MODULE 4: CLASSICAL BANACH SPACE

Unit 1: Sets of Measure Zero
Unit 2: $L^p -$ Spaces

UNIT 1: SETS OF MEASURE ZERO

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1.0 INTRODUCTION

In this unit, we discuss the property that holds almost everywhere of the general measure space with some example.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- know what it means to say that a property named P occurs almost everywhere on X;
- know some examples of properties that occurs almost everywhere on X; and
- know how to prove basic results on properties that occurs almost everywhere on X.

3.0 MAIN CONTENT

3.1 Properties that Hold Almost Everywhere

Definition 3.1.1: let $(X, \mathcal{M}, \mu)$, be a measure space. A property named P on X is said to hold almost everywhere (a.e) if the set of points in X at which it fails to hold is a set of measure zero. That is if there is a set N that belongs to $\mathcal{M}$, such that $\mu(N) = 0$, and N contains every point at which the property fails to hold. More generally, if E is a subset of X, then a property is said to hold almost everywhere on E if the set of points in E at which it fails to hold is a set of measure zero.

Consider a property that holds almost everywhere, and let F be the set of points in X at which it fails to hold. Then it is not necessary that F belong to $\mathcal{M}$, it is only necessary that there exist a set N that belongs to $\mathcal{M}$, includes F, and satisfies $\mu(N) = 0$.

We give some examples.

Example 3.1.2: Suppose f and g are functions on X. Then f = g almost everywhere if the set of points x at which $f(x) \neq g)x)$ is a set of measure zero. Also, $f \leq g$ on X almost everywhere means the set of points or elements of X for which f > g have measure zero and $f \geq g$ almost everywhere if the set of points x at which f(x) < g(x) is a set of measure zero. If $\{f_n\}$ A is a sequence of functions on X and f is a function on X, then $\{f_n\}$ converges to f almost everywhere if the set of points x at which $f(x) = \lim_n f_n(x)$ fails to hold is a set of measure zero.

Proposition 3.1.3: Let $(X, \mathcal{M}, \mu)$, be a measure space, and let f and g be extended real-valued functions on X that are equal almost everywhere. If μ is complete and if f is measurable, then g is measurable.

Proof: Let t be a real number and let N be a set that belongs to $\mathcal{M}$, satisfies $\mu(N)$, and is such that f and g agree everywhere outside N. Then

$$\{x \in X: g(x) \leq t\} = (\{x \in X: f(x) \leq t\} \cap N^C) \cup (\{x \in X: g(x) \leq t\} \cap N) \quad (3.1.1)$$

The completion of μ implies $\{x \in X: g(x) \leq t\} \cap N$ belongs to $\mathcal{M}$ and so equation (3.1.1) implies $\{x \in X: g(x) \leq t\}$ belongs to $\mathcal{M}$. Since t is arbitrary, the measurability of g follows.

Corollary 3.1.4: Let $(X, \mathcal{M}, \mu)$, be a measure space, let $\{f_n\}$ be a sequence of extended real-valued functions on X, and let f be an extended real-valued function on X such that $\{f_n\}$ converges to f almost everywhere. If μ is complete and if each f_n is measurable, then f is measurable.

Proof: By Proposition 3.3.6 of module 2, unit 3, the function $liminf_n f_n$ is measurable. Since f and $liminf_n f_n$ agree almost everywhere, Proposition 3.1.3 above implies that f is measurable.

Definition 3.1.5: We say that f and g are equivalent if f and g differ on a set of measure zero. We write f ~ g to denote that f and g are equivalent. If f ~ g, we say that f = g almost everywhere on X.

Consider $L'(\mu)$, an identify two functions if they are equivalent. Write [f] for the class of functions equivalent to f. That [f] = {g: f ~ g}. Since ~ is a proper equivalent relation on $L'(\mu)$, it splits $L'(\mu)$ into collection of mutually disjoint equivalent classes [f], [g]. We do this because it is possible for a function f which is not zero everywhere but which has $\int_x f d\mu = 0$.

The equation $\int_x |f| d\mu = 0$ does not imply that f = 0. But if f = 0, then $\int_x f d\mu = 0$

Theorem 3.1.6: Let $\{f_n\}$ be a sequence of measurable functions. $f_n: X \to C$ a.e. Suppose that $\sum_{n=1}^{\infty} \int_x [f_n] d\mu < \infty$ then $\sum_{n=1}^{\infty} f_n(x)$ converges to f(x) a.e on X.

That is,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \ a.e \ on \ X \ and \int_x f d\mu = \sum_{n=1}^{\infty} \int_x f_n d\mu$$

Proof: The S_n be the set on which f_n is defined such that $\mu({S_n}^c) = 0$. Let us define

$\varphi(x) = \sum_n |f_n(x)|$ for all x in S = $\cap_n S_n$, we have $\mu(S^c) = 0$

By Theorem 3.1.5 $\int_s \varphi(x) < \infty$. If $E = \{x \in X: \varphi(x) < \infty\}, \mu(E^c) = 0, \sum_n |f_n(x)|$ converges for any $x \in X$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ ∀×∈ E, then we obtain
$|f_n(x)| \leq \varphi(x)$ ∀×∈ E

By Theorem 3.2.4, we obtain

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

This is equivalent to

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

because $\mu(E^c)$

4.0 CONCLUSION

The concept of sets of measure zero has been studied in this unit. This allows us to define the concept of a property that hold almost everywhere. It also allows us to know when two functions on a measurable space are equivalent, which is very useful in the space of Lebesgue integrable functions.

5.0 SUMMARY

In this unit we have learnt:

(i) the definition of a property that occurs almost everywhere with examples;
(ii) how to prove some results on some properties that occur almost everywhere.

