MODULE 2: GENERAL MEASURE SPACE (X, $\mathcal{M}, \mu$)

UNIT 1: ALGEBRAS AND SIGMA - ALGEBRAS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall discuss Algebras and Sigma - Algebras with examples and some basic results on them.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what the algebra and sigma - algebras are;
- know what measurable space and measurable set are;
- understand the concept of Borel sigma - algebra on the real line; and
- prove some basic results on measurable space.

3.0 MAIN CONTENT

3.1 Algebras and Sigma - Algebras

Definition 3.1.1: Let X be an arbitrary set. A collection Ω of subsets of X is called an algebra if

(i) X $\epsilon\ \mathcal{M}$
(ii) for any set A $\epsilon\ \mathcal{M}$, the set $A^c \epsilon\ \mathcal{M}$
(iii) for each finite sequence $A_1, A_2, \dots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcup_{i=1}^{n} A_i \epsilon \mathcal{M}$

(iv) for each finite sequence $A_1, A_2, \dots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcap_{i=1}^{n} A_i \epsilon \mathcal{M}$

Of course, in conditions (ii), (iii) and (iv), we have required $\mathcal{M}$ be closed under complementation, under the formation of finite unions and under the formation of finite intersections.

Remark 3.1.2:

(i) It is easy to check that closure under complementation and closure under the formation of finite unions together imply closure under the formation of finite intersection (using the fact that $\bigcap_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i^c)^c$). Thus, we could have defined an algebra (Definition 3.1.1) using only conditions (i), (ii) and (iii).

(ii) $\emptyset \epsilon \mathcal{M}$, since $\emptyset = X^c$ and $X \epsilon \mathcal{M}$

Definition 3.1.3: Let X be an arbitrary set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra if

(i) X $\epsilon \mathcal{M}$
(ii) for any set A ϵ $\mathcal{M}$, the set $A^c \epsilon \mathcal{M}$
(iii) for each infinite sequence $A_1, A_2, \dots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcup_{i=1}^{\infty} A_i \epsilon \mathcal{M}$
(iv) for each infinite sequence $A_1, A_2, \dots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcap_{i=1}^{\infty} A_i \epsilon \mathcal{M}$

Thus a σ- algebra on X is a family of subsets of X that contains and is closed under complementation, under the formation of countable unions and under the formation of countable intersections.

Remark 3.1.4:

(i) As in the case of algebra (Definition 3.1.1), we could have used only conditions (i), (ii) and (iii) or only conditions (i), (ii) and (iv) in our Definition 3.1.3, since
$\bigcap_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{\infty} A_i^c)^c$.

(ii) $\emptyset \epsilon \mathcal{M}$, since $\emptyset = X^c$ and $X \epsilon \mathcal{M}$

(iii) Let $A_1, A_2, \ldots, A_n$ of sets that belong to $\mathcal{M}$, the set $\bigcap_{i=1}^{n} A_i \epsilon \mathcal{M}$
We take $A_{n+1} = A_{n+2} = A_{n+3} = \cdots = \emptyset$ in (ii) of Definition2.1.3.
(iv) Let A, B ϵ $\mathcal{M}$ then A ∩ B^c since A ∩ B^c = $(A^c \cup B)^c$
(v) Each σ- algebra on X is an algebra on X.

Example 3.1.5:

(i) Let X be any set and let $\mathcal{M}$ = P(X), the power set of X. Then $\mathcal{M}$ is a σ-algebra on X.
(ii) Let X be any set, then $\mathcal{M}$ = {∅ , X} is a σ- algebra on X.
(iii) Let X = N = [1, 2, 3, 4,…}. Then $\mathcal{M}$ = {∅, X {1, 3, 5, 7,…}, {2, 4, 6, 8,…}} is a σ- algebra on X.
(iv) Let X be an infinite set, and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathcal{M}$ is an algebra on X, but it is not a σ-algebra on X, since it is not closed under the formation of countable unions.
(v) Let X be any set, and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is countable. Then $\mathcal{M}$ is a σ- algebra on X.
(vi) Let X = {a, b, c, d, e, f} and $\mathcal{M}$ = {∅, X, {a, b, c}, {f}}. Then $\mathcal{M}$ is not a σ-algebra on X.
(vii) Let X be an infinite set, and let $\mathcal{M}$ be the collection of all finite subsets of X. Then $\mathcal{M}$ does not contain X and is not closed under complementation, and so is not an algebra (or a σ- algebra) on X.
(viii) Let $\mathcal{M}$ be the collection of all subsets of that are unions of finitely many intervals of the form (a, b], (a, +∞) or (-∞, b]. It is easy to check that each set that belongs to $\mathcal{M}$ is the union of a finite disjoint collection of intervals of the type listed above, and then to check that $\mathcal{M}$ is an algebra on ℝ. $\mathcal{M}$ is not a σ- algebra on ℝ, for example, the bounded open subintervals of ℝ are unions of sequence of sets in $\mathcal{M}$, but do not themselves belong to $\mathcal{M}$.

