MODULE 1: LEBESGUE MEASURE OF SUBSET OF $\mathbb{R}$

UNIT 1: MEASURE OF A BOUNDED OPEN SET

CONTENT

1.0 Introduction

The concept of measure m(E) of a set E is a generalization of such concept as

(i) The length L(Δ) of a line segment Δ.
(ii) The area A(f) of a plane figure f.
(iii) The volume V(G) of a space figure G.
(iv) The increment f(b) - f(a) of a non-decreasing function f(t) over the close and bounded interval [a, b].
(v) The integral of non negative functions over a set on the real line.

We shall extend the notion of set to more complicated set rather than an interval. Thus, we want a function *m* define on a subset E of $\mathbb{R}$, if possible to have the following properties:

(i) $m(E_1 \cup E_2) = m(E_1) + m(E_2)\ for\ E_1 \cap E_2 = \emptyset$

(ii) $m(\emptyset)$

(iii) $m(E + x) = m(E_1)$, where E+ x means the translation of E distance x

(iv) if $E_1 \subseteq E_2$, then $m(E_1) \leq m(E_1)$

(v) $m(\cup_{n=1} E_n) = \sum_{n=1} m(E_n),\ E_i \cap E_j = \emptyset\ for\ i \neq j$

If we have such an m, then we will be able to integrate more functions.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what the measure of a bounded open set is;
- understand some basic properties of measure of bounded open sets; and
- prove some basic results on measures of bounded open sets.

3.0 MAIN CONTENT

3.1 Measure of a Bounded Open Set

We recall that a set is bounded if it is contained within a ball. Since open sets possesses a very simple structure, it is customary to begin the study of measure with open sets.

Definition 3.1.1: The measure m(G) of a non empty bounded open set G is the sum of the lengths of all its component intervals. i.e. m(G) = $\sum_k m(I_k)$, where I_k are component intervals in G.

Clearly m(G) <∞.

Lemma 3.1.2: If a finite number of pairwise disjoint open intervals $I_1, I_2, I_3, \ldots, I_n$ are contained in an open interval G, then

$$m(G) \geq \sum_{k=1}^{n} m(I_k)$$

Proof: G = (A, B) and let $I_k = (a_k, b_k), k = 1, 2, \ldots, n,$ with $a_1 < \cdots < a_n$. Then $b_k < b_{k+1}$, k = 1,2,…, n-1 (since otherwise, I_k and I_{k+1} will have some common points). Hence the representation

$$Q = (B - b_n) \cup (a_n - b_{n-1}) \cup (a_{n-1} - b_{n-2}) \cup \ldots \cup (a_2 - b_1) \cup (a_1 - A)$$

is non empty. It follows that

B - A = m(G) = $\sum_{k=1}^{n} m(I_k) + m(Q)$

And since Q ≠ ∅, then m(Q) is non - zero and so m(G) ≥ $\sum_{k=1}^{n} m(I_k)$

Corollary 3.1.3: If a denumerable (countable) family of disjoint open intervals I_k, k = 1, 2, 3, …, are contained in an open interval G, then m(G) ≥ $\sum_{k=1}^{n} m(I_k)$

Proof: This **f**ollows from Lemma 3.1.2 as n tends to infinity.

Theorem 3.1.4: Let G_1, G_2 be open sets such that $G_1 \subseteq G_2$, then $m(G_1) \leq m(G_2)$

Proof: Let I_i , i = 1, 2, 3,… and I_k, k = 1, 2 , 3, …

be the components intervals of G_1 and G_2 respectively. Then we know that each of the interval I_i is contained in one and only one of the interval J_k, hence the family of I_i can be divided into pairwise disjoint sub – families A_k, where we put A_k in if $A_i \subseteq J_k$Then by Definition 3.1.1,

we have

$$m(G_1) = \sum_i m(I_i) = \sum_k \sum_{i:\, I_i \subseteq A_k} m(I_i)$$

$$\leq \sum_k m(J_k) = m(G_1).$$

Corollary 3.1.5: The measure of a bounded open set G is the greatest lower bound of the measures of all bounded open sets containing G.

That is

M(G) = inf {m(E): G ⊆E, E is open and bounded}

Proof: Exercise

Theorem 3.1.6: If the bounded open set G is the union of finite or denumerable family of pairwise disjoint open sets (that is, $G = \cup_k G_k$, $G_i \cap G_j = \emptyset$ for $i \neq j$), then $m(G) = \sum_k m(G_k)$.

