MODULE 3

UNIT 1 STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

CONTENTS

1.0 INTRODUCTION

In this unit, you shall be introduced to a special kind of boundary value problem known as a Sturm-Liouville Problem. Your study of this type of problem will introduce you to several important concepts including characteristic function, orthogonality, and Fourier series (which are beyond the scope of this book). These concepts are frequently employed in the applications of differential equations to physics and engineering.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define and give examples of Sturm-Liouville problems; and
- know the meaning of characteristic values and characteristic functions.

3.0 MAIN CONTENT

3.1 Sturm-Liouville Problems

3.1.1 Definition and Examples

The first concern in this unit is a study of the special type of two-point boundary value problem given in the following definition:

Definition 3.1 Consider a boundary value problem which consists of

1. a second-order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \qquad (1)$$

where p, q and r are real functions such that p has continuous derivative, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$; and λ is a parameter independent of x; and

2. two supplementary conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0, \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned} \qquad (2)$$

where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero.

This type of boundary-value problem is called a *Sturm-Liouville Problem (or Sturm-Liouville System).*

Two important special cases are those in which the supplementary conditions (2) are either of the form

$$y'(a) = 0, \quad y(b) = 0 \qquad (3)$$

or of the form

$$y'(a) = 0, \quad y'(b) = 0. \qquad (4)$$

Example 3.1 The boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad (5)$$

$$y(0) = y(\pi) = 0 \qquad (6)$$

is a Sturm-Liouville problem. The differential equation (5) may be written

$$\frac{d}{dx}\left[1 \cdot \frac{dy}{dx}\right] + [0 + \lambda \cdot 1]y = 0$$

and hence is of the form (1), where p(x) = 1, q(x) = 0, and r(x) = 1. The supplementary conditions (6) are of the special form (3) of (2).

Example 3.2 The boundary-value problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + [12x^2 + \lambda x^3]y = 0 \qquad (7)$$

$$3y(1) + 4y'(1) = 0,$$

$$5y(2) - 3y'(2) = 0 \qquad (8)$$

is a Sturm-Liouville Problem. The differential equation (7) is of the form of (1), where $p(x) = x$, $q(x) = 2x^2$, and $r(x) = x^3$. The conditions (8) are of the form (2), where a = 1, b = 2, $A_1 = 3$, $A_2 = 4$, $B_1 = 5$, and $B_2 = -3$.

You are now due to be introduced to what is involved in solving a Sturm-Liouville Problem. You must find a function f which satisfies both the differential equation (1) and the two supplementary conditions (2). Clearly one solution of any problem of this type is the trivial solution ϕ such that $\phi(x) = 0$ for all values of x. Equally clear is the fact that this trivial solution is not very useful. You should therefore focus you attention on the search for *nontrivial* solutions of the problem. That is, you should attempt to find functions, *not identically zero*, which satisfies both the differential equation (1) and the two conditions (2). You shall see that the existence of the nontrivial solutions depends upon the value of the parameter λ in the differential equation (1). To illustrate this, you have to return to the Sturm-Liouville Problem of Example (1) and attempt to find nontrivial solutions

Example 3.3 Find nontrivial solutions of the Sturm-Liouville Problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad (5)$$

$$y(0) = 0, y(\pi) = 0. \qquad (6)$$

Solution

You would need to consider three cases according as λ = 0, λ < 0 and λ > 0. In each case you should first find the general solution of the differential equation (5). You shall then attempt to determine the two arbitrary constants in this solution so that the supplementary conditions (6) will also be satisfied.

Case I: (λ = 0). In this case the differential equation (5) reduced at once to

$$\frac{d^2y}{dx^2}$$

and so the general solution is

$$y = c_1 + c_2x. \qquad (9)$$

You can now apply conditions (6) to the solution (9). Applying the first condition $y(0) = 0$, you obtain $c_1 = 0$. Applying the second condition $y(\pi) = 0$, you find that $c1 + c_2\pi = 0$. Hence, since $c_1 = 0$, you must have also that $c_2 = 0$. Thus in order for the solution (9) to satisfy the conditions (6), you must have $c_1 = c_2 = 0$. But then the solution (9) becomes the solution y such that $y(x) = 0$ for all values of x. Thus if the parameter $\lambda = 0$, the only solution of the given problem is the trivial solution.