Definition 3.1.6: Let X be any non - empty set and let $\mathcal{M}$ be a σ- algebra on X. The pair (X, $\mathcal{M}$) is called a measurable space. The members of $\mathcal{M}$ are called measurable sets

Proposition 3.1.7: Let (X, $\mathcal{M}$) be a measurable space. Then

(i) $\emptyset \in \mathcal{M}$
(ii) $\mathcal{M}$ is an algebra of sets
(iii) If $\{A_i\}_{i=1} \epsilon \mathcal{M}, \ then \ \bigcap_{i=1}^{n} A_i \in \mathcal{M}$
(iv) If A, B ϵ $\mathcal{M}$, then A ∩ B^c $\epsilon\mathcal{M}$

Proof: See Remarks 3.1.2 and 3.1.4

Proposition 3.1.8: Let X be any non - empty set. Then the intersection of an arbitrary non - empty collection of σ- algebras on X is a σ- algebra on X.

Proof: Denote $\{\mathcal{M}_\alpha\}_{\alpha\in\Omega}$ the non – empty collection of σ – algebra on X. We show that $\bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$ is a σ – algebra on X. It is enough to check that $\bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$ contains X, is closed under complementation, and is closed under the formation of countable unions.

$X \in \bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$, since $X \in \mathcal{M}_\alpha\ \forall\ \alpha \in \Omega$. Now suppose $A \in \bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$, then $A^c \in \mathcal{M}_\alpha$ $\forall\ \alpha \in \Omega$ and so $A^c \in \bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$. Finally, suppose $\{A_i\}$ is a sequence of sets that belong to $\bigcap_{\alpha\in\Omega}\mathcal{M}_\alpha$, then $\{A_i\} \in \mathcal{M}_\alpha\ \forall\ \alpha \in \Omega$, and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}_\alpha\ \forall\ \alpha \in \Omega$.

Thus, $\bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha\epsilon\Omega}\mathcal{M}_\alpha$

Corollary 3.1.9: Let X be any non - empty set, and let Ω be a family of subsets of X. Then there is a smallest σ- algebra on X that includes Ω.

Proof: Let $\{\mathcal{M}_\alpha\}_{\alpha\epsilon\Omega}$ be the collection of all σ- algebras on X that includes Ω. Then $\{\mathcal{M}_\alpha\}_{\alpha\epsilon\Omega}$ is non - empty since P(X) $\epsilon\{\mathcal{M}_\alpha\}_{\alpha\epsilon\Omega}$. Then the intersection of all σ-algebras on X that belong to $\{\mathcal{M}_\alpha\}_{\alpha\epsilon\Omega}$ is a σ- algebra on X by Proposition2.1.6, that is $\cap_{\alpha\epsilon\Omega}\mathcal{M}_\alpha$ is a σ- algebra on X and $\cap_{\alpha\epsilon\Omega}\mathcal{M}_\alpha$ includes Ω and is included in every σ-algebra on X that included Ω.

Remark 3.1.10: To say that $\mathcal{M}$ is the smallest σ- algebra on X that includes Ω is to say that $\mathcal{M}$ is a σ- algebra on X that includes Ω, and that every σ- algebra on X that includes Ω also includes $\mathcal{M}$. This smallest σ-algebra on X that includes Ω is clearly unique, it is called the σ- algebra generated by Ω, and it is often denoted by σ (Ω).

We now use the preceding corollary to define an important family of σ- algebras.

Definition 3.1.11: The Borel σ- algebra on $\mathbb{R}^n$ is the σ- algebra on generated by the collection of open subsets of $\mathbb{R}^n$, and it is denoted by **B**($\mathbb{R}^n$). The Borel subsets of $\mathbb{R}^n$ are those that belong to **B**($\mathbb{R}^n$). In the case n = 1, we generally writes B($\mathbb{R}$)

Proposition 3.1.12: The σ- algebra **B**($\mathbb{R}^n$) of Borel subsets of $\mathbb{R}$ is generated by each of the following collections of sets;

(i) the collection of all closed subsets of $\mathbb{R}$

(ii) the collection of all subintervals of $\mathbb{R}$ of the form (-∞ , b]

(iii) the collection of all subintervals of $\mathbb{R}$ of the form (a, b]

Proof: Exercise

For general Borel sets, we have the following.