(This is called the countable additive property of m).

Proof: Exercise

Lemma 3.1.7: Let the open interval J be the union of finite or denumerable family of open sets (that is, $J = \cup_k G_k$). Then

$$m(J) \leq \sum_k m(G_k)$$

Proof: Exercise

Theorem 3.1.8: Let the bounded open set G be the union of finite or denumerable number of open sets G_k (that is, G = U_k G_k). Then

$$m(G) \leq \sum_k m(G_k)$$

Proof: Let I_k , i = 1, 2, 3, ... be the component intervals of the open set G. Then

m(G) = $\sum_i m(I_i)$. However $I_i = I_i \cap (U_k G_k) = U_k(I_i \cap G_k)$. Hence by Lemma 3.1.7, we have m(I_i) ≤ $\sum_k m(I_i \cap G_k)$, and so, we have

$$m(G) \leq \sum_i [\sum_k m(I_i \cap G_k)] \qquad (3.1.1)$$

On the other hand, $G_k = G_k \cap U_i I_i = U_i(I_i \cap G_k)$ Each terms of the right hand side of the last equality are pairwise disjoint (since $I_i \cap I_j = \emptyset$ for i ≠ j). Hence, by applying Theorem 3.1.6, we have

$$m(G_k) \leq \sum_i m(I_i \cap G_k) \qquad (3.1.2)$$

From (3.1.1) and (3.1.2), we have

$$m(G) \leq \sum_i [\sum_k m(I_i \cap G_k)]$$

$$= \sum_k [\sum_i m(I_i \cap G_k)] = \sum_k m(G_k).$$

4.0 CONCLUSION

The notion of measure is a generalization of concepts such as length, volume, area, and so on. In this unit, we have extended the notion of set to more complicated set rather an interval and define a function on this set, called measure which satisfies some properties. This will enable us to integrate more functions.

5.0 SUMMARY

In this unit we have learnt:

i. the definition of measure of a bounded open set;
ii. the basic properties of measure of bounded open sets; and
iii. how to prove some basic results on measures of bounded open sets.

6.0 TUTOR MARKED ASSIGNMENT

1. Find the length (measure) of the set

$$\cup_{k=1}\left\{x: 0 \leq x < \frac{1}{k}\right\}$$

2. Find the length of the set

$$\cup_{k=1}\left\{x: 0 \leq x < \frac{1}{3^k}\right\}$$

7.0 FURTHER READING AND OTHER RESOURCE

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UNIT 2: MEASURE OF A BOUNDED CLOSED SET

CONTENTS

1.0 INTRODUCTION

In this unit, we discuss the measure of a bounded close set and some of its basic properties.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what the measure of a bounded closed set is;
- understand some basic properties of measure of bounded closed sets; and
- prove some basic results on measures of bounded closed sets.

3.0 MAIN CONTENT

3.1 Measure of a Bounded Closed Set

Let F be a non - empty closed set and let S be the smallest closed interval containing the set F. The set $C_sF = [a, b] - F$ is open and hence has a definite measure $m(C_sF)$. $S = [a, b]$, C_sF is the complement of F with respect to S, and so is open since F is closed. This leads to the following definition.

Definition 3.2.1: The measure of a non - empty bounded closed set F is the number $m(F) = b - a - m(C_sF)$, where $S = [a, b]$ is the smallest closed interval containing the set F.

Example 3.2.1

(i) if $F = [a , b]$, then $S = [a , b]$ and $C_sF = \emptyset$. Hence

$$\mathbf{m(F) = b - a - m(C_sF) = b - a - 0 = b - a}$$

thus, the measure of a closed interval is equal to its length.

(ii) If F is the union of a finite number of pairwise disjoint closed interval.

$F = [a_1 b_1] \cup [a_2 b_2] \cup \ldots \cup [a_n b_n]$

We may consider the closed intervals as being enumerated in the order of increasing left end points, then $b_k < a_{k+1}$, $k = 1, 2, \ldots, n-1$, it follows that S= $[a_1, b_n]$ and

$C_s F = (b_1 a_2) \cup (b_2 a_3) \cup \ldots \cup (b_{n-1} a_n)$. Hence,

$$m(F) = b_n - a_1 - \sum_{k=1}^{n-1} (a_{k+1} - b_k) = \sum_{k=1}^{n} (b_k - a_k)$$

That is, the measure of a union of a finite number of pairwise disjoint closed intervals equals the sum of the length of these intervals.