Case II: $(\lambda < 0)$. The auxiliary equation of the differential equation (5) is $m^2 + \lambda = 0$ and the roots $\pm\sqrt{-\lambda}$. Since in this case $\lambda < 0$, these roots are real and unequal. Denoting $\sqrt{-\lambda}$ by α, you can see that for $\lambda < 0$ the general solution of (5) is of the form

$$y = c1e^{\alpha x} + c2e^{-\alpha x} \qquad (10)$$

Applying the conditions (6) to the solution (10) starting with the first, gives you

$$c_1 + c_2 = 0 \qquad (11)$$

Applying the second condition $y(\pi) = 0$, you find that

$$c_1e^{\alpha\pi} + c_2e^{-\alpha\pi} = 0 \qquad (12)$$

You must thus determine c_1 and c_2 such that the system consisting of (11) and (12) is satisfied. Thus in order for the solution (10) to satisfy the conditions (6), the constants c_1 and c_2 must satisfy the system (11) and (12). Obviously $c_1 = c_2 = 0$ is a solution of this system; but these values of c_1 and c_2 would only give the trivial solution of this given problem. You must therefore seek nonzero values of c_1 and c_2 which satisfy (11) and (12). By some theorems of ODE, this system has nonzero solutions only if the determinant of the coefficient is zero. Therefore you must have

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{\alpha\pi} \end{vmatrix} = 0.$$

But this implies that $e^{\alpha\pi} = e^{-\alpha\pi}$ and hence that α. Thus in order for a nontrivial function of the form (10) to satisfy the conditions (6) you must have $\alpha = 0$. Since $\alpha = \sqrt{-\lambda}$, you must have $\lambda = 0$. But $\lambda < 0$ in this case. Thus there are no nontrivial solutions of the given problem in the case $\lambda < 0$.

Case III: $(\lambda > 0)$. Since $\lambda > 0$, here, the roots $\pm\sqrt{-\lambda}$of the auxiliary equation of (5) are the conjugate complex numbers $\pm \sqrt{\lambda_i}$. Thus in this case the general solution of (5) is of the form

$$y = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x. \qquad (13)$$

Applying now the conditions (6) to this general solution, beginning from the first condition y(0)=0, you obtain

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

and hence c_2. Applying the second condition $y(\pi) = 0$, you would find that

$$c_1 \sin \sqrt{\lambda}\pi + c_2 \cos \sqrt{\lambda}\pi = 0$$

Since $c_2 = 0$, this reduces at once to

$$c_1 \sin \sqrt{\lambda}\pi = 0 \qquad (14)$$

You must therefore satisfy (14). At first glance it appears that you can do this in either of two ways:

you can set $c_1 = 0$ or you can set $\sin \sqrt{\lambda}\pi = 0$. However, if you set $c_1 = 0$, then (since $c_2 = 0$ also) the solution (13) reduces immediately to the unwanted trivial solution. Thus to obtain a *nontrivial* solution you cannot set $c_1 = 0$ but rather you must set

$$\sin \sqrt{\lambda}\pi = 0 \qquad (15)$$

If $k > 0$, then $\sin k\pi = 0$ only if k is a positive integer $n = 1, 2, 3, ...$ Thus in order that the differential equation (5) have a nontrivial solution of the form (13) satisfying the conditions (6), you must have

$$\lambda = n^2, \text{ where } \quad n = 1, 2, 3, ... \qquad (16)$$

In other words, the parameter λ in (5) must be a member of the infinite sequence

$$1, 4, 9, 16, ..., n^2, ...$$

You can now summarize you result as follows. If $\lambda < 0$ the Sturm-Liouville problem consisting of (5) and (6) does *not* have a nontrivial solution; if $\lambda > 0$, a nontrivial solution can exist only if λ is one of the values given by (16). You now note that if λ is one of the values (16), then the problem does have nontrivial solutions. Indeed, from (13) you see that nontrivial solutions corresponding to $\lambda = n^2 (n = 1, 2, 3, ...)$ are given by

$$y = c_n \sin nx (n = 1, 2, 3, ...), \qquad (17)$$

where $c_n (n = 1, 2, 3, ...)$ is an arbitrary nonzero constant. That is, the functions defined by $c_1 \sin x$, $c_2 \sin 2x$, $c_3 \sin 3x$, ..., where $c_1, c_2, c_3, ...$ are arbitrary nonzero constants, are non trivial solutions of the given problem.