Definition 3.1.13: Let (X, τ) be a topological space. (By Corollary3.1.9), there exists a smallest σ- algebra B on X which contain the family of open subsets of X. This is called the **B**orel

σ- algebra on X. We shall denote this by **B**(X). For any U ∈ **B**(X), U is called a Borelian or a Borel set.

In particular;

(i) All the closed subsets are Borelian (Since U ∈ B(X) imply U^c ∈ B(X) and \$^R$ is closed.

(ii) (X, **B**(X)) is a measurable space.

4.0 CONCLUSION

The concept of algebras and sigma-algebras with some of their basic properties has been discussed in this unit. This led to the introduction of Borel sigma-algebra on the set of real numbers was shown to be generated by the collection of closed subsets of real line. This allows us to integrate functions on the whole of real line.

5.0 SUMMARY

In this unit we have learnt:

i. the definition of algebra and sigma algebra with examples;

ii. the concept of measurable space and measurable sets;

iii. some results on Borel sigma algebra;

iv. how to prove some results on measurable space.

6.0 TUTOR MARKED ASSIGNMENT

1. Show by example that the union of a collection of σ- algebras on a set X can fail to be a σ- algebra. (**Hint**: There are examples in which X is a small finite set.)

2. Find an infinite collection of subsets of $\mathbb{R}$ that contains $\mathbb{R}$, is closed under the formation of countable unions, and is closed under the formation of countable

intersections, but is not a σ- algebra.

3. Find the σ- algebra on that is generated by the collection of all one - point subsets of $\mathbb{R}$.
4. Show that **B**($\mathbb{R}$) is generated by the collection of all compact subsets of $\mathbb{R}$.
5. Let (X, $\mathcal{M}$) be a measurable space and let Y be a topological space. Let f: X→Y. If

   $$\Omega = \{E \subseteq Y : F^{-1}(E) \in \mathcal{M}\},$$

   show that Ω is a σ- algebra on Y.
6. Let (X, $\mathcal{M}$, μ) be a measure space and
   $\mathcal{M}^*$ = {E ϵX:∃ $A, B \in \mathcal{M}$, with A ⊆ E ⊆ B and
   $A \subseteq E \subseteq B \; and \; \mu(B - A) = 0\}$
   Show that $\mathcal{M}^*$ is a σ- algebra on X.

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UNIT 2: GENERAL MEASURES SPACE (X, $\mathcal{M}, \mu$)

CONTENTS

1.0 INTRODUCTION

In this unit, we shall discuss measures and some of their basic properties. We start by discussing set functions, additive and countable additive set functions with some examples.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what the additive and countably additive set functions are;
- know what a measure is and also give some examples of measures;
- know some basic results on measures; and
- prove some basic results on measures.

3.0 MAIN CONTENT

3.1 Measures

Definition 3.2.1: A set function μ is a function whose domain is a collection of sets.

Definition 3.2.2: Let (X, $\mathcal{M}$) be a measurable space. A set function μ whose domain is the σ- algebra $\mathcal{M}$ is called

(i) Additive if whenever A, B $\in \mathcal{M}$ and A∩B = ∅, then
$\mu\ (A \cup B) = \mu(A) + \mu(B)$.

(ii) Count ably additive if for and $\{A_n\}$ which are members of $\mathcal{M}$ and $A_i \cap A_j = \emptyset$

whenever $i \neq j$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

In the case (ii), we take the values of μ to belong to the extended half - line [0, +∞].

Remark: Since $\mu(A_n)$ is non - negative for each n, the sum $\sum_{n=1}^{\infty} \mu(A_n)$ always exists, either as a real number as +∞.

Definition 3.2.3: Let (X, $\mathcal{M}$) be a measurable space. A measure (or a countable additive measure) on $\mathcal{M}$ is a function: $\mathcal{M}$→[0 , +∞] that satisfy μ (∅) = 0 and is countable additive.

Definition 3.2.4: Let $\mathcal{M}$ be an algebra (not necessarily a σ- algebra) on the set X. A function μ whose domain is $\mathcal{M}$ is called finitely additive if
$\mu(\bigcup_{n=1}^{k} A_k) = \sum_{n=1}^{k} \mu(A_k)$ for each finite sequence $A_1, A_2, \ldots, A_n$ of disjoint sets that belong to $\mathcal{M}$.

