MODULE 5: PRODUCT MEASURES AND FUBINI'S THEOREM

UNIT 1: PRODUCT SPACES

CONTENT

1.0 INTRODUCTION

This unit is devoted to measures and integrals on product spaces. We shall study the basic facts about product measures and about the evaluation of integrals on product spaces.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- Know what the product Spaces are;
- Some basic facts about product spaces;
- know what the product measures are;
- know how to evaluate integrals on product spaces.

3.0 MAIN CONTENT

3.1 Product Measures and Product Spaces

Definition 3.1.1: Let (X, $\mathcal{M}_x$) and (Y, $\mathcal{M}_y$) be measurable spaces and let X × Y be the Cartesian product of the sets X and Y. A subset of X × Y is called a rectangle with measurable sides if it has the form A × B for some A in $\mathcal{M}_x$ and B in $\mathcal{M}_y$; the $\sigma -$ algebra on X × Y generated by the collection of all rectangles with measurable sides is called the product of the $\sigma -$ algebra on $\mathcal{M}_x$ and $\mathcal{M}_y$ and is denoted by $\mathcal{M}_x \times \mathcal{M}_y$.

Example 3.1.2: Consider the space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We show that the product $\sigma -$ algebra $\mathbf{B}(\mathbb{R}) \times \mathbf{B}(\mathbb{R})$ is equal to the $\sigma -$algebra $\mathbf{B}(\mathbb{R}^2)$ of Borel subsets of $\mathbb{R}^2$. We recall that $\mathbf{B}(\mathbb{R}^2)$ is generated by the collection of all sets of the form (a, b] × (c,d]. Thus $\mathbf{B}(\mathbb{R}^2)$ is generated by a subfamily of the $\sigma -$ algebra $\mathbf{B}(\mathbb{R}) \times \mathbf{B}(\mathbb{R})$, and so is included in $\mathbf{B}(\mathbb{R}) \times \mathbf{B}(\mathbb{R})$. For the reverse inclusion, let $\pi_1$and π_2 be projections of $\mathbb{R}^2$ onto $\mathbb{R}$, defined by π_1(x, y) = x and π_2 (x, y) = y.

$\pi_1$and π_2 are continuous, and hence Borel measurable. It follows from this and the identity

$$A \times B = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) = {\pi_1}^{-1}(A) \cap {\pi_2}^{-1}(B)$$

that if A and B belongs to $\mathbf{B}(\mathbb{R})$, then A × B belongs to $\mathbf{B}(\mathbb{R}^2)$. Since $\mathbf{B}(\mathbb{R}) \times \mathbf{B}(\mathbb{R})$ is the $\sigma -$ algebra generated by the collection of all such rectangles A × B, it must be included in $\mathbf{B}(\mathbb{R}^2)$. Thus $\mathbf{B}(\mathbb{R}) \times \mathbf{B}(\mathbb{R}) = \mathbf{B}(\mathbb{R}^2)$.

We next introduce some terminology and notation. Suppose X and Y are sets and E is a subset of X × Y. Then for each x in X and each y in Y, the sections E_x and E^y are the subsets of Y and X given by

$$E_x = \{y \in Y: (x,y) \in E\}$$

and

$$E^y = \{x \in X: (x,y) \in E\}.$$

If f is a function on X × Y, then the sections f_x and f^y are the functions on Y and X given by

$$f_x(y) = f(x,y)$$

and

$$f^y(x) = f(x,y)$$

Proposition 3.1.3: Let (X, $\mathcal{M}_x$) and (Y, $\mathcal{M}_y$) be measurable spaces.

(i) If E is a subset of X × Y that belongs to $\mathcal{M}_x \times \mathcal{M}_y$ then each section E_x belongs to $\mathcal{M}_y$ and each section E^y belongs to $\mathcal{M}_x$.

(ii) If f is an extended real-valued (or a complex-valued) $\mathcal{M}_x \times \mathcal{M}_y$ - measurable function on X × Y, then each section f_x is $\mathcal{M}_y$ -measurable and f^y is $\mathcal{M}_x$-measurable.

Proof: Exercise

Proposition 3.1.4: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_y, v$) be σ −finite measure spaces. If E belongs to the σ −algebra $\mathcal{M}_x \times \mathcal{M}_y$, then the function $x \to v(E_x)$ is $\mathcal{M}_x$- measurable and the function $y \to \mu(E^y)$ is $\mathcal{M}_y$ - measurable.

