac{dz}{z^2} = 0,$

where C is the unit circle. This result does not follow from the Cauchy's theorem, because $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$. Hence the condition that f be analytic in D is sufficient rather than necessary for (1) to be true.

Example 13

$\int_C \frac{dz}{z^2} = 2\pi i,$

The integration being taken around the unit circle in the clockwise sense. C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $\frac{1}{z}$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence the condition that the domain *D* be simply connected is quite essential.

Example 14

$$\int_C \frac{7z-6}{z^2-2z}dz = \int_C \frac{7z-6}{z(z-2)}dz = \int_C \frac{3}{z}dz + \int_C \frac{4}{z-2}dz = 3\cdot 2\pi i + 0$$
$$= 6\pi i$$

(*C* the unit circle, counterclockwise) by partial fraction reduction.

Cauchy's proof under the condition that $f'(z)$ ***I continuous***

From (5) we have

$$\int_C f(z)dz = \int_C (udx - vdy) + \int_C (udy + vdx).$$

Since $f(z)$ is analytic in *D*, its derivative $F'(z)$ exists in *D*. Since $F'(z)$ is assume to be continuous, (4) and (5) in previous section imply that *u* and *v* have continuous partial derivatives in *D*. Hence Green's theorem with *u* and –*v* instead of F_1 and F_2 is applicable and gives

$$\int_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

where *R* is the region bounded by *C*. The second Cauchy-Rieman integration shows that the integrand on the right is identically zero.

Hence, the integral on the left is zero. In the same fashion it follows by the use of the first Cauchy-Rieman equation that the last integral in the above formula is zero. This complete Cauchy's proof.

3.3.2 Independence of Path, Deformation of Path

We shall now discuss an important consequence of Cauchy's integral theorem that has great practical interest, proceedings as follows. If we subdivided the path, *C* in Cauchy's theorem into two arcs C_1^* and C_2, then (1) takes the form

$$(2') \qquad \int_{C_1} fdz + \int_{C_2} fdz = 0\,.$$

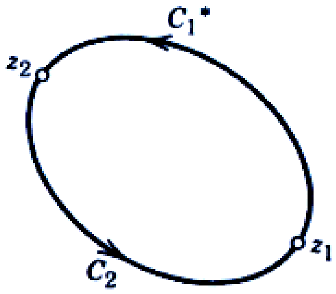


Fig. 29: Formula (2')

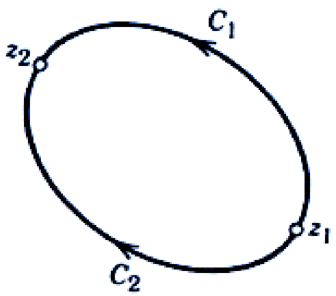


Fig. 30: Formula (2)

If we now reverse the sense of integration along C_1^*, then the integral over C_1^* is multiplied by -1. Denoting C_1^* with its new orientation by C_2, we thus obtain from $(6')$.

2. $$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz.$$

Hence, if *f* it is analytic in *D*, C_1^* and C_2 are any path in *D* joining two points in *D* and having no further points in common, then (2) holds.

If those paths C_1^* and C_2 have finitely many points in common, then (2) continues to hold. This follow by apply previous result to the portion of C_1 and C_2 between each pair of consecutives point of intersection.

If it is even true that (2) *holds for any paths that join ant points* z_1 *and* z_2 *and lie entirely in the simply connected domain D in which* $f(z)$ *is analytic.*

To express this we may say *that the integral of* $f(z)$ is **independent of path in *D*. (**Of course the value of the integral depends on the choice of z_1 *and* z_2.**)**

The proof may require additional consideration of the case in which C_1 and C_2 have infinitely many points of intersection, and is not presented here.