3.1.2 Characteristic Values and Characteristic Functions

Example 12.3 shows that the existence of nontrivial solution of a Sturm-Liouville Problem does indeed depend upon the value of the parameter λ in the differential equation of the problem. Those values of the parameter for which nontrivial solutions do exist, as well as the corresponding nontrivial solutions themselves, are singled out by the following definition.

Definition 3.2 Consider the Sturm-Liouville Problem consisting of the differential equation (1) and the supplementary conditions (2). The values of the parameter λ in (1) for which there exist nontrivial solutions, of the problem are called the ***Characteristic values*** of the problem. The corresponding nontrivial solutions themselves are called the ***characteristic functions*** of the problem.

Example 3.4 Consider again the Sturm-Liouville Problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad (5)$$

$$y(0) = 0,\ y(\pi) = 0. \qquad (6)$$

In Example 12.3 you found that the values of λ in (5) for which there exist nontrivial solutions of this problem are the values

$$\lambda = n^2, \text{ where } n = 1, 2, 3, \ldots \qquad (16)$$

These then are the characteristics values of the problem under consideration. The characteristic function of the problem at the corresponding nontrivial solutions

$$y = c_n \sin nx \ (n = 1, 2, 3, \ldots) \qquad (17)$$

where $c_n (n = 1, 2, 3, \ldots)$ is an arbitrary nonzero constant.

Example 3.5 Find the characteristic values and the characteristic functions of the Sturm-Liouville Problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \qquad (18)$$

$$y'(1) = 0,\ y'(e^{2\pi}) = 0 \qquad (19)$$

where it is assumed that the parameter λ in (18) is nonnegative.

Solution

Consider separately the cases λ = 0 and λ > 0. If λ = 0, the differential equation (18) reduces to

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = 0,$$

The general solution of this differential equation is

$$y = C \ln|x| + C_0,$$

where C and C0 are arbitrary constants. If you apply the conditions (19) to this general solution, you will find that both of them require that C = 0 but neither of them imposes any arbitrary constant. These are nontrivial solutions for all choices of $C_0 \neq 0$. Thus λ = 0 is a characteristic value and the corresponding characteristic functions are given by y = C_0, where C_0 is an arbitrary nonzero constant.

If λ > 0, you see that for x ≠ 0 this equation is equivalent to the Cauchy-Euler Equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0 \qquad (20)$$

Letting x = e^t, then equation (20) transforms into

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad (21)$$

Since λ > 0, the general solution of (21) is of the form

$$y = c_1 \sin\sqrt{\lambda}t + c_2 \cos\sqrt{\lambda}t$$

Thus for λ > 0 and x > 0 the general solution of (18) may be written

$$y = c_1 \sin(\sqrt{\lambda}\ln x) + c_2 \cos(\sqrt{\lambda}x). \qquad (22)$$

Differentiating (22) and Applying the supplementary conditions (19) gives you that

$$\frac{dy}{dx} = \frac{c_1\sqrt{\lambda}}{x}\cos(\sqrt{\lambda}x) - \frac{c_2\sqrt{\lambda}}{x}\sin(\sqrt{\lambda}\ln x) \qquad (23)$$

for x > 0. Applying the first condition $y'(0) = 0$ of (19) to (23), you would have

$$c_1\sqrt{\lambda}\cos(\sqrt{\lambda}x) - c_2\sqrt{\lambda}\sin(\sqrt{\lambda}\ln 1) = 0$$

or simply $c_1 \sqrt{\lambda} = 0$. Thus you must have

$$c_1 = 0.$$

Applying the second condition $y'(e^{2\pi}) = 0$ of (19) to (23), you obtain

$$c_1\sqrt{\lambda}e^{-2\pi} \cos(\sqrt{\lambda}\ln e^{2\pi}) - c_2\sqrt{\lambda}\, e^{-2\pi}\sin(\sqrt{\lambda}\ln e^{2\pi}) = 0$$