Proof: Exercise

Theorem 3.1.5: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_y$, v) be $\sigma -$ finite measure spaces. Then there is a unique measure $\mu \times v$ on the σ −algebra $\mathcal{M}_x \times \mathcal{M}_y$, such that

$$(\mu \times v)(A \times B) = \mu(A)v(B)$$

holds for each A in $\mathcal{M}_x$ and B in $\mathcal{M}_y$. Furthermore, the measure under $\mu \times v$ of an arbitrary set E in $\mathcal{M}_x \times \mathcal{M}_y$ is given by

$$(\mu \times v)(E) = \int_X v(E_x)\ \mu(dx) = \int_Y \mu(E^y)v(dy).$$

The measure $\mu \times v$ is called the product of μ and v

Proof: Exercise

4.0 CONCLUSION

The concepts of product measures and product spaces with some of their basic properties have been studied in this unit. These led to the information about integration on product spaces.

5.0 SUMMARY

In this unit we have learnt:

(i) definition of product spaces and product measure with example;
(ii) some basic results on product spaces and product measures;
(iii) the evaluation of integrals on product spaces

6.0 Further Reading and Other Resources

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)
D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

G. B.Folland, , Real Analysis: Modern Techniques and Their Applications, John Wiley and Sons, ISBN 0471317160 Second edition. (1999).

H.L. Royden, Real Analysis, third edition, Macmillan Publishing Company New York, 1988.

H.S. Bear, A Primer of Lebesgue Integration, San Diego: Academic Press, ISBN 978-0120839711(2001)

J. Thomas, Set Theory: The Third Millennium Edition, Revised and Expanded, Springer Verlag, ISBN 3-540-44085-2(2003)

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R. M. Dudley, Real Analysis and Probability. Cambridge University Press.(2002).

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Walter Rudin, Principle of Mathematical Analysis, Second Edition, McGraw Hill, New York, 1976.

Walter Rudin, Real and Complex Analysis, Third Edition, McGraw - Hill, New York, 1987.

UNIT 2: FUBINI'S THEOREM

CONTENT

1.0 INTRODUCTION

In this unit, we discuss a proposition and the Fubini's theorem. These results enable us to evaluate integrals with respect to product measures in terms of iterated integrals.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- know the Funini's theorem; and
- how to evaluate integrals with respect to product measures in terms of the iterated integrals.

3.0 MAIN CONTENT

3.1 Fubini's Theorem

Proposition 3.2.1: (X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_y, v$) be $\sigma -$ finite measure spaces, and let $f: X \times Y \to [0, +\infty]$ be $\mathcal{M}_x \times \mathcal{M}_y$ -
measurable. Then

(i) The function $x \to \int_Y f_x dv$ is $\mathcal{M}_x$ - measurable and the function $y \to \int_X f^y d\mu$ is $\mathcal{M}_y$ - measurable, and

(ii) f satisfies

$$\int_{X \times Y} f \, d(\mu \times v) = \int_Y \left(\int_X f^y d\mu\right) v(dy)$$

$= \int_X (\int_Y f_x dv)\, \mu(dx).$

Remark: We note that the functions f_x and f^y are non-negative and measurable (see Proposition 3.1.3, Module 5, unit 1); thus the expression $\int_Y f_x dv$ is defined for each x in X and the expression $\int_X f^y d\mu$ is defined for each y in Y.

Proof: First suppose that E belongs to $\mathcal{M}_x \times \mathcal{M}_y$ and that f is the characteristic function on E. Then the sections f_x and f^y are the characteristic functions of the sections E_x and E^y, and so the relations $\int f_x dv = v(E_x)$ and $\int f^y d\mu = \mu(E^y)$ hold for each x and y. Thus Proposition 3.1.4 and Theorem 3.1.5 of Module 5, unit 1, imply that conditions (i) and (ii) hold if f is a characteristic function. The additivity and homogeneity of the integral now imply that they hold for non-negative simple $\mathcal{M}_x \times \mathcal{M}_y$ - measurable functions, and so, they hold for arbitrary non-negative $\mathcal{M}_x \times \mathcal{M}_y$ - measurable functions.

Theorem 3.2.2 (Fubini's Theorem)
(X, $\mathcal{M}_x, \mu$) and (Y, $\mathcal{M}_y, v$) be σ −finite measure spaces, and let
$f: X \times Y \to [-\infty, +\infty]$, be $\mathcal{M}_x \times \mathcal{M}_y$ – measurable and $\mu \times v$ – integrable.
then

(i) for μ -almost every x in X the section f_x is v -integrable and for v -almost every y in Y the section f^y is μ-integrable,

(ii) the relation

$$\int_{X \times Y} f \, d(\mu \times v) = \int_Y (\int_X f^y d\mu)\, dv$$

$$= \int_X (\int_Y f_x dv)\, d\mu$$

holds.

Proof: Exercise

4.0 CONCLUSION

The Fubini's theorem and some of it consequences have been studied in this unit. In particular, it enables us to evaluate integrals with respect to product measures in terms of iterated integrals.

5.0 SUMMARY

In this unit we have learnt:

(i) the Fubini's theorem;

(ii) how to evaluate integrals with respect to product measures in terms of the iterated integrals.