We may imagine that the path C_2 in (2) was obtained from C_1 by a continuous deformation. It follows that in a given integral we may impose a continuous deformation on the path of integration (keeping the endpoint fixed); as long as we do not pass through a point where $f(z)$ is not analytic, the value of the integral will not change under such deformation. This is often called the **principle of deformation of path.**

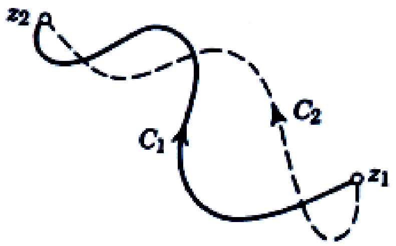


Fig. 31: Paths having finitely Many Intersections

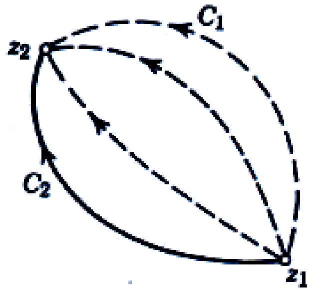


Fig. 32: Continuous Deformation of Path

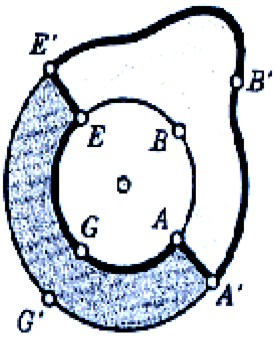


Fig. 33: Unit Circle and Path C

Example 15

$\int_C \frac{dz}{z} = 2\pi i,$ (Counterclockwise integration) now follow from example (1), for any simple closed path C whose interior contains 0.The figure above gives the idea: first deform ABE continuously into the path $AA'B'E'E$. The heavy curve in the figure shows the resulting deformed path. Then deform $E'EGAA'$ and $E'G'A'$.

There is more general systemic approach to problem of this kind, as we shall now see.

3.3.3 Cauchy Theorem for Multiple Connected Domains

A multiplys connected domain D^* can be cut so that the resulting domain (that is, D^* without the point of the cut or cuts) become simply connected.

For doubly connected domain D^* we need one cut $\tilde{C}$ (figure below).If $f(z)$ is analytic in D^* and at each point of C_1 and C_2 then, since C_1, C_2 and C bound a simply connected domain, it follows from Cauchy's theorem that the integral of f taken over $C_1, \tilde{C}, C_2$ in the sense indicated by the arrows in the figure has the value zero. Since we integrate along $\tilde{C}$ in both directions, the corresponding integrals cancel out, and we obtain

(3*) $\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$

where one of the curve is traversed in the counterclockwise sense and the other in the opposite sense. Reversing the sense of integration on one of the curves, we may write this

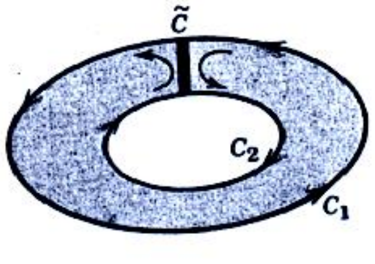


Fig 34: Doubly Connected Domain

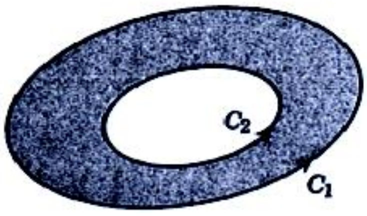


Fig. 35: Paths in (3)

3. $\int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

where curve now traversed in the same sense (the figure above). We remember that (3) holds under the assumption that $f(z)$ is analytic in the domain bounded by C_1 and C_2 and at each point of C_1 and C_2.

Can you see how the result in Example (7) now follows immediately from our present consideration?

For more complicated domains we may need more than one cuts, but the basic idea remains the same as before. For instance, for the triply connected domain in figure below,

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0$$

where C_2 and C_3 are traversed in the same sense and C_1 is traversed in the opposite sense.

