MODULE 3: THEORY OF INTEGRATION

UNIT 1: INTEGRATION OF POSITIVE FUNCTIONS

CONTENT

1.0 INTRODUCTION

In this unit, we shall discuss the Lebesgue integrals of positive functions over the general measure space and their basic properties. The monotone convergence and Beppo Levi's theorems with Fatou's lemma will also be discussed.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

(i) define the Lebesgue integral of positive function over a general measure space;
(ii) know some basic properties of Lebesgue integrals;
(iii) prove some basic results on Lebesgue integrals;
(iv) know and prove monotone convergence theorem, Beppo Levi's theorem and Fatou's lemma.

3.0 MAIN CONTENT

3.1 Integration of Positive Functions

Definition 3.1.1: Let (X, $\mathcal{M}$, μ) be a measure space and let S be a simple measurable, non - negative function on X with representation S(x) = $\sum_{i=1}^{n} S_i \, \chi A_i(x)$ where $A_i = \{x \in X: S(x) = s_i\}$. Let E ∈ $\mathcal{M}$, we define

$$\int_E S\, d\mu = \sum_{i=1}^{n} s_i\ \mu(A_i \cap E), \quad A_i \subseteq X$$

Let X → [0, ∞] be measurable function, $\int_E f\, d\mu$ is defined by

$$\int_E f\, d\mu = sup_{0 \le S \le f} \int_E S\, d\mu = \sup \int_E S\, d\mu$$

Where S is a simple measurable function on S such that $0 \le S \le f$.

$\int_E f\, d\mu$ is called the Lebesgue integral on f defined on E with respect to μ.

We note that $0 \le \int_E f\, d\mu \le \infty$.

Theorem 3.1.2:

(i) If $0 \le f \le g$, then $\int_E f\, d\mu \le \int_E g\, d\mu$

(ii) If A ⊆ B and $f \ge 0$, $\int_E f\, d\mu \le \int_E g\, d\mu$

(iii) If f(x) = 0 for every x in E, then $\int_E f\, d\mu = 0$ even if $\mu(E) = +\infty$

(iv) If $f \ge 0$, and $0 \le c < \infty$, $\int_E cf\, d\mu = c\int_E f\, d\mu$.

(v) If $\mu(E) = 0$, $\int_E f\, d\mu = 0$, even if $f(x) = \infty$ for every x in E.

(vi) If $f \ge 0$, then $\int_E f\, d\mu = \int_E \chi_E f\, d\mu$

Proof: (i) - (v) as exercise. For the prove of (vi):

$$\int_X \chi_E f\, d\mu = \int_E \chi_E f\, d\mu + \int_{X\setminus E} \chi_E f\, d\mu$$

$$= \int_E f\, d\mu + 0 = \int_E f\, d\mu$$

Theorem 3.1.3: Let s and t be two simple measurable functions defined on X. For every E in $\mathcal{M}$, let $\varphi(E) = \int_E s\, d\mu$. Then φ is a measure on $\mathcal{M}$ and

$$\int_X (s+t)\, d\mu = \int_X s\, d\mu + \int_X t\, d\mu.$$

Proof: Clearly φ is a set function. We shall show that φ is a measure on $\mathcal{M}$ by showing that $\varphi(E) > 0$ and for $E_1, E_2, E_2, \dots, \in \mathcal{M}$ such that

$E_i \cap E_j = \emptyset$ for i ≠ j, $\varphi(\bigcup_{i\ge1} E_i) = \sum_{i\ge1} \varphi(E_i)$.

$$\varphi(E) = \int_E s\, d\mu = \sum_{i=1}^{n} s_i \mu(A_i \cap E) \geq 0$$

Also,

$$\varphi(U_{j\geq 1}E_j) = \int_{U_{j\geq 1}E_j} s\, d\,\mu = \sum_{i=1}^{n} s_i \mu(A_i \cap (U_{j\geq 1}E_j)) \geq 0$$

But

$A_i \cap (U_{j\geq 1}E_j) = U_{j\geq 1}(A_i \cap E_j) = U_{j\geq 1}E_{ij}$

where $E_{ij} = A_i \cap E_j$ and $E_{ij} \cap E_{ip} = \emptyset$, for $j \neq p$

Thus,

$\varphi(\cup_{j\geq 1} E_j) = \sum_{i=1}^{n} s_i\ \mu(\cup_{j\geq 1}(E_{ij})) = \sum_{i=1}^{n} s_i \sum_{j\geq 1} \mu(E_{ij})$

$\sum_{j\geq 1} \sum_{i=1}^{n} s_i\ \mu(E_{ij}) = \sum_{j\geq 1} \sum_{i=1}^{n} s_i\ \mu(A_i\ \cap\ E_j)$

$= \sum_{i\geq 1} \varphi\,(E_i)$.