A finitely additive measure on the algebra $\mathcal{M}$ is a function: $\mathcal{M}$→[0 , +∞] that satisfy μ (∅) = 0 and is finitely additive.

It is easy to check that every countably additive measure is finitely additive.

Remark: Countably additive measures seem to be sufficient for almost all applications, and support a much powerful theory of integration than do finitely additive measures. Thus we shall devote all our attention to countably additive measures. We shall emphasize that a measure will always be a countably additive measure.

If (X, $\mathcal{M}$) be a measurable space and μ is a measure on $\mathcal{M}$, then the triple (X, $\mathcal{M}$, μ) is called a measure space. Thus, a measure space is a measurable space in which a measure is defined.

We shall assume that there exists at least one A ∈ $\mathcal{M}$, such that μ(A) < ∞.

Examples 3.2.5:

(i) Let (X, $\mathcal{M}$) be a measurable space. Define a function: $\mathcal{M}$→ [0, +∞] by letting μ(A) be n if A is a finite set with n elements, and letting μ(A) = +∞ if A is an infinite set.
Then μ is a measure, it is often called counting measure on (X, $\mathcal{M}$).

(ii) Let (X, $\mathcal{M}$) be a measurable space. Let x ∈ X.

Define

$$\delta_x(A) = \begin{cases} 1 & if\ x\ is\ an\ element\ of\ A \\ 0 & if\ x\ is\ no\ an\ element\ of\ A \end{cases}$$

Then δ_x is a measure, it is called a point mass concentrated at x.

(iii) Let X = ℝ and $\mathcal{M}$ = **B**(ℝ) be the Borel σ- algebra on ℝ . Let: **B**(ℝ)→[0, +∞] be defined by μ(A) = length of A , where A is a subinterval of ℝ . Then μ is a measure, it is known as the Labesgue measure.

(iv) Let X = ℕ the set of natural numbers and let $\mathcal{M}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathcal{M}$ is an algebra, but not a σ-algebra.

Define : $\mathcal{M}$ → [0 , +∞] as

$$\mu_A = \begin{cases} 1\ if\ A\ is\ infinite \\ 0\ \ \ if\ A\ is\ finite \end{cases}$$

Then μ is a finitely additive measure.

(v) Let (X, $\mathcal{M}$) be a measurable space. Define μ: $\mathcal{M}$ → [0, +∞] by

$$\mu_A = \begin{cases} 1 & if\ A\ \neq \emptyset \\ 0 & if\ A\ = \emptyset \end{cases}$$

Then μ is not a measure, nor even a finitely additive measure, for if A, B ∈ Q and A ∩ B = ∅, then $\mu(A \cup B) = 1$, while $\mu(A) + \mu(B) = 1 + 1 = 2$

Proposition 3.2.6: Let (X, $\mathcal{M}$) be a measure space, and let A and B be subsets of X that belong to $\mathcal{M}$ and satisfy A⊆ B. Then $(A) < \mu(B)$. If in addition A satisfies $\mu(A) < +\infty$, then

$$\mu(B - A) = \mu(B) - \mu(A).$$

Proof: Since A∩ (B - A) = ∅ and B = A ∪ (B - A), then by additivity of μ, we have

$$\mu(B) = \mu(A) + \mu(B - A) \qquad (3.2.1)$$

And since $\mu(B - A) \geq 0$, then

$$\mu(A) < \mu(B)$$

In the case $\mu(A) < +\infty$, then we have

$$\mu(B) - \mu(A) = \mu(B - A),$$

from equation (3.2.1).

Definition 3.2.7: Let μ be a measure on a measurable space $(X, \mathcal{M})$.

(i) μ is a finite measure if $\mu(X) < +\infty$

(ii) μ is a $\sigma -$ finite measure if X is the union of a sequence $A_1, A_2, \ldots$, of sets that belong to $\mathcal{M}$ and satisfy $\mu(A_i) < +\infty$ for each i.

(iii) A set in $\mathcal{M}$ is $\sigma -$finite under μ if it is the union of a sequence of sets that belong to $\mathcal{M}$ and have finite measure under μ.

(iv) The measure space $(X, \mathcal{M}, \mu)$ is called finite if μ is finite.

(v) The measure space $(X, \mathcal{M}, \mu)$ is called $\sigma -$finite if μ is $\sigma -$finite.

We note that the measure defined in Example 3.2.5 (i) above is finite if and only if the set X is finite. The measure defined in Example 3.2.5 (ii) is finite. The Lesbesgue measure defined in Example2.2.5 (iii) is $\sigma -$finite, since $\mathbb{R}$ is the union of sequence of bounded intervals.

