MODULE 1

UNIT 1 ORDINARY DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

An ordinary differential equation (ODE) is an equation that contains one or several derivatives of an unknown function, which could be called y(x) (or sometimes y(t) if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example

- $y' = \cos x$
- $y'' + 9y = 0$
- $x^2y'''y' + 2e^xy'' = (x^2 + 2)y^2$

are ordinary differential equations (ODEs).

An ODE is said to be of order ***n*** if the nth derivative of the unknown function ***y*** is the highest derivative of ***y*** in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) is of second order, and (3) of third order.

In this unit, you shall be introduced to first, and second order ordinary differential equations and also you shall have a brief grasp of systems of ordinary differential equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- solve first order ordinary differential equations of different kinds;
- solve second order ordinary differential equations, both homogeneous and non homogeneous equations; and
- solve systems of ordinary differential equations.

3.0 MAIN CONTENT

3.1 First Order ODEs

A first order ordinary differential equation is an equation that contains only the first derivative y' and may contain y and any given function of x. Such equations can be of the following forms

$$y' = f(x, y) \qquad (1)$$

when written explicitly, and

$$F(x, y, y') = 0 \qquad (2)$$

in its implicit form. For instance, the implicit first order ODE $x^{-3}y' - 4y^2 = 0\ (x \neq 0)$ can be written explicitly as $y' = 4x^3y^2$

3.1.1 Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (1) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that

$y' = h'(x)$

Example 3.1 Verify that $y = h(x) = c/x$, $x \neq 0$ is a solution of $xy' = -y$, where c is an arbitrary constant.

Solution

To verify this, you have to differentiate, $y' = h'(x) = -c/x^2$, and multiply by x to get $xy' = -c/x = -y$. Thus $xy' = -y$, the given ODE.

Example 3.2 The ODE $y' = \frac{dy}{dx} = \cos x$ can be solved directly by integration on both sides. Indeed by calculus, you have

$y = \int \cos x\, dx = \sin x + c$

where c is an arbitrary constant.

Observe that in each of the ODEs given in the above examples, the solution contain arbitrary constant. Such a solution containing arbitrary constant ***c*** is called **general solution** of the ODE. While a solution of an ODE that does not contain an arbitrary constant ***c*** is called a **particular solution** of the ODE. For instance in Example 2, if you fix $c = 3$, then $y = \sin x + 3$ is a particular solution of the ODE $y' = \cos x$.

3.1.2 Initial Value Problem (IVP)

The value

$$y(x_0) = y_0 \qquad (3)$$

given at an initial value *x0* of *x* in the interval $a < x < b$ is called an inital condition. An ODE

$$y' = f(x, y), \qquad y(x_0) = y_0 \qquad (4)$$

with an initial condition is called an **initial value problem (IVP).**
Example 3.3 Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y \qquad y(0) = 5.7$$

Solution.

The general solution to the above differential equation is $y = ce^{3x}$. From this solution an the inital condition, you have $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has $y(x) = 5.7e^{3x}$ as the particular solution.

In what follows, you will learn different approaches to solving different kinds first order ordinary differential equations.

3.1.3 Separable ODEs

Many practically useful ODEs can be reduced to the form

$$g(y)y' = f(x) \qquad (5)$$

by purely algebraic manipulations. Then you can integrate on both sides with respect to x, to obtain

$$\int g(y)y'dx = \int f(x)dx + c. \qquad (6)$$

But $y' = \frac{dy}{dx}$, so that $y'dx = dy$, and by substitution in (6) you have

$$\int g(y)dy = \int f(x)dx + c \qquad (7)$$

If **f** and **g** are continuous functions, the integral in (7) exist, and by evaluating them you obtain a general solution of (5). This method of solving ODEs is called the **method of separating variables,** and (5) is called a **separable equation,** because in (7) the variables are now separated; **x** appears only on the right and **y** only on the left.

Example 3.4 Solve the ODE $y' = 1 + y^2$.

Solution

This ODE is separable because it can be written as

$$\frac{dy}{1+y^2} = dx. \quad \text{Thus by integration,} \quad \int \frac{dy}{1+y^2} = \int dx + c$$

you obtain, arctan $y = x + c$ or $y = \tan(x + c)$

Example 3.5 Solve the initial value problem

$$y' = ky, \qquad y(0) = y_0$$

where k is a constant.

Solution

By separation of varibles and integrating, you have

$$\frac{dy}{y} = k dx, \quad \int \frac{dy}{y} = \int k dx, \quad \text{so that} \quad \ln|y| = kt + \tilde{c}, \quad \text{i.e.,} \quad y = ce^{kt}$$

Using the initial condition, you have that $C = y_0$. Hence

$$C = y_0 j\, y = y_0 e^{kx}$$

is the solution of the initial value problem.

Reduction to Separable Form

Certain nonseparable ODEs can be made separable to transformations that introduce for y a new unknown function. This technique is discussed for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right) \qquad (8)$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE,* a term that shall be reserved for a more important purpose.)

For this form of an ODE, you shall set $y/x = u$; thus,

$y = ux$ and by product differentiation, $y' = u'x + u$

Substituting into $y' = f(y/x)$ then gives

$$u'x + u = f(u) \qquad \text{or} \qquad u'x = f(u) - u.$$

You can see that this can be separated as follows;

$$\frac{du}{f(u) - u} = \frac{dx}{x} \qquad (9)$$

Example 3.6 Solve $2xyy' = y^2 - x2$

Solution

To get the usual explicit form, divide the given equation by 2xy,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$$

Now let $y = ux$, and as before, $y' = u'x + y$ Thus substiting for **y** and **y**$'$ and then simplifying by subtracting u on both sides gives you

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}$$

You see that in the last equation you can now separate the variables,

$\frac{2udu}{1+u^2} = -\frac{dx}{x}$. **By integration,** $\ln(1 + u^2) = -\ln|x| + \tilde{c} = \ln\left|\frac{1}{x}\right| + \tilde{c}$.

Take exponets on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2 to obtain

$x^2 + y^2 = cx$. **Thus** $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$.

3.1.4 Exact ODEs

You remember from calculus that if a function u(x,y) has continuous partial derivatives, its differential (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

From this it follows that if $u(x, y) = c = const$, then $du = 0$

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)dx + 3x^2y^2dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^{3'}}{3x^2y^2}$$

an ODE that you can solve by going backward. This idea leads to poyourful solution method as follows.

A first-order ODE $M(x, y) + N(x, y)y' = 0$**,** written as (use $dy = y'dx$)

$$M(x, y)dx + N(x, y)dy = 0 \qquad (10)$$

is called an **exact differential equation** if the differential form ***M(x, y)dx*** + ***N(x, y)dy*** is exact, that is, this form is the differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial u}dy \qquad (11)$$

of some function u(x,y). Then (10) can be written

$$du = 0.$$

By integration you immediately obtain the general solution of (1) in the form

$$u(x, y) = c. \qquad (12)$$

This is called an **implicit solution**, in contrast with a solution $y = \varphi(x)$ as defined earlier, which is also called an *explicit solution*, for distinction.

Comparing (10) and (11), you see that (10) is an exact differential equation if there is some function $u(x, y)$ such that

$$\textbf{(a)} \quad \frac{\partial u}{\partial x} = M, \qquad \textbf{(b)} \quad \frac{\partial u}{\partial y} = N \qquad (13)$$

From this, you can derive a formula for checking whether (10) is exact or not, as follows.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections. Then by partial differentiation of (13),

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

By the assumption of continuity, the two second partial derivatives are equal. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (14)$$

This condition is not only necessary but also sufficient for (10) to be an exact differential equation.

If (10) is exact, the function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ can be found by inspection or in the following systematic way. From (13a) you have by integration with respect to x

$$u = \int M dx + k(y) \qquad (15)$$

in this integration, $\boldsymbol{y}$ is to be regarded as a constant, and $k(y$ plays the role of a "constant" of integration. To determine $k(y)$, you have to derive *au/ay* from (15), use (13b) to get *dk/dy*, and integrate *dk/dy* to get k.

Formula (15) was obtained from (4a). Instead of (4a), you may equally use (13b). Then instead of (15) you will first have by integration with respect to y that

$$u = \int N\,dy + l(x). \qquad (15^*)$$

To determine $l(x)$, you can derive *au/ax* from (6^*), use (4a) to get *dl/dx*, and integrate. The following are examples for illustration

Example 3.7 Solve

$$M = \cos(x + y)dx + (3y^2 + 2y + \cos(x + y))dy = 0 \qquad (16)$$

Solution

Step 1. Test for exactness. Our equation is of the form (10) with

$$\boldsymbol{M} = \cos(\boldsymbol{x} + \boldsymbol{y}) \qquad \boldsymbol{N} = 3y^2 + 2y + \cos(x + y)$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x+y) \qquad \frac{\partial N}{\partial x} = -\sin(x+y)$$

From this and (14), you see that (16) is exact.

Step 2. Implicit general solution. From (15) you obtain by integration

$$u = \int M dx + k(y) = \int \cos(x + y)dx + k(y) = \sin(x + y) + k(y) \qquad (17)$$

To find $\boldsymbol{k}(\boldsymbol{y})$**,** differentiate this formula with respect to $\boldsymbol{y}$ and use formula (13b), to obtain

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x+y)$$

Hence dk/dy = $3\mathbf{y}^2 + 2\mathbf{y}$. By integration, k = $y^3 + y^2 + c$ Inserting this result into (17) and observing (12), you obtain the ansyour

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c$$

Step 3. Checking an implicit solution. You can check by differentiating the implicit solution $u(x, y) = \mathbf{c}$ implicitly and see whether this leads to the given ODE (16):

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \cos(x+y)dx + (\cos(x+y) + 3y^2 + 2y)dy = 0 \qquad (18)$$

This completes the check.

Example 3.8 WARNING! Breakdown in the Case of Nonexactness

The equation $-ydx + xdy = 0$ is not exact because $M = -y$ and $N = x$, so that in (14), $\partial M/\partial y = -1$ but $\partial \mathbf{N}/\partial \mathbf{x} = 1$. You can show that in such a case the present method does not work. From (15),

$$u = \int M dx + k(y) = -xy + k(y), \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $\mathbf{N} = \mathbf{x}$, by (13b). Hoyouver, this is impossible because $k(y)$ can depend only on y. Try (*15**); it will also fail. Solve the equation by another method that is discussed below.

Reduction to Exact Form Integrating Factors

The ODE in Example 3.8 is $-ydx + xdy = 0$. It is not exact. Hoyouver, if you multiply it by $1/x^2$, you will get an exact equation [check exactness by (14)!],

$$\frac{-ydx + xdy}{x^2} = -\frac{y}{x^2}dx + \frac{1}{x}dy = d\left(\frac{y}{x}\right) = 0 \qquad (19)$$

Integration of (19) then gives the general solution $y/x = c = const.$

This example gives the idea. All you did was multiply a given nonexact equation, say,

$$P(x, y)dx + Q(x, y)dy = 0 \qquad (20)$$

by a function F that, in general, will be a function of both **x** and **y.** The result was an equation

$$FPdx + FQdy = 0 \qquad (21)$$

that is exact, so you can solve it as just discussed. Such a function $F(x, y)$ is then called an **integrating factor of (20).**

Example 3.9 Integrating Factor

The integrating factor in (19) is $F = 1/x^2$. Hence in this case the exact equation (21) is

$$FPdx + FQdy = \frac{-ydx + xdy}{x^2} = d\left(\frac{y}{x}\right) = 0. \qquad \textbf{Solution} \qquad \frac{y}{x} = c.$$

These are straight lines **y** = **cx** throught the origin.
It is remarkable that you can readily find other integrating factors for the equation $-ydx + xdy = 0$, namely, $1/y^2$, $1/(xy)$, and $1/(x^2 + y^2)$, because

$$\frac{-ydx + xdy}{y^2} = d\left(\frac{x}{y}\right), \quad \frac{-ydx + xdy}{xy} = -d\left(\ln\frac{x}{y}\right), \quad \frac{-ydx + xdy}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right). \qquad (22)$$

How to Find Integrating Factors

In simpler cases you may find integrating factors by inspection or perhaps after some trials, keeping 22 in mind. In the general case, the idea is the following.

For $M\,dx + N\,dy = 0$ the exactness condition (13) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FPdx + FQdy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \qquad (23)$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_yP + FP_y = F_xQ + FQ_x.$$

In the general case, this would be complicated and useless. So following the *Golden Rule: If you cannot solve your problem, try to solve a simpler one* - the result may be useful (and may also help you later on). Hence you look for an integrating factor depending only on *one*; fortunately, in many practical cases, there are such factors, as you shall see. Thus, let $F = F(\mathbf{x})$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (23) becomes

$$FP_y = F'Q + FQ_x$$

Dividing by FQ and reshuffling terms, you have

$$\frac{1}{F}\frac{dF}{dx} = R, \qquad \textit{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

This proves the following theorem.

Theorem 3.1 Integrating Factor *F(x) If (20) is such that the right side R of (24), depends only on* **x,** *then (20) has an integrating factor F = F(x), which is obtained by integrating (24) and taking exponents on both sides,*

$$F(x) = \exp \int R(x)dx. \qquad (25)$$

Similarly, if $F^* = F^*(y)$, then instead of (24) you get

$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \textit{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \qquad (26)$$

and you have the companion

Theorem 3.2 Integrating Factor $F^*(y)$ *If (20) is such that the right side* R^* *of (26) depends only of* **y**, *then (20) has an integrating factor* $F^* = F^*(y)$ *which is obtained from (26) in the form*

$$F^*(y) = \exp \int R^*(y)dy. \qquad (27)$$

Example 3.10 Application of Theorems 1 and 2. Initial Value Problem
Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0, \quad y(0) = -1 \qquad (28)$$

Solution

Step 1. Nonexactness. The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^y) = e^{x+y} + e^y + ye^y \quad \text{but} \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y - 1) = e^y$$

Step 2. Integrating factor. General solution. Theorem 1 fails because ***R*** [the right side of (16)] depends on both *x* and *y*,

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (26) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1$$

Hence (27) give the integrating factor $\boldsymbol{F^*(y) = e^{-y}}$. From this result and (28) you get the exact equation

$$(e^x + y)dx + (x - e^{-y})dy = 0.$$

Test for exactness: you will get on both sides of the exactness condition. By integration, using (13a).

$$u = \int (e^x + y)\,dx = e^x + xy + k(y)$$

Differentiate this with respect to y and use (13b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

Step 3. Particular solution. The initial condition $\boldsymbol{y}(0) = 1$ gives $u(0, -1) = 1 + 0 + \boldsymbol{e} = 3.72$. Hence the ansyour is $e^x + xy + e^{-y} = 1 + e = 3.72$.

Step 4. Checking. Check by substitution that the ansyour satisfies the given equation as youll as the initial condition.

3.1.5 Linear ODEs. Bernoulli Equation

Linear ODEs or PDEs that can be transformed to linear form are models for various phenomena, for instance, in physics, biology, population dynamics, and ecology, as you shall see.

A first-order ODE is said to be *linear* if it can be written

$$y' + p(x)y = r(x) \qquad (29)$$

The defining feature of this equation is that it is linear in both the unknown function ***y*** and its derivative $y' = dy/dx$, whereas p and r may be *any* given functions of x. If in an application the independent variable is time, you should write *t* instead of *x*.

If the first term is $f(x)y'$ (instead of y'), divide the equation by $f(x)$ to get the " standard form" (1), with y' as the first term, which is practical.

For instance, $y' \cos x + y \sin x = x$ is a linear ODE, and its standard form is $y + y\tan x = x\sec x$.

The function *r(x)* on the right may be a force, and the solution *y(x)* a displacement in a motion or an electrical current or some other physical quantity. In engineering, ***r(x)*** is frequently called *input*, and ***y(x)*** is called the *output* or the response to the input (and, if given, to the initial condition).

Homogeneous Linear ODE

You are expected to solve (29) in some interval $a < x < b$, call it *I.* Two cases are possible for $r(x) = 0$; i.e., either $r(x) \equiv 0$ or $r(x) \neq 0$.

If r(x)=0, then the ODE (29) becomes

$$y' + p(x)y = 0 \qquad (30)$$

and is called homogeneous. By separating variables and integrating you obtain

$$\frac{dy}{y} = -p(x)dx, \qquad \textbf{thus} \qquad \ln|y| = -\int p(x)dx + c*.$$

Takin exponents on both sides, you obtain the general solution of the homogeneous ODE(30),

$$y(x) = ce^{-\int p(x)dx} \qquad (c = \pm e^{c*} \text{ when } y \gtrless 0) \qquad (31)$$

here you may choose c = 0 and obtain the **trivial solution** $y(x) = 0$ for all ***x*** in the interval I.

Nonhomogeneous Linear ODE

The next is to consider the case $r(x) \neq 0$ in (29) for all x in the interval I considered. In this case, the ODE (29) is called nonhomogeneous. It turns out that in this case, (29) has a pleasant property; namely, it has an integrating factor depending only on x. You can find this factor $F(x)$ by theorem 3.1 in the last section. For this purpose you write (29) as

$$(py - r)dx + dy = 0$$

This is $Pdx + Qdy = 0$, where $P = py - r$ and $Q = 1$. Hence the right side of (24) is simply 1(p-0)=p, so that (24) becomes

$$\frac{1}{F}\frac{dF}{dx} = p(x)$$

Separation and integration gives

$$\frac{dF}{F} = p\,dx \qquad \text{and} \qquad \ln|F| = \int p\,dx.$$

Taking exponents on both sides, gives the desired integrating factor F(x),

$$F(x) = e^{\int p dx}$$

Now, multiply (29) on both sides by this F. Then by the product rule,

$$e^{\int p dx}(y' + py) = (e^{\int p dx}y)' = e^{\int p dx}r.$$

By integrating the second and third of these three expressions with respect to x, you get

$$e^{\int p dx}y = \int e^{\int p dx}r\,dx + c.$$

Dividing this equation by $e^{\int p dx}$ and denoting the exponent $\int p dx$ by h, you obtain

$$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p\,dx. \qquad \textbf{(32)}$$

(The constant of integration in h does not matter.) Formula (32) is the general solution of (29) in the form of an integral. Thus, solving (29) is now reduced to the evaluation of an integral.

Example 3.11 First-Order ODE, General Solution

Solve the linear ODE

$$y' - y = e^{2x}$$

Solution

Here

$$p = -1, \qquad r = e^{2x}, \qquad h = \int p dx = -x$$

and from (32) you obtain the general solution

$$y(x) = e^x(\int e^{-x}e^{2x}dx + c = e^x(e^x + c) = ce^x + e^{2x}.$$

In simpler cases, such as the present, you may not need the general formula (32), but may wish to proceed directly, multiplying the given equation by $e^h = e^{-x}$. This gives

$$(y' - y)e^{-x} = (ye^{-x})' = e2xe{-}x = e^x.$$

Integrating on both sides, you obtain the same result as before;

$$ye^{-x} = e^x + c, \text{hence } y = e^{2x} + ce^x$$

Example 3.12 First-Order ODE, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Here $p = \tan x$, $r = \sin 2\mathbf{x} = 2 \sin \mathbf{x} \cos \mathbf{x}$, and

$$\int p dx = \int \tan x dx = \ln |\sec x|.$$

From this you see that in (32),

$e h = \sec x$, $\quad e^{-h} = \cos x$, $\quad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x$,
and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x dx + c\right) = c \cos x - 2\cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$, thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos \mathbf{x} - 2 \cos^2 x$.

Reduction to Linear Form. Bernoulli Equation

Numerous application can be modeled by ODE's that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the Bernoulli equation.

$$y' + p(x)y = g(x)y^a \qquad a \in R \qquad (33)$$

If $a = 0$ or $a = 1$, Equation (33) is linear. Otherwise it is nonlinear. Then you will set

$$u(x) = [y(x)]^{1-a}.$$

Differentiating this and substituting $\mathbf{y}'$ from (33), you obtain

$$u' = (1 - a)y\_^a y' = (1 - a)y\_^a(gy^a - py).$$

Simplification gives

$$u' = (1 - a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that you get the linear ODE

$$u' + (1 - a)pu = (1 - a)g. \qquad (34)$$

Example 3.13 Logistic Equation

Solve the following Bernoulli equation, known as the logistic equation (or Verhulst equation)

$$y' = Ay - By^2 \qquad (35)$$

Solution

Write (34) in the form (35) in the form (33), that is,

$$y' - Ay = B - By^2$$

to see that $a = 2$, so that $u = y^{r-a} = y^{-}{}'$. Differentiating this u and substitute y^0 from (35),

$$u' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-'}$$

The last term is $-Ay^{-}{}' = -Au$. Hence you have obtained the linear ODE

$$u' + Au = B$$

The general solutionis [by (32)]

$$u = ce^{-At} + B/A.$$

Since u=1/y, this gives the solution of (35),

$$y = \frac{1}{u} = \frac{1}{ce^{At} + B/A} \qquad (36)$$

Directly from (35) you see that y 0 ($y(t) = 0$ for all t) is also a solution.