Hence φ is countably additive and so, it is a measure on $\mathcal{M}$.

Let s = $\sum_{i=1}^{n} \alpha_i\, \chi_{A_i}$ and t $\sum_{j=1}^{n} \beta_j\, \chi_{B_j}$ and let E ∈ $\mathcal{M}$. Then

$$\int_E s\, d\mu + \int_E t\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) + \sum_{j=1}^{n} \beta_i \mu(B_j \cap E)$$

$= \sum_{i=1}^{n} \alpha_i\, [\int_{E\cap A_i} \chi_{A_i}\, d\,\mu + \int_{E\cap A_i^c} \chi_{A_i}\, d\,\mu] +$

$\sum_{j=1}^{n} \beta_j\, [\int_{E\cap B_j} \chi_{B_j}\, d\,\mu + \int_{E\cap B_j^c} \chi_{B_j}\, d\,\mu]$

$= \sum_{i=1}^{n} \alpha_i \int_{E\cap A_i} \chi_{A_i}\, d\,\mu + \sum_{j=1}^{n} \beta_j \int_{E\cap B_j} \chi_{B_j}\, d\,\mu$

$$= \int_E \sum_{i=1}^{n} \alpha_i \chi_{A_i} \, d\mu + \int_E \sum_{j=1}^{n} \beta_j \chi_{B_j} \, d\mu$$

$$= \int_E \left(\sum_{i=1}^{n} \alpha_i \chi_{A_i} + \sum_{j=1}^{n} \beta_j \chi_{B_j}\right) d\mu$$

$$= \int_E (s+t) \, d\mu.$$

Theorem 3.1.4: (Lebesgue Monotone Convergence)

Let $\{f_n\}$ A be a sequence of extended real valued measurable functions defined on X such that

(i) $0 \le f_1(x) \le f_2(x) \le \ldots \le \infty$ for every x in X

(ii) For any x in X, $f_n(x) \to f(x)$ as $n \to \infty$.

Then f is measurable and $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$. That is

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu = \int_X f \, d\mu.$$

Proof: By hypothesis $\{f_n\}$ is an increasing sequence, and so $f_n \le f_{n+1}$.

Thus by Theorem3.1.2 (i), we have $\int_X f_n d\mu \le \int_X f_{n+1} d\mu$ and there exist $\alpha \epsilon [0, \infty]$ such that $\int_X f_n d\mu \to \alpha$ as $n \to \infty$.

By Theorem 3.3.6, f is measurable since $f = sup_n f_n$.
We prove that

$$\alpha \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu \qquad (3.1.1),$$

since if equation (3.1.1) holds, then the result follows.

As $f_n \le f$, then

$$\int_X f_n \, d\mu \le \int_X f \, d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu,$$

and this implies,

$$\int_X (\lim_{n\to\infty} f_n) d\mu \le \int_X f \, d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu \qquad (3.1.2).$$

Let us now take a simple measurable function s such that $0 \le s \le f$. Also let C be a constant such that 0 < C < 1. Define $E_n = \{x \in X: f_n(x) \geq Cs(x)\}$, n = 1, 2… Then E_n are measurable sets since $E_n = \{x \in X: f_n(x) - Cs(x) \geq 0\}$ and $E_1 \subseteq E_2 \subseteq \ldots$.

Moreover $X = \bigcup_{n=1}^{\infty} f_n(x)$ and so, we have

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} Cs \, d\mu \text{ for n = 1, 2, … .}$$

By taking limit as $n \to \infty$, we have

$$\lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} C \int_{E_n} s \, d\mu$$

$$\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq C \int_{E_n} s \, d\mu$$

as $c \to 1$, we have

$$\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq \int_X s \, d\mu$$

Thus,

$$\alpha \geq sup_{0 \leq s \leq f} \int_X s \, d\mu = \int_X f \, d\mu \qquad (3.1.3).$$

From (3.1.2) and (3.1.3), the result follows.