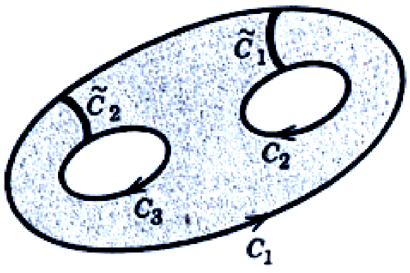


Fig. 36: Triply Connected Domain

Example 16

From (3), Example 2, it now follows that

$$\oint_C (z - z_0)^m \, dz = \begin{cases} 2\pi i (m = -1) \\ 0 (m \neq -1 and \text{ integer}) \end{cases}$$

For counterclockwise integration around any simple closed path containing z_0 in its interior.

In the next section, using Cauchy integral theorem, we prove the existence of indefinite integrals of analytic functions. This will also justify our earlier method of indefinite integration.

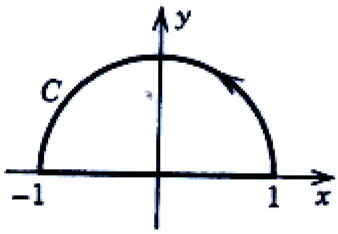


Fig. 39: Problem 29

3.4 Existence of Indefinite Integral

This section includes an application of Cauchy's integral theorem. It relates to Theorem 2 in section 3.2 on the evaluation of line integrals by indefinite integration and substitution of the limits of integration:

1. $\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0)$ $[F'(z) = f(z)]$,

Where $F(z)$ is an indefinite integral of *f(z),* that is $F'(z) = f(z)$, as indicated.
In most applications, such a $F(z)$ can be found from differentiation formulas.

Theorem 1 (Existence of an Indefinite Integral)

If $f(z)$ *is analytic in a simply connected domain D, then there exists an indefinite integral* $F(z)$ *of f(z)in D, which is analytic in D joining two points* z_0 *and* z_1 *in D, the integral of f(z) from* z_0 *and* z_1 *can be evaluated by formula (1).*

Proof

The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from z_0 in D to any z *in D* is independent of path in D. We keep z_0 fixed. Then this integral becomes a function of z, which we denote by $F(z)$:

2. $$F(z) = \int_{z_0}^{z_1} f(z^*)dz^*.$$

We show that this $F(z)$ is analytic in D and that $F'(z) = f(z)$. The idea of doing this is as follows. We form the differential quotient

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)dz^* - \int_{z_0}^{z+\Delta z} f(z^*)dz^*\right]$$
$$= \frac{1}{\Delta z}\int_{z_0}^{z+\Delta z} f(z^*)dz^*,$$

Subtract $f(z)$ from it and show that expression obtained approaches zero as $\Delta z \to 0$; this is done by using the continuity of $f(z)$. We now give the details.

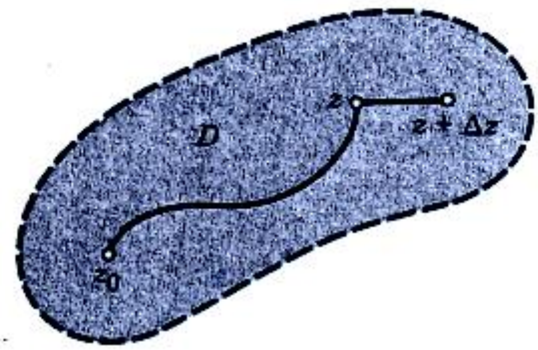


Fig. 38: Path of Integration

We keep z fixed. Then we choose $z + \Delta z$ in D. This is possible since D is a domain; hence D contains a neighbourhood of z. See figure above. The segment we use as the path of integration in the previous formula. We now subtract $f(z)$. This is a constant, since z is kept fixed. Hence

$$\int_{z}^{z+\Delta z} f(z)dz^* = f(z)\int_{z}^{z+\Delta z} dz^* = f(z)\Delta z.$$