6.0 TUTOR MARKED ASSIGNMENT

1. Let Å be Lebesgue measure on $(\mathbb{R}, \mathbf{B}(\mathbb{R}))$, let μ be counting measure on $(\mathbb{R}, \mathbf{B}(\mathbb{R}))$, and let $f: \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function on the line $\{(x, y) \epsilon \mathbb{R}^2 : y = x\}$. Show that

$$\iint f(x,y)\mu(dy)\ \lambda(dx) \neq \iint f(x,y)\lambda(dx)\ \mu\lambda(dy).$$

2. Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\text{f(x,y)} = \begin{cases} 1 & if\ x \geq 0\ \ and\ x \leq y < x+1 \\ -1 & if\ x \geq 0\ \ and\ x+1 \leq y < x+2 \\ 0 & otherwise \end{cases}$$

Show that $\iint\ (x, y)\text{-}1(dy)\ \text{-}1(dx) \neq \iint\ (x, y)\text{-}1(dx)\ \text{-}1(dy)$. Why does this not contradict the Fubini's theorem.

7.0 FURTHER READING AND OTHER RESOURCES

B. Nicolas, Integration I, Springer Verlag, ISBN 3-540-41129-1 Chapter 3.(2004)

D. H. Fremlin, Measure Theory. Torres Fremlin. (2000).

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SELF ASSESSMENT QUESTIONS

1. (a) i. Define the following terms:

   A. measure of a bounded open set G,

   B. measure of a bounded closed set F,

   C. inner measure of a bounded set E, and

   D. outer measure of a bounded set E.

   ii. Let A and B be bounded sets such that $A \subset B$, prove that $m_*(A) \leq m_*(B)$ where $m_*(A)$ and $m_*(B)$ are the inner measure of A and B respectively.

   (b) Show that the measure of a union of a finite number of pairwise disjoint closed intervals equal the sum of the length of these intervals.

2. (a) Define the following terms:

   i. σ-algebra,

   ii. Borel σ-algebra on $\mathfrak{R}$,

   iii. measurable space,

   iv. set function,

   v. additive set function,

   vi. countably additive set function,

   vii. measure and

   viii. measure space.

   (b) i. Let (X, M, μ) be a measure space and let $A, B \in \mathrm{M}$ such that $A \subset B$, show that

   $\mu(A) \leq \mu(B)$

ii. Let (X, M) be a measure space. If $A, B \in \mathsf{M}$, show that $A - B \in \mathsf{M}$.

iii. Let $X = \{1, 2, 3, ...\}$ and $\mathsf{M} = \{\Phi, X, \{1, 3, 5, ...\}, \{2, 4, 6, ...\}\}$.

Prove or disprove that M is a σ-algebra on X.

3. (a) Let (X, M, μ) be a measure space. When is μ said to be

i. finite?

ii. σ-finite?

If $\{A_k\}$ is an increasing sequence of sets that belong to **M.** show that

$$\mu\left(\bigcup_k A_k\right) = lim_k(A_k)$$

(b) let (X, M, μ) be a measure space. If $E^k \in \boldsymbol{M}, \mu(E_k) < \infty$ and $E_{k+1} \subset E_k$, show that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = lim_{k\to\infty}(E_k)$$

4. (a) let (X, M) be a measure space and let $A \in \mathsf{M}$.

When is a function $f: A \to [-\infty, +\infty]$ said to be measurable?
Let f and g be $[-\infty, +\infty]$ - valued measurable functions on A. show that

i. Max (f, g) is measurable
ii. Min (f, g) is measurable

(b) i. let $(X, \mathsf{M}_x,)$ and let $(Y, \mathsf{M}_y,)$, be a measure space. When is a function $f: (X, \mathsf{M}_x) \to (Y, \mathsf{M}_y,)$ said to be measurable?

ii. let $(X, \mathsf{M}_x,)$, $(Y, \mathsf{M}_y,)$ and $(Z, \mathsf{M}_z,)$, be a measure space and let $f: (Y, \mathsf{M}_y,) \to (Z, \mathsf{M}_z,)$ and $g: (X, \mathsf{M}_x) \to (Y, \mathsf{M}_y,)$ be measurable. Show that $fog: (X, \mathrm{M}_x) \to (\mathrm{Z}, \mathrm{M}_z)$ is measurable.

iii. Let (X, M) be a measurable space and (Y, τ) be a topological space. Let $f: X \to Y$ show that if f is measurable and E is any close subset of Y, then $f^{-1}(E) \in M$.

5. (a) i. Give a brief description of the way in which the Lebesgue integral is defined. Start with the definition of step function and progress through

functions in $L^1(\Re)$. State the general $L^p(\Re)$ and the norm that is defined on it for $1 < p < \infty$.

ii. State the Monotone convergence Theorem and the Dominated Convergence Theorem for integrals.