Theorem 3.1.5: (Beppo Levi's Theorem)

Let $\{f_n\}$ be a sequence of measurable functions, $f_n: X \to [0, \infty]$ and assume that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every x in X, then $\int_X f \, d\mu = \sum_{n=1}^{\infty} f_n \, d\mu$

That is, the integral of sum of functions is the sum of the integral of the functions.

Proof: At first, there exist simple functions s_i' and s_i'' such that $s_i' \to f_1$ and $s_i'' \to f_2$

Let $s_i = s_i' + s_i''$, we have $s_i \to f_1 + f_2$ and by Theorem 31.3 and Theorem 3.14, we have

$$\int_X (f_1 + f_2) \, d\mu = \int_X (f_1) \, d\mu + \int_X (f_2) \, d\mu$$

The result for n > 2, can be prove by induction.

Theorem 3.1.6: (Fatou's Lemma)

Let $f_n : X \rightarrow [0, \infty]$X] C0, ∞D be measurable functions for every n = 1, 2, …, then
$\int_X (\lim_{n\to\infty} \inf f_n) d\mu \leq \lim_{n\to\infty} \inf \int_X f_n \, d\mu$.

Proof: Let $g_k = \inf_{i \geq k} f_i$, k = 1, 2, … . That is $g_k = \inf\{f_k, f_{k+1}, f_{k+2}, \ldots\}$.

Then $g_k \leq f_n$ for k $\leq n$. Therefore $\int_X g_k \, d\mu \leq \int_X f_n \, d\mu$ for k $\leq n$.

By keeping k fixed, we have

$$\int_X g_k \, d\mu \leq \lim_{n\to\infty} \inf \int_X f_n \, d\mu \qquad (3.1.4).$$

Now the sequence $\{f_n\}$ is monotone increasing, and has to converge to
$sup_k g_k = \lim_{n\to\infty} \inf f_n$.

Hence, by Theorem 3.1.4, we have

$$\int_X (\lim_{n\to\infty} \inf f_n) d\mu = \lim_{k\to\infty} \int_X f_n \, d\mu$$

$$\leq \lim_{n\to\infty} \inf \int_X \, d\mu$$

by (3.1.4) and the proof is complete.

4.0 CONCLUSION

The concept of Lebesgue integrals of positive functions over the general measure space with some of its basic properties has been discussed in this unit. The monotone convergence theorem and the dominated convergence theorem have been shown to be very useful and main tools in evaluating the integral of measurable functions over measure spaces.

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of Lebesgue integral of a positive function over the general measure space;

(ii) the basic properties of Lebesgue integrals;

(iii) how to prove some basic results on Lebesgue integrals;

(iv) how to prove monotone convergence theorem. Beppo Levi's theorem and Fatou's lemma.

6.0 TUTOR MARKED ASSIGNMENT

(1) Prove or disprove: A real valued function f defined on X is Lebesgue integrable if and only if it is Riemann integrable.

(2) Let

$$I_n = \int_0^n (1 - \frac{x}{n})^n) e^{\frac{x}{2}} \, dx.$$

show that $\lim_{n\to\infty} I_n$

(3) Let

$$f_m = \begin{cases} \chi_E & if\ m\ is\ even \\ 1 - \chi_E & if\ m\ is\ odd \end{cases}$$

where E ⊆ X and X is a measurable space. Examine Fatou's lemma.

7.0 FURTHER READING AND OTHER RESOURCES

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)

D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

G. B.Folland, , Real Analysis: Modern Techniques and Their Applications, John Wiley and Sons, ISBN 0471317160 Second edition. (1999).

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

H.S. Bear, A Primer of Lebesgue Integration, San Diego: Academic Press, ISBN 978-0120839711(2001)

J. Thomas, Set Theory: The Third Millennium Edition, Revised and Expanded, Springer Verlag, ISBN 3-540-44085-2(2003)

N. Weaver, Measure Theory and Functional Analysis. World Scientific Publishing. (2013).

R. M. Dudley, Real Analysis and Probability. Cambridge University Press.(2002).

T. Tao, An Introduction to Measure Theory. American Mathematical Society. (2011).

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw - Hill, New York, 1987.

UNIT 2: INTEGRATION OF COMPLEX FUNCTIONS

CONTENT

1.0 INTRODUCTION

In this unit, we discuss the Lebesgue integral of complex function over the general measure space and also prove the dominated convergence theorem.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define the Lebesgue integral of complex function over a general measure space;
- know some basic properties of Lebesgue integrals of complex functions; and
- know and prove the dominated convergence theorem.