Thus

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^*$$

This trick permits us to write a single integral:

$$\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)=\frac{1}{\Delta z}\int_{z_0}^{z+\Delta z}\left[f(z^*)-f(z)\right]dz^*$$

$f(z)$ is analytic, hence continuous. An $\epsilon>0$ being given, we can thus find a $\delta>0$ such that

$$\left|f(z^*)-f(z)\right|<\epsilon \qquad \text{when } \left|z^*-z\right|<\delta$$

Consequently, letting $|\Delta z|<\delta$, we see that the *ML*-inequality yields

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right|=\frac{1}{|\Delta z|}\left|\int_{z_0}^{z+\Delta z}\left[f(z^*)-f(z)\right]dz^*\right|\le\frac{1}{|\Delta z|}\epsilon\ |\Delta z|=\epsilon;$$

that is, by the definition of a limit and of the derivative,

$$F'(z)=\lim_{\Delta z}\frac{F(z+\Delta z)-F(z)}{\Delta z}=f(z).$$

Since z is any point in *D*, this proves that $F(z)$ is analytic in *D* and is an indefinite integral or antiderivative of $f(z)$ in *D*, written

$$F(z)=\int f(z)dz.$$

Also, if $G'(z)=f(z)$, then $F'(z)-G'(z)\equiv 0$ in D; hence $F(z)-G(z)$ is constant in *D*. That is, two indefinite integrals of $f(z)$. This proves the theorem.

See section 3.2 for examples and problems on indefinite integration.

The theorem in this section followed from Cauchy's integral theorem. A much more fundamental consequence is **Cauchy's integral formula** for evaluating integrals over close curves, which we discuss in the next section.

3.5 Cauchy's Integral Formula

The most important consequences of Cauchy's integral theorem is Cauchy's integral formula. This formula is useful for evaluating integrals (see example below). More importantly, it plays a key role in providing the surprising fact that analytic function have derivative of all orders (see section 3.6), In establishing Taylor series representations and so on. Cauchy's integral formula and its conditions of validity may be stated as follows.

Theorem 1 (Cauchy's Integral Formula)

Let $f(z)$ is analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D which encloses z_0 (fig. below),

1. $$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

(Cauchy's integral formula)

The integration being taken in the counterclockwise sense.

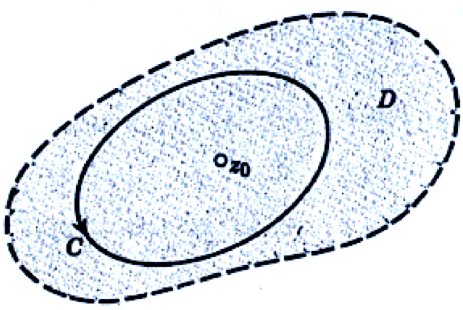
Proof

By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. We insert this into (1) on the left and can take constant factor $f(z_0)$ out from under the integral sign. Then

2. $$\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz.$$

The first on the right hand equals $f(z_0)\cdot 2\pi i$ (see Example 8 in sec. 3.3, with m=-1). This proves this theorem, provided the second integral on the right is zero. This is what we are now going to show. It's integrand is analytic, except at z_0. Hence by the principle of deformation of path (sec. 3.3) we replace C by a small circle K of radius ρ and centre z_0 (figure below), without altering the value of the integral. Since $f(z)$ is analytic, it is continuous. Hence, an $\epsilon>0$ being given, we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \qquad \text{for all } z \text{ in the disk } |z - z_0| < \delta$$



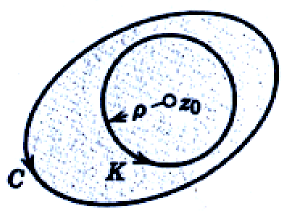


Fig. 39: Cauchy's Integral **Fig. 40: Proof of Formula**

Cauchy's Integral Formula

Choosing the radius ρ of *k* smaller than δ , we thus have the inequality

$$\left|\frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\varepsilon}{\rho}$$

At each point of *k*. The length of *k* is $2\pi\rho$ .Hence by *ML*-inequality in sec. 3.2,

$$\left|\oint_K \frac{f(z)-f(z_0)}{z-z_0}\right| dz < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon.$$

Since ϵ(>0) can be choosing arbitrarily small, it follows that the last integral on the right-hand side of (2) has the value zero, and the theorem is proved.