3.2 Second Order Linear ODEs

A second-order ODE is called linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = r(x) \qquad (37)$$

and nonlinear if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in y and its derivativs*, whereas the function p, q and r on the right may be any given functions of x. If the equation begins with say $f(x)y''$ then divide by $f(x)$ to have the standard form (37) with y'' as the first term, which is practical.

If $r(x)$ 0 (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (37) is reduced to

$$y'' + p(x)y' + q(x)y = 0 \qquad (38)$$

and is called homogeneous. If $r(x) \neq 0$, then (1) is called nonhomogeneous.

For instance, a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x}\cos x$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0, \qquad \text{in standard form} \qquad y'' + \frac{1}{x}y' + y = 0.$$

An example of a non linear ODE is

$$y''y + y'^2 = 0$$

The functions p and q in (37) and (38) are called the coefficients of the ODEs. Solutions are defined similarly as for first-order ODEs in section. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if you replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''. Examples are given below.

Homogeneous Linear ODEs: Superposition Principle

Linear ODEs have a rich solution structure. For the homogeneous equation the back bone of this structure is the *superposition principle* or *linearity principle*, which says that you can obtain further solutions from given ones by adding them or by multiplying them with any constants. Of course, this is a great advantage of homogeneous linear ODEs. The following is an example.

Example 3.14 Homogeneous Linear ODEs: Superposition of Solutions

The functions y = cos x and y = sin x are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all x. You can verify this by differentiation and substitution. This gives you $(\cos x)'' = -\cos x$; hence

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Similarly for y = sin x. You can go an important step further. You multiply cos x by any constant, for instance, 4.7 and sinx by say, −2, and take the sum of the results, claiming that it is a solution. Indeed differentiation and substitution gives.

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x) = -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0$$

In this example you have obtained from $y_1(=\cos x)$ and $y_2(=\sin x)$ a function of the form

$$y = c_1y_1 + c_2y_2 \ (c_1, c_2 \text{ arbitrary constants}). \qquad (39)$$

This is called a **linear combination** of y_1 and y_2**.** In terms of this concept you can now formulate the result suggested by your example, often called the **superposition principle or linearity principle**.

Theorem 3.3 Fundamental Theorem for the Homogeneous Linear ODE (38)

For a homogeneous linear ODE (38), any linear combination of two solutions on an open interval I is again a solution of (38) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Proof. Let y_1 and y_2 be solutions of (38) on *I.* Then by substituting $y = c_1y_1 + c_2y_2$ and its derivatives into (38), and using the familiar rule of derivatives, you get

$$\begin{aligned} y'' + py' + qy &= (c1y1 + c2y2)'' + p(c1y1 + c2y2)' + q(c1y1 + c2y2) \\ &= c1y'' + c2y'' + p(c1y'_1 + c2y'_2) + q(c1y1 + c2y2) \\ &= c1(y'' + py'_1 + qy1) + c2(y''_2 + py'_2 + qy2) = 0 \end{aligned}$$

since in the last line, $(\cdots) = 0$ because *y1* and *y2* are solutions, by assumption. This shows that *y* is a solution of (38) on *I.*

Remark 3.1 You should not forget that this highly important theorem holds for*homogeneous linear* ODEs only but ***does not hold*** for nonhomogeneous linear or nonlinear ODEs, as the following example illustrate.

Example 3.15 A Nonhomogeneous Linear ODE Verify by substitution that the functions **y** = 1 + cos **x** and **y** = 1 + sin **x** are solutions of the nonhomogeneous linear ODE

$$y'' + y = 1$$

but their sum is not a solution. Neither is, for instance $2(1 + \cos x)$ *or* $5(1 + \sin x)$**.**

Example 3.16 A Nonlinear ODE Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy' = 0$$

but their sum is not a solution. Neither is $-x^2$**,** so you cannot even multiply by −1!

Initial Value Problem. Basis. General Solution

Recall that from section 3.1, that for a first-order ODE, an *initial value problem* consists of the ODE and one *initial condition* $y(x_0) = y_0$. The initial condition is used to determine the *arbitrary constant* **c** in the *general solution* of the ODE. This results in a unique solution as you need it in most applications. That solution is called a particular solution of the ODE. These ideas extend to second-order equations as follows.

For a second-order homogeneous linear ODE (38) an initial value problem consists of (38) and two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1. \qquad (40)$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (40) are used to detem the two arbitrary constants c_1 and c_2 in general solution

$$y = c_1y_1 + c_2y_2 \qquad (41)$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE, with "suitable" to be explained after the next example. This results in a unique solution, passing through the point (x_0**,** K_0) with K_1 as the tangent direction (the slope) at that point. That solution is called a particular soluion of the ODE (38).

Example 3.17 Initial Value Problem

Solve the initial value problem

$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$

Solution

Step 1. General solution. The functions cos **x** and sin **x** are solutions of the ODE (by example 3.14), and you can take

$$y = c1 \cos x + c_2 \sin x$$

This will turn out to be the general solution as defined below.

Step 2. Particular solution. You need the derivative $y' = c_1 \sin x + c_2 \cos$ **x.** From this and the initial values you will obtain, since cos 0 = 1 and sin 0 = 0**,**

$$y(0) = c_1 = 3.0 \qquad and \qquad y'(0) = c_2 = -0.5.$$

This gives as the solution of your initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x.$$

Definition 3.1 General Solutin, Basis, Particular Solution
A general solution of an ODE (38) on an open interval I is a solution (41) in which y_1 and y_2 are solutions of (38) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called the basis (or a fundamental system) of solutions of (38) on I.

A particular solution of (38) on I is obtained if you assign specific values to c_1 and c_2 in (41). Note that y_1 and y_2 are called *proportional* on I if for all x on I,

$$\text{(a) } y_1 = ky_2 \qquad or \qquad \text{(b) } y_2 = ly_1 \qquad (42)$$

where k and l are numbers, zero or not. (Note (a) implies (b) if and only if $k \neq 0$).

Actually, you can reformulate your definition of a basis by using a concept of general importance. Namely, two functions y_1 and y_2 are called linearly independent on an interval *I* where they are defined if

$k_1y_1(x) + k_2y_2(x) = 0$ *everywhere on I implies* $k_1 = 0$ *and* $k_2 = 0$. (43)

And y_1 and y_2 are called linearly dependent on *I* if (43) holds for some constants k_1, k_2 not both zero. Then if $k_1 \neq 0$ $k_2 \neq 0$, you can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1}y_2 \qquad \text{or} \qquad y_2 = -\frac{k_1}{k_2}y_1.$$

In contrast, in the case of linear *independence* these functions are not proportional because then you cannot divide in (43). This gives the following

Definition 3.2 Basis (Reformulated)

A basis of solutions of (38) on an open interval *I* is a pair of linearly independent solutions of (38) on *I*.

If the coefficients *p* and *q* of (38) are continuous on some open interval *I*, then (38) has a general solution. It yields the unique solution of any initial value problem (38), (40). It includes all solutions of (38) on *I*; hence (38) has no *singular solutions* (solutions not obtainable from the general solution).

Example 3.18 cos ***x*** and sin ***x*** in example 3.17 from a basis of solutions of the ODE $y'' + y = 0$ for all *x* because their quotient is cot $x \neq const$ (or tan $x \neq const$). Hence $y = c_1 \cos x + c_2 \sin \boldsymbol{x}$ is a general solution. The solution $y = 3.0 \cos x - 0.5 \sin x$ of the initial value problem is a particular solution.

Example 3.19 Verify by substitution that $y_1 = e^X$ and $y_2 = e^{-X}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y'' - y = 0, \qquad y(0) = 6 \qquad y'(0) = -2.$$

Solution

$(e^X)'' - e^X = 0$ and $(e^{-X})'' - e^{-X} = 0$ shows that e^X and e^{-X} are solutions. They are not proportional, because $e^X/e^{-X} \neq const$. Hence *eX*, e^{-X} form a basis for all *x*. You now write down the corresponding general solution and its derivatives and equate their values at 0 to the given initial conditions,

$y = c_1e^x + c_2e^{-x}$, $y' = c_1e^x - c_2e^{-x}$, $y(0) = c_1 + c_2 = 6$, $y'(0) = c1 - c2 = -2$

By addition and subtraction, $c_1 = 2$, $c_2 = 4$, so that the *ansyour* is $y = 2e^X + 4e^{-X}$. This is the particular solution satisfying the two initial conditions.

3.2.1 Homogeneous Linear ODEs with Constant Coefficients

In this section, you shall study second-order homogeneous linear ODEs whose coefficients *a* and *b*,

$$y'' + ay' + by = 0. \qquad (44)$$

How to solve (44)? You remember that the solution of the first-order linear ODE with a constan coefficient *k*

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives you the idea to try as a solution of (44) the function

$$y = e^{\lambda x}. \qquad (45)$$

Substitutin (45), and its derivative

$$y' = \lambda e^{\lambda x} \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x}$$

into your equation (44), you obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important characteristic equation (or *auxiliary equation*)

$$\lambda^2 + a\lambda + b = 0 \qquad (46)$$

then the exponential function (45) is a solution of the ODE (44). Now from elementary algebra you recall that the roots of this quadratic equation (46) are

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \qquad \lambda_1 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}) \qquad (47)$$

(46) and (47) will be basic because our derivation shows that the functions

$$y_1 = e^{\lambda 1x} \qquad \text{and} \qquad y_2 = e^{\lambda 2x} \qquad (48)$$

are solutions of (44). Verify by substituting (48) into (44).

From algebra you further know that the quadratic equation (46) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,
(Case I) Two real and distinct roots if $a^2 - 4b > 0$,
(Case II) A real and repeated root if $a^2 - 4b = 0$,
(Case III) Complex conjugate roots if $a^2 - 4b < 0$.

Case I. Two Distinct Real Roots λ_1 and λ_2

In this case, a basis of solutions of (44) on any interval is

$$y_1 = e^{\lambda 1x} \text{ and } y_2 = e\lambda_2 x$$

because y_1 and y_2 are defined (and real) for all **x** their quotient is not constant. The corresponding general equation is

$$y = c_1 e^{\lambda 1x} + c_2 e^{\lambda 2x} \qquad (49)$$

Example 3.20 You can now solve $y'' - y = 0$ in Example 3.19 systematically. The characteristic equation is $\lambda_2 - 1 = 0$. Its roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence a basis of solutions is e^x and e^{-x} and gives the same general solution as before.

$$y = c_1 e^x + c_2 e^{-x}.$$

Example 3.21 Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5$$

Solution

Step 1. General solution. The characteristic equation is

$$\lambda_2 + \lambda - 2 = 0.$$

Its roots are

$$\lambda_1 = {}_2{}^1(-1 + \sqrt{9}) = 1 \text{ and } \lambda_2 = ½\,(-1 - \sqrt{9}) = -2$$

so that you obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular solution. Since $y'(x) = c_1 e^x - 2c_2 e^{-2x}$, you obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4$$
$$y'(0) = c_1 - 2c_2 = -5$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the ansyour $y = e^x + 3e^{-2x}$.

Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, you see directly from (47) that you get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/_2$, hence only one solution,

$$y_1 = e^{-(a/2)x}.$$

To obtain a second independent solution y_2 (needed for a basis), you use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$. Substituting this and its derivatives $y'_2 = u'y1 + uy'_1$ and y''_2 into (44), you first have

$$(u''y1 + 2u'y'_1 + uy''_1) + a(u'y1 + uy'_1) + buy1 = 0$$

Collecting terms in u'', u', and u, as in the last section, you obtain

$$u''y_1 + u'(2y'_1 + ay_1) + u(y''_1 + ay'_1 + by1) = 0.$$

The expression in the last parentheses is zero, since y_1 is a solution of (44). The expression in the first parentheses is zero, too, since

$$2y'_1 = -ae^{-ax/2} = -ay1.$$

You are thus left with $u''y_1 = 0$. Hence u" = 0. By two integrations, $u = c_1x + c_2$. To get a second independent solution $y_2 = uy_1$, you can simply choose $c_1 = 1$, $c_2 = 0$ and take u = x. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis. Hence in the case of a double root of (46) a basis of solutions (44) on any interval is

$$e^{-ax/2}, \quad xe^{-ax/2},$$

The corresponding general solution is

$$y = (c_1 + c_2x)e^{-ax/2} \qquad (50)$$

Example 3.22 The characteristic equation of the ODE $y'' + 6y' + 9y$ is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. It has the double root $\lambda = -3$. Hence a basis is e^{-3x} and xe^{-3x}. The corresponding general solution is $y = (c_1 + c_2x)e^{-3x}$.

Example 3.23 Solve the initial value proble

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, y'(0) = -3.5$$

Solution

The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2x)e^{-0.5x}.$$

You need it derivative

$$Y' = c_2e^{-0.5x} - 0.5(c_1 + c_2x)e^{-0.5x}.$$

From this and the initial conditions you obtain

$$y(0) = c_1 = 3.0, y'(0) = c_2 - 0.5c_1 = -3.5; \qquad hence \quad c_2 = -2.$$

The particular solution of the initial value problem is
$y = (3 - 2x)e^{-0.5x}$

Case III. Complex Roots $-\frac{1}{2}a + i\omega$ **and** $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (46) is negative. In this case, the roots of (46) and thus the solutions of the ODE (44) come at first out complex. Hoyouver, you show that from them you can obtain a basis of *real* solutions

$$y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{ax/2} \sin \omega x \qquad (\omega > 0) \qquad (51)$$

where $\omega^2 = b - \frac{1}{4}a^2$**.** It can be verified by substitution that these are solutions in the present case. You can derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient cot **ωx** is not constant. Hence a real general solution in Case III is

$$y = e^{ax/2}(A \cos \omega x + B \sin \omega x) \qquad (A, B, \text{arbitray}). \qquad (52)$$

Example 3.24 Solve the inital valued problem

$$y'' + 0.4' + 0.4y = 0, \qquad y(0) = 0 \qquad y'(0) = 3.$$

Solution

Step 1. General solution. The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and the genral solution (52) is $y = e^{-0.2x}(A \cos 3x + B \sin 3x)$.

Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expresion is $y = Be{-}0.2x \sin 3x$. You need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + {}^{3e-0.2x} \cos 3x).$$

From this and the second initial condition, you should obtain $\mathbf{y}^0(0) = 3B = 3$. Hence B = 1. Your solution is then

$$y = e^{-0.2x} \sin 3x.$$

Example 3.25 Complex Roots

A general solution of the ODE

$$y'' + \omega^2 y = 0 \qquad (\omega \text{ constant, not zero})$$

is

$$y = A \cos \omega x + B \sin \omega x.$$

With $\omega = 1$**,** this comfirms example 3.17.

3.2.2 Nonhomogeneous ODEs

Method of Undetermined Coefficients

In this section you will be introduced to nonhomogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = r(x) \qquad (53)$$

where $r(x) \neq 0$**.** You will see that a "general solution" of (53) is the sum of a general solution of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (54)$$

and a "particular solution" of (53). These two new terms "general solution of (53)" and a particular solution of (53)" are defined as follows.

Definition 3.3 General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (53) on an open interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x); \qquad (55)$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (54) on I and y_p is any solution of (53) on I containing no arbitrary constants.

A Particular solution of (53) on I is a solution obtained from (55) by assigning specific values to the arbitrary constants c_1 and c_2 in yh.

Your task is now two fold, first to justify these definitions and then to develop a method for finding a solution y_p of (53).

Accordingly, you should first show that a general solution as just defined satisfied (53) and that the solution of (53) and (54) are related in a very simple way.

Theorem 3.4 Relations of Solutions of (53) to Those of (54)

(a) *The sum of a solution y of (53) on some open interval I and a solution $\tilde{y}$ of (54) on I is a solution (53) on I. In particular, (55) is a solution of (53) on I.*

(b) *The difference of two solutions of (53) on I is a solution of (54) on I.*

Proof.

(a) Let $L[y]$ denote the left side of (53). Then for any solutions y of (53) and $\tilde{y}$ of (54) on I, $L[y + \tilde{y}] = L[y]+L[\tilde{y}] = r + 0 = r$.

(b) For any solutions y and y^* of (53) on I you have

$$L[y - y^*] = L[y] - L[y^*] = r - r = 0.$$

■

Now for homogeneous ODEs (54) you know that general solutions include all solutions. You show that the same is true for nonhomogeneous ODEs (53).

Theorem 3.5 A General Solution of a Nonhomogeneous ODE includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (53) are continuous on some open interval I, then every solution of (53) on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution (55) on I.

Proof. Let y^* be any solution of (53) on I and $x0$ and x in I. Let (55) be any general solution of (53) on I This solution exists Indeed, $y_h = c_1y_1 + c_2y_2$ exists (why?) because of the continuity assumption, and y_p exists according to a construction (how?) Now, by theorem 1(b) just proved, the difference $Y = y^* - y_p$ is a solution of (54) on I. At x_0 you have

$$Y(x_0) = y^*(x_0) - y_p(x_0), \qquad Y'(x_0) = y^{*'}(x_0) - y'_p(x_0).$$

Existence and uniqueness theorem (proved in the next unit) implies that for these conditions, as for any other initial conditions in I, there exists a unique particular solution of (54) obtained by assigning suitable values to c_1, c_2 in yh. From this and $y^* = Y + y_p$ the statement follows.

Method of Undetermined Coefficients

Your discussion suggests the following. *To solve the nonhomogeneous ODE (53) or an initial value problem for (53), you have to solve the homogeneous ODE (54) and find any solution y_p* of (53), so that you obtain a general solution (55) of (53)
How can you find a solution y_p of (53)? One method is the so-called **method of undetermined coefficients**. It is much simpler than another, more general method, (which may not be discussed in this book). Since it applies to models of vibrational systems and electric circuits. It is frequently used in engineering.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with **constant coefficients a and b**

$$y'' + ay' + by = r(x) \qquad (56)$$

when $r(x)$ is an exponential function, a poyour of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea. You choose a form for y_p similar to $r(x)$, but with unknown coefficients to determined by substituting that y_p and its derivatives into the ODE. Table 1 shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Table 1: Method of Undetermined Coeffients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \ldots)$	$K_nx^n + K_{n-1}x^{n-1} + \cdots + K_1x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Example 3.26 Application of the Basic Rule (a)
Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5. \qquad (57)$$

Solution.

Step 1. General solution of the homogeneous ODE. The ODE $y'' + y = 0$ has the general solution $y_h = A \cos x + B \sin x$.

Step 2. Solution y_p ***of the nonhomogeneous ODE***. First try $y_p = Kx^2$. Then$y''_p = 2K$. By substitution. By substitution, $2K + Kx^2 = 0.001x^2$. For this to hold for all x. the coefficient of each poyour of x (x^2 and x^0) must be the same on both sides; thus $K = 0.001$ and $2K = 0$, a contradiction.

The second line in Table 1 suggests the choice

$y_p = K_2x^2 + K1x + K0$. Then $y''_p + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2$.

Equating the coefficients of x^2, x, x^0 on both sides, you have $K_2 = 0.001$, $K_1 = 0.2K_2 + K_0 = 0$. Hence $K_0 = -2K_2 = -0.002$. This gives $y_p = 0.001x^2 - 0.002$, and

$y = yh + y_p = A \cos x + B \sin x + 0.001x^2 - 0.002$.

Step 3. Solution of the initial value problem. Setting x=0 and using the first initial condition gives y(0)=A-0.002=0, hence A=0.002. By differentiation and from the second initial condition.

$y' = y'_h + y'_p = -A \sin x + B \cos x + 0.002x$ and $y'(0) = B = 1.5$.