The following propositions give some elementary but useful properties of measures.

Proposition 3.2.8: Let (X, Q, μ) be a measure space. If $\{A_k\}$ is an arbitrary sequence of sets that belong to $\mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

Proof: Define a sequence $\{B_k\}$ of subsets of X by letting $B_1 = A_1$, and $B_k = A_k - \cup_{i=1}^{k-1} A_i$ for k > 1. Then each $B_k \in \mathcal{M}$ and $B_k \subseteq A_k$ and so $\mu(B_k) \leq \mu(A_k)$.

Since in addition the sets B_k are disjoint and satisfy $\cup_k B_k = U_k A_k$, it follows that

$$\mu(\bigcup_k A_k) = \mu(\bigcup_k B_k) = \sum_k \mu(B_k) \leq \sum_k \mu(A_k)$$

$$\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

Thus,

Remark: The above proposition shows that the countable additivity of μ implies the countable subadditivity of μ.

Proposition 3.2.9: Let (X, $\mathcal{M}$, μ) be a measure space.

(i) If $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$. Then

$$\mu(\bigcup_k A_k) = \lim_k \mu(A_k).$$

(ii) If $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{M}$, and if $\mu(A_n) < +\infty$ for some *n*, then

$$\mu\left(\bigcap_k A_k\right) \lim_k \mu(A_k)$$

Proof:

(i) Suppose that $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$ and define a sequence $\{B_k\}$ of sets by letting $B_1 = A_1$, and $B_j = A_j - A_{j-1}$ for j > 1. The sets B_j are disjoint and belong to $\mathcal{M}$, and satisfy $A_k = \bigcup_{j=1}^{k} B_j$. It follows that $U_k A_k = U_j B_j$

$$\mu(\bigcup_k A_k) = \mu(\bigcup_j B_j) = \sum_j \mu(B_j)$$

$$= \lim_k \sum_{j=1}^{k} \mu(B_j) = \lim_k \mu(\bigcup_{j=1}^{k} B_j)$$

$$= \lim_k \mu(A_k).$$

(ii) Now suppose that $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{M}$, and let $\mu(A_n) < +\infty$ holds for some n. We can assume n = 1. For each k, let $C_k = A_1 - A_k$. Then $\{C_k\}$ is an increasing sequence of sets that belong to $\mathcal{M}$, and $U_k C_k = A_1 - (\bigcap_k A_k)_j$. It follows from (i) that

$$\mu\left(\bigcup_k C_k\right) \lim_k \mu(C_k)$$

and so

$$\mu(A_1 - (\cap_k A_k)) = \lim_k \mu(A_1 - A_k) \qquad (3.2.2)$$

By using Proposition 3.2.6 and the assumption that $\mu(A_1) < +\infty$ equation (3.2.2) gives

$$\mu(A_1 - (\cap_k A_k)) = \mu(A_1) - \lim_k \mu(A_k)$$

Thus,

$$\mu\left(\bigcap_k A_k\right) \lim_k \mu(A_k)$$

The preceding proposition has the following partial converse, which is sometimes useful for checking when a finitely additive measure is in fact countably additive.

Proposition 3.2.10: Let (X, $\mathcal{M}$) be a measurable space, and let μ be a finitely additive measure on (X, $\mathcal{M}$). Then μ is a measure if either

(i) $\lim_k \mu(A_k)$ holds for each increasing sequence $\{A_k\}$ of sets that belong to $\mathcal{M}$. Or

(ii) $\lim_k \mu(A_k)$ 0 holds for each decreasing sequence $\{A_k\}$ of sets that belong to $\mathcal{M}$ and satisfy $\cap_k A_k = \emptyset$

Proof:
We need to verify the countable additivity of μ. Let $\{B_j\}$ be a disjoint sequence of sets that belong to $\mathcal{M}$, we shall prove that $\mu(U_j B_j) = \sum_j \mu B_j$.

First assume that condition (i) holds, and for each k, let $A_k = \cup_{j=1}^{k} B_j$ then the female additivity of μ implies that

$$\mu(A_k) = \sum_{j=1}^{k} \mu(B_j)$$

while condition (1) implies that

$$\mu(\cup_{k=1}^{\infty} A_k) = \lim_k \mu(A_k)$$

since $\cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} A_j$, it follows that

$$\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j)$$

Now assume that condition (ii) holds, and for each k, let $A_k = \cup_{j=1}^{\infty} B_j$. Then the finite additivity of μ implies that

$$\mu(\cup_{j=1}^{\infty} B_j) = \sum_{j=1}^{k} \mu(B_j) + \mu(A_{k+1}) \qquad (3.2.3).$$

while (ii) implies that $\lim_k \mu(A_{k+1}) = 0$, hence

$$\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j)$$

by taking limit over k in equation (3.2.3).