Example 17

Cauchy's Integral Formula

$$\oint_C \frac{e^z}{z-2}dz = 2\pi e^z \Big|_{z=2} = 2\pi e^2$$

For any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem).

Example 18

Cauchy's Integral Formula

$$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{z^3-3}{2z-\frac{1}{2}i}dz = 2\pi\left[\tfrac{1}{2}z^3-3\right]\Big|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside C}).$$

Example 19

Integration Around Different Contour

$$g(z) = \frac{z^2+1}{z^2-1}$$

in the counterclockwise sense around a circle of radius 1 with centre at the point

a. $z = 1$ (b) $z = \frac{1}{2}$ (c) $z = -1 + \frac{1}{2i}$, (d) $z = i$.

Solution

To see what is going on, locate the point where *g(z)* is not analytic and sketch them along with the contours (figure below) . These points are -1 and1. We see that (b) will give the same result as (a), by the principle of deformation of path. And (d) gives zero, By Cauchy's integral theorem. We consider (a) and afterward (c).

Here $z_0 = 1$, so that $z - z_0 = z - 1$in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z^2-1} = (\frac{z^2+1}{z+1})(\frac{1}{z-1}); \qquad \text{thus} \qquad f(z) = \frac{z^2+1}{z^2-1},$$

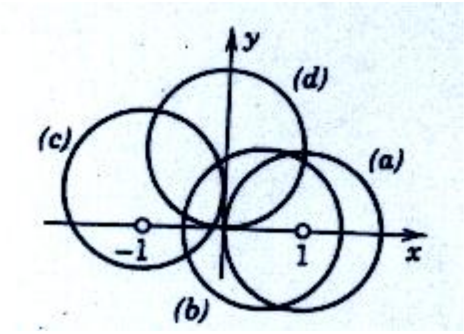


Fig. 41: Example 3

Looking back, we point to a chain of basic results. The beginning was Cauchy's integral theorem in sec. 3.3. From it followed Cauchy's integral formula (1) in this section. From it follows the existence of all higher derivatives of an analytic function, in the next section. This is the probably the most exciting link of our chain. From it follows in the Taylor series for analytic functions.

3.6 Derivative of Analytic Functions

From the assumption that a real function of a real variable is once differentiable, nothing follows about the existence of derivatives of higher order**.** We shall now see that from the assumption that a complex function has a first derivative in a domain *D*, there follows the existence of derivative of all orders in *D*. This means that in this respect complex analytic functions behave much more simply than real functions that are once differentiable.

Theorem 1 (Derivative of Analytic Function)

If $f(z)$ *is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic function in D. The value of these derivatives at a point* z_0 *in D are given by the formulas*

(1′) $$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

(1″) $$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$$

and in general

(1) $$f^n(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1,2,\cdots);$$

here C is any simple closed path in D that encloses z_0 *and whose full interior belongs to D; And we integrate counterclockwise around C(*figure below*).*

Comment

For memorizing (1), it is useful to observe that these formulas are obtained formally by differentiating the Cauchy formula (1), Sec. 3.5, under the integral sign with respect to z_0.