3.0 MAIN CONTENT

3.1 Integration of Complex Functions

We recall that if f is measure, then $|f|$ is measurable.

Definition 3.2.1: A measurable function f on X is said to be Lebesgue measurable (or summable) if $\int_X |f| \, d\mu < \infty$.

We write $L'(\mu)$ to denote the set of all Lebesgue integrable functions.
Definition 3.2.2: $L'(\mu) = \left\{f\colon X \to C / \int_X |f| \, d\mu < \infty\right\}$, where μ is a positive measure on X is an arbitrary measurable space.

More generally, if p is non - negative real number, we put $L^p(\mu)$ functions f such that $\int_X |f|^p \, d\mu < \infty$. That is

$$L^p(\mu) = \{f: X \to C \; / \; \int_X \; |f|^p \; d\,\mu < \; \infty\}.$$

A function f in $L^p(\mu)$ is said to be p^{th} power summable (integrable).

Definition 3.2.3: Let f be an element in $L^1(\mu)$ such that f = u + iv, where u and v are real measurable functions on X, for any measurable set E, we define

$$\int_E \; f\, d\,\mu = \; \int_E \; u^+ \, d\,\mu - \; \int_E \; u^- \, d\,\mu + i \int_E \; v^+ \, d\,\mu \text{ - i} \int_E \; v^- \, d\,\mu.$$

Theorem 3.2.3: $L^1(\mu)$ is a vector space over the real field. Moreover,

$$\left|\int_E \; f\; d\,\mu\right| \leq \; \int_E \quad |f|\; d\,\mu.$$

Proof: Exercise

Theorem 3.2.4: (Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions on X. $f_n: X \to C$ such that for x in X, f(x) = $\lim_{n\to\infty} f_n(x)$. If there exists a function $g \in L'(\mu)$ such that $|f_n(x)| \leq g(x)$, n = 1, 2, … then $f \in L'(\mu)$ and $\lim\limits_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$

That is

$$\lim_{n \to \infty} \int_X \; f_n \; d\,\mu = \; \int_X \; f\, d\,\mu.$$

Proof: Since f is measurable and $|f| \leq g \; (since |f_n(x)| \leq g(x))$, the fact that $f \in L'(\mu)$ follows from the fact that $g \in L'(\mu)$

(since $f_n - f$ and $|f_n(x)| \leq g(x)$, we have $|f| \leq \lim\limits_{n\to\infty} |f_n(x)| \leq g(x)$. From this, we obtain $|f_n - f| \leq |f_n| + |f| \leq 2g$. Consider the sequence $h_n = 2g - |f_n - f|$. By Fatou's Lemma, we have

$$\int_X \; (\lim_{n \to \infty} inf h_n \,) d\,\mu \; \leq \; \lim_{n \to \infty} inf \; \int_X \; h_n \, d\,\mu$$

Which gives

$$\int_X 2g\, d\mu \leq \int_X 2g\, d\mu + \lim_{n\to\infty} inf\ (-\int_X |f_n - f| d\mu).$$

And since $g \in L'(\mu)$, we have

$$0 \leq \lim_{n\to\infty} inf\ (-\int_X |f_n - f| d\mu),$$

And so,

$$0 \leq -\lim_{n\to\infty} sup\ (\int_X |f_n - f| d\mu).$$

This implies

$$\lim_{n\to\infty} sup\ (\int_X |f_n - f| d\mu) \leq 0 \qquad (3.2.1).$$

The left hand side of (3.2.1) must be zero.

Hence

$$\lim_{n\to\infty} \int_X |f_n - f|\ d\mu = \lim_{n\to\infty} sup\ (\int_X |f_n - f| d\mu = 0 \qquad (3.2.2).$$

But by Theorem 3.2.3, we have

$$\left|\int_X (f_n - f)\, d\mu\right| \leq \int_E |f_n - f|\, d\mu.$$

In view of (3.3.2), we have

$$\left|\lim_{n\to\infty} \int_X (f_n - f)\, d\mu\right| = 0$$

and so,

$$\lim_{n\to\infty} \int_X (f_n - f)\, d\mu = 0.$$

4.0 CONCLUSION

The integration of complex functions with some of its basic properties has been discussed in this unit. This led to the introduction of some important function spaces, e.g. the Lp – spaces. The importance of dominated convergence theorem in evaluating these integrals was also established.