4.0 CONCLUSION

The concept of finitely additive measures and countably additive measure with some of their basic properties have been discussed in this unit. Countably additive measures seem to be sufficient for almost all applications, and support a more powerful theory of integration than finite additive measures.

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of additive and countably additive set functions;
(ii) the definition of measure with examples;
(iii) some basic results on measures;
(iv) how to prove some basic results on measures.

6.0 TUTOR MARKED ASSIGNMENT

(1) Suppose that μ is a finite measure on (X, $\mathcal{M}$)

(i) Show that if A, B $\in \mathcal{M}$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

(ii) Show that if A, B, C $\in \mathcal{M}$, then

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B)$$

$$- \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C)$$

(iii) Find and prove a corresponding formula for the measure of the union of n sets

(2) Let (X, $\mathcal{M}$) be a measurable space. Suppose μ is a non - negative countably additive function on $\mathcal{M}$. Show that if $\mu(A)$ is finite for some A $\in \mathcal{M}$, then $\mu(\emptyset) = 0$. (Thus μ is a measure).

(2) Let (X, $\mathcal{M}$) be a measurable space, and let x, y $\in$ X. Show that the point masses ∂_x and ∂_y are equal if and only if x and y belong to exactly the same sets in $\mathcal{M}$.

(4) Let (X, $\mathcal{M}, \mu$), be a measure space, and define μ^*: $\mathcal{M} \to [0, \infty]$ by

$$\mu^*(A) = \sup \{ \mu(B) : B \subseteq A, \mathrm{B} \in \mathcal{M} \text{ and } \mu(B) < +\infty\}$$

(i) Show that μ^* is a measure on (X, $\mathcal{M}$).

(ii) Show that if μ is σ- finite, then $\mu^* = \mu$.

(iii) Find μ^* if X is non - empty and μ is a measure defined by

$$\mu(A) \begin{cases} +\infty \;\; if \;\; A \in \mathcal{M} \; and \; A \neq \emptyset \\ 0 \;\; if \; A = \emptyset \end{cases}$$

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UNIT 3: MEASURABLE FUNCTIONS (X, $\mathcal{M}, \mu$)

CONTENT

1.0 INTRODUCTION

In this section we introduce measurable functions and study some of their basic properties.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define what a measurable function is with examples;
- know some basic results on measurable functions; and
- prove some basic results on measurable functions.

3.0 MAIN CONTENT

3.1 MEASURABLE FUNCTIONS

We begin with the following result.

Proposition 3.3.1: Let (X, $\mathcal{M}$) be a measurable space, and let A be a subset of X that belongs to $\mathcal{M}$. For a function f: $\mathcal{M}\rightarrow$ [-∞, +∞] the following conditions are equivalent:

(i) for each real number t the set $\{x \in A: f(x) \leq t\}$ belong to $\mathcal{M}$
(ii) for each real number t the set $\{x \in A: f(x) < t\}$ belong to $\mathcal{M}$
(iii) for each real number t the set $\{x \in A: f(x) \geq t\}$ belong to $\mathcal{M}$
(iii) for each real number t the set $\{x \in A: f(x) > t\}$ belong to $\mathcal{M}$

Proof: (i) ⇒ (ii). This follows from the identity
and the fact that arbitrary union of members of $\mathcal{M}$ is in $\mathcal{M}$.

(ii) ⇒ (iii). This follows from the identity

$$\{x \in A : f(x) \geq t\} = A - \{x \in A : f(x) < t\}.$$

(iii) ⇒ (iv). This follows from the identity

$$\{x \in A : f(x) > t\} = \bigcap_n \{x \in A : f(x) \geq t + \frac{1}{n}\}$$

and the fact that arbitrary intersection of members of $\mathcal{M}$ is in $\mathcal{M}$.

(iv) ⇒ (i). This follows from the identity

$$\{x \in A : f(x) \leq t\} = A - \{x \in A : f(x) > t\}$$

Definition 3.3.2: Let (X, $\mathcal{M}$) be a measurable space and let A be a subset of X that belongs to $\mathcal{M}$. A function f: A →[-∞, +∞] is said to be measurable with respect to $\mathcal{M}$ if it satisfies one, and hence all of the conditions of Proposition 3.3.1.