Since $c_1 = 0$ by (24), and $\ln e^{2\pi} = 2\pi$, this reduces at once to

$$c_2 \sqrt{\lambda}e^{-2\pi} \sin(2\pi\sqrt{\lambda}) = 0.$$

Since $c_1 = 0$, the choice c2 = 0 would lead to the trivial solution. Thus you must have $\sin(2\pi\sqrt{\lambda} = 0$ and hence $2\pi\sqrt{\lambda} = 0\ n\pi$, where $n = 1, 2, 3, ...$ Thus in order to satisfy the second condition (19) nontrivially you must have

$$\lambda = \frac{n^2}{4}(n = 1, 2, 3, ...) \qquad (24)$$

Corresponding to these values of λ you obtain for x > 0 the nontrivial solutions

$$y = C_n \cos\left(\frac{n \ln x}{2}\right), \quad (n = 1, 2, 3, ...) \qquad (25)$$

where $C_n(n = 1, 2, 3, ...)$ are arbitrary nonzero constants.

Thus the values

$$\lambda = 0, \frac{1}{4}, 1, \frac{9}{4}, 4, \frac{25}{4}, ..., \frac{n^2}{4}, ...,$$

given by (25) for $n \geq 0$, are the characteristic values of the given problem. The functions

$$C_0, C_1 \cos\left(\frac{\ln x}{2}\right), C_2 \cos(\ln x), C_3 \cos\left(\frac{3 \ln x}{2}\right), ...,$$

given by (26) for $n \geq 0$, where $C_0, C_1, C_2, C_3, ...$ are arbitrary nonzero constants, are the corresponding characteristic functions.

For each of the Sturm-Liouville Problems of Examples (3.3) and (3.5), you must have found an infinite number of characteristic values. You could observe that in each of these problems the infinite set of characteristic values thus found can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdot \cdot \cdot$$

such that $\lambda_n \to +\infty$ as n $\to \infty$. For example, the characteristic values of the problem in example (3.3) can be arranged in the monotonic increasing sequence

$$1 < 4 < 9 < 16 < \cdots \quad (26)$$

such that $\lambda_n \to +\infty$ as n $\to +\infty$. You also note that in each problem there is a one-parameter of characteristic functions corresponding to the same characteristic value are merely nonzero constant multiples of each other. For example, in the problem of example 3.3, the one-parameter family of characteristic functions corresponding to the characteristic value n^2 is $c_n \sin nx$, where $c_{n \neq} 0$ is the parameter.

You might now inquire whether or not all Sturm-Liouville Problems of the type under consideration possess characteristic values and characteristic functions having the properties noted in the preceding paragraph. You can ansyour in the affirmative by stating the following important theorem.

Theorem 3.1 *Consider the Sturm-Liouville Problem consisting of*

(i) *the differential equation*

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0, \quad (27)$$

where p, q *and* r *are real functions such that* p *has a continuous derivative,* q *and* r *are continuous and* $p(x) > 0$ *and* $r(x) > 0$ *for all* x *on the real interval* $a \leq x \leq b$; *and* λ *is a parameter independent of* x; *and*

(ii) *the conditions*

$$A_1 y(a) + A_2 y'(a) = 0 \quad (28)$$

$$B_1 y(b) + B_2 y'(b) = 0$$

where A_1, A_2, B_1 *and* B_2 *are real constants such that* A_1 *and* A_2 *are not both zero and* B_1 *and* B_2 *are not both zero.*

Then

(i) There exists an infinite number of characteristic values λ_n can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that $\lambda_n \to +\infty$ *as* n $\to +\infty$.

(ii) corresponding to each characteristic value λ_n there exists a one-parameter family of characteristic functions ϕ_n. Each of these characteristic functions is defined on a < x < b, and any two characteristic functions corresponding to the same characteristic value are nonzero constant multiples of each other.