Proof of Theorem

We prove (1′).
We start from the definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

On the right we represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula (1), sec. 3.5; we can combine the two integrals into a single integral by taking the common denominator and simplifying the numerator (where $z - z_0$ drops out and only $f(z)\Delta z$ remains):

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}dz - \oint_C \frac{f(z)}{z - z_0}dz\right]$$

$$= \frac{1}{2\pi i \Delta z}\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}dz \cdot$$

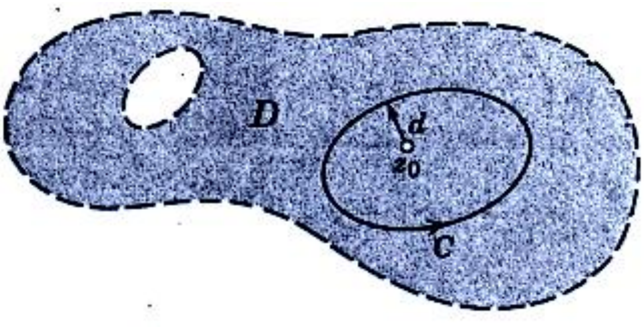


Fig. 42: Theorem 1 and its Proof

Clearly, we can now establish $(1')$ by showing that, as $\Delta z \to 0,$ the integral on the right approaches the integral in $(1')$. To do this, we consider the difference between these two integrals. We can write this difference as a single integral by taking the common denominator and simplifying. This gives

$$\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} dz - \oint_C \frac{f(z)}{(z-z_0)^2} dz$$
$$= \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} dz$$

We show by *ML*-inequality (Sec. 3.2) that this difference approaches zero as $\Delta z \to 0,$.

Being analytic, the function $f(z)$ is continuous on C, hence bounded in absolute value, say, $|f(z)| \le K.$ Let d be the smallest distance from z_0 to the points of *C*(see fig. below). Then for all $|z|$ on *C*,

$$|z-z_0|^2 \ge d^2,$$

hence

$$\frac{1}{|z-z_0|^2} \le \frac{1}{d^2}.$$

Furthermore, if $|\Delta z| \le d/2,$ then for all *z* on *C* we also have

$$|z-z_0-\Delta z| \ge \frac{d}{2}, \quad \text{hence} \quad \frac{1}{|z-z_0-\Delta z|} \le \frac{2}{d}.$$

Let *L* be the length of *C*. Then by *ML*-inequality, if $|\Delta z| \le d/2,$

$$\left|\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz\right| \le K|\Delta z|\frac{2}{d}\cdot\frac{1}{d^2}\cdot$$

This approaches zero as $\Delta z \to 0$, Formula $(1')$ is proved.

Note that we used Cauchy's integral formula (1), Sec. 3.5, but if all we had known about $f(z_0)$ is the fact that it can be represented by (1), Sec. 3.5, our argument would have established the existence of the derivative $f'(z_0)$ of $f(z)$. This is essential to continuation and completion of this proof, because it implies that $(1'')$ can be proved by similar argument, with f replaced by f', and that the general formula (1) then follows by induction.

Example 20

Evaluation of Line Integrals

From $(1')$, for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}dz = 2\pi i(\cos z)'\Big|_{z=\pi i}$$
$$= 2\pi i \sin \pi i = 2\pi \sinh \pi$$

Example 21

From $(1'')$, for any contour enclosing the point -1(counterclockwise)

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz = \pi i(z^4-3z^2+6)''\Big|_{z=-i}$$
$$= \pi i\left[12z^2-6\right]_{z=-i} = -18\pi i$$

Example 22

By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1}$$
$$= 2\pi i\,\frac{e^z(z^2+4)-e^z 2z}{(z^2+4)^2}\Bigg|_{z=1}$$

$$= \frac{6e\pi}{25} i = 2.050i.$$

3.6.1 Moreras's Theorem

If $f(z)$ *is continuous in a simply connected domain D and if*

2. $\oint_C f(z)dz = 0$

for every closed path in D, then $f(z)$ *is analytic in D.*

Proof

In sec.3.4 it was shown that if $f(z)$

$$F(z) = \int_{z_0}^{z} f(z^*)dz^*$$

is analytic in *D* and $F'(z) = f(z)$. In the proof we use only the continuity of $f(z)$ and the property that its integral around every close path in *D* is zero; from the assumptions we concluded that $F(z)$ is analytic. By theorem 1, the derivative of $F(z)$ is analytic, that is $f(z)$ is analytic in *D*, and Morera's theorem is proved.