In case X = $\mathbb{R}^n$, a function that is measurable with respect to **B**($\mathbb{R}^n$)is called Borel measurable or a Borel function.

Examples 3.3.3:

(i) Let f: $\mathbb{R}^n \to \mathbb{R}$ be continuous. Then for each real number t, the set $\{x \in \mathbb{R}^n : f(x) < t\}$ is open, and so is a Borel set. Thus f is Borel measurable.

(ii) Let I be subinterval of and let f: I→ $\mathbb{R}$ be non - decreasing. Then for each real number t the set $\{x \in \mathbb{R}^n : f(x) < t\}$is a Borel set (it is either an interval, a set consisting of only one point, or the empty set). Thus f is Borel measurable.

(iii) Let (X, $\mathcal{M}$) be a measurable space, and let B be a subset of X. Then χ_B, the characteristic function of B is measurable if and only if B ∈ $\mathcal{M}$.

(iv) Let (X, $\mathcal{M}$) be a measurable space, let f : X →[- ∞, + ∞] be simple and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the values of f. Then f is measurable if and only if $\{x \in X: f(x) = \alpha_i\} \in \mathcal{M}$ holds of i = 1, 2, …, n.

We recall the following definition

Definition 3.3.4:

(i) A real value function S: X → ℝ is called a simple function if it has a finite number of values.

(ii) Let X be any set and let A be a subset of X. Denote by χ_A by setting

$$\chi_A(x) = \begin{cases} 1 & if\ x\ is\ in\ A \\ 0 & if\ \ x\ is\ not\ in\ A \end{cases}$$

Then the function χ_A is called the characteristic function of the set A.

Let X be a measurable space and let S: X →[0, ∞] be a simple function. Let $S_1, S_2, \dots, S_n$ be the distinct values of S and let $A_i = \{x \in X: S(x) = s_i\}$, i = 1,2,…, n. Then S can be written as a finite linear combination of characteristic functions of the set A_i . That is

$$S(x) = \sum_{i=1}^{n} s_i\, \chi_{A_i}(x)$$

Let f and g be [-∞, ∞] - valued functions having a common domain A. The maximum and minimum of f and, written as f ∨ g and f ∧ g defined by

$$(f \vee g)(x) = \max(f(x), g(x))$$

and

$$(f \wedge g)(x) = \min(f(x), g(x))$$

If {n} is a sequence of [-∞,∞] - valued functions on A, then $\sup_n f_n,\ inf_n f_n,\ lim\ \sup_n f_n, lim\ inf_n f_n\ and\ lim_n f_n$, are defined in a similar way.

For example

$$\sup_n f_n(x) = \sup_n(f_n(x))$$

The domain of $lim_n f_n$ consists of those points in A at which $lim\ \sup_n f_n$ and $lim\ inf\, f_n$ agree; the domain of each of the other four functions is A. Each of these functions can have infinite values, in particular, $lim\ inf_n f_n$ can be +∞ or -∞.

We also recall the following

$$f^+ = \max(f, 0)$$

$$f^- = \max(-f, 0) = -\min(f, 0)$$

$$|f| = f^+ + f^-$$

$$\max(f, g) = \frac{|f-g| + f + g}{2}$$

$$\min(f, g) = \frac{-|f-g| + f + g}{2}$$

Proposition 3.3.5: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, and let f and g be [-∞, +∞] - valued measurable functions on A. Then f ∨ g and f ∧ g are measurable.

Proof: The measurability of f g follows from the identity

$$\{x \in A: (f \vee g)(x) \leq t\} = \{x \in A: (x) \leq t\} \cap \{x \in A: g(x) \leq t\}$$

and the measurability of f ∧ g follows from the identity

$$\{x \in A: (f \wedge g)(x) \leq t\} = \{x \in A: (x) \leq t\} \cup \{x \in A: g(x) \leq t\}$$

Proposition 3.3.6: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, and let $\{f_n\}$ be a sequence of [-∞, +∞] - valued measurable functions on A. Then

(i) the functions $\sup_n f_n$ and $inf_n f_n$; are measurable
(ii) the functions $lim \sup_n f_n$ and $lim\, inf_n f_n$ are measurable
(iii) the function $lim_n f_n$ is measurable.