6.0 TUTOR MARKED ASSIGNMENT

1. Show that ~ is an equivalent relation.

2. Let f and g be continuous real-valued functions on the real line. Show that if f = g holds almost everywhere with respect to the Lebesgue measure on the real line, then f = g everywhere.

7.0 FURTHER READING AND OTHER RESOURCES

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)

D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

G. B. Folland, , Real Analysis: Modern Techniques and Their Applications, John Wiley and Sons, ISBN 0471317160 Second edition. (1999).

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H. S. Bear, A Primer of Lebesgue Integration, San Diego: Academic Press, ISBN 978-0120839711(2001)

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UNIT 2: L^p − SPACES

CONTENT

1.0 Introduction
2.0 Objectives
3.0 Main Content
 3.1 L^p- Spaces
4.0 Conclusion
5.0 Summary
6.0 Tutor Marked Assignment
7.0 Further Reading and Other Resources

1.0 INTRODUCTION

In this unit we study some spaces of functions – the L^p- Spaces and the norm defined on them. We also study the inequalities involving the norm defined on the L^p-spaces. The Holder's and Minkowski. The Minkowski inequality gives the sub-additivity of the norm.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- Know what the - Spaces are;
- know that the - Spaces with the norm defined on them are Banach spaces; and
- know the Minkonski and Holder's inequalities for function spaces

3.0 MAIN CONTENT

3.1 L^p − SPACES

Definition 3.2.1: Let p and q be two positive numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then p and q are said to be conjugate exponents

Definition 3.2.2: Let (X, $\mathcal{M},\mathcal{M}$) be a measure space, for 0 < p < ∞, let f: X → C be a measurable function, we set

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

$L^p(\mu) = \{$ f:X $\rightarrow C, measurable\ functions\ such\ that\ \int_X |f|^p\, d\mu < \infty\}$.

The norm on $L^p(\mu)$ is defined as

$\|f\|_p = (\int_X |f|^p\, d\mu)^{\frac{1}{p}}$ for f $\in L^p(\mu)$.

If $\mu = m$ (Lebesgue measure on $\mathbb{R}^n$), then $L^p(u) = L^p(\mathbb{R}^n)$.
$L^p(u)$ is also denoted by $L^p(X)$ and $L^p(X), \|.\|_p$ is a Banach space, it is reflexive and separable for 1 < p < ∞.

Two functions f and g in $L^p(X)$ are said to be equal if $\int_x |f - g|\, d\mu = 0$. Thus $L^p(X)$ is a set of equivalent classes with norm $\|.\|_p$.

Theorem 3.2.3: (Holder's Inequality)

Let p and q be conjugate exponents where $1 \le p \le \infty$ If $f \in L^p(X)$ $g \in L^q(X)$ and, then

fg $\in L^1(X)$ and $\|fg\|_1 \le \|f\|_p . \|g\|_q$.

That is,

$\int_X |fg| d\mu \le (\int_X |f|^p\, d\mu)^{\frac{1}{p}} . (\int_X |g|^q\, d\mu)^{\frac{1}{q}}$.

Proof: Exercise

Remark: The case of Theorem 4.2.3, for p = q = 2 is called Cauchy Schwarz inequality.

Theorem 3.2.4: (Minkonski Inequality)

Suppose $1 \le p \le \infty$ and $f, g \in L^p(X)$, then (f + g) ∈ $L^p(X)$,

$\|f + g\|_p \le \|f\|_p + \|g\|_p$.

That is,

$(\int_X |f + g|^p\, d\mu)^{\frac{1}{p}} \le (\int_X |f|^p\, d\mu)^{\frac{1}{p}} + (\int_X |g|^p\, d\mu)^{\frac{1}{p}}$.

Proof: Exercise

(1) Let p_1, p_2 be positive real numbers such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and f_1, f_2 be functions such that $f_1 \epsilon L^{p_1}(X)$ and $f_2 \epsilon L^{p_2}(X)$, prove that the function $f = f_1 f_2$ is in $L^p(X)$, and

$$\|f\|_p \leq \|f_1\|_{p_1} . \|f_2\|_{p_2}.$$

(2) Prove that if $f \in L^p(X) \cap L^q(X)$ with $1 \leq p \leq q$, then for any $p \leq r \leq q$, we have

$$\|f\|_r \leq \|f\|_p{}^{\alpha} \|f\|_q{}^{1-\alpha} \text{ with } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

4.0 CONCLUSION

The spaces of functions is studied is in this unit and the norms defined on them. These spaces with the norms defined on them are complete in the sense that every Cauchy sequence in them converges to some point in them, and so they form Banach spaces. The inequalities involving the norms defined on the spaces are very important, for example the Holder's and Minkowski's inequalities which are very useful in establishing the sub-additivity of the norm.

5.0 SUMMARY

In this unit we have learnt:

(i) definition of L^p- Spaces
(ii) that L^p- Spaces with the norm defined on them are Banach spaces
(iii) the Minkonski and Holder's inequalities

6.0 TUTOR MARKED ASSIGNMENT

(1) Let p_1, p_2 be positive real numbers such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and f_1, f_2 be functions such that $f_1 \epsilon L^{p_1}(X)$ and $f_2 \epsilon L^{p_2}(X)$, prove that the function $f = f_1 f_2$ is in $L^p(X)$, and

$$\|f\|_p \leq \|f_1\|_{p_1} . \|f_2\|_{p_2}.$$

(2) Prove that if $f \in L^p(X) \cap L^q(X)$ with $1 \leq p \leq q$, then for any $p \leq r \leq q$, we have

$$\|f\|_r \leq \|f\|_p{}^{\alpha} \|f\|_q{}^{1-\alpha} \text{ with } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

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