Theorem 3.2.3: The measure of a bounded closed set F is non – negative.

$C_s F \subseteq (a, b)$ and so m(C_sF)≤ $(a, b) = b - a$

but m(F) = b – a – m(C_sF). Hence, m(F) ≥ $b - a - (b - a) = 0$

Lemma 3.2.5: Let F be a bounded closed set contained in the open interval I. Then

m(F) = m(I) - m(I - F).

Proof: Exercise

Theorem 3.2.6: Let F be a closed set and let G be a bounded open set. If F G, then

m(F) ≤ m(G).

Proof: Let I be an interval containing G. Then I = G∪(I - F) and applying Theorem 3.1.8, we have

m(F) = m(I) - m(I - F) ≤ m(G) + m(I - F) - m(I - F) = m(G).

Theorem 3.2.7: The measure of a bounded open set G is the least upper bound of the measures of all closed sets contained in G.

Proof: Exercise

Theorem 3.2.8: The measure of a bounded closed set F is the greatest lower bound of the measures of all possible bounded open sets containing F.

Proof: Exercise

Theorem 3.2.9: Let the bounded closed set F be the union of a finite number of pairwise disjoint closed sets, F = $\bigcup_{i=1}^{n} F_i$, $F_i \cap F_j = \emptyset$ for $i \neq j$. Then

$$m(F) = \sum_{i=1}^{n} m\,(F_i)$$

Proof: Exercise

4.0 CONCLUSION

The concept of measure of bounded closed set with some of its basic properties has been discussed in this unit. This properties are very useful when integrating functions over bounded closed set.

5.0 SUMMARY

In this unit we have learnt:

i. the definition of measure of a bounded closed set;
ii. the basic properties of measure of bounded closed sets; and
iii. how to prove some basic results on measures of bounded closed sets.

6.0 TUTOR MARKED ASSIGNMENT

Prove Theorem 3.2.4, Theorem 3.2.7, Theorem 3.2.8 and Theorem 3.2.9

7.0 FURTHER READING AND OTHER RESOURCE

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)
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UNIT 3: THE OUTER AND INNER MEASURES OF BOUNDED SETS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the outer and inner measures of bounded sets and also establish some basic results on outer and inner measures.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what the outer and inner measures of bounded sets are;
- know what a measurable set is;
- understand some basic properties of outer and inner measures of bounded sets; and
- prove some basic results on outer and inner measures of bounded sets.

3.0 MAIN CONTENT

3.1 The Outer and Inner measures of Bounded Sets

Definition 3.3.1: The outer measure $m^*(E)$ of a bounded set E is the greatest lower bound of the measures of all bounded open sets containing the set E.

That is,

$$m^*(E) = \inf\{m(G) : E \subseteq G, G \text{ is open and bounded}\}$$

Clearly for any bounded set E, $m^*(E)$ is well defined and $0 \leq m^*(E) < +\infty$.

Definition 3.3.2: The inner measure $m^*(E)$ of a bounded set E is the least upper bound of the measures of all closed sets contained in the set E.

That is,

$m_*(E)$ = sup{m(F) : F ⊆ E and F is closed}.

Also, for any bounded set E, $m_*(E)$ is well defined and $0 \leq m_*(E) < +\infty$.

Theorem 3.3.3: Let G be a bounded open set. Then $m^*(E) = m_*(E)$ = m(G).

Proof: This follows from Corollary 3.1.5 and Theorem 3.2.7.

Theorem 3.3.4: Let F be a bounded closed set. Then $m^*(E) = m_*(E)$ = m(F).

Proof: This follows from Theorem 3.2.4 and Theorem 3.2.8.

We recall that a set is measurable if its outer measure equals its inner measure.

Theorem 3.3.5: For any bounded set E, $m_*(E) \leq m^*(E)$

Proof: Let G be a bounded open set containing the set E, for any closed subset F of the set E, we have F ⊆ G, and so, m(F) ≤ m(G). Hence $m_*(F) \leq m(G)$. Since this is true for every bounded open set G containing E, we have that $m_*(E) \leq m^*(E)$.