Proof:

(i) The measurability of $\sup_n f_n$ follows from the identity

$$\{x \in A : (sup_n f_n)(x) \leq t\} = \cap_n \{x \in A : f_n(x) \leq t\}$$

The measurability of $inf_n f_n$ follows from the identity

$\{x \in A : (sup_n f_n)(x) \leq t\} = \cup_n \{x \in A : f_n(x) \leq t\}$

(ii) For each positive integer k, define the functions g_k and h_k by

$g_k = sup_{n\geq k} f_n$ and $h_k = inf_{n\geq k} f_n$

From (i), we have that each g_k is measurable and each h_k is measurable, and that $inf_k g_k$ and $\sup_k h_n$ are measurable. Since $lim \sup_n f_n$ and $lim\, inf_n f_n$ are equal to $inf_k g_k$ and $\sup_k h_n$, then they are measurable.

(iii) Let A_0 be the domain of $lim_n f_n$. Then

$A_0 = \{x \in A : (limsup_n f_n)(x) = (liminf_n f_n)(x)\}$,

and so by Proposition 3.3.1, A_0 belongs to $\mathcal{M}$. Since

$\{x \in A_0 : lim_n f_n(x) \leq t\} = A_0 \cap \{x \in A : limsup_n f_n(x) \leq t\}$,

The measurability of $lim_n f_n$ follows.

Proposition 3.3.7: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$, let f and g be measurable real - valued functions on A, and let α be a real number. Then αf,

f + g, f - g, fg, f/g , $|f|, f^+, f^-$ are measurable.

Proof: Exercise.

Proposition 3.3.8: Let (X, $\mathcal{M}$) be a measurable space, let A be a subset of X that belongs to $\mathcal{M}$. For a function f: A→ℝ, the following are equivalent:

(i) f is measurable
(ii) for each open subset U of ℝ, the set $f^{-1}(U) \in \mathcal{M}$
(iii) for each closed subset C of ℝ, the set $f^{-1}(C) \in \mathcal{M}$
(iv) for each Borel subset B of ℝ, the set $f^{-1}(B) \in \mathcal{M}$

Proof: Exercise

Lastly in this section, we give a general definition of measurable functions.

Definition 3.3.9: Let (X, $\mathcal{M}_X$) and (Y, $\mathcal{M}_Y$) be measurable spaces. A function f: X→Y is said to measurable with respect to $\mathcal{M}_X$ and $\mathcal{M}_Y$ if for each $B \in \mathcal{M}_Y$ the set $f^{-1}(B) \in \mathcal{M}_X$.

Instead of saying that f is measurable with respect to $\mathcal{M}_X$ and $\mathcal{M}_Y$, we shall sometimes say that f is measurable from $\mathcal{M}_X$ and $\mathcal{M}_Y$ or simply f: (X, $\mathcal{M}_X$) →(Y, $\mathcal{M}_Y$) is measurable.

Likewise if A belongs to $\mathcal{M}_X$, a function f: A→Y is measurable if $f^{-1}(B) \in \mathcal{M}_X$ holds whenever $B \in \mathcal{M}_Y$.

Proposition 3.3.10: Let (X, $\mathcal{M}_X$), (Y, $\mathcal{M}_Y$) and (Z, $\mathcal{M}_Z$) be measurable spaces and let f: (Y, $\mathcal{M}_Y$) →(Z, $\mathcal{M}_Z$) and g: (X, $\mathcal{M}_X$)→$(Y, \mathcal{M}_X)$ be measurable. Then the composition f o g : (X, $\mathcal{M}_X$)→$(Z, \mathcal{M}_Z)$ is measurable.

Proof: Exercise

4.0 CONCLUSION

The concept of measurable functions defined on measurable space with values in extended real line with some of its basic properties has been discussed in this unit. this led to the introduction of Borel functions on the real line. measurable functions are the basic for the integration on the real line

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of a measurable function with examples;
(ii) the basic properties of measure functions;
(iii) how to prove some basic results on measure functions.

6.0 TUTOR MARKED ASSIGNMENT

(1) Let (X, $\mathcal{M}_X$) be a measurable space and let A be a measurable subset of X, show that χ_A the characteristics function of A is a measurable function on A.

(2) Show that the supremum of an uncountable family of [-∞, +∞] - valued Borel measurable functions on ℝ can fail to be Borel measurable.

(3) Show that if f: ℝ→ℝ is differentiable everywhere on ℝ , then its derivative f' is Borel measurable.

(4) (X,$\mathcal{M}$) be a measurable space, and let $\{f_n\}$ A be a sequence of [-∞, +∞] – valued measurable functions on X. Show that {x ∈ X: $lim_n f_n(x)$} exists and is finite} belongs to $\mathcal{M}$.

(5) If $|f'|$ is a measurable function, is it true to claim that f is measurable? If not give a counter example.

7.0 FURTHER READING AND OTHER RESOURCES

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