(iii) Each characteristic function ϕ_n corresponding to the characteristic value λ_n (n = 1, 2, 3, ...) has exactly (n − 1) zeros in the open interval a < **x** < b.

Example 3.6 Consider again the Sturm-Liouville Problem of Examples 3.3 and 3.4

$$\frac{d^2y}{dx^2} + \lambda y = 0 \tag{29}$$

$$y(0) = 0,\ y(\pi) = 0. \tag{30}$$

You have already noted the validity of conclusions (i) and (ii) of theorem 3.1 for this problem. The infinite number of characteristic values $\lambda_n = n^2 (n = 1, 2, 3, ...)$ can be arranged in the unbounded monotonic increasing sequence indicated by (27); and the characteristic functions $c_n \sin nx (c_n \neq 0)$, corresponding to $\lambda_n = n^2$ possess the properties stated.

Conclusion (iii) is illustrated by showing that each function $c_n \sin nx$ corresponding to $\lambda_n = n^2$ has exactly (n − 1) zeros in the open interval $0 < x < \pi$. You know that sin nx = 0 if and only if nx = $k\pi$, where k is an integer. Thus the zeros of $c_n \sin nx$ are given by

$$x = \frac{k\pi}{n},\ (n = 0, \pm1, \pm2, ...) \tag{31}$$

The zeros (28) which lie in the open interval $0 \leq x \leq \pi$ are precisely those for which k = 1, 2, 3, ..., n − 1. Thus, just as conclusion (iii) asserts, each characteristic functions $c_n \sin nx$ has precisely (n − 1) zeros in the open interval $0 < x < \pi$.

4.0 CONCLUSION

In this unit, you have studied the two point Sturm-Liouville problem. You saw some examples, and learnt how to find the solutions to these problems by obtaining the Characteristic values and the Characteristic functions.

5.0 SUMMARY

Having gone through this unit, you now know:

- what is meant by a two point Sturm-Liouville boundary value problem.
- the meaning of a characteristic value and the corresponding characteristic function of a Sturm-Liouville problem.

6.0 TUTOR MARKED ASSIGNMENT

Exercise 6.1

Find the characteristic values and characteristic functions of each of the following Sturm-Liouville Problems:

1. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) = 0, \\ y\left(\dfrac{\pi}{2}\right) = 0. \end{cases}$$

2. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) = 0, \\ y'(\pi) = 0. \end{cases}$$

3. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) = 0, \\ y(L) = 0, \quad where \quad L > 0 \end{cases}$$

4. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y'(0) = 0, \\ y'(L) = 0, \quad where \quad L > 0. \end{cases}$$

5. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) = 0, \\ y^{(}\pi) - y'(\pi) = 0. \end{cases}$$

6. $$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) - y'(0) = 0, \\ y^{(}\pi) - y'(\pi) = 0. \end{cases}$$

7. $$\begin{cases} \frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0 \\ y(1) = 0, \\ y(e^{\pi}) = 0. \end{cases}$$

8. $$\begin{cases} \frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0 \\ y(1) = 0, \\ y'(e^{\pi}) = 0. \end{cases}$$

9. $$\begin{cases} \frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] + \frac{\lambda}{x^2+1}y = 0 \\ y(0) = 0, \\ y(1) = 0. \end{cases}$$

[Hint. Let $x = \tan t$.]

0. $$\begin{cases} \frac{d}{dx}\left[\frac{1}{3x^2+1}\frac{dy}{dx}\right] + +\lambda(3x^2+1)y = 0 \\ y(0) = 0, \\ y(\pi) = 0. \end{cases}$$

[Hint: Let $t = x^3 + x$]

UNIT 2 NONLINEAR EQUATIONS

CONTENTS

1.0 INTRODUCTION

The mathematical formulation of numerous physical problems results in differential equations where are actually nonlinear. In many cases it is possible to replace such a non-linear equation by a related linear equation which approximates the actual nonlinear equations closely enough to give useful results. However, such a "linearization" is not always feasible; and when it is not, the original nonlinear equation itself must be considered. While the general theory and methods of linear equations are highly developed, very little of a general character is known about nonlinear equations. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation. In this unit, you shall be introduced briefly to certain of these methods.