This gives the ansyour

$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$

Example 3.27 Application of the Modification Rule (b)

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, y(0) = 1, y'(0) = 0. \qquad (58)$$

Solution.

Step 1. General solution of the homogeneous ODE. The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2)e^{-1.5x}.$$

Step 2. Solution $\mathbf{y_p}$ ***of the nonhomogeneous ODE.*** The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But you see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule you have to multiply your choice function by x^2. That is, you choose

$$y_p = Cx^2e^{-1.5x} \qquad \text{Then} \qquad y'_p = C(2x - 1.5x^2)e^{1.5x},$$
$$y''_p = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}.$$

Substituting these expressions into the give ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0$, $0 = 0$, $2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^2e^{-1.5x}$. Hence the given ODE has the general solution

$$y = yh + y_p = (c1 + c2)e^{-1.5x} - 5x^2e^{-1.5x}.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ in y and using the first initial condition, you obtain $y(0) = c1 = 1$. Differentiation of y gives

$$y' = (c2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}$$

From this and the second initial condition you have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the ansyour

$$y = (1 + 1.5x)e-1.5x - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

Example 3.28 Application of the Sum Rule (c)

Solve the initial value problem

$$y'' + 2y' + 5y = e0.5x + 40 \cos 10x - 190 \sin 10x, \qquad y(0) = 0.16y'(0) = 40.08 \qquad (59)$$

Solution.

Step 1. General solution of the homogeneous ODE

The characteristic equation $\lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i) = 0$

shows that a real general solution of the homogeneous ODE is

$$y_h = e^{-x}(A \cos 2x + B \sin 2x)$$

Step 2. Solution of the nonhomogeneous ODE. Write $y_p = y_{p1} + y_{p2}$, where y_{p1} corresponds to the exponential term and y_{p2} to the sum of the order two terms. Set

$$y_{p1} = Ce^{0.5x} \qquad \text{then} \qquad y'_{p1} = 0.5Ce^{0.5x} \qquad \text{and} \qquad y''_{p1} = 0.25Ce^{0.5x}.$$

Substitution into the given ODE and omission of the exponential factor gives $(0.25+2.05+5)C = 1$, hence $C = 1/6.25 = 0.16$, and $y_{p1} = 0.16e^{0.5x}$.

Now set $y_{p2} = K \cos 10x + M \sin 10x$, as in Table 1, and obtain

$y'_{p2} = -10K \sin 10x + 10M \cos 10x$, $y''_{p2} = -100K \cos 10x - 100M \sin 10x$. Substitution into the given ODE gives for the cosine terms and for the sine terms
$-100K + 2 \cdot 10M + 5K = 40$, $-100M - 2 \cdot 10K + 5M = -190$

The solution is $K = 0$, $M = 2$. Hence $y_{p2} = 2 \sin 10x$. Together,

$y = y_h + y_{p1} + y_{p2} = e^{-x} (A \cos 2x + B \sin 2x) + 0.16e^{0.5x} + 2 \sin 10x.$

Step 3. Solution of the initial value problem. From y and the first initial condition, $y(0) = A+0.16 = 0.16$, hence $A = 0$. Differentiation gives
$y' = e^{-x}(-A \cos 2x - B \sin 2x - 2A \sin 2x + 2B \cos 2x) + 0.08e^{0.5x} + 20 \cos 10x.$

From this and the second initial condition you have $y'0 = -A + 2B + 0.08 + 20 = 40.08$, hence $B = 10$. This gives the solution

$10e^{-x} \sin 2x + 0.16e^{0.5x} + 2 \sin 10x.$

3.2.3 Solution by Variation of Parameters

Here is a continuation of the discussion of nonhomogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = r(x). \qquad (60)$$

In previous sections you have seen that a general soluton of (60) is the sum of the general solution y_h of the corresponding homogeneous ODE and any particular solution y_p of (60). To obtain y_p when r(x) is not too complicated, you can often use the *method of undetermined coefficients*, as you have shown in the last section.

Hoyouver, since this method is restricted to functions r(x) whose derivatives are of a form similar to r(x) itself (poyours, exponential functions, etc.), It is desirable to have a method valid for more general ODEs (53), which you shall now develop. It is called the **method of variation of parameters** and is credited to Lagrange. Here p, q, r in (1) may be variable (given functions of x), but you should assume that they are continuous on some open interval I.

Lagrange's method gives a particular solution y_p of (53) on I in form

$$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \qquad (61)$$

where y1, y2 form a basis of solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (62)$$

on I, and W is the Wronskian of y_1, y_2,

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_1' \end{pmatrix} = y_1 y_1'$$

Example 3.29 Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}$$

Solution

A basis of solutions of the homogeneous ODE on any interval is y1 = cos x, y2 = sin x. This gives the Wronskian

$W(y_1, y_2) = \cos x \cos x - \sin x(-\sin x) = 1.$

From (61), choosing zero constant of integration, you get the particular solution of the given ODE

$y_p = -\cos x \int \sin x \sec x\, dx + \sin x \int \cos x \sec x\, dx$

$= \cos x \ln |\cos x| + x \sin x$

From y_p and the general solution $y_h = c_1y_1 + c_2y_2$ of the homogeneous ODE you obtain the ansyour

$y = y_h + y_p = (c_1 + \ln |\cos x|) \cos x + (c_2 + x) \sin x.$

Had you included integration constants $-c_1$, c_2 in (61), then (61) would have given the additional c1 cos x + c_2 sin x = $c_1y_1 + c_2y_2$, that is, a general solution of the given ODE directly from (61). This will always be the case.

3.3 Higher Order Linear ODEs

Recall that an ODE is of nth order if the nth derivative $y^{(n)} = d^ny/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Thus the ODE is of the form

$F(x, y, y', ..., y^{(n)}) = 0 \qquad \left(y^{(n)} = \frac{d^n y}{dx^n}\right)$

where loyour order derivatives and y itself may or may not occur. Such an ODE is called linear if it can be written

$$y(n) + p_n-1(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \qquad (64)$$

(For n = 2 you return to second order equation with p1 = p and p0 = q). The **coefficients** $p_0, ..., p_{n-1}$ and the function r on the right are any given functions of x, and y is unknown.

$y^{(n)}$ has coefficient 1. This is practical, and it is called the standard form. (If you have $p_n(x)y^{(n)}$), divide by $\boldsymbol{p}$ ($\boldsymbol{x}$) to get this form.) An ***n***th-order ODE that cannot be written in the form (64) is called nonlinear.

If $r(x)$ is identically zero, $r(x) \equiv 0$ (zero for all $\boldsymbol{x}$ considered, usually in some open interval I), then (64) becomes

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad (65)$$

and is called **homogeneous**. If $r(x)$ is not identically zero, then the ODE is called **nonhomogeneous**.

A **solution** of an nth-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that is defined and n times differentiable on I and is such that the ODE becomes an identity if you replace the unknown function y and its derivatives by h and its corresponding derivatives.

The extension of the concepts and methods of section 3.2 for linear ODEs from order n=2 to arbitrary order n is easily obtained. This is straightforward and needs no new ideas. Hoyouver, the formulas become more involved, the variety of roots of the characteristic equation becomes much larger with increasing n, and the Wronskian plays a more prominent role. For a detailed study on higher order ODE, see unit 3.

4.0 CONCLUSION

In this unit, you have studied ordinary differential equations (ODEs) of **first order, second order** and of **higher order** ($n > 2$). You have used different approach in solving some first order differential equation and second order ordinary differential equation. You have also proved some important theorem of higher order differential equation, which are obvious extensions of the theorems proved for second order ordinary differential equations.

5.0 SUMMARY

Having gone through this unit, you now know that

- **first order** ODEs are equations of the form

  $F(x, y, y') = 0$ or in explicit form $y' = f(x, y)$

  involving the derivative $y' = dy/dx$ of an unknown function y, given functions of x, and, perhaps, y itself. If the independent variable x is time, you denote it by t.
- A general solution, of a first-order ODE is a solution involving an arbitrary constant, which you denoted by c.
- unique solutions can be found by determining a value of $\boldsymbol{c}$ from an initial condition $y(x_0) = y_0$.
- An ODE together with an initial condition is called an initial value problem

  $y' = f(x, y)$, $y(x_0) = y_0$ (x_0, y_0 given numbers) and its solution is a particular solution of the ODE.

- A separable ODE is one that you can put into the form

$$g(y)dy = f(x)dx$$

and solve by integrating both sides

- An **exact ODE** is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $Mdx + Ndy$ is the **differential**

$$du = u_x dx + u_y dy$$

of a function $u(x, y)$, so that from du=0 you immediately get the implicit general solution $u(x, y) = c$. This methods extend to nonexact ODEs that can be made exact by multiplying then by some function $F(x, y)$, called the **integrating factor**

- **Linear ODEs** are such of the form

$$y' + p(x)y = r(x)$$

The **Bernoulli equation** are equations of the form

$$y' + p(x)y = g(x)y^a$$

- A **second-order** ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x)$$

- it is **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval; this is written $r(x) \equiv 0$. Then

$$y'' + p(x)y' + q(x)y = 0.$$

- If $r(x) \neq 0$ for some x considered, then the second order linear ODE is called **nonhomogeneous**.
- **by superposition principle** the linear combination $y = ky_1 + ly_2$ of two solutions y_1, y_2 is again a solution.
- Two linearly independent solutions y_1, y_2 of a homogeneous ODE on an open interval I form **a basis** (or **fundamental system**) of solutions on I and $y = c_1y_1 + c_2y_2$ with arbitrary constants c_1, c_2 is a **general solution** of the homogeneous ODE on I. From it you can obtain a **particular solution** if you specify numeric values (numbers) for $c1$ and $c2$, usually by prescribing two **initial conditions**

  $y(x_0) = K_0$ $y'(x_0) = K1$ (x_0, K_0, K_1 are given numbers)

  a second order ODE together with an initial conditions form an **initial value problem**.

- the **general solution** of a nonhomogenous ODE is of the form

$$y = y_h + y_p$$

where y_h is a general solution of the homogeneous part and y_p is a particular solution of the nonhomo-geneous part obtained by a general method, **variation of parameters**, or in many practical cases by the **method of undetermined coefficients**. The latter applies when second order ODE has constant coefficients ***p*** and ***q*,** and ***r***(***x***) is a poyour of ***x*,** sine, cosine, etc. Then you can write the second order ODE as

$$y'' + ay' + by = r(x)$$

The corresponding homogeneous ODE $y'' + ay' + by = 0$ has solutions $y = e^{\lambda x}$, where λ is a root of

$$\lambda 2 + a\lambda + b = 0.$$

Hence there are three cases which are

Case	Type of Roots	General Solution
I	Distinct real λ_1, λ_2	$y = c_1e^{\lambda 1x} + c_2e^{\lambda 2x}$
II	Double $-\frac{1}{2}a$ Complex	$y = (c_1 + c_2x)e^{-ax/2}$
III	$-\frac{1}{2}a \pm i\omega^*$	$y = e^{-ax/2}(A\cos \omega^* x + B \sin \omega^* x)$

6.0 TUTOR-MARKED ASSIGNMENT

Exercise 6.1 First-Order ODEs

Find the general solution using suitable method.

1. $y' = x^2(1 + y^2)$
   (a) $y = \frac{1}{2}\tan(2x + c)$
   (b) $y = \tan(x^3 + c)$
   (c) $y = 2\tan(x^3 - c)$
   (d) $y = tan\left(\frac{x^3}{3} + c\right)$

2. $yy' + xy^2 = x$
   (a) $y = \sqrt{1 + ce^{-x2}}$
   (b) $y = 1 + ce^{-x2}$
   (c) $y = 1 + ce^{-x}$
   (d) $y = \sqrt{1 + ce^{-x}}$

3. $y' + y \sin x = \sin x$

(a) $y = ce^{\cos x} + 1$

(b) $y = ce^{-\cos x} + 1$

(c) $y = ce^{\sin x} + 1$

(d) $y = ce^{-\sin x} + 1$

4. $3 \sin 2y dx + 2x \cos 2y dy = 0$

(a) $\sin 2\mathbf{y} = -\mathbf{cx}^3$

(b) $\sin y = cx^3$

(c) $\sin 2y = cx^{-3}$

(d) $\sin 2y = \frac{c}{2}x^3$

5. $(y \cos xy - 2x)dx + (x \cos xy + 2y)dy = 0$

(a) $\sin y - x^2 + xy = c$

(b) $\sin xy - x^2 + y^2 = c$

(c) $\sin xy - x^2 + y = c$

(d) $\sin y^2 - xy + y^2 = c$

6. $\sin(y - x)dx + [\cos(y - x) - \sin(y - x)]dy = 0$

(a) $e^x \sin(y - x) = c$

(b) $e' \cos(y - x) = c$

(c) $e' \sin(y - x) = c$

(d) $e^x \cos(y - x) = c$

Solve the following initial value problems using suitable method in each case.

7. $yy' + x = 0,\ y(3) = 4$

(a) $y^2 + x^2 = 16$

(b) $y^2 + x^2 = 25$

(c) $y^2 - x^2 = 16$

(d) $y^2 - x^2 = 25$

8. $y' = 1 + y^2,\ y(\frac{1}{4}\pi) = 0$

(a) $y = \tan(x - \frac{1}{4}\pi)$

(b) $y = \tan(x + \frac{1}{4}\pi)$

(c) $y = \tan(2x - \frac{1}{4}\pi)$

(d) $y = tan(2x + {}_4{}^1 x)$

9. $(2xy^2 - \sin x)dx + (2 + 2x^2y)dy = 0,\ y(0) = 1$

(a) $x^2y + \sin x + 2y = c$

(b) $x^2y + \cos x + 2y = c$

(c) $x^2y^2 + \sin x + 2y = c$

(d) $x^2y^2 + \cos x + 2y = c$

Exercise 6.2 Second Order ODEs

Find a general solution in the following

1. $y'' - 2y' - 8y = 52 \cos 6x$

(a) $c1e^{4x} + c2e^{-2x} - 1.1 \cos 6x + 0.3 \sin 6x$

(b) $c1e^{4x} + c2e^{-2x} + 1.1 \cos 6x - 0.3 \sin 6x$

(c) $c1e^{4x} + c2e^{-2x} - 1.1 \cos 6x - 0.3 \sin 6x$

(d) $c1e^{4x} + c2e^{-2x} + 1.1 \cos 6x + 0.3 \sin 6x$

2. $y'' + 8y' + 25y = 26 \sin 3x$

(a) $e^{-4x}(A \cos 3x + B \sin 3x) - 34 \cos 3x - 21 \sin 3x$

(b) $e^{-4x}(A \cos 3x + B \sin 3x) - 34 \cos 3x + {}^{21} \sin 3x$

(c) $e^{-4x}(A \cos 3x + B \sin 3x) - 34 \sin 3x + {}^{21} \cos 3x$

(d) $e^{-4x}(A \cos 3x + B \sin 3x) + {}^{34} \sin 3x - 21 \cos 3x$

Solve the following initial value problems.

3. $y'' + 5y' - 14y = 0,\ y(0) = 6,\ y'(0) = -6$

(a) $y = 4e^{2x} - 2e^{-7x}$

(b) $y = 4e^{2x} + 2e^{-7x}$

(c) $y = 4e^{2x} + 2e^{7x}$

(d) $y = 4e^{2x} - 2e^{7x}$

4. $x^2y'' - xy' - 24y = 0,\ y(1) = 15,\ y'(1) = 0$

(a) $y = 9x^{-4} + 6x^6$

(b) $y = 9x^4 + 6x^6$

(c) $y = 9x^{-4} + 6x^{-6}$

(d) $y = 9x^4 + 6x^{-6}$

5. $y'' + 5y' + 6y = 108x^2$, $y(0) = 18$, $y'(0) = -26$

(a) $y = e^{-2x} - 2e^{-3x} + 18x^2 - 30x + 19$

(b) $y = e^{-2x} + 2e^{-3x} + 18x^2 - 30x + 19$

(c) $y = e^{-2x} - 2e^{-3x} + 18x^2 + 30x + 19$

(d) $y = e^{-2x} - 2e^{-3x} + 18x^2 - 30x - 19$

Exercise 6.3 Higher-Order ODEs

Solve the given ODE.

1. $4x^2y'' + 12xy'' + 3y' = 0$

(a) $c1 + c2x^{1/2} + c3x^{-1/2}$

(b) $c1 + c2x^{1/2} + c3x^{1/2}$

(c) $c1 + c2x^{-1/2} + c3x^{-1/2}$

(d) $c1 + c2x^{-1/2} + c3x^{1/2}$

2. $8y''' + 12y'' - 2y' - 3y = 0$

(a) $c1e^{-0.5x} + c2e^{0.5x} + c3e^{1.5x}$

(b) $c1e^{0.5x} + c2e^{-0.5x} + c3e^{-1.5x}$

(c) $c1e^{-0.5x} + c2e^{0.5x} + c3e^{-1.5x}$

(d) $c1e^{0.5x} + c2e^{0.5x} + c3e^{1.5x}$

Solve the given initial value problem.

3. $x3y''' + 7x^2y'' - 2xy' - 10y = 0$, $y(1) = 1$, $y'(1) = -7$, $y''(1) = 44$

(a) $0.5x^{-1} - 1.5x^{-5}$

(b) $0.5x^{-1} + 1.5x^{-5}$

(c) $-0.5x^{-1} - 1.5x^{-5}$

(d) $-0.5x^{-1} + 1.5x^{-5}$

UNIT 2 EXISTENCE AND UNIQUENESS THEOREM

CONTENTS

1.0 INTRODUCTION

In Unit 1, you were introduced to basic methods of obtaining solutions of some nth order (n - 1) ordinary differential equation. In this unit, your attention shall be directed to the more theoretical aspects of differential equation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- say when a first order ODE has a solution, a unique solution or no solutions;
- say when a function of two variables satisfies a Lipschitz condition on the second variable;
- approximate a solution of an ODE using the Picard’s iteration;
- describe the dependence of a solution on initial condition on the function; and
- state and apply the existence theorem for linear differential equations.

3.0 MAIN CONTENT

3.1 Existence and Uniqueness Theorem of First-Order Equations

In order to fully understand the proof of this theorem and those which follow, you will need to be familiar with certain concepts of real function theory. Since you may not be familiar with all these topics, the first section will be devoted to a brief survey of some of them.

3.1.1 Some Concepts from Real Function Theory

Uniform Convergence

Definition 3.1 (Convergent Sequence of Real Numbers) A sequence $\{x_n\}$ of real numbers is said to *converge* to the limit x if, given any $\epsilon > 0$, there exists a positive number N such that

$$|x_n - x| < \epsilon$$

for all $n > N$. This you can indicate by writing $\lim_{n\to\infty} x_n = x$.

Definition 3.2 (Pointwise Convergence) A sequence $\{f_n\}$ of real valued functions $f_n : I \subset \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$ defined on an interval I of $\mathbb{R}$ is said to converge to a real valued function $f : I \to \mathbb{R}$ if given any $\in > 0$, and $x \in I$, there exists $N = N(\epsilon, x) \in$ N such that

$$|f_n(x) - f(x)| \leq \in$$

for all $n \geq N$

Definition 3.3 (Uniform Convergence) A sequence $\{f_n\}$ of real valued functions $f_n : I \subset \mathbb{R} \to \mathbb{R}(n \geq 1)$ defined on an interval I of $\mathbb{R}$ is said to converge to a real valued function $f : I \to$ II if given any $\epsilon > 0$, there exists $N = N(\epsilon) \in$ N such that

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $n \geq N$ and for all $x \in I$.