Theorem 3.3.6: Let A and B be bounded sets such that A ⊆ B. Then

$$m_*(A) \leq m_*(B) \text{ and } m^*(A) \leq m^*(B)$$

Proof: We prove the first part and leave the second part as assignment. Let S be the set of numbers consisting of the measures of all closed subsets of the set A and let T be the analogous set for the set B. Then $m_*(A) = \sup S$ and $m_*(B) = \sup T$. If F is a closed subset of A, then F is necessarily a subset of B since A ⊆ B. It follows thus that S ⊆ T and so $m_*(A) \leq m_*(B)$

Theorem 3.3.7: If a bounded set E is the union of a finite or denumerable number of sets $= E_k U_k$. Then

$$m^*(E) \leq \sum_k m^*(E_k)$$

Proof: If the series $\sum_k m^*(E_k)$ diverges the result is trivial. Suppose the series $\sum_k m^*(E_k)$ converges. Pick an arbitrary $\varepsilon > 0$, we can find bounded open set G_k such that $E_k \subseteq G$,

$$m(_{G_k}) < m^*(E_k) + \frac{1}{2^k}, k = 1, 2, \ldots$$

Denote by I an open interval containing the set E. Then

$$E \subseteq I \cap (U_k G_k)$$

$$m^*(E) \leq \mathrm{m}(\mathrm{I} \cap (\cup_k G_k)) = \mathrm{m}(\cup_k (G_k \cap \mathrm{I}))$$

$$\leq \sum_k m(G_k \cap \mathrm{I}) \leq \sum_k m(G_k) \leq \sum_k m^*(E_k) + \epsilon.$$

The result follows since M is arbitrary.

Theorem 3.3.8: If a bounded set E is the union of a finite or denumerable number of pairwise disjoint sets E_k, (that is E = $U_k E_k$, $E_i \cap E_j = \emptyset$ for i ≠ j). Then

$$m_*(E) \geq \sum_k m_*(E_k).$$

Proof: Consider the first n sets $E_1, E_2, \ldots, E_n$ For an arbitrary M > 0, there exist closed sets F_k such that $F_k \subseteq U_k, m(F_k) > m_*(E_k) - \frac{\epsilon}{n}$k = 1,2,…n. The sets F_k are pairwise disjoint and their union $U_k F_k$ is closed. Hence, by applying Theorem 3.2.9, we have

$$m_*(E) \geq m(\cup_{k=1}^{n} F_k) = \sum_{k=1}^{n} m(F_k)$$

$$> \sum_{k=1}^{n} m_*(E_k) - \epsilon.$$

Since M is arbitrary, it follows that

$$\sum_{k=1}^{n} m_*(E_k) \leq m_*(E).$$

This proves the theorem for finite case. For the denumerable case, by noting that the number n is arbitrary, we can establish the convergence of the series $E_k m_*(E_k)$ and the inequality

$$\sum_{k=1}^{\infty} m_*(E_k) \leq m_*(E)$$

Remark: The above theorem is not generally true if we omit the condition that the sets have no common points (It may be false if E_k have some points in common).

For example, if $E_1 = [0,1], E_2 = [0,1]$ and $E = E_1 \cup E_2$. Then $m_*(E_1) = 1$ while

$$m_*(E_1 \cup E_2) = 1 + 1 = 2$$

4.0 CONCLUSION

The concept of outer and inner measure of a bounded set with some of their basic properties have been discussed in this unit. These properties are very useful when integrating functions over bounded set. These concepts enable us to know when a set is measurable and allows us to integrate functions over measurable sets

5.0 SUMMARY

In this unit we have learnt:

i. the definition of outer and inner measures of a bounded sets;
ii. the definition of measurable sets;
iii. the basic properties of outer and inner measures of bounded sets; and
iv. how to prove some basic results on outer and inner measures of bounded sets.

6.0 TUTOR MARKED ASSIGNMENT

1. Let A and B be bounded sets such that A $\subseteq$ B. Show that

   $m^*(A) \leq m^*(B)$

2. Show that the outer measure of an interval is its length.
3. Show that outer measure is translation invariant.

7.0 FURTHER READING AND OTHER RESOURCES

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