(b) Let k be a positive constant. Define $f: \Re \to \Re$ by

$$f(x) = xe^{-kx} \ \ if \ \ x \geq 0 \ \ and \ \ f(x) = 0 \ \ if \ \ x < 0$$

and

$$g(x) = \frac{x}{e^x - 1} if \ x \geq 0 \ and \ g(x) = 0 \ if \ x < 0$$

i. Use the Monotone Convergence Theorem to show that $f \in L^1(\Re)$, and evaluate f. ii. Show that, for $x > 0$,

$g(x) = x(e^{-x} + e^{-2x} + e^{-3x} + ...)$ iii. Deduce that $g \in L^1(\Re)$, and evaluate g.

6. (a) i. Let μ be a measure on a σ-algebra M. When do we say that a property P occurs almost everywhere on $E \in$ M?

ii. Let f and g be two measurable functions, when do we say that f and g are equivalent?

iii. Prove that if $f \in L^p(\Omega) \cap L^q(\Omega$with) $1 \leq p \leq q$, then for any $p \leq r \leq q$, we have $\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$

(b) Let p_1 and p_2 be positive real numbers such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and f_1, f_2 are functions such that $f_1 \in L^{p_1}(\Omega)$ and $f_2 \in L^{p_2}(\Omega)$. Show that

i. $f = f_1 f_2 \in L^p(\Omega)$ and

ii. $\|f\|_P \leq \|f\|_{P1} . \|f\|_{P2}$

7. (a) Define the term null set.

Prove that every countable subset of $\Re$ is a null set.

(b) State the monotone convergence theorem for integrals.

Define $f : \Re \to \Re$ by

$$f(x) = xe^{-2x} \ \ if \ \ x \geq 0, \ \ f(x) = 0 \ \ if \ \ x < 0.$$

Use monotone convergence theorem to evaluate $\int f$.

(c) State Lebesgue's dominated convergence theorem.

Let the function $f : \Re \to \Re$ be such that, for all $n \in \mathbf{N}$, $f|_{[-n,n]}$ is Riemann integrable and $\lim_{n\to\infty} \int_{-n}^{n} |f(x)|dx$ exists. Prove that $f \in L^1(\Re)$.

Define $g : \Re \to \Re$ by

$$g(x) = \frac{\cos 3x}{1+x^2} \ \ if \ x \geq 0 \quad g(x) = 0 \ \ if \ x < 0.$$

Show that $g \in L^1(\Re)$.

8. (a) State monotone convergence theorem. By considering the sequence of partial sums, show that the real function f defined by the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{(1+n^2x^2)^2} \quad (0 \leq x \leq 1)$$

is in $L^1[0, 1]$ and that $$\int_0^1 f(x)dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

(b) State the Dominated Convergence Theorem. Let $f \in L^1[0, 1]$ and F be the function on $[0, 1]$ defined by

$$F(x) = \int_0^x f(t)dt (= \int_{[0,1]} f\chi_{[0,1]}dt).$$

If $\{x_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} x_n = \frac{1}{2}$, show that $\chi_{[0,x_n]}(t) \to \chi_{[0,\frac{1}{2}]}(t)$ as $n \to \infty$ for $t = \frac{1}{2}$. Hence show that $F(x_n) \to F(\frac{1}{2})$ as $n \to \infty$.

9. (a) State the Monotone Convergence Theorem. Use it to prove that the function f given by

$$f(x) = \frac{x}{1+x^4} \text{ if } x \geq 0 \quad f(x) = 0 \text{ if } x < 0$$

is in $L^1(\Re)$.

(b) State the Dominated Convergence Theorem. Use it to show that

$$\lim_{n\to\infty} \int_0^{\infty} \frac{x\sin(\frac{x}{n})}{1+x^4}dx = 0.$$

(c) Show that $\int_0^1 x^n \log(x)dx = \frac{(-1)^n}{(n+1)^2}$ for integers $n \geq 1$. Show that

$$\int_0^1 \frac{x\log(x)}{1+x}dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2},$$

stating clearly which results of integration are used to justify the calculation.

10. (a) State the Monotone Convergence Theorem. By considering the sequence of partial sums, show that the real function f defined by

$$f(x) = \sum_{n=1}^{\infty} x^{n^2} \quad (0 \geq x < 1) \quad f(x) = 0 \text{ for } x \text{ not-in-}[0,1)$$

is in $L^1(\Re)$ and that $$\int_{\Re} f dx = \sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

(b) State the Dominated Convergence Theorem. Use it to show that if f is a bounded continuous real function on $\Re$, then

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{\Re} \frac{f\left(\frac{x}{n}\right)}{1+x^2}dx = f(0)$$