Example 3.1 Consider the sequence of functions $\{f_n\}$ defined for all x on the real interval $0 \leq x \leq 1$ by

$$f_n(x) = \frac{nx^2}{nx + 1} \quad 0 \leq x \leq 1, \ (n = 1, 2, 3, ...)$$

Solution

For $x = 0$, The corresponding sequence $\{f_n(0)\}$ of real numbers is 0, 0, 0, ... and $\lim_{n\to\infty} f_n(0) = 0$. For every x such that $0 < x \leq 1$, you have

$$f_n(x) = \frac{nx^2}{nx+1} \ and \ \lim_{n\to\infty} f_n(x) = x$$

Thus the sequence $\{f_n\}$ converges pointwise on $0 \leq x \leq 1$ to the limit function f defined by $f(x) = x$, $0 \leq x \leq 1$. Futher the convegence is uniform on $0 \leq x \leq 1$. To this, consider

$$|f_n(x) - f(x)| = \left|\frac{nx^2}{nx+1} - x\right| = \frac{x}{nx+1}$$

Given $\epsilon > 0$, you have $\frac{x}{nx+1} \leq \epsilon$ provided $n > \frac{1}{\in} - \frac{1}{x} -$. But for x such that $0 \leq x \leq 1$, you have $\frac{1}{\in} - \frac{1}{x} \leq \frac{1}{\in} - 1$. Thus if you choose $N = \frac{1}{\in} - 1$, you have that n > $\frac{1}{\in} - \frac{1}{x}$ for all n > N. hence, given ∈ > 0, there exists N = $\frac{1}{\in} - 1$ (depending only upon ∈ and not on x) such that $|fn(x) - f(x)| < \in$ for all $n > N$ for every x such that $0 \leq x \leq 1$. In other words, the convergence is uniform on $0 \leq x \leq 1$.

The following are two important theorems on uniformly convergent sequences which shall be used in the proof of existence theorem. You can find the proofs of this theorem in most texts on advanced calculus and real analysis.

Theorem 3.1 *Let* $\{f_n\}$ *be a sequence of real valued functions which converges uniformly to* f *on the interval* $a \leq x \leq b$. *Suppose each function* $f_n(n = 1, 2, 3, ...)$ *is continuous on* $a \leq x \leq b$. *Then the limit function* f *is continuous on* $a \leq x \leq b$.

Example 3.2 In Example 3.1 you saw that the sequence of functions $\{f_n\}$ defined on the real interval $0 \leq x \leq 1$ by

$$f_n(x) = \frac{nx^2}{nx+1}, \; (n = 1, 2, 3, \ldots)$$

converges uniformly to a limit function f on $0 \leq x \leq 1$. Further, each function $f_n(n = 1, 2, 3, ...)$ is continuous on $0 \leq x \leq 1$. By theorem 3.1, you could conclude at once that the limit function f is also continuous on $0 \leq x \leq 1$. Indeed, in this example, the limit function f is that defined by $f(x) = x$, $0 \leq x \leq 1$, and clearly this function f is continuous on $0 \leq x \leq 1$

Theorem 3.2 *Let* $\{f_n\}$ *be a sequence of real functions which converges uniformly to f on the interval* $a \leq x \leq b$. *Suppose each function* $f_n(n = 1, 2, 3, ...)$ *is continuous on* $a \leq x \leq b$. *Then for every* α *and* β *such that* $a \leq \alpha < \beta \leq b$,

$$\lim_{n\to\infty} \int_\alpha^\beta f_n(x)dx = \int_\alpha^\beta \lim_{n\to\infty} f_n(x)dx$$

Example 3.3 You could illustrate this theorem by again considering the sequence of functions $\{f_n\}$ discussed in Examples 3.1 and 3.2 and defined by

$$f_n(x) = \frac{nx^2}{nx+1}, \; 0 \leq x \leq 1, \; (n = 1, 2, 3, \ldots)$$

The hypothesis of Theorem 3.2 is identical with that of theorem 3.1, and you have already noted in Example 3.2 that the sequence under consideration satisfies this hypothesis on $0 \leq x \leq 1$. Thus you could conclude that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\beta} f_n(x)dx$$

Since $\lim_{n\to\infty} f_n(x)dx = \boldsymbol{x}$ in this example, you conclusion here would be that

You could verify this directly,

$$\int_0^1 \frac{nx^2}{nx+1}dx = \int_0^1 \left[x - \frac{1}{n} + \frac{1}{n(nx+1)}\right] dx$$

$$= \frac{x^2}{2} - \frac{x}{n} + \frac{1}{n^2}\ln(nx+1)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{n} + \tfrac{\ln(n+1)}{n^2}$$

This $$\lim_{n\to\infty} \int_0^1 \frac{nx^2}{nx+1}dx = \lim_{n\to\infty} \left[\frac{1}{2} - \frac{1}{n} + \frac{\ln(n+1)}{n^2}\right] = \frac{1}{2}$$

Clearly $\int_0^1 x dx = \frac{1}{2}$ also, and your conclusion is thus verified.

Next is to consider briefly the uniform convergence of an infinite *series* of real functions, each of which is defined on a real interval $a \le x \le b$.

Definition 3.4 Consider the infinite series $\sum_{n-1}^{\infty} u_n$of real functions $u_n(n = 1, 2, 3, \ldots)$, each of which is defined on a real interval $a \le x \le b$. Consider the sequence $\{f_n\}$ of so-called *partial sums* of this series, defined as follows:

$$\begin{cases} f_1 = u_1 \\ f_2 = u_1 + u_2 \\ f_3 = u_1 + u_2 + u_3 \\ \ldots\ldots\ldots \\ f_n = u_1 + u_2 + u_3 + \cdots + u_n \\ \ldots\ldots\ldots \end{cases}$$

The infinite series $\sum_{n-1}^{\infty} u_n$ is said to *uniform uniformly* to f on $a \le x \le b$ if its seqeunce of partial sums $\{f_n\}$ converges uniformly to f and $a \le x \le b$.

The following theorem gives you a very useful test for uniform convergence of series.

Theorem 3.3 (Youierstrass M-Test) *Let $\{M_n\}$ be a sequence of positive constants such that the series of constants $\sum_{n-1}^{\infty} M_n$ converges. Let$\sum_{n-1}^{\infty} u_n$ be a series of real functions such that $|u_n(x)| \le M_n$, for all x such that $a \le x \le b$ and for $n = 1, 2, 3, \ldots$. Then the series $\sum_{n-1}^{\infty} u_n$ converges uniformly on $a \le x \le b$.*

Example 3.4 Consider the series$\sum_{n-1}^{\infty} \frac{\sin nx}{n^2}$ on the interval $0 \leq x \leq 1$. The sequence $\left\{\frac{1}{n^2}\right\}$ is a sequence of positive constants which is such that the series $\sum_{n-1}^{\infty} \frac{1}{n^2}$ is convergent.

You can take $Mn = \frac{1}{n^2}$ and observe that

$$|u_n(x)| = \left|\frac{\sin nx}{n^2}\right| = \frac{1}{n^2} = M_n$$

for all x such that $0 \leq x \leq 1$ and for each $n = 1, 2, 3, \ldots$ Thus by theorem 3.3, the series $\sum_{n-1}^{\infty} \frac{\sin nx}{x}$converges uniformly on $0 \leq x \leq 1$.

Functions of Two Real Variables; the Lipschitz Condition

Definition 3.5

1. A set of points *A* of the *xy* plane will be called ***connected*** if any two points of *A* can be joined by a continuous curve which lies entirely in *A*.
2. A set of points *A* of the *xy* plane is called ***open*** if each point of *A* is the center of a circle whose interior lies entirely in *A*.
3. A open, connected set in the *xy* plane is called a ***domain***.
4. A point *P* is called a ***boundary point*** of a domain *D* if every circle about *P* contains both points in *D* and points not in *D*.
5. A domain plus its boundary points will be called a ***closed domain***.

Example 3.5 The set of all points (x, y) lying within the ellipse $x^2+2y^2 = 1$ and characterized by $x^2+y^2 < 1$ is a domain *D*. The boundary points of *D* are the points of the ellipse itself. The set of points (x, y) such that $x^2 + 2y^2 \leq 1$ within and on the ellipse is a closed domain.

It is assumed that you are somewhat familiar with functions *f* of two real variables *x* and *y*, defined on a domain of the *xy* plane or on such a domain plus its boundary. The following are few concepts and results.

Definition 3.6 Let *f* be a real function defined on a domain *D* of the *xy* plane, and let (x_0, y_0) be an (interior) point of *D*. The function *f* is said to be *continuous* at (x_0, y_0) if, given any $c > 0$, there exists a $8 > 0$ such that

$|f(x, y) - f(x_0, y_0)| < \in$

for all $(x, y) \in D$ such that

$|x - x_0| < \delta$ and $|y - y_0| < \delta$

Definition 3.7 Let f be a real function defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to be ***bounded*** on D if there exists a positive number M such that $|f(x, y)| \leq M$ for all (x, y) in D.

Theorem 3.4 *Let* f *be defined and continuous on a closed rectangle*

$R : a \leq x \leq b,\ c \leq y \leq d$

Then the function f *is bounded on* R.

Example 3.6 The function f defined by $f(x, y) = x^2 + y^2$ is continuous on the closed rectangle $R : 0 \leq x \leq 1, 0 \leq y \leq 2$. Thus by theorem 3.4, the function f is bounded on R. In fact, you have $|f(x, y)| = |x^2 + y^2| \leq 5$ for all $(x, y) \in R$.

Having disposed of these preliminaries, you would now be introduced to a concept which will be of paramount importance in the existence and uniqueness proof.

Definition 3.8 Let f be defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to satisfy a *Lipschitz Condition* (with respect to y) in D if there exist a constant $k > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2| \quad (1)$$

for every (x, y_1) and (x, y_2) which belong to D. The constant k is called the *Lipschitz Constant.*

The following theorem will help you to determine when a function f satisfies the Lipschitz condition.

Theorem 3.5 *Let* f *be such that* $\frac{\partial f}{\partial y}$ *exists and is bounded for all* $(x, y) \in D$*, where* D *is a domain or closed domain such that the line segment joining any two points of* D *lies entirely within* D. *Then* f *satisfies a Lipschitz Condition (with respect to* y*) in* D, *where the Lipschitz Constant is given by*

$$k = \underset{(z,y)\in D}{\text{sump}} \left|\frac{\partial f(x, y)}{\partial y}\right|$$

Example 3.7 Consider the function f defined by $f(x, y) = y^2$, where D is the rectangle defined by $|x| \leq a$, $|y| \leq b$. Then $\frac{\partial f(x,y)}{\partial y} = 2y$, and so $\frac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$. Thus by theorem 3.5, the function f satisfies a Lipschitz Condition in D, where the Lipschitz Constant k is given by $2b$. If you directly apply the definition of Lipschitz condition instead of theorem 3.5, you would find that

$$|f(x, y_1) - f(x, y2)| = |y^2_1 - y^2_2| = |y_1 + y_2||y_1 - y_2| \leq 2b|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

Note that the sufficient condition of theorem 3.5 is not necessary for f to satisfy a Lipschitz condition in D. That is, there exist functions f such that f satisfies a Lipschitz condition (with respect to y) in D but such that the hypothesis of theorem 3.5 is not satisfied.

Example 3.8 Consider the function f defined by $f(x, y) = x|y|$, where D is the rectangle defined by $|x| \leq a$, $|y| \leq b$. Note that

$|f(x, y1) - f(x, y2)| = |x|y1| - x|y2|| \leq |x||y1 - y2| \leq a|y1 - y2|$ for all $(x, y1), (x, y2) \in D$. Thus f satisfies a Lipschitz Condition (with respect to y) in D. Hoyouver thepartial derivative $\frac{\partial f}{\partial y}$ does not exist at any point $(x, 0) \in D$ for which $x \neq 0$.

3.1.2 Existence and Uniqueness of Solutions

The Basic Problem and a Basic Lemma

The basic problem with which this unit is primarily concerned is formulated below as follows.

Let D be a domain in the xy plane and let (x_0, y_0) be an (interior) point of D. Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \qquad (2)$$

where f is a continuous real valued function defined on D. Consider the following problem. Your wish is to determine:

1. an interval $\alpha < x < \beta$ of real **x** axis such that $\alpha < x0 < \beta$, and
2. a differentiable real function ϕ defined on this interval $[\alpha, \beta]$ satisfying the following three requirements:
   (a) $(x, \phi(x)) \in D$, an thus $f(x, \phi(x))$ is defined, for all $x \in [\alpha,\beta]$.

   (b) $\frac{d\phi(x)}{dx} = f[x,\phi(x)]$thus _ satisfies the differential equation (2), for all $x \in [\alpha, \beta]$.

   (c) $\phi(x_0) = y_0$

You should call this problem the *initial-value problem* associated with the differential equation (2) and the point $(\mathbf{x}0, \mathbf{y}0)$. And denote it briefly by

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0, \end{cases} \qquad (3)$$

and call a function **Ø** satisfying the above requirements on an interval *[α, β]* a *solution* of the problem on the interval *[α, β]*.

If ϕ is a solution of the problem on *[*α, β*]*, then the requirement (b) shows that Ø has a continuous first derivative Ø$'$ on *[*α, β*]*.

In ordet to investigate this problem you shall need the following basic lemma.

Lemma 3.1 *Let* f *be a continuous function defined on a domain* D *of the* xy *plane. Let* Ø *be a continuous function defined on a real interval* α ≤ x ≤ β *and such that* [x, ϕ(x)*]* E D *for all* x E *[*α, β*]*. *Let* x0 *be any real number such that* α < x < β. *Then* Ø *is a solution of the differential equation* $\frac{dy}{dx}$ = f *(*x, y*) on [*α, β*] and is such that* Ø(x0*)* = y0 *if and only if* ϕ *satisfies the integral equation*

$$\emptyset(x) = y_o + \int_{xo}^{x} f[t, \emptyset(t)]dt \qquad (4)$$

for all x ∈ *[*α, β*]*.

Proof. If Ø satisfies the differential equation $\frac{dy}{dx}$ = f*(*x, y*)* on *[*α, β*]*, then $\frac{d\emptyset(x)}{dx}$ = *f[x, ϕ(x)]* for all *x ∈ [α, β]* and the integration yields at once

$$\emptyset(x) = \int_{xo}^{x} f[t, \emptyset(t)]dt + c$$

If also $\phi(x_0) = y_0$, then you have **c** = **y**0 and so ϕ satisfies the integral equation (4) for all *x ∈ [α, β]*. Conversely, if ϕ satisfies the integral equation (4) for all *x ∈ [α, β]*, then differentiation yields

$$\frac{d\emptyset(x)}{dx} = f[x, \phi(x)]$$

for all *x ∈ [α, β]* and so **Ø** satisfies the differential equation $\frac{dy}{dx}$ = *f (x, y)* on *[α, β];* also the equation (4) shows that $\phi(x0) = y0$.

The Existence and Uniqueness Theorem

The following is the statement of the main theormem of this chapter.

Theorem 3.6 *Let D be a domain of the xy plane, and let f be a real function satisfying the following two requirements*

(i) *f is continuous in D; and*
(ii) *f satisfies a Lipschitz Condition (with respect to y) in D; that is, there exists a constant k>0 such that*

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

for all (**x, y**1*)*, (**x, y**2*) in* **D.**

Let (x_0, y_0) be an (interior) point of D; let a and b be such that the rectangle R : $|x - x_0| \leq a$, $|y - y_0| \leq b$, lies in D; let $M = \max |f(x, y)|$ for $(x, y) \in R$; and let $h = \left(a, \frac{b}{M}\right)$.

Then there exists a unique solution ϕ of the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases} \qquad (5)$$

on the interval $|x - x_0| \leq h$. That is, there exists a unique differentiable real function ϕ defined on the interval $|x - x_0| \leq h$ which is such that

(i) $\dfrac{d\phi(x)}{dx} = f[x, \phi(x)]$ *for all x on this interval; and*

(ii) $\phi(x0) = y0$

Remark 3.1 *Since R lies in D, f satisfies the requirements (i) and (ii) of the Hypothesis in R. In paticular, since f is thus continuous in the rectangular closed domain R, the constant M defined in the second hypothesis actually does exist. If you examine more closely the number h defined in the second hypothesis of the theorem, you would discover if $a < \frac{b}{M}$, then h = a and the solution Ø is defined for all x in the interval $|x - x_0| \leq a$ used in defining the rectangle R. If, hoyouver, $\frac{b}{M} < a$, then $h = \frac{b}{M} < a$ and so the solution ϕ is assured only for all x in the smaller interval $|x - x_0| \leq h < a$ associated with the smaller rectangle R_1 defined by $|x - x_0| \leq h < a$, $|y - y_0| < b$.*

Due to the lengthy nature of the proof of this theorem, you shall be refered to more advanced textbooks on Ordinary Differential Equations. The ones given in the reference could be of help. But you shall be given the method and steps of proof.

Method of Proof

You shall prove this theorem by the method of *successive approximations.* Let **x** be such that $|x - x0| < h$. You could define the sequence of functions

$\phi 1, \phi 2, \phi 3, \ldots, \phi_n, \ldots$

called the *successive approximations* or ***(Picards Iterants)*** as follows:

$$\begin{cases} \phi_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0]dt \\ \phi_2(x) = y_0 + \int_{x_0}^{x} f[t, \phi_1(t)]dt \\ \phi_3(x) = y_0 + \int_{x_0}^{x} f[t, \phi_2(t)]dt \\ \qquad \ldots\ldots\ldots\ldots \\ \phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \phi_{n-1}(t)]dt \end{cases} \qquad (6)$$

The proof is divided into five main steps in which you would show the following

1. The functions $\{\phi_n\}$ defined by (6) actually exist, have continuous derivatives, and satisfy the inequality $|\phi_n(x) - y0| < b$ on $|x - x0| < h$; and thus $f\,[x, \phi_n(x)]$ is defined on this interval.
2. The functiions $\{\phi_n\}$ satisfy the inequlity

$$|\phi_n(x) - \phi_{n-1}(x)| \leq \frac{M}{k} \cdot \frac{(kh)^n}{n!} \qquad \text{on} \qquad |x - x_0| \leq h$$

3. As $n \to \infty$, the sequence of functions $\{\phi_n\}$ converges uniformly to a continuous function ϕ on $|x - x0| < h$.
4. The limit function ϕ satisfies the differential equation $\frac{dy}{dx} = f(x, y)$ on $|x - x_0| < h$ and is such that $\phi(x_0) = y0$.

5. This function **Ø** is the only differentiable function on $|x - x_0| < h$ which satisfies the differential equation $\frac{dy}{dx} = f(x, y)$ and is such that $\phi(x_0) = y_0$.

Remarks and Examples

Notice carefully that Theorem 3.6 is both an existence theorem and a uniqueness theorem. It tells you that if

(i) f is continuous, and
(ii) f satisfies a Lipschitz condition (with respect to **y**) in the rectangle R,

Then

(a) there exists a solution ϕ of $\frac{dy}{dx} = f(x, y)$, defined on $|x - x_0| \leq h$, which is such that $\phi(x_0) = y_0$; and
(b) this solution ϕ is *unique* solution satisfying these conditions.

Example 3.9 Consider the initial-value problem

$$\begin{cases} \frac{dy}{dx} = y^{\frac{1}{2}} \\ y(0) = 0. \end{cases} \qquad (7)$$

Here $f(x, y) = y^1/_2$ is continuous in the rectangle $R : |x| \leq a, |y| \leq b$ about the origin. Thus there exists at leat one solution of $\frac{dy}{dx} = y^1/_2$ on $|x| \leq h$ such that $y(0) = 0$.

Being assured of the existence of a solution of problem (7), you can now examine the uniqueness aspect. If f satisfies a Lipschitz Condition on R, then Theorem 3.6 will apply and uniqueness will also be assured. You have

$$\left|\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}\right| = \left|\frac{y_1^{\frac{1}{3}}-y_2^{\frac{1}{3}}}{y_1-y_2}\right|$$

If you choose $y1 = \delta > 0$ and $y2 = -\delta$, this becomes

$$\left|\frac{\delta^{\frac{1}{3}}-(-\delta^{\frac{1}{3}})}{\delta-(-\delta)}\right| = \frac{1}{\delta^{\frac{2}{3}}}$$

Since this becomes unbounded as δ approches zero, you see that ***f*** does not satisfy a Lipschitz Condition throughout any domain containing the line $y = 0$, and hence not in ***R.*** Thus you can not apply theorem 3.6 to obtain a uniqueness conclusion here. On the other hand, you must not conclude that uniqueness is impossible simply because a Lipschitz Condtion is not satisfied. The simple truth is that at this point you can draw no conclusion one way or the other about uniqueness in this problem.