Theorem 1 also yields a basic inequality that has many applications. To get it, all we have to do is to choose for *C* in (1) a circle of radius *r* and centre z_0 and apply *ML*-inequality (Sec. 3.2); with $|f(z) \leq M|$ on *C* we obtain from (1)

$$\left| f^{(n)} z_0 \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{)z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r.$$

This yields **Cauchy's inequality**

3. $\left| f^{(n)} z_0 \right| \leq \dfrac{n! M}{r^n}.$

To gain first impression of the importance of this inequality, let us prove a famous theorem on entire functions (functions that are analytic for all z; cf.Sec. 2.6)

3.6.2 Liouville's Theorem

If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be a constant.

Proof

By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using (3), we see that $|f'(z_0)| < K/r$. Since this is true for every *r*, we can take *r* as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, and $f(z)$ is a constant.

This completes the proof.

This is the end of section on complex integration, which gave us a first impression of the methods that have no counterpart in real integral calculus. We have seen that these methods result directly or indirectly from Cauchy's integral theorem (Sec.3..3) More on integration follows in the next section.

In the next section, we consider **power series,** which play a great role in complex analysis, and we shall see that the Taylor series of calculus have a complex counterpart, so that e^z, $\cos z, \sin z$ etc. have Maclaurin series that are quite similar to those in calculus.

4.0 CONCLUSION

In conclusion, we state that if a function is analytic, it has derivative of all orders.

5.0 SUMMARY

The complex line integral of a function $f(z)$ taken over a path C is donated by (sec. 3.1)

$$\int_C f(z)dz \quad \text{or, if } C \text{ is closed, also by} \quad \oint_C f(z)dz.$$

Such an integral can be evaluated by using the equation *z=z(t)* of *C*, where $a \le t \le b$ (se. 3.2):

1. $\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)$ $\qquad \left(i = \frac{dt}{dt} \right)$

   As another method, if $f(z)$ is analytic (sec.2.4) in a simply connected domain *D*, then there exists an $F(z)$ in *D* such that

$F'(z) = f(z)$ and for every path C in D from a point z_0 to a point z_1 we have

2. $\int_C f(z)dz = F(z_1) - F(z_0)$ $\quad [F'(z) = f(z)]$.

Cauchy integral theorem states that if $f(z)$ is analytic in as simply connected domain D, then for every closed path C in D

3. $\oint_C f(z)dz = 0$.

If $f(z)$ is as in Cauchy's integral theorem, then for any z_0 in its interior we have C**auchy integral formula**

4. $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$

Furthermore, then $f(z)$ has derivative of all orders in D that are themselves analytic functions in D and (sec. 3.6)

5. $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz.$ $\quad (n = 1, 2, \cdots)$.

6.0 TUTOR-MARKED ASSIGNMENT

i. Show that $\oint_C \frac{dz}{z} = 2\pi i$ (C the unit circle clockwise)

ii. Evaluate $\oint_C e^z dz$ by the method in theorem 1 and compare the result by method in theorem 2.
(C is the line segment from 0 to $1 + \frac{\pi i}{2}$)

iii. For what contour C will it follow from Cauchy's theorem that
(a) $\oint_C \frac{dz}{z} = 0,$ (b) $\oint \frac{e^{-z}}{(z^5 - z)}dz = 0$?

iv. Evaluate the following integrals
(a) $\int_i^{2i} (z^2 - 1)^3 dz$ (b) $\int_0^{\pi i} z \cos z dz$

v. State and prove Morera's theorem

vi. State and prove Liouville's theorem

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