2.0 OBJECTIVES

At the end of this unit, you should be able to;

- define phase plane, paths and critical points;
- describe types of critical points;
- define and describe stability of a critical point;
- determine the critical points of linear system;
- describe the nature of the critical point (0, 0);
- describe the stability of the critical point (0, 0); and

- linearize a nonlinear differential equation and describe the nature and stability of the critical point (0, 0).

3.0 MAIN CONTENT

3.1 Phase Plane, Paths, and Critical Points

3.1.1 Basic Concepts and Definitions

For simplicity, you should be concerned with second-order nonlinear differential equations of the form

$$\ddot{x} = F(x, \dot{x}) \qquad (1)$$

where x=x(t). As a specific example of such equation you have the important van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad (2)$$

where μ is a positive constant. For the time being, you could observe that you can put (2) in form (1), where

$$F(x, \dot{x}) = -\mu(x^2 - 1)\dot{x} - x$$

Suppose that the differential equation (1) describes a certain dynamical system having on degree of freedom. The state of this system at time t is determined by the values of x (position) and $\dot{x}$ (velocity). The plane of the varibles x and $\dot{x}$ is called a *phase plane.*

If you let y = $\dot{x}$, you can replace the second-order equation (1) by the equivalent system

$$\begin{cases} \dot{x} = y \\ \dot{y} = F(x, y) \end{cases} \qquad (3)$$

You can determine information about the equation (1) from a study of the system (1). In particular you should be interested in the configuration formed by the curves which the solutions of (3) define. You should regard t as a parameter so that these curves will appear in the *xy* plane. Since $y = \dot{x} = dx/dt$, this *xy* plane is simply the *x*, *dx/dt*- phase plane mentioned in the preceeding paragraph.

More generally, you should consider the system of the form

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \qquad (4)$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t appears only in the differentials dt of the left members and not explicitly in the functions P and Q on the right, is called an *autonomous system*. You shall now proceed to study the configurations formed in the xy-phase plane by the curves which are defined by the solutions of (4).

From the existence theorem, it follows that given any number t_0 and any pair (x_0, y_0) of real numbers, there exists a unique solution

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad (5)$$

of the system (5) such that

$$\begin{cases} f(t_0) = x_0 \\ g(t_0) = y_0 \end{cases}$$

If f and g are not both constant functions, then (5) defines a curve in the xy plane which you shall call a *path* of the system (4).

If the ordered pair of functions defined by (5) is a solution of (4) and t_1 is any real number, then it is easy to see that the ordered pair of functions defined by

$$\begin{cases} x = f(t - t_1) \\ y = g(t - t_1) \end{cases} \qquad (6)$$

is also a solution of (4). Assuming that f and g in (5) are not both constant functions and that $t_1 \neq 0$, the solutions defined by (5) and (6) are two *different solutions* are simply different parametrizations of the *same path*. You can observe that the terms *solution* and *path* are not synonymous. On the one hand, a *solution* of (4) is an ordered *pair of functions* (f, g) such that $x = f(t)$, $y = g(t)$ simultaneously satisfy the two equations of the system (4) identically; on the other hand, a *path* of (4) is a *curve* in the xy-phase plane, which may be defined parametrically by more than one solution of (4).

Through any point of the xy-phase plane there passes at most one path of (4). Let C be a path of (4) and consider the totality of different solutions of (4) which define this path C parametrically. For each of these defining solutions, C is traced out in the *same direction* as the parameter t increases. Thus with each path C there is associated a definite direction, the direction of increase of the parameter $t$in the various possible parametric representations of C by the corresponding solutions of the system. In your figures, you shall use arrows to indicate this direction associated with a path.

Eliminating t betweeen the two equations of the system (4), you obtain the equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \qquad (7)$$

This equation gives the slope of the tangent to the path of (4) passing through the point (x, y), provided the functions P and Q are not both zero at this point. The general solution of (7) thus provides the one-parameter family of paths of (4). However, the description (7) does not indicate the directions associated with these paths.