In fact, the problem does not have a unique solution; for you can actually exhibit two solutions. Indeed, the functions ***0***1 and ***0***2 defined, respectively, by $\phi1(x) = 0$ for all x and

$$\phi_1(x) = \begin{cases} \left(\frac{2}{3}x\right)^{\frac{3}{2}}, & x \geq 0 \\ 0, & x \leq 0, \end{cases}$$

are both solutions of problem (7) on the interval $-\infty < x < \infty$

You can observe that theorem 3.6 is a "local" existence theorem and an existence theorem "in the small." For it states that if ***f*** satisfies the given hypothesis in a domain ***D*** and if $(x0, y0)$ is a point of ***D,*** then there exists a solution ϕ of

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

defined on an interval $|x - x_0| \leq h$, where h is *sufficiently small.* It does not assert that ϕ is defined for all x, even if ***f*** satisfies the given hypotheses for all (x, y). Existence "in the large" can not be asserted unless additional, very specialized restrictions are placed upon ***f.*** If ***f*** is linear in ***y,*** then an existence theorem in the large may be proved.

Example 3.10 Consider the initial value problem

$$\begin{cases} \frac{dy}{dx} = y^2 \\ y(1) = -1 \end{cases} \qquad (8)$$

Here $f(x, y) = y^2$ and $\frac{\partial f(x,y)}{\partial y}\partial y = 2y$ are both continuous for all (x, y)**.** Thus using theorem 3.5 you observe that f satisfies the hypothesis of theorem 3.6 in every rectangle

$|x - 1| \leq a, \quad |y + 1| \leq b,$

about the point (1, −1). As in theorem 3.6, let $M = \max |f(x, y)|$ for $(x, y) \in R$, and $h = \min\left(a, \frac{b}{M}\right)$. Then theorem 3.6 asserts that the initial-value problem (8) possesses a unique solution defined on $|x - 1| \leq h$.

Now in this case $M = (-1 - b)^2 = (b + 1)^2$ and so

$$h = \min\left[a, \frac{b}{(b+1)^2}\right]$$

Now consider $F(b)\ \frac{b}{(b+1)^2}$. From $F'(b) = \frac{1-b}{(b+1)^3}$, you see that the maximum value of $F(b)$ for $b > 0$ occurs at $b = 1$; and you find $F(1) = \frac{1}{4}$. Thus if $a \geq \frac{1}{4}$, $\frac{b}{(b+1)^2} \leq a$ for all $b > 0$ and so $h = \frac{b}{(b+1)^2} \leq \frac{1}{4}$, regardless of the value of a. If, hoyouver, a < $\frac{1}{4}$, then certainly h < $\frac{1}{4}$. Thus in any case h $\leq \frac{1}{4}$. For $b = 1$, $a \geq \frac{1}{4}$, $h = \min\left[a, \frac{b}{(b+1)^2}\right] = \min(a, \frac{1}{4}) = \frac{1}{4}$.

This is the “best possible” ***h,*** according to the theorem. That is, at best theorem 3.6 assures us that the initial-value problem (8) possesses a unique solution on an interval 34 $\leq x \leq {}^5{}_4$. Thus, although the hypotheses of theorem 3.6 are satisfied for all (x, y), the theorem only assures us of a solution to our problem on the “small” interval $|x - 1| \leq \frac{1}{4}$.

On the other hand, this not necessarily mean that the actual solution of problem (8) is defined only on this small interval and nowhere outside of it. It may actually be defined on a much larger interval which includes the small $|x - 1| \leq 41$ on which it is guaranteed by the theorem. Indeed, the solution of problem (8) is readily found to be $y = -1\ _x$, and this is actually defined and possesses a continuous derivatives on the interval $0 < x < \infty$.

Theorem 3.7 *Let **f** be continuous in the unbounded domain*
$D : a < x < b, -\infty < y < +\infty$. *Let **f** satisfy a Lipschitz Condition (with respect to **y**) in this unbounded domain. That is, assume there exists **k** >* 0 *such that*

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

*Then a solution ϕ of $\frac{dy}{dx} = f(x, y)$ such that $\phi(x_0) = y_0$, where (x_0, y_0) is any point of D, is defined on the entire open interval $a < x < b$. In particular, if $a = -\infty$ and $b = +\infty$, then **0** is defined for all x, $-\infty < x < +\infty$.*

For example, a solution of the initial-value problem

$\frac{dy}{dx} = f(x)y, \quad y(x_0) = y_0,$

where F is continuous for $-\infty < x < +\infty$, is 44 defined for all x, $-\infty < x < +\infty$

3.1.3 Dependence of Solutions on Initial Condition and on the Function *f*

You shall be introduce to how the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ depends upon a slight change in the initial conditions or upon a slight change int function f. It would seem that slight changes would cause only slight changes in the solution.

In the first place, you shall consider the result of a slight change in the initial condition $y(x_0) = y_0$. Let f be continuous and satisfy a Lipschitz Condition with respect to y in a domain *D*, and let (x_0, y_0) be a fixed point of *D*. Then by theorem 3.6 the initial-value problem

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = Y_0 \end{cases}$$

has a unique solution **Ø** defined on some sufficiently small interval $|x - x0| < h0$. Now suppose the initial **y** value is changed from **y**0 to **Y0.** Our first concern is whether or not the new initial-value problem.

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = Y_0 \end{cases} \qquad (9)$$

also has a unique solution on some sufficiently small interval $|x - x_0| \leq h_1$. If Y_0 is such that $|Y_0 - y_0|$ is sufficiently small then you can be certain that the problem (9) does posses a unique solution on some such interval $|x - x_0| \leq h_1$. In fact, let the rectangle $R : |x - x_0| < a$, $|y - y_0| < b$, lie in *D* and let *Y0* be such that $|Y_0 - y_0| \leq \frac{b}{2}$. Then an application of theroem 3.6 to problem (9) shows that this problem has a unique solution ψ which is defined and contained in *R* for $|x - x_0| < h_1$, where $h1 = \min a, \left(a, \frac{b}{2M}\right)$ and $M = \max |f(x, y)|$ for $(x, y) \in R$. Thus you may assume that there exists $\delta > 0$ and $h > 0$ such that for each *Y0* satisfying $|Y_0 - y_0| < \delta$ problem (9) possesses a unique solution $Ø(x, Y_0)$ on $|x - x_0| < h$
The following is the basic theorem concerning the dependence of solutions on initial conditions. You can obtain the proofs of these theorems in most advanced book on Ordinary differential Equations.

Theorem 3.8 *Let f be continuous and satisfy a Lipschitz Condition with respect to* y, *with Lipschitz Constant* k, *in a domain* D *of the* xy *plane; and let* (x_0, y_0) *be a fixed point of* D.

Assume there exists δ > 0 *and* h > 0 *such that for each* Y_0 *satisfying* $|Y_0 - y0| \leq \delta$ *the initial-value problem*

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = Y_0 \end{cases} \qquad (10)$$

possesses a unique solution $\phi(x, Y_0)$ *defined and contained in* D *on* $|x - x_0| \leq h$.
If ϕ *denotes the unique solution of (10) when* Y0 = y0, *and* Ø *denotes the unique solution of (10) when* $Y0 = \tilde{y}0$, *where* $|\tilde{y}0 - y0| = \delta 1 < \delta$, *then*

$$|\tilde{\varnothing}(x) - \varnothing(x)| \leq \delta 1 e^{kh} \quad on \quad |x - x_0| \leq h.$$

Thus the solution $\varnothing(x, Y_0)$ *of problem (10) is a continuous function of the initial value* Y_0 *at* $Y_0 = y_0$.

Thus under the conditions stated, if the initial values of the two solutions *Ø* and $\tilde{\varnothing}$ differ by a sufficiently small amount, then their values will differ by an arbitrary small amount at every point of $|x - x_0| < h$.

The following shows how the solution of $\frac{dy}{dx} = f(x, y)$ will change if the function *f* is slightly changed. In this connection you have following theorem.

Theorem 3.9

1. *In a domain D of the xy plane, assume that*
   - *(i)* *f is continuous and satisfies a Lipschitz Condition with respect to y, with Lipschitz constant k.*
   - *(ii)* *F is continuous.*
   - *(iii)* $|F(x, y) - f(x, y)| < \epsilon$ *for* $(x, y) \in D$.

2. *Let* $(x0, y0)$ *be a point of D, and let*

   (i) *Ø be the solution of the initial-value problem*

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = Y_0 \end{cases}$$

   (ii) ψ *be a solution of the initial-value problem*

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = Y_0 \end{cases}$$

(iii) *[x, Ø(x)] and [x, ψ(x)] in D for* $|x - x_0| < h$.

Then

$|Ø(x) - \psi(x)| < \frac{\epsilon}{k}(e^{kh} - 1)$ *on* $|x - x0| < h$.

Thus, under the hypothesis stated, if *6* is sufficiently small, the difference betyouen the solutions Ø and ψ will be arbitrary small on $|x - x_0| < h$. The following example illustrates how this result can be used to advantage.

Example 3.11 Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = x^2 + y^2 + y + 1, \\ y(0) = 0 \end{cases} \qquad (11)$$

The differential equation of this problem may not be solved explicitly by any of the methods which you know, but the differential equation $\frac{dy}{dx} = y + 1$ can be. If *x* and *y* are sufficiently small, the difference
$|(x^2 + y^2 + y + 1) - (y + 1)| = |x^2 + y^2|$ will be less than or equal to any given $\epsilon > 0$. Thus the solution of problem (11) will differ from that of the problem

$$\begin{cases} \dfrac{dy}{dx} = y + 1 \\ y(0) = 0 \end{cases} \qquad (12)$$

by an arbitrarily small amount if ***x*** and ***y*** are sufficiently small. You can thus use the explicit solution of problem (12) to obtain information about the solution of the problem (11) in a sufficiently small neighbour-hood of *(0, 0)*.

3.2 Existence and Uniqueness Theorem of Linear Differential Equations

This section is an extension of what you have studied in the previous section about existence and uniqueness theorem of first order equation to that of nth other Linear Differential Equations.

3.2.1 The Basic Existence Theorem

The first concern here is the basic existence theorem for an initial value problem involving the nth order linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} = +a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x) \qquad (13)$$

where a_0, a1, ..., an_1, an, and b are continuous on a real interval

$a \leq x \leq b$, and $a_0(x) = \epsilon\ 0$ on $a < x < b$. Recall that in the last section you obtained an existence theorem for the general first-order initial value dy problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$, by means of the method of successive approximations. As a first step toward obtaining an existence theorem for above mentioned existence theorem of the section 3.1 to obtain a similar theorem concerning a system of *n* first-order differential equations in *n* unknowns.

Specifically, you have to consider a system of n first-order differential equations of the form

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n) \\ \vdots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n) \end{cases} \tag{14}$$

the n unknowns $y_1, y_2, \ldots, y_n$, where $f_1, f_2, \ldots, f_n$ are n continuous real functions defined in some domain D of real $(n + 1)$—dimensional $x, y_1, y_2, \ldots, y_n$ space.

Definition 3.9 By a *solution* of the system you would mean an ordered set of n differential real functions

$(\varnothing_1, \varnothing_2, \ldots, \varnothing_n)$

defined on some real x interval $a \leq x \leq b$ such that

$[x, \varnothing_1(x), \varnothing_2(x), \ldots, \varnothing_n(x)] \in D$

and

$$\begin{cases} \frac{d\phi_1(x)}{dx} = f_1[x, \phi_1(x), \phi_2(x), \ldots, \phi_n(x)] \\ \frac{d\phi_2(x)}{dx} = f_2[x, \phi_1(x), \phi_2(x), \ldots, \phi_n(x)] \\ \vdots \\ \frac{d\phi_n(x)}{dx} = f_n[x, \phi_1(x), \phi_2(x), \ldots, \phi_n(x)] \end{cases}$$

for all x such that $a < x < b$.

Corresponding to theorem 3.6, you have the followiing theorem dealing with the system (14). Since its proofs parallels that of theorem 3.6 you would merely have the outline of the major steps of the proof and omit the details.

Theorem 3.10 *Let the functions* f_1, f_2,..., f_n *be continuous in the* (n+1)−*dimensional rectangle* R *defined by*

$|x - x_0| < a, |y_1 - c_1| < b_1, ..., |y_n - c_n| < b_n$

where $(x_0, c_1, ..., c_n)$ *is a point of real* (*n*+1)−*dimensional* $(x, y_1, ..., y_n)$ *space and* a, b1, ..., b_n *are positive constants.*

Let M *be such that* $|fi(x, y_1, y_2, ..., y_n)| < M$ *for* i = 1, 2, ..., n *for all* $(x, y_1, y_2, ..., y_n)$ E R. *Let* $h = \min\left(a\ \frac{b_1}{M}, \frac{b_2}{M}, ..., \frac{b_n}{M}\right)$.

Let the function fi(i = 1, 2, ..., n) *satisfy a Lipschitz condition with Lipschitz constant* k *in* R. *That is, assume there exists a constant* k > 0 *such that*

$$|f_i(x, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n) - f_i(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)| \leq k\ (|\bar{y}_1 - \tilde{y}_1| + |\bar{y}_2 - \tilde{y}_2| + \cdots + |\bar{y}_n - \tilde{y}_n|) \qquad (15)$$

for any two points $(x, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n)$, $(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_{n}n) \in R$, *and for i* = 1, 2, ..., n.

There exists a unique solution

$\varnothing_1, \varnothing_2, ..., \varnothing_n$

of the system

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n) \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_n) \\ \quad\vdots \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, ..., y_n) \end{cases} \qquad (16)$$

such that

$\varnothing_1(x_0) = c_1, \varnothing_2(x_0) = c_2, ..., \varnothing_n(x_0) = c_n,$

defined for $|x - x_0| \leq h$.

Outline of Proof.

First of all define functions $\varnothing_{i,j}$ by

$\varnothing_{i,0}(x) = c_i \qquad (i = 1, 2, ..., n)$

and

$$\emptyset_{i.j}(x) = c_i + \int_{xo} f_i[t, \emptyset_{1,j-1}(t) \dots, \emptyset_{n,j-1}(t)]dt$$

(i = 1, 2, ..., n; j = 1, 2, 3, ...).

Then prove by mathematical induction that all of the functions **Ø**i,j so defined are continuous and satisfy the relations

$$\left|\emptyset_{i.j}(x) - \emptyset_{i.j-1}(x)\right| \le \frac{M(kn)^{j-1}|x - x_0|^j}{j!}$$

(i = 1, 2, 3, ..., n; j = 1, 2, 3, ...; $|x - x_0| \le h$). Thus also

$$\left|\emptyset_{i.j}(x) - \emptyset_{i.j-1}(x)\right| \le \frac{M}{kn}\frac{(knh)^j}{j!} \qquad (17)$$

(i = 1, 2, 3, ..., n; j = 1, 2, 3, ...). This would enable you to conclude that for each i = 1, 2, ..., n, the sequence $\{\emptyset_{i,j}\}$ defined by

$$\emptyset_{i.j}(x) = \emptyset_{i.0}(x) + \sum_{p=1}^{j}\left|\emptyset_{i.j}(x) - \emptyset_{i.p-1}(x)\right|, (j = 1,2,3, \dots)$$

converges uniformly to a continuous function $\emptyset_i$. You may then show that each $\emptyset_i$(i = 1, 2, ..., n) satisfy the integral equation

$$\emptyset_i(x) = c_i + \int_{xo}^{x} f_i[t, \emptyset_1(t), \dots, \emptyset_n(t)]dt,$$

on $|x - x0| < h$. From this you have at once that

$$\frac{d\emptyset(x)}{dx} = f_i[t, \emptyset_1(t), \dots, \emptyset_n(t)]$$

on $|x - x_0| < h$ and $\emptyset_i(x0) = c_i$(i = 1, 2, ..., n).

Our outline of the existence proof is thus completed. It is clear that it parallels that for the case n = 1 given for theorem 3.6. Th proof of the uniqueness in the present case also parallels that of theorem 3.6, and you expected to make necessary changes and complete the present outline.

Theorem 3.10 can be used to obtain an existence and uniqueness theorem for the basic initial-value problem associated with an nth order differential equation of the form

$$y^{(n)} = f[x, y, y', \dots, y^{(n-1)}] \qquad (18)$$

Definition 3.10 Consider the differential equation

$$y^{(n)} = f[x, y, y', ..., y^{(n)}], \qquad (19)$$

where f is a continuous real function defined in a domain *D* of real (*n*+1)−dimensional (*x*, *y*, *y*',..., $y^{(n-1)}$)-space. By a *solution* of the equation (10) you mean a real function Ø having an *nth* derivative (and hence all loyour ordered derivatives) on a real interval $a \leq x \leq b$ such that

$$f[x, Ø(x), Ø`(x), ..., Ø^{(n-1)(x)}] \in D$$

and $Ø^{(n)}(x) = f[x, Ø(x), Ø'(x),..., Ø^{(n-1)}1(x)]$ for all *x* such that $a \leq x \leq b$.

Theorem 3.11 *Consider the differential equation*

$$y^{(n)} = f[x, y, y', ..., y^{(n)}] \qquad (20)$$

where the functiion f *is continuous and satisfies a Lipschitz Condition of the form (15) in a domain* D *of real* (n + 1)−*dimensional* (x, y, y', ..., $y^{(n-1)}$)-*space. Let* (x_0, c_0, c_1, ..., c_{n-1}) *be a point of* D.
Then there exists a unique solution Ø of the nth-order differential equation such that

$$Ø(x_0) = c_0, Ø'(x_0) = c_1,..., Ø^{(n-1)}(x0) = c_{n-1}, \qquad (21)$$

defined on some interval $|x - x_0| \leq h$ *about* $x = x_0$.

3.2.2 Basic Existence and Uniqueness theorems on Linear Systems

Now, your attention will be shifted to the linear system

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + b_1(x) \\ y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + b_2(x) \\ \qquad \cdots\cdots\cdots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + b_n(x) \end{cases} \qquad (22)$$

where the coefficients a_{ij} and the functions b_i are continuous on the interval $a < x < b$. The following lemma is of great importance.

Lemma 3.2 *Let the functions* aij *and* b_i(i = 1, 2, ..., n; j = 1, 2, ..., n) *be continuous on the interval* $a \leq x \leq b$.

Then the functions f_i *defined by*

$$fi(x, y_1, y_2, ..., y_n) = a_{i1}(x)y_1 + a_{i2}(x)y_2 + \cdots + a_{in}(x)y_n + b_i(x),$$

(i = 1, 2, ..., n), satisfy a Lipschitz Condition on

$a \leq x \leq b, -\infty < y_i < +\infty (i = 1, 2, ..., n)$.

That is, there exists a constant k > 0 *such that*

$$|f(x, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n) - f(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)| \leq k(|\bar{y}_1 - \tilde{y}_1| + |\bar{y}_2 - \tilde{y}_2| + \cdots + |\bar{y}_n - \tilde{y}_n|)$$

for all x *such that* $a \leq x \leq b$ *and any two sets of real numbers* $\bar{y}_1, \bar{y}_2, ..., \bar{y}_n$ *and* $\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n$ (i = 1, 2, ..., *n*).

Proof. Since each of the functions aij is continuous on $a \leq x \leq b$, corresponding to each of these functions there exists a constant kij such that $|a_{ij}(x)| \leq k_{ij}$ for all $x \in [a, b]$, $(i = 1, 2, ..., n; j = 1, 2, ..., n)$. Let
$k = \max\{k_{ij}\}$ for $i = 1, 2, ..., n; j = 1, 2, ..., n$. Then $|a_{ij}(x)| \leq k$ for all $x \in [a, b]$. Then for every $x \in [a, b]$ and any two sets of real numbers
$\bar{y}_1, \bar{y}_2, ..., \bar{y}_n$ and $\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n$, you have

$$\begin{aligned}
|f_i(\bar{y}_1, \bar{y}_2, ..., \bar{y}_n) - f_i(\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)| &\leq |a_{i1}(x)\bar{y}_1 + a_{i2}(x)\bar{y}_2 + \cdots + a_{in}(x)\bar{y}_n + b_i(x) \\
&\quad -a_{i1}(x)\tilde{y}_1 - a_{i2}(x)\tilde{y}_2 - \cdots - a_{in}(x)\tilde{y}_n - b_i(x)| \\
&= |a_{i1(x)}[\bar{y}_1 - \bar{y}_1] + a_{i2(x)}[\bar{y}_2 - \bar{y}_2] + \cdots + a_{in(x)}[\bar{y}_n - \bar{y}_n]| \\
&\leq |a_{i1}(x)||\bar{y}_1 - \tilde{y}_1| + |a_{i2}(x)||\bar{y}_2 - \tilde{y}_2| + \cdots + |a_{in}(x)||\bar{y}_n - \tilde{y}_n| \\
&\leq k(|\bar{y}_1 - \tilde{y}_1| + |\bar{y}_1 - \tilde{y}_1| + \cdots + |\bar{y}_1 - \tilde{y}_1|).
\end{aligned}$$

The following gives you the existence theorem concerning the linear system (22).