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of Lebesgue integral of a complex function over the general measure space;

(ii) the basic properties of Lebesgue integrals of complex functions;

(iii) how to prove the dominated convergence theorem.

6.0 FURTHER READING AND OTHER RESOURCES

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)

D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

G. B.Folland, Real Analysis: Modern Techniques and Their Applications, John Wiley and Sons, ISBN 0471317160 Second edition. (1999).

H. L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

H. S. Bear, A Primer of Lebesgue Integration, San Diego: Academic Press, ISBN 978-0120839711(2001)

J. Thomas, Set Theory: The Third Millennium Edition, Revised and Expanded, Springer Verlag, ISBN 3-540-44085-2(2003)

N. Weaver, Measure Theory and Functional Analysis. World Scientific Publishing. (2013).

R. M. Dudley, Real Analysis and Probability. Cambridge University Press.(2002).

T. Tao, An Introduction to Measure Theory. American Mathematical Society. (2011).

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw - Hill, New York, 1987.

UNIT 3: LEBESGUE INTEGRATION OF REAL - VALUED FUNCTIONS DEFINED ON $\mathbb{R}^n$

CONTENT

1.0 INTRODUCTION

Here, we discuss Lebesgue integration of real - valued functions defined on subsets of $\mathbb{R}^n$, we restrict ourselves to n = 1, that $\mathbb{R}$. We first give a brief description of the way in which the Lebesgue integral is defined on subsets of $\mathbb{R}$.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define the Lebesgue integral of a real - valued function defined on the real line;
- know the equivalent form of monotone convergence theorem and dominated convergence theorem with respect to the real line instead of the general measure space;
- know how to use these theorems to evaluate some complicated or complex integrals.

3.0 MAIN CONTENT

3.1 Lebesgue Integration of Real - Valued Functions Defined on $\mathbb{R}$

Definition 3.3.1: A step function $f = \sum_{r=1}^{n} \beta_r \chi_{E_r}$ is a (finite) linear combination of the characteristic functions where each for such E_r is a bounded interval. For such f we define

$$\int f = \sum_{r=1}^{n} \beta_r \mathrm{m}(E_r)$$

Where $\mathrm{m}(E_r)$ is the length of E_r.

A function f: $\mathbb{R} \rightarrow \mathbb{R}$ is in L^1 if there is an increasing sequence $\{f_n\}$ of step functions such that $\{f_n\}$ is bounded above and $f_n \rightarrow f$ almost everywhere (a.e.). For such an f we define

$$\int f = \lim_{n\to\infty} \int f_n.$$

A function f:$\mathbb{R} \rightarrow \mathbb{R}$ is in A function f:$\mathbb{R} \rightarrow \mathbb{R}$ is in $L^1(\mathbb{R})$ if f = p - q, where p, q For such an f, we define

$$\int f = \int p - \int q.$$

Theorem 3.3.2: (Monotone Convergence Theorem)

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $\{f_n\}$ is bounded above (below) and $\{f_n\}$ is increasing (decreasing) a.e. Then there exists f $\in L^1(\mathbb{R})$ such that $\int f = \lim_{n\to\infty} \int f_n$.

Proof: Exercise

Theorem 3.3.3 (Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ and g $\in L^1(\mathbb{R})$. Let f be a real function defined on $\mathbb{R}$ such that $f_n(x) \rightarrow f(x)$ for almost all x and for all n and all x $\in \mathbb{R}$, $|f_n(x)| \leq g(x)$. Then f $\in L^1(\mathbb{R})$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proof: Exercise

Example 3.3.4: Define f: $\mathbb{R} \rightarrow \mathbb{R}$ and g: $\mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} xe^{-2x} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

Use the monotone convergence theorem to evaluate $\int f$.

Solution: For any natural number n, define f_n: $\mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} xe^{-2x} & if\ 0 \leq x \leq n \\ 0 & otherwise \end{cases}$$

Then $\{f_n\}$ is a sequence in $L^1(\mathbb{R})$ and $\{\int f_n\}$ is bounded above since

$$\int f_n = \int_0^n xe^{-2x}\,\mathrm{dx} = \frac{-ne^{-2n}}{2} - \frac{e^{-2n}}{4} + \frac{1}{4} < \frac{1}{4}.$$

Since for all $x \in \mathbb{R}$, $f_n(x) \to f(x)$, it follows from the monotone convergence theorem that $f \in L'(\mathbb{R})$ and $\int f = \lim_{n\to\infty} f_n = \frac{1}{4}$

Example 3.3.5: Let the function f:ℝ → ℝ be such that, for all natural number n, f restricted to [-n, n] is Riemann integrable and $\lim_{n\to\infty} \int_{-n}^{n} |f(x)|\, dx$ exists. Prove that $f \in L'(\mathbb{R})$.

Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{cos3x}{1+x^2} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

Show that $g: \mathbb{R} \to \mathbb{R}$ by

Proof: For a natural number n, define

$$f_n(x) = \begin{cases} f(x) & if\ -n \leq x \leq n \\ 0 & otherwise \end{cases}$$

Then $\{f_n\}$ and $|\{f_n\}|$ are sequences in $L'(\mathbb{R})$. The sequence $\{\int |f_n|\}$ is bounded above, since $\int |f_n| = \int_{-n}^{n} |f(x)|\, dx$ converges, and for all $x \in \mathbb{R}$, $\{|f_n(x)|\}$ is increasing and converges to $|f(x)|$. Hence, by the monotone convergence theorem, $|f| \in L'(\mathbb{R})$. Also, since $f_n(x) \to f(x)$ $(x \in \mathbb{R})$ and $|f_n(x)| \leq |f|(x)$ for all n and for all $x \in \mathbb{R}$, the dominated convergence theorem, applies to give $f \in L'(\mathbb{R})$.

For each natural number n, g restricted to [-n, n] is Riemann integrable and

$$\int_{-n}^{n} |g(x)|\, dx = \int_0^n \left|\frac{cos3x}{1+x^2}\right| \mathrm{dx}$$
$$\leq \int_0^n \left|\frac{1}{1+x^2}\right|\, dx = tan^{-1}\mathrm{n} < \frac{\pi}{2}.$$

Since these integrals are increasing,

$$\lim_{n\to\infty}\int_{-n}^{n}|g(x)|\,dx$$

Exists and hence $g \in L'(\mathbb{R})$.

4.0 CONCLUSION

A detailed description on how Lebesgue integral is defined on the real line and subset of the real line was given in this unit with some its basic properties. This allows us to give the equivalent form of the monotone convergence theorem and dominated convergence theorem on the real line, instead of the general measure space. The examples given in this unit shows how to apply monotone and dominated convergence theorems to evaluate complicated integrals over the real line or subsets of the real line.

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of Lebesgue integral of a real-valued function over the real line;
(ii) the equivalent version of monotone convergence theorem and dominated convergence theorem with respect to the real line instead of the general measure space;
(iii) how to use the monotone convergence theorem and the dominated convergence theorem to evaluate some complicated or complex integrals .

6.0 TUTOR MARKER ASSIGNMENT

1. Let k be a positive constant. Define $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} xe^{-kx} \ if\ x \geq 0 \\ 0 \ \ if\ x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{x}{e^x} - \ 1 \ \ if\ x > 0 \\ 0 \ \ if\ x \ \leq 0 \end{cases}$$

(i) Use the monotone convergence theorem to show that $f \in L'(\mathbb{R})$
(ii) Evaluate $\int f$
(iii) Show that for x > 0, $g(x) = x(e^{-x} + e^{-2x} + e^{-3x} + \cdots)$
(iv) Deduce that $g \in L'(\mathbb{R})$

2. By considering the sequence of partial sums, show that the real function f defined by the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{(1+n^2x^2)^2} \qquad (0 \le x \le 1)$$

is in $L'([0,1])$ and that

$$\int_0^1 f(x)dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

7.0 FURTHER READING AND OTHER RESOURCE

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3. (2004)

D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

G. B.Folland, , Real Analysis: Modern Techniques and Their Applications, John Wiley and Sons, ISBN 0471317160 Second edition. (1999).

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

H.S. Bear, A Primer of Lebesgue Integration, San Diego: Academic Press, ISBN 978-0120839711(2001)

J. Thomas, Set Theory: The Third Millennium Edition, Revised and Expanded, Springer Verlag, ISBN 3-540-44085-2(2003)

N. Weaver, Measure Theory and Functional Analysis. World Scientific Publishing. (2013).

R. M. Dudley, Real Analysis and Probability. Cambridge University Press. (2002).

T. Tao, An Introduction to Measure Theory. American Mathematical Society. (2011).

Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw - Hill, New York, 1987.