Theorem 3.12 *Let the coefficients* aij *and the functions* b_i, *(i, j* = 1, 2, ..., *n) in the linear system (22) be continuous on the real interval* $a \leq x \leq b$.

Let x_0 *be a point of the interval* $a \leq x \leq b$, *and let* $c_1, c_2, ..., c_n$ *be a set of n real constants.*

Then there exists a unique solution

$$\varnothing_1, \varnothing_2, ..., \varnothing_n$$

of the system (22) such that

$$\varnothing_1(x_0) = c_1, \varnothing_2(x_0) = c_2, ..., \varnothing_n(x_0) = c_n,$$

and this solution is defined on the entire interval $a \leq x \leq b$.

Outline of Proof

The system (22) is a special case of the system (16) with which theorem 3.10 is concerned, and the present outline of proof parallels that given for theorem 3.10. You would first of all define the functions $\varnothing_{i,j}$ by

$$\varnothing_{i,0}(x) = c_i \quad (i = 1, 2, ..., n)$$

and

$$\varnothing_{i,j}(x) = c_i + \int_{xo}^{x}\left[a_{i1}(t)\varnothing_{1,j-1}(t) + \cdots + a_{in}(t)\varnothing_{n,j-1}(t) + b_i(t)\right]dt \quad (23)$$

$(i = 1, 2, ..., n; j = 1, 2, 3, ...)$ on $a \leq x \leq b$.

The functions Øi,j so defined are continuous on the entire interval $a \leq x \leq b$. Also, by hypothesis there exists $M > 0$ such that $|a_{i1}(x)_{c1} + \cdot\cdot\cdot + a_{in}(x)c_n + b_i(x)| \leq M$, $(i = 1, 2, ..., n)$, $a \leq x \leq b$.

By the lemma the functions defined by

$$a_{i1}(x)y_1 + a_{i2}(x)y_2 + \cdot\cdot\cdot + a_{in}(x)y_n + b_i(x)$$

satisfy a Lipschitz condition on $a \leq x \leq b$. You can thus use the formulas (23) and this Lipschitz condition to obtain by induction the inequality

$$|\varnothing_{i,j}(x) - \varnothing_{i,j-1}(x)| \leq \frac{M(k_n)j^{-1}|x - x_0|j}{j!}$$

$(i = 1, 2, ..., n; j = 1, 2, 3, ...)$ on the entire interval $a \leq x \leq b$. Thus also

$$|\varnothing_{i,j}(x) - \varnothing_{i,j-1}(x)| \leq \frac{M}{kn}\frac{(knH)^j}{j!} \quad (24)$$

$(i = 1, 2, ..., n; j = 1, 2, 3, ...)$, $a \leq x \leq b$, where $H = \max(|a - x_0|, |b - x_0|)$. The inequality (24) here corresponds to the inequality (17) in the proof of theorem 3.10. The remainder of the proof outlined for theorem 3.10 now carries over to the present case for $a \leq x \leq b$ and you would obtain the desired conclusion.

Now you are in a position to obtain the basic existence theorem for the initial value problem associated with the nth-order linear differential equation (13).

Theorem 3.13 *Consider the differential equation of (13) where* $a_0, a_1, ..., a_{n-1}, a_n$, *and* b *are continuous on the interval* $a \leq x \leq b$ *and* $a_0(x) \neq 0$ *on* $a \leq x \leq b$.

Let x_0 *be a point of the interval* $a \leq x \leq b$, *and let* $c_0, c_1, ..., c_{n-1}$ *be a set of n real constants.*

Then there exists a unique solution Ø of (13) such that

$$\varnothing(x_0) = c_0,\ \varnothing'(x_0) = c_1,\ \ldots,\ \varnothing^{(n-1)}(x_0)c_{n-1}, \qquad (25)$$

and this solution is defined over the entire interval $a \le x \le b$.

Proof Let

$$y1 = y,\ y2 = \frac{dy}{dx},\ \ldots,\ y_n \quad \frac{d^{n-1}}{dx^{n-1}}.$$

Then the nth order linear differential equation (13) is equivalent to the linear system

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = -\dfrac{a_n(x)}{a_0(x)}y_1 - \dfrac{a_n(x)}{a_0(x)}y_1 - \cdots - \dfrac{a_n(x)}{a_0(x)}y_1 - \dfrac{b(x)}{a_0(x)} \end{cases} \qquad (26)$$

If $\varnothing$ is a solution of (13) which satisfies the conditions (25), then the ordered set of functions $\varnothing_1, \varnothing_2, \ldots, \varnothing_n$, where $\varnothing_1 = \varnothing, \varnothing_2 = \varnothing', \ldots, \varnothing_n = \varnothing^{(n-1)}$, is a solution of the linear system (26) which satisfies the conditions

$$\varnothing_1(x0) = c_0,\ \varnothing_2(x_0) = c_1,\ \ldots,\ \varnothing_n(x_0) = c_{n-1}, \qquad (27)$$

Conversely, if $\varnothing_1, \ldots, \varnothing_n$ is a solution of (26) which satisfies (27), then the function $\varnothing = \varnothing_1$ is a solution of the differential equation (13) which satisfies conditions (25).

The system (26) is simply a special case of the linear system (22) to which theorem 3.12 applies. Thus the system (26) possesses a unique solution $\varnothing_1, \ldots, \varnothing_n$ defined on the entire interval $a < x < b$ which satisfies the conditions (27). Thus if you set $\varnothing = \varnothing_1$**,** the above-noted equivalence of (13) and (25) with (26) and (27) gives the desired conclusion.

Example 3.12

Consider the initial-value problem:

$$\begin{cases} (x^2 - x - 6)\dfrac{d^2y}{dx^2} + (x^2+4)\dfrac{dy}{dx} + \dfrac{1}{2x+3}y = e^{-x} \\ y(2) = 0 \\ y(2) = 4 \end{cases}$$

The coefficient of y is continuous except at $x = —3/2$. The remaining coefficients and the non-homogeneous term are continuous for all values of x, $— \infty < x < \infty$. The leading coefficients $(x^2 — x — 6)$ equals zero at $\mathbf{x} = —2$ and $x = 3$. Thus the hypothesis of theorem 3.13 is satisfied in every closed interval $a \le x \le b$ such that $—3/2 < a < x_0 = 2 < b < 3$. Therefore the given initial-value problem has a unique solution, and you are assured that this solution is defined over every such closed interval $a \le x \le b$.

An important corollary to this theorem concerning the homogeneous equation

$$a_0(x) \frac{d^{(n)}y}{dx^n} + a_1(x) \frac{d^{(n-1)}y}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = 0 \qquad (28)$$

This corollary, is stated and proved below

Corollary 3.1 *The function Ø is a solution of the homogeneous equation such that*

$$Ø(x_0) = 0, Ø'(x_0) = 0, \ldots, Ø^{(n-1)}(x_0) = 0, \qquad (29)$$

where x_0 is a point of an interval $a \le x \le b$ on which the coefficients a_0, a1, ..., a_n are all continuous and $a_0(x) \neq 0$.

Then $Ø(x) = 0$ for all x such that $a \le x \le b$.

Proof: First not that Ø such that $Ø(x) = 0$ for all $x \; E \; [a, b]$ is indeed a solution of the differential equation (28) which satisfies the initial conditions (29). But by theorem (3.13) the initial-value problem composed of Equation (28) and conditions (29) has a *unique* solution on $a \le x \le b$. Hence the stated conclusion follows.

4.0 CONCLUSION

In this unit you have studied the Existence and Uniqueness theorem of Ordinary Differential Equations of First order and Linear System. This has enabled you to know when a given Ordinary differential equation has *a solution*, *a unique solution* or *no solutions.*

5.0 SUMMARY

Having gone through this section, you are now able to:

(i) say when a first order ODE has a solution, a unique solution or no solutions.
(ii) say when a function of two variables satisfies a Lipschitz condition on the second variable.
(iii) approximate a solution of an ODE using the Picard's iteration.
(iv) describe the dependence of a solution on initial condition on the function f.
(v) state and apply the existence theorem for linear differential equations.

6.0 TUTOR-MARKED ASSIGNMENTS

Exercise 6.1

1. Consider the initial-value problem

$$\frac{dy}{dx} = y^{4/3}, \quad y(x_0) = y_0$$

(a) Discuss the existence of a solution of this problem.
(b) Discuss the uniqueness of a solution of this problem.

2. For each of the following initial-value problems show that there exists a unique solution of the problem if **y**0≠ 0**.** In each case discuss the existence and uniqueness of a solution if **y**0 = 0**.**

(a) $\frac{dy}{dx} = y^{\frac{2}{3}}, \quad y(x_0) = y_0$

(b) $\frac{dy}{dx} = \sqrt{|y|}, \quad y(x_0) = y_0$

3. For each of the following initial-value problems find the largest interval $|x| \leq h$ on which theorem 3.6 guarantees the existence of a unique solution. In each case find the unique solution and show that it actually exists over a larger interval that that guaranteed by the theorem.

(a) $\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0$

(b) $\frac{dy}{dx} = e^{2y}, \quad y(0) = 0$

4. Show that theorem 3.6 guarantees the existence of a unique solution of the initial-value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0$$

on the interval $|x| \leq \frac{\sqrt{2}}{2}$

5. Which of the following sequences of functions $\{f_n\}$ defined on $0 \leq x \leq 1$ does not converge uniformly on $0 \leq x \leq 1$**.**

(a) $f_n(x) = \frac{1}{x+n}, \; 0 \leq x \leq 1, \; (n = 1, 2, 3, \ldots).$

(b) $f_n(x) = x - \frac{x^n}{n}, \; 0 \leq x \leq 1, \; (n = 1, 2, 3, \ldots).$

(c) $f_n(x) = \frac{1}{nx+1}, \; 0 \leq x \leq 1, \; (n = 1, 2, 3, \ldots).$

(d) $f_n(x) = \frac{nx^2}{nx+1}, \; 0 \leq x \leq 1, \; (n = 1, 2, 3, \ldots).$

6. Which of the following functions does not satisfy a Lipschitz Condition in the rectangle *D* defined by $|x| \le a$, $|y| \le b$.

   (a) $f(x, y) = x^2 + y^2$
   (b) $f(x, y) = x \sin y + y \cos x.$
   (c) $f(x, y) = x^2 e^{x+y}.$
   (d) $f(x, y) = y^{\frac{2}{3}}$

7. Consider the third-order differential equation

$$\frac{d^3y}{dx^3} = x^2 + y\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2$$

of the form (20) of the text.

(a) Does there exist a unique solution **Ø** of the given equation such that

$$Ø(0) = 1, Ø'(0) = -3, Ø''(0) = 0?$$

Explain precisely why or why not.

(b) Find the system of three first-order equations to which the given third order equation is equivalent.

8. Does there exist a solution of the initial value problem

$$\begin{cases} (x^2 - 4)\dfrac{d^4y}{dx^4} + 2x\dfrac{d^2y}{dx^2} + (\sin x)y = 0? \\ y(0) = 0, y'(0) = 1, y''(0) = 1, y'''(0) = -1? \end{cases}$$

If so, is the solution unique and over what interval are you assured that it is defined? Explain precisely.

9. Give that each of the functions f_1 and f_2 defined for all *x* by

$$f_1(x) = \sin x \qquad \text{and} \qquad f_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

are solutions of the initial-value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(0) = 0, \ y'(0) = 1 \end{cases}$$

For all *x*, $-\infty < x\infty$, what theorem enables us to conclude that *f1(x)* = *f2(x)* for all *x*, $-\infty < x < \infty$? Explain.

10. Consider the differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad (30)$$

where a_0, a_1 and a_2 are continuous for all x, $-\infty < x < \infty$, and $a_0(x) \neq 0$ for all values of x.

(a) Let f be a nontrivial solution of differential equation (30), let f' denote the derivatives and let $x0 \in [a, b]$. Prove that if $f(x0) = 0$, then $f'(x0) \neq 0$.

(b) Let f and g be two distinct solutions of differential equation (30), and suppose there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0) = 0$. Prove that there exists a constant **c** such that $f = cg$. [Hint: Observe that the function h defined by $h(x) = Af(x) - B_g(x)$, where $A = g'(x_0)$ and $B = f'(x_0)$, is also a solution of differential equation (30).]

UNIT 3 PROPERTIES OF SOLUTIONS OF LINEAR DIFFEREINTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

The subject of linear ordinary differential equations is one of great theoretical and practical importance. Theoretically, the subject is one of simplicity and elegance. Practically, linear differential equations originate in a variety of applications to science and engineering. Fortunately many of the linear differential equations which thus occur are of a special type, linear with constant coefficients, for which explicit methods of solution are available.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use certain methods to obtain solutions of linear ordinary differential equations with constant coefficient; and
- know some basic theorems which could be used to solve such problems.

3.0 MAIN CONTENT

3.1 Basic Theory of Linear Differential Equations

3.1.1 Definition and Basic Existence Theorem

Definition 3.1 A *Linear differential equation of order n* is an equatio of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = b(x) \qquad (1)$$

where a_0 is not identically zero. It shall be assumed that a_0, a_1, ..., a_n and b are continuous real functioins on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$.

The right hand member is called the *nonhomogeneous term.* If b is identically zero the equation reduces to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n-1y' + a_n(x)y = 0$$

and is then called *homogeneous*.

Example 3.1 $y'' + 3xy' + x^3y = e^x$ is a linear differential equation of the second order.

Example 3.2 $y''' + xy'' + 3x^2y' - 5y = \sin x$ is a linear differential equation of the third order.

You can recall from the last unit the following basic existence theorem for initial-value problems associated with an nth order linear differential equation:

Theorem 3.1 *Consider*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = b(x) \qquad (1)$$

where a_0, a_1, ..., a_n and bare continuous real functions of a real interval $a \leq x \leq b$ and $a_0(x) \neq x$ for any x on $a \leq x \leq b$.

Let x_0 be any point of the interval $a \leq x \leq b$, and let c_0, c_1,..., c_{n-1} be n arbitrary constants.

Then there exists a unique solution f of (1) such that $f(x_0) = c_0$ $f'(x_0) =$ c_1, ..., $f^{(n-1)}(x_0) = c_{n-1}$, and this solutiion is defined over the entire interval $a \leq x \leq b$.

Example 3.3 Consider the initial-value problem

$$\begin{cases} y'' + 3xy' + x^3y = e^x \\ y(1) = 2 \\ y'(1) = 5 \end{cases}$$

The coefficients 1, $3x$ and x^2 as youll as the nonhomogeneous term e^x, in this second-order differential equation are all continuous for all values of x, $-\infty < x < \infty$**.** The point x_0 here is the point 1, which certainly belongs to this interval; and the real numbers c_0 and c_1 are 2 and −5**,** respectively. Thus theorem 3.1 tells you that a solution of the given problem exists, is unique, and is defined for all x, $-\infty < x < \infty$.

Example 3.4 Consider the initial-value problem

$$\begin{cases} 2y''' + xy'' + 3x^2y' - 5y = \sin x \\ y(4) = 3 \\ y'(4) = 5 \\ y''(4) = -\dfrac{7}{2} \end{cases}$$

Here you have a third-order problem. The coeffiecients 2, x, $3x^2$, and -5, as youll as the nonhomogeneous term sin x, are all continuous for all x, $-\infty < x < \infty$. The point $x_0 = 4$ certainly belongs to this interval; the real numbers c_0, c_1 and c_2 in this problem are 3, 5 and – $^7/_2$, respectively. Theorem 3.1 tells you that this problem has a unique solution which is defined for all x, $-\infty < x < \infty$.

A useful corollary to theorem 3.1 is the following:

Corollary 3.1 *Let* **y** *be a solution of the* ***homogeneous*** *equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}y' + a_n(x)y = 0 \qquad (2)$$

such that $y(x_0) = 0$, $y'(x_0) = 0$, ..., $y^{(n-1)}(x_0) = 0$, *where x0 is a point of the interval* $a \leq x \leq b$ *in which the coefficients a0, a1, ...,* a_n *are all continuous and* $a_0(x) \neq 0$.

Then $y(x) = 0$ *for all* x *on* $a \leq x \leq b$.

Example 3.5 The solution **y** of the third-order homogeneous equation

$$y''' + 2y'' + 4xy' + x^2y = 0$$

which is such that

$y(2) = y'(2) = y''(2) = 0$

is the trivial solution y such that $y(x) = 0$ for all x.

3.1.2 The Homogeneous Equations

Here you shall be considering the fundamental results concerning the homogeneous equations (2). First is the statement of the basic theorem:

Theorem 3.2 Basic Theorem on Linear Homogeneous Differential Equations Let $y_1, y_2, ..., y_m$ *be any m solutions of the homogeneous linear differential equation (2).*

Then $c_1y_1 + c_2y_2 + \cdots + cmym$ *is also a solution of (2), where* $c_1, c_2, ..., c_m$ *are arbitrary constants.*

You could put this theorem in a very simple form by means of the concept of linear combination, which is now introduced to you.

Definition 3.2 If $y_1, y_2, ..., y_m$ are m given functions, and $c_1, ..., c_m$ are constants, the the expression

$c_1y_1 + c_2y_2 + \cdots + c_my_m$

is called a *linear combination* of $y_1, y_2, ..., y_m$.

In terms of this concept, theorem (3.2) may be stated as follows:

Theorem 3.3 (Theorem 3.2 restated) *Any linear combination of solutions of the homogeneous linear differential equation is also a solution of (2)*

Example 3.6 You could verify that sin **x** and cos **x** are solutions of
$y'' + y = 0$.

Theorem 3.2 states that $c_1 \sin x + c_2 \cos x$ is also a solution for any constants c_1 and c_2. For example,

$5 \sin x + 6 \cos x$
is a solution.

Example 3.7 You should be able to verify that e^x, e^{-1}, and e^{2x} are solutions of

$$y''' - 2y'' - y' + 2y = 0$$

Theorem 4.2 states that $y(x) = c_1e^x + c_2e^{-x} + c_3e^{2x}$ is a solution for any constants c_1, c_2 and c_3. For example,

$$y(x) = 2e^x - 3e^{-x} + \frac{2}{3}e^{2x}$$

is a solution.

Here you shall consider what constitutes the general solution of 4.2. To understand this, you would first be introduced to the concepts of *linear dependence* and *linear independence.*

Definition 3.3 Then n functions $y_1, y_2, ..., y_n$ are called *linearly dependent* on $a \leq x \leq b$ if there exists constants, $c_1, c_2, ..., c_n$, *not all zero* such that

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) = 0$$

for all x such that $a \leq x \leq b$.

In particular, two functions y_1 and y_2 are *linearly dependent* on $a \leq x \leq b$ if there exists constants c_1, c_2, *not both zero,* such that

$$c_1y_1(x) + c_2y_2(x) = 0$$

for all x such that $a \leq x \leq b$.

Example 3.8 You could observe that x and 2_x are linearly dependent on the interval $0 \leq x \leq 1$. For there exists constants c_1 and c_2 not both zero such that

$$c_1x + c_2(2_x) = 0$$

for all x on the interval $-1 \leq x \leq 2$. For example, let $c_1 = 2, c_2 = -2$.

Example 3.9 You could observe that sin x, 3 sin x, and − sin x are linearly dependent on the interval $-1 \leq x \leq 2$ for there exists constants c_1, c_2, c_3, *not all zero,* such that

$$c_1 \sin x + c_2(3 \sin x) + c_3(-\sin x) = 0$$

for all x on the interval $-1 \leq x \leq 2$. For example, let $c_1 = 1$, $c_2 = 1$, $c_3 = 4$.

Definition 3.4 The n functions $y_1, ...y_n$ are called *linearly independent* on the interval $a \leq x \leq b$ if they are *not* linearly dependent there. That is, the function $y_1, ..., y_n$ are *linearly independent* on $a \leq x \leq b$ if the relation
$c_1y_1 + \cdots + c_ny_n = 0$

for all x such that $a \leq x \leq b$ implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

(in other words, the only identically vanishing linear combination of $y_1, ..., y_n$ is the “trivial” linear combination

$$0 \cdot y_1 + 0 \cdot y_2 + \cdots + 0 \cdot y_n).$$

Example 3.10 You could observe that x and x^2 are linearly independent on $0 \leq x \leq 1$, since $c_1x+c_2x^2 = 0$ for all x on $40 \leq x \leq 1$ implies that both $c_1 = 0$ and $c_2 = 0$.

Theorem 3.4 *The nth order homogeneous linear differentia equation (2) always possesses n solutions which are linearly independent. Further, if $y_1, y_2, ..., y_n$ are n linearly independent solutions of (2), then every solution* **y** *of (2) can be expressed as a linear combination*

$$y = c_1y_1 + \cdots + c_ny_n$$

of these n linearly independent solutions by proper choice of the constants, $c_1, ..., c_n$

The above theorem helps us to formulate the meaning of a general solution of an nth order homogeneous linear differential equation as follows:

Definition 3.5 (General Solution) If $y_1, y_2, ..., y_n$ *are n* linearly independent solutions of the nth order homogeneous linear differential equation (2) on $a \leq x \leq b$, then the function y defined by

$$y(x) = c_1y_1(x) + \cdots + c_ny_n(x),\ a \leq x \leq b,$$

where $c_1, ..., c_n$ are arbitrary constants, is called a *general solution* of (2) on $a \leq x \leq b$.

Thus if you can find n linearly independent solutions of (2), you can at once write the general solution of (2) as a linear combination of these n solutions.

Example 3.11 You have observed that sin x and cos x are solutions of $y'' + y = 0$ for all x, $-\infty < x < \infty$. Further, you can verify that these two solutions are linearly independent. Thus the general **y** solution may be expressed as the linear combination

$$y(x) = c_1 \sin x + c_2 \cos x$$

where $c1$ and $c2$ are arbitrary constants.

Example 3.12 The solutions e^x, e^{-1} and e^{2x} of

$$y''' - 2y'' - y' + 2y = 0$$

may be shown to be linearly independent for all x, $-\infty < x < \infty$. Thus the general solution y may be expressed as the linear combination

$$y(x) = c_1e^x + c_2e^{-x} + c_3e^{2x},$$

where c_1, c_2, and c_3 are arbitrary constants.

The next theorem gives you a simple criterion for determining whether or not n solutions of (2) are linearly independent. Before that, you need the following concept.

Definition 3.6 Let $y_1, ..., y_n$ be n real functions each of which has an $(n-1)st$ derivative on a real interval $a \leq x \leq b$. The determinant

$$W(y_1, y_2, ..., y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

in which primes denote derivatives, is called the ***Wronskian*** of these n functions. You can observe that $W(y_1, y_2, ..., y_n)$ is itself a real function defined on $a \leq x \leq b$. Its value at **x**is denoted by $W(y_1, y_2, ..., y_n)(x)$.

Theorem 3.5 *The n solutions $y_1, y_2, ..., y_n$ of the nth order homogeneous linear differential equation (2) are linearly independent on $a \leq x \leq b$ if and only if the Wronskian of $y_1, y_2, ..., y_n$ is different from zero for some x on the interval $a \leq x \leq b$.*

You have further:

Theorem 3.6 *The Wronskian of n solutions $y_1, y_2, ..., y_n$ of (2) is either identically zero on $a \leq x \leq b$ or else is never zero on $a \leq x \leq b$.*

Thus if you can find n solutions of (2), you can apply the theorems (3.5) and (3.6) to determine whether or not they are linearly independent. If they are linearly independent, then you can form the general solution as a linear combination of these n linearly independent solutions.

In the case of the general *second*-order homogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0,$$

the Wronskian of two solutions y_1 and y_2 is the second-order determinant

$$\begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = y_1 y_2^1 - y_2^1 y_2$$

Example 3.13 You can apply theorem 3.5 to show that the solutions sin ***x*** and cos *x* of

$$y'' + y = 0$$

are linearly independent.

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Thus since $W(\sin x, \cos x) \neq 0$, you would conclude that sin x and cos x are indeed linearly independent.

Example 3.14 The solutions e^X, e^{-X}, and e^{2X} of

$$y''' - 2y'' - y' + 2y = 0$$

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0.$$

3.1.3 Nonhomogeneous Equation

You shall now consider breifly the nonhomogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdot\cdot\cdot + a_{n-1}(x)y' + a_n(x)y = b(x) \qquad (1)$$

The basic theorem dealing with this equation is the following

Theorem 3.7 *Let v be a any soluton of the given (nonhomongeneous) nth-order linear differential equation (1). Let u be any solution of the corresponding homogeneous equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdot\cdot\cdot + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (2)$$

Then y=u+v is a solution of the given (nonhomogeneous) equation (1).

Example 3.15 Observet that $y = x$ is a solution of the nonhomogeneous equation

$$y'' + y = x$$

and that $y = \sin \boldsymbol{x}$ is a solution of the corresponding homogeneous equation

$$y'' + y = 0$$

The by theorem 3.7, the sum

$$y = \sin x + x$$

is also a solution of the given nonhomogeneous equation

$$y'' + y = x.$$

You can check that this is indeed true.

Definition 3.7 Consider the *nth*-order (nonhomogeneous) linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdot \cdot \cdot + a_{n-1}(x)y' + a_n(x)y = b(x) \quad (1)$$

and the corresponding homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdot \cdot \cdot + a_{n-1}(x)y' + a_n(x)y = 0 \quad (2)$$

1. The general solution of (2) is called the *complementary* function of equation (1) and is denoted by $\boldsymbol{y}_c$
2. Any particular solution of (1) involving no arbitrary constants is called a *particular integral* of (1), denoted by $\boldsymbol{y}_p$
3. The solution $\boldsymbol{y} = \boldsymbol{y}_c + \boldsymbol{y}_p$ of (1), where $\boldsymbol{y}_c$ is the complementary function and $\boldsymbol{y}_p$ is a particular integral of (1) is called the *general solution* of (1)

Thus to find the general solution of (1), you need merely find:

(a) The complementary function, that is the general linear combination of n linearly independent solutions of the corresponding homogeneous equation (2); and

(b) a particular integral, that is, any particular solution of (1) involving no arbitrary constants.

Example 3.16 Consider the differential equation.

$$y'' + y = x$$

The complementary function is the general solution

$$y_c = c_1 \text{ Sin } x + c_2 \cos x$$

of the corresponding homogeneous equation

$$y' + y = 0.$$

In particular integral is given by

$$y_p = x.$$

Thus the general solutionof the given equation may be written

$$y = y_c + y_p = c_1 \text{ Sin } x + c_2 \cos x + x.$$

The remaining sections of this unit, shall be devoted to study methods. of obtaining the two constituents parts of the general solutions.

3.2 General Theory for Linear Differential Equations with Constant Coefficent

The general form of an *nth* order linear differential equation is given by

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdot\cdot\cdot + a_{n-1}y' + a_ny = b(x) \qquad (3)$$

where $a_0, a_1, ..., a_{n-1}, a_n$ are constants and b is a function of x. If $b(x) \equiv 0$ then (3) becomes of

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdot\cdot\cdot + a_{n-1}y' + a_ny = 0 \qquad (4)$$

and is called *homogeneous* otherwise, it is said to be *nonhomogeneous.*

3.2.1 Homogeneous Linear Equations with Constant Coefficents

As you have already said, the general form of a *homogeneous linear Ordinary Differential Equation with constant coefficients* is given by (4) i.e.,

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdot\cdot\cdot + a_{n-1}y' + a_ny = 0 \qquad (4)$$

where a_0, a1, ..., a_{n-1}, a_n are constants.

In order to solve this equation, you would first of all assume a trial solution of the form

$$y(x) = e^{\lambda x} \qquad (5)$$

Differentiating and substituting this in (4), gives you

$$a_0\lambda^n e^{\lambda x} + a_1\lambda^{n-1}e^{\lambda x} + \cdot\cdot\cdot + a_{n-1}\lambda e^{\lambda x} + a_ne^{\lambda x} = 0$$

or

$$e^{\lambda x}(a_0\lambda^n + a_1\lambda^{n-1} + \cdot\cdot\cdot + a_{n-1}\lambda + a_n) = 0.$$

Since $e^{\lambda x} \neq 0$, you obtain that the polynomial equation in the unknown λ :

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdot\cdot\cdot + a_{n-1}\lambda + a_n = 0 \qquad (6)$$

This equation is called the ***auxilliary equation*** or the ***characteristic equation*** of the given differential equation (6). If $y = e^{\lambda x}$ is a solution of (4) then you see that the constant λ satisfy (6). Hence to solve (5), you write the auxiliary equation (6) and solve it for λ. Observe that (6) is formally obtained form (4) by merely replacing the kth derivative in (4) by λ^k(k = 1, 2, ..., n). Three cases arises, according as the roots of (6) are real and distinct, real and repeated, or complex.

3.2.2 Case I. Distinct Real Roots

Suppose the roots of (6) are the n distinct real numbers

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

Then

$$e^{\lambda 1x}, e^{\lambda 2x}, \ldots, e\lambda x$$

are n distinct solutions of (4). Further, using the Wronskian determinant one may show that these n solutions are linearly independent. Thus you have the following result.

Theorem 3.8 *Consider the nth-order homogeneous linear differential equation (4) with constant coefficients. If the auxilliary equation (6) has the n distinct real roots* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *then the general solution of (6) is*

$$y = c_1e^{\lambda 1x} + c_2e^{\lambda 2x} + \cdots + c_ne^{\lambda nx}$$

where $c_1, c_2, \ldots, c_n$ *are arbitrary constants.*

Example 3.17 $y'' - 3y' + 2y = 0$

The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 2) = 0, \lambda_1 = 1, \lambda_2 = 2.$$

The roots are real and distinct. Thus $\mathbf{e}^x$ and $\mathbf{e}^{2x}$ are solutions and the general solution may be written

$$y = c_1e^x + c_2e^{2x}$$

You can verify that e^x and e^{2x} are indeed linearly independent. Their Wronskian is

$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$. Thus by theorem 3.5, you are sure of their linear independence.

Example 3.18 $y''' - 4y'' + y' + 6y = 0$. The auxilliary equation is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0.$$

You could observe that $\lambda = -1$ is a root of this equation. By sythetic division, you obtain by factorization

$$(\lambda + 1)(\lambda^2 - 5\lambda + 6) = 0$$

or

$$(\lambda + 1)(\lambda - 2)(\lambda - 3) = 0$$

Thus the roots are the distinct real numbers

$$\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3$$

and the general solution is

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}.$$

3.2.3 Case II. Repeated Real Roots

For a better understanding, you can begin the study of this case by considering a simple example.

Example 3.19 An introductory Example. Consider the differential equation

$$y'' - 6y' + 9y = 0. \qquad (7)$$

the auxilliary equation is

$$\lambda^2 - 6\lambda + 9 = 0$$

or

$$(\lambda - 3)^2 = 0$$

The roots of this equation are

$$\lambda_1 = 3,\ \lambda_2 = 3.$$

(real but not distinct). Corresponding to the root λ_1, you have the solution e^{3x}, and corresoponding to λ_2 you have the same solution e^{3x}. The linear combination $c_1e^{3x} + c_2e^{3x}$ of these "two" solutions is clearly not the general solution of the differential equation (7), for it not a linear combination of two *linearly independent* solutions. Indeed, you may write the combination $c_1e^{3x} + c_2e^{3x}$ as simply c_0e^{3x}, where $c_0 = c_1 + c_2$; and clearly $y = c_0e^{3x}$, involving one arbitrary constant, is not the general solution of the given *second*-order equation.

You must find a linearly independent solution; but how can you proceed to do so? Since you already know the one solution e^{3x}, you may set

$$y = e^{3x}u \qquad (8)$$

where u is to be determined. Then

$y' = e^{3x}u6/ + 3e^{ex}u.$
$y'' = e^{3x}u'' + 6e^{3x}u^{l} + 9e^{3x}u.$

Substituting into equation (7) you have

$(e^{3x}u'' + 6e^{3x}u^{0} + 9e^{3x}u) - 6(e^{3x}u6/ + 3e^{ex}u) + 9e^{3x}u.$

or

$$e3xy'' = 0$$

Letting $w = u'$, you have the first-order equation

$$e^{3x}w' = 0$$

or simply

$$w' = 0$$

The general solution of this first-order equation is simply $w = c$, where **c** is an arbitrary constant. Choosing the particular solution $w = 1$ and recalling that $u' = w$, you find

$$u = x + c_0$$

where c_0 is an arbitrary constant. For any constant c_0, you could verify that $ue^{3x} = (x + c_0)e^{3x}$ is a solution of the given second order equation (7). Now you can also verify that this solution and the previously known solution e^{3x} are linearly independent. Choosing $c_0 = 0$, you obtain the solution

$$y = xe^{3x},$$

and thus corresponding to the double root 3 you find the linearly independent solutions

$$e^{3x} and xe^{3x}$$

of equation (7) Thus the general solution of equation (7) may be written

$$y = c_1e^{3x} + c_2xe^{3x} \qquad (9)$$

or

$$y = (c_1 + c_2x)e^{3x} \qquad (10)$$

With this example as a guide, you can return to the general nth-order equation (4). If the auxilliary equation (6) has the double real root λ, you would surely expect that $e^{\lambda x}$ and $xe^{\lambda x}$ would be the corresponding linearly independent solutions. This is indeed the case. Specifically, suppose the roots of (6) are the double real root **λ** and the $(n - 1)$ distinct real

$$\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$$

Then linearly independent solutions of (4) are

$$e^{\lambda x}, xe^{\lambda x}, e^{\lambda 1x}, e^{\lambda 2x}, \ldots, e^{\lambda n-2x}$$

and the general solution may be written

$$y = c_1e^{\lambda x} + c_2xe^{\lambda x} + c_3e^{\lambda 1x} + c_4e^{\lambda 2x} + \cdots + c_ne^{\lambda n-2x.}$$

or

$$y = (c_1 + c_2x)e^{\lambda x} + c_3e^{\lambda 1x} + c_4e^{\lambda 2x} + \cdots + c_ne^{\lambda n-2x.}$$

In like manner, if the auxilliary equation (6) has the triple real root λ, corresponding linearly independent solutions are

$$e^{\lambda x}, xe^{\lambda x} and x^2e^{\lambda x}.$$

The corresponding part of the general solution may be written

$(c_1 + c_2x + c_3x^2)e^{\lambda x}$.

Proceeding further in like manner, you could summarize case II in the following theorem:

Theorem 3.9 *(i) Consider the nth order homogeneous linear differential equation (4) with constant coefficients. If the auxilliary equation (6) has the real root λ recurring k times, then the part of the general solution of (4) corresponding to this k-fold repeated root is*

$$(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{\lambda x}.$$

1. *(ii) If, further, the remaining roots of the auxiliary equation (6) are the distinct real numbers* λk+1, ..., λ_n, *then the general solution of (4) is*

$$y = (c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{\lambda x} + c_{k+1}e^{\lambda k+1x} + \cdots + c_ne^{\lambda nx}.$$

2. *If, hoyouver any of the remaining roots are also repeated, then the parts of the general solution of (4) corresponding to each of these other repeated roots are expressions similar to that corresponding to λ in part (i)*

Here are some examples

Example 3.20 Find the general solution of

$$y''' - 4y'' - 3y' + 18y = 0$$

The auxilliary equation.

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$$

has roots 3, 3, −2. The general solution is

$$y = c_1e^{3x} + c_2xe^{3x} + c_3e^{-2x}$$

or

$$y = (c_1 + c_2x)e^{3x} + c_3e^{-2x}.$$

Example 3.21 Find the general solution of

$$y(iv) - 5y''' + 6y'' + 4y' - 8y = 0.$$

The auxilliary equation is

$$\lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = 0$$

with roots 2, 2, 2, −1. The part of the general solution corresponding to the threefold root 2 is

$$y = (c_1 + c_2x + c_3x^2)e^{2x}$$

and that corresponding to the simple root −1 is simply

$$y = c_4e^{-x}$$

Thus the general solution is

$$y = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x}$$

3.2.4 Case III. Conjugate Complex Roots

Now suppose that the auxiliary equation has the complex number $a + bi$ $(a, b$ real, $i^2 = -1,\ b \neq 0)$ as a nonrepeated root. Then, sinc the coefficients are real, the conjugate complex number $a - bi$ is also a nonrepeated root. The corresponding part of the general solution is

$$k_1e^{(a+bi)x} + k_2e^{(a-bi)x}$$

where k_1 and k_2 are arbitrary constants. The solutions defined by $e(a+bi)x$ $_{and}$ $e(a-bi)x$ are complex functions of the real variable x. It is desirable to replace these by two *real* linearly independent solutions. This can be accomplished by using Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which holds for all real θ.

Using this you have:

$$k_1e^{(a+bi)x} + k_2e^{(a-bi)x} = k_1e^{ax}e^{bix} + k_2e^{ax}e^{-bix}$$

$$= e^{ax}[k_1e^{ibx} + k_2e^{-ibx}]$$

$$= e^{ax}[k_1(\cos bx + i \sin bx) + k_2(\cos bx - i \sin bx)]$$

$$= e^{ax}[(k_1 + k_2) \cos bx + i(k_1 - k_2) \sin bx]$$

$$= e^{ax}[c_1 \sin bx + c_2 \cos bx]$$

where $c_1 = i(k_1 - k_2)$, $c_2 = k_1 + k_2$ are two new arbitrary constants. Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots $a + bi$ is

$$e^{ax}[c_1 \sin bx + c_2 \cos bx]$$

Combining this with the results of case II, you have the following theorem covering case III.

Theorem 3.10
1. Consider the nth order homogeneous linear differential equation (4) with constant coefficients. If the auxiliary equation (6) has the conjugate complex roots a+bi and a-bi, neither repeated, then the corresponding part of the general solution of (4) may be written

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx).$$

2. *If, hoyouver, a+bi and a-bi are each k-fold roots of the auxiliary equation (6), then the corresponding part of the general solution of (4) may be written*

$$y = e^{ax}[(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1}) \sin bx + (c_{k+1} + c_{k+2}x + c_{k+3}x^2 + \cdots + c2kx^{k-1}) \cos bx]$$

Here are some several examples.

Example 3.22 Find the general solution of

$$y'' + y = 0$$

You have already used this equation to illustrate the theorems of section 3.1. You could now obtain its solution using theorem 3.10. The auxiliary equation $\lambda^2 + 1 = 0$ has the roots $\lambda = \pm i$. this are the pure imaginary complex numbers $a \pm bi$, where $a = 0, b = 1$. The general solution is thus

$$y = r^{0x}(c_1 \sin 1 \cdot x + c_2 \cos 1 \cdot x),$$

which is simply

$$y = c_1 \sin x + c_2 \cos x$$

Example 3.23 Find the general solution of

$$y'' - 6y' + 25y = 0$$

The auxiliary equation is $\lambda^2 - 6\lambda + 25 = 0$.

Solving it, you find

$$\lambda = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

Here the roots are the conjugate complex numbers $a \pm bi$, where $a = 3, b = 4$. The general solution may be

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x).$$

Example 3.24 Find the general solution of

$$yiv - 4y''' + 14y'' - 20y' + 25y = 0$$

The auxiliary equation is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$$

The solution of this equation presents some ingenuity and labor. Since our purpose in this example is not to display your mastery of the solution of algebraic equation but rather to illustrate the above principles of determining the general solution of differential equation, you can verify that the roots are

$$1 + 2i,\ 1 - 2i,\ 1 + 2i,\ 1 - 2i$$

Since each pair of conjugate roots is double, the general solution is

$$y = e^x[(c_1 + c_2) \sin 2x + (c_3 + c_4) \cos 2x]$$

3.2.5 An Initial-Value Problem

Here is an application of the results concerning general solutions of homogeneous linear equation with constants to an initial-value problem involving such an equation.

Example 3.25 Solve the initial-value problem

$$y'' - 6y' + 25y = 0 \qquad (11)$$

$$y(0) = -3 \qquad (12)$$

$$y'(0) = -1 \qquad (13)$$

First note that by theorem 3.1, this problem has a unique solution defined for all x, $-\infty < x < \infty$. You can now proceed to find this solution; that is, you seek the particular solution of the differential equation (11) which satisfies the two initial conditions (12) and (13). You have already found the general solution of the differential equation (11) in example 3.23. It is

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x) \qquad (14)$$

From this, you find

$$y' = e^{3x}[(3c_1 - 4c_2) \sin 4x + (4c_1 + 3c_2) \cos 4x] \qquad (15)$$

You can now apply the initial conditions. Applying condition (12),
$y(0) = -3$**.** to equation (14), you find

$$-3 = e^0(c_1 \sin 0 + c_2 \cos 0)$$

which reduces at once to

$$c_2 = -3 \qquad (16)$$

Applying condition (13), $y'(0) = -1$, to Equation (15), you obtain

$$-1 = e^0[(3c_1 - 4c_2) \sin 0 + (4c_1 + 3c_2) \cos 0]$$

which reduces to

$$4c_1 + 3c_2 = -1 \qquad (17)$$

Solving Equation (16) and (17), you find

$$\begin{cases} c_1 = 2 \\ c_1 = -3 \end{cases}$$

Replacing **c**1 and $\mathbf{c}_2$ in Equation (4.19) by these values, you obtain the unique solution of the given initial-value problem in the form

$$y = e^{3x}(2 \sin 4x - 3 \cos 4x)$$

You may write this in an alternate form by first multiplying and dividing by $\sqrt{(22) + (-3)^2} = \sqrt{13}$ to obtain

$$y = \sqrt{13e^{3x}} \left[\frac{1}{\sqrt{13}} \sin 4x - \frac{3}{\sqrt{13}} \cos 4x\right]$$

from this you may express the solution in the alternate form

$y = \sqrt{13}e^{3x} \sin(4x + \emptyset)$,

where the angle $\emptyset$ is defined by the equations

$$\begin{cases} \sin\emptyset = -\dfrac{3}{\sqrt{13}} \\ \cos\emptyset = \dfrac{2}{\sqrt{13}} \end{cases}$$

3.2.6 The Method of Undetermined Coefficients

You will be dealing with the (nonhomogeneous) differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots a_{n-1}y' + a_ny = b(x) \qquad (18)$$

where the coefficients $a_0, a_1, ..., a_n$ are constants but where the nonhomogeneous term b is (in general) a nonconstant function of x. Recall that the general solution of (18) may be written

$$y = y_c + y_p$$

where $\mathbf{y}_c$ is the *complementary function*. That is the general solution of the corresponding homogeneous equation (equation (18) with b replaced by 0), and $\mathbf{y}_p$ is a *particular integral*, that is, any solution of (18) containing no arbitrary constants. In last section, you learnt how to find the complementary function, now you will consider methods of determining a particular integral.

You shall first consider the method of *undetermined coefficients.* Mathematically speaking, the class of functions b to which this method applies is actually quite restricted; but this mathematically narrow class includes functions of frequent occurrence and considerable importance in various physical applications. And this method has one distinct advantage - when it does apply, it is relatively simple.

The following are some preliminary definitions

Definition 3.8 You should call a function *UC function* if it is *either*

1. A function defined by any of the following

   (a) x^n, where n is a positive integer or zero.

   (b) e^{ax}, where a is a constant $\neq 0$.

   (c) sin(bx + c), where b and c are constants, b $\neq$ 0.

   (d) cos(bx + c), where b and c are constants, b $\neq$ 0.

or

2. A function defined as a finite product of two or more functions of these four types.

The method of undetermined coefficients applies when the nonhomogeneous function *b* in the differential equation is a finite linear combination of UC functions. Observe that given a UC function *f*, each successive derivative of **f** is either itself a constant multiple of a UC function or else a linear combination of UC functions.

Definition 3.9 Consider a UC function *f.* The set of functions consisting of *f* itself and all linearly independent UC functions of which the succesive derivatives of *f* are either constant multiples or linear combinations will be called the *UC set of f.*

Example 3.26 The function *f* defined for all real *x* by $f(x) = x^3$ is a UC function. Computing derivatives of *f*, you find

$$f'(x) = 3x^2, f''(x) = 6x, f''' = 6,$$

$$f^{(n)}(x) = 0 \text{ forn} > 2$$

Examples

Below are a few illustrative examples which gives you the procedure for finding the particular integral using the method of undetermined coefficients.

Example 3.27

$$y'' - 2y' - 3y = 2ex - 10 \sin x.$$

The corresponding homogeneous equation is

$$y'' - 2y' - 3y = 0$$

and the complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}$$

The nonhomogeneous term is the linear combination $2e^x - 10 \sin x$ of the two UC functions given by e^x and sin *x*.

1. Form the UC set for each of these two functions. You find

$$S_1 = \{e^x\}$$
$$S_2 = \{\sin x, \cos x\}.$$

2. Note that neither of these sets is identical with nor included in the other; hence both are retained.

3. Furthermore, by examining the complementary function, you see that none of the functions e^x, sin x, cos *x* in either of these sets is a solution of the corresponding homogeneous equation. Hence neither set needs to be revised.

4. Thus the original set S_1 and S_2 remain intact in this problem, and you form the linear combination

$$Ae^x + B\sin x + C\cos x$$

of the three elements e^x, $\sin x$, $\cos x$ of S_1 and S_2, with the undetermined coefficients A, B, C.

5. You can determine these unknown coefficients by substituting the linear combination formed in step (4) into the differential equation and demanding that it satisfies the differential equation identically.

That is, you take

$$y_p = Ae^x + B\sin x + C\cos x$$

is a particular solution. Then

$$y'_p = Ae^x + B\cos x - C\sin x$$
$$y''_p = Ae^x - B\sin x - C\cos x$$

Substitution, gives you

$$[Ae^x - B\sin x - C\cos x] - 2[Ae^x + B\cos x - C\sin x] - 3[Ae^x + B\sin x + C\cos x] = 2e^x - 10\sin x$$

or

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x.$$

Equating coefficients of like terms, you should obtain the equations

$$\begin{cases} -4A = 2 \\ -4B + 2C = -10 \\ -2C - 2B = 0 \end{cases}$$

From these equations, you find that

$$\begin{cases} A = -\dfrac{1}{2} \\ B = 2 \\ C = -1 \end{cases}$$

and hence you obtain the particular integral

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$$

Thus the general solution of the differential equation under consideration is

$$y = y_c + y_p = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x$$

Example 3.28 Initial-Value Problem This section will be closed by applying the results to the solution to the initial-value problem

$$\begin{cases} y'' - 2y' - 3y = 2e^x - 10\sin x \\ y(0) = 2 \\ y'(0) = 4. \end{cases}$$

By theorem 3.1, this problem has a unique solution, defined for all x, $-\infty < x < \infty$; So you can now proceed to find it. In example 3.27, you found that the general solution of the differential equation is

$$y = c_1e^{3x} + c_2e^{-x} - \frac{1}{2} + 2\sin x - \cos x.$$

From this, you have

$$y' = 3c_1e^{3x} - c_2e^{-x} - \frac{1}{2}ex + 2\cos x + \sin x$$

Applying the initial conditions to the last two equations, respectively, you have

$$\begin{cases} 2 = c_1e^0 + c_2e^0 - \frac{1}{2}e^0 + 2\sin 0 - \cos 0 \\ 4 = 3c_1e^0 - c_2e^0 - \frac{1}{2}e^0 + 2\cos 0 + \sin 0 \end{cases}$$

These equations simplify at once to the following:

$$\begin{cases} c_1 + c_2 = \frac{7}{2} \\ 3c_1 - c_2 = \frac{5}{2} \end{cases}$$

From these two equations you obtain

$$\begin{cases} c_1 = \frac{3}{2} \\ c_2 = 2. \end{cases}$$

Substituting these values for $\mathbf{c}_1$ and $\mathbf{c}_2$ into the general solution you obtain the unique solution of the given initial-value problem in the form

$$y = \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

3.2.7 Variation of Paramenters

The Method

While the process of carrying out the method of undetermined coefficients is quite straightforward (involving only techniques of college algebra and differentiation, the method applies in general to a rather small class of problems. For example, it does not apply to the apparently simple eqution

$$y'' + y = \tan x$$

You thus need a method of finding a particular integral which applies in all cases of variable coefficients) in which the complementary function is known. Such a method is called *method of variation of parameters*, which you would now consider.

This method shall be developed with the general second order differential equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x) \qquad (19)$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad (20)$$

Then the complementary function of equation (19) is

$$c_1y_1 + c_2y_2$$

where c_1 and c_2 are arbitray constants. The procedure in the method of variation of parameters is to replace the arbitray constants c_1 and c_2 in the complementary function by respective functions v_1 and v_2 which will be determined so that the resulting function

$$v_1y_1 + v_2y_2 \qquad (21)$$

will be a particular integral of equation (19) (hence the name, *variation* of parameters). *conditions* that (21) be a solution of (19). Since you have two functions but only one conditions on them, you are thus free to impose a second condition, provided this second condition does not violate the first one. You shall see when and how to impose this additional condition as you proceed.

You thus assume a solution of the form (21) and write

$$y_p = v_1y_1 + v_2y_2 \qquad (22)$$

Differentiating (22) you would have

$$y'_p = v_1y'_1 + v_2y'_2 + v_iy_1 + v'_2y_2 \qquad (23)$$

At this point, you should impose the condition;

$$v'_1y_1 + v'_2y_2 = 0 \qquad (24)$$

With this condition imposed, (33) reduces to

$$= v_1y'_1 + v_2y'_2. \qquad (25)$$

Now differentiating (25), you should obtain

$$y''_p = v_1y''_1 + v_2y''_2 + v'_1y'_1 + v'_2y'_2. \qquad (26)$$

Now impose the basic condition that (22) be a solution of equation (19). Thus you substitute (22), (25), and (26) for y, y', and y'', respectively, in equation (19) and obtain the identity

$$a_0[v_1y''_i + v_2y'_2 + v'_1y'_1 + v'_2y'_2] + a_1[v_1y'_1 + v_2y'_2] + a_2[v'_1y'_1 + v'_2y'_2] = b$$

This can be written as

$$v_1[a_0y''_1 + a_1y'_1 + a_2y_1] + v_2[a_0y''_2 + a_1y_2a_2y_2] + a_0[v'_1y'_1 + v'_2y'_2] = b \qquad (27)$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (20), the expressions in the first two brackets in (27) are identically zero. This leaves merely

$$v'_1y'_1 + v'_2v'_2\frac{b}{a_0} \qquad (28)$$

This is actually what the basic conditioin demands. Thus the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

$$\begin{cases} y_1v'_1 + y_2v'_2 = 0 \\ y'_1v'_1 + y'_2v'_2 = \dfrac{b}{a_0} \end{cases} \qquad (29)$$

is satisfied. The determinant of coefficients of this system is precisely

$$w(y_1, y_2 = \begin{vmatrix} y'_1 & y'_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous differential equations (20) you know that $W(y_1, y_2) \neq 0$. Hence the system (29) has a unique solution. Actually solving the sytem you obtain

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{b}{a_0} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{by_2}{a_0 W(y_1, y_2)}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{b}{a_0} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{by_1}{a_0 W(y_1, y_2)}$$

Thus you obtain the functions v_1 and v_2 given by

$$v_1(x) = -\int^x \frac{b(t)y_2(t)dt}{a_0(t)W[y_1(t), y_2(t)]|}$$

$$v_2(x) = \int^x \frac{b(t)y_1(t)dt}{a_0(t)W[y_1(t), y_2(t)]} \qquad (30)$$

Therefore a particular integral of equation (29) is

$$y_p = v_1 y_1 + v_2 y_2,$$

where v_1 and v_2 are defined by (30)

Examples

Example 3.29

$$y'' + y = \tan x \qquad (31)$$

The complementary function is

$$y_c = c_1 \sin x + c_2 \cos x.$$

Assume

$$y_p = v_1 \sin x + v_2 \cos x, \qquad (32)$$

where the functions v_1 and v_2 will be determined such that this is a particular integral of the differential equation (31). Then using the formulas above, you obtain

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x}{-1} = \sin x$$

$$v_1' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\sin x \tan x}{-1} = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Integrating you find:

$$\begin{cases} v_1 = -\cos x + c_3 \\ v_2 = \sin x - \ln|\sec x + \tan x| + c_4 \end{cases} \tag{33}$$

Substituting (33) into (32) you have

$$\begin{aligned} y_p &= [-\cos x + c3] \sin x + [\sin x - \ln \sec x + \tan x + c_4] \cos x \\ &= -\sin x \cos x + c_3 \sin x + \sin x \cos x - \ln \sec x + \tan x (\cos x) + c_4 \cos x \\ &= c_3 \sin x + c_4 \cos x - (\cos x)[\ln \sec x + \tan x] \end{aligned}$$

Since a particular integral is a solution free of arbitrary constants, you may assign any particular values A and B to c_3 and c_4, respectively, and the result will be the particular integral

$A \sin x + B \cos x - (\cos x)[\ln \sec x + \tan x]$

Thus $y = y_c + y_p$ becomes

$y = c_1 \sin x + c_2 \cos x + A \sin x + B \cos x - (\cos x) \ln | \sec x + \tan x|$

which you may write as

$y = C_1 \sin x + C_2 \cos x - \cos x - (\cos x) \ln | \sec x + \tan x|$,

where $C_1 = c_1 + A$, $C_2 = c_2 + B$.

Thus you see that you might as youll have chosen the constants c_3 and c_4 both equal to 0 in (33). For essentially the same result, $y = c_1 \sin x + c_2 \cos x - (\cos x) \ln \sec x + \tan x$, would have been obtained. This is the general solution of the differential equation (31).

The method of variation of parameters extends to higher order linear equations. The proof of the validity of this method for the general nth-order equation will not be given in this work; you can find it in advanced text of ODE.

4.0 CONCLUSION

In this unit, you have studied the basic theory of linear differential equations and have used the explicit methods described in this unit to obtain the general and particular solutions of differential equations with constant coefficients.

5.0 SUMMARY

Having gone through this unit, you now know;

- the basic theory of nth order linear ordinary differential equations.
- how to obtain the solutions to a given ODE using the explicit method described in this unit.

6.0 TUTOR-MARKED ASSIGNMENTS

Exercise 6.1

1. Theorem 3.1 applies to one of the following problems but not to the other. Determine to which the problems applies and state precisely the conclusion which can be drawn in this case. Explain why the theorem does not apply to the remaining problem.

(a) $$\begin{cases} y'' + 5y' + 6y = e^x \\ y(0) = 5 \\ y'(0) = 7. \end{cases}$$

(b) $$\begin{cases} y'' + 5y' + 6y = e^x \\ y(0) = 5 \\ y'(1) = 7. \end{cases}$$

2. Ansyour orally: What is the solution of the following initial-value problem? Why?

$$\begin{cases} y'' + xy' + x^2y = 0 \\ y(1) = 0 \\ y'(1) = 0. \end{cases}$$

3. Consider the differential equation

$$y'' - 5y' + 6y = 0.$$

(a) Show that e^{2x} and e^{3x} are linearly independent solutions of this equation on the interval $-\infty < x < \infty$.

(b) Write the general solution of the given equation.

(c) Find the solution which satisfies the condition $y(0) = 2$, $y'(0) = 3$. Explain why this solution is unique.

4. Consider the differential equation

$$x^2y^{ll} + xy^l - 4y = 0$$

(a) Show that $\mathbf{x}^2$ and $_{x21}$ are linearly independent solutions of this equations of this equation on the interval $0 < x < \infty$.
(b) Write the general solution of the given equation.
(c) Find the solution which satisfies the conditions $y(2) = 3$, $y'(2) = -1$. Over what interval is this solution defined?

5. The functions e^x and e^{4x} are both solutions of the differential equation

$$y'' - 5y' + 4y = 0.$$

(a) Show that these solutions are linearly independent on the interval $-\infty < x < \infty$.
(b) What theorem enables you to conclude at once that $2e^x - 3e^{4x}$ is also a solution of the given differential equation?
(c) Show that the solution of part (b) and the solution e^x are also linearly independent on $-\infty < x < \infty$.

6. Given that $e{-}x$, e^{3x} and $\mathbf{e}^{4x}$ are all solutions of

$$y''' - 6y'' + 5y' + 12y = 0,$$

show that they are linearly independent on the interval $-\infty < x < \infty$ and write the general solution.

Exercise 6.2

Find the general solution of each of the differential equations in the following exercises.

1. $y'' - 5y' + 6y = 0.$
2. $y'' - 2y' - 3y = 0.$
3. $4y'' - 12y' + 5y = 0.$
4. $3y'' - 14y' - 5y = 0.$
5. $y''' - 3y'' - y' + 3y = 0$
6. $y''' - 6y''+5y' + 12y = 0$
7. $y'' - 8y' + 16y = 0$
8. $4y'' + 4y' + y = 0$
9. $y'' - 4y' + 13y = 0$
10. $y'' + 6y' + 25y = 0$
11. $y'' + 9y = 0$
12. $4y'' + y = 0$

13. $y''' - 5y'' + 7y' - 3y = 0$

14. $4y''' + 4y'' - 7y' + 2y = 0$

15. $y''' - 6y'' + 12y' - 8y = 0$

16. $y''' + 4y'' + 5y' + 6y = 0$

17. $y''' - y'' + y' - y = 0$

18. $y^{(iv)} + 8y'' + 16y = 0$

19. $y^{(v)} - 2y^{(iv)} + y''' = 0$

20. $y^{(iv)} - y''' - 3y'' + y' + 2y = 0$

21. $y^{(iv)} - 3y''' - 2y'' + 2y' + 12y = 0$

22. $y^{(iv)} + 6y''' + 15y'' + 20y' + 12y = 0$

23. $y^{(iv)} + y = 0.$

24. $y^{(v)} = 0$

Solve the initial-value problems in the exercises that follow.

25. $\begin{cases} y'' - y' - 12y = 0 \\ y(0) = 3 \\ y'(0) = 5 \end{cases}$

26. $\begin{cases} 9y'' - 6y' + y = 0 \\ y(0) = 3 \\ y'(0) = -1 \end{cases}$

27. $\begin{cases} y'' - 4y' + 29y = 0 \\ y(0) = 0 \\ y'(0) = 5 \end{cases}$

28. $$\begin{cases} 4y'' + 4y' + 37y = 0 \\ y(0) = 2 \\ y'(0) = -4 \end{cases}$$

29. $$\begin{cases} y''' - 6y'' + 11y' - 6y = 0 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 2. \end{cases}$$

30. $$\begin{cases} y''' - 2y'' + 4y' - 8y = 0 \\ y(0) = 2 \\ y'(0) = 0 \\ y''(0) = 0. \end{cases}$$

31. The roots of the auxiliary equation, corresponding to a certain 10th-order homogeneous linear differential equation with constant coefficients, are
$4, 4, 4, 4, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i$

Write the general solution.

32. Given that sin *x* is a solution of

$$y^{(iv)} + 2y''' + 6y'' + 2y' + 5y = 0,$$

find the general solution.

Exercise 6.3

Find the general solution of each of the differential equations following

1. $y''' - 3y' + 2y = 4x^2$
2. $y'' - 2y' - 8y = 4x^2 - 21e^{-3x}$.
3. $y'' + 2y' + 5y = 6 \sin 2x + 7 \cos 2x$
4. $y''' + 2y'' - 3y' - 10y = 8e^{-2x}$.
5. $y''' + y'' + 3y' - 5y = 5 \sin 2x + 10x^2 - 3x + 7$.
6. $y^{(iv)}) - 3y''' + 2y'' = 3e^{-x} + 6e^{2x} - 6x$
7. $y^{(iv)} - 5y''' + 7y'' - 5y' + 6y = 5 \sin x - 12 \sin 2x$

Solve the initial-value problem in Exercises 18 through 21.

8. $$\begin{cases} y'' - 4y' + 3y = 9x^2 + 4. \\ y(0) = 6 \\ y'(0) = 8. \end{cases}$$

9. $$\begin{cases} y'' + 4y' + 13y = 5\sin 2x. \\ y(0) = 1 \\ y'(0) = -2. \end{cases}$$

10. $$\begin{cases} y'' + y = 3x^2 - 4\sin x. \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

11. $$\begin{cases} y''' - 4y'' + y' + 6y = 3xe^x + 2e^x - \sin x. \\ y(0) = \frac{33}{40} \\ y'(0) = 0 \\ y''(0) = 0 \end{cases}$$