At a point (x_0, y_0) at which both P and Q are zero, the slope of the tangent to the path, as defined by (7), is indeterminate. Such points are singled out in 136 the following definition.

Definition 3.1 Given the autonomous system

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \qquad (4)$$

a point (x_0, y_0) at which both

$P(x_0, y_0) = 0$ *and* $Q(x_0, y_0) = 0$

is called a *critical point* of (4).

Example 3.1 Consider the linear autonomous system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \qquad (8)$$

Solving this, using the methods developed in unit 4, you would find that the general solution of the system may be written

$$\begin{cases} x = c_1 \sin t - c_2 \cos t \\ y = c_1 \cos t - c_2 \sin t \end{cases}$$

where c1 and c2 are arbitrary constants. The solution satisfying the conditions x(0) = 0, y(0) = 1 is readily found to be

$$\begin{cases} x = \sin t \\ y = \cos t \end{cases} \quad (9)$$

This solution defines a path C1 in the *xy* plane. The solution satisfying the conditions $x(0) = -1$, $y(0) = 0$ is

$$\begin{cases} x = \sin(t - \pi/2 \\ y = \cos(t - \pi/2 \end{cases} \quad (10)$$

The solution (10) is different from the solution (9), but (10) also defines the same path C_1. That is, the ordered pairs of functions defined by (9) and (10) are two different solutions of (8) which are different parametrizations of the *same path* C1. Eliminating t from either (9) or (10) you obtain the equation $x^2 + y^2 = 1$ of the path C_1 in the *xy* phase plane. Thus the path C_1 is the circle with center at (0, 0) and radius 1. From either (9) or (10) you see that the direction associated with C_1 is the *clockwise* direction.

Eliminating t between the equations of the system (8) you obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \quad (11)$$

which gives the slope of the tangent to the path of (8) passing through the point (x, y), provided $(x, y) \neq (0, 0)$.

The general solution

$$x^2 + y^2 = c^2$$

of equation (11) gives the one-parameter family of paths in the *xy* phase plane. Several of these are shown in figure 1. The path C_1 referred to above is of course that for which c = 1

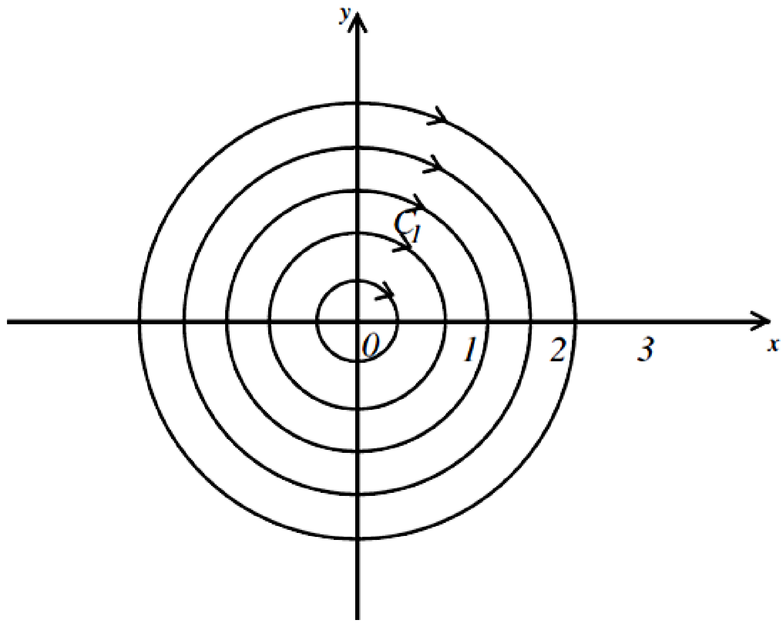

Figure 1:

Looking back at the system (8), you see that $P(x, y) = y$ and $Q(x, y) = -x$. Therefore the only critical point of the system is the origin (0, 0). Given any real number t_0, the solution $x = f(t), y = g(t)$ such that $f(t_0) = g(t_0) = 0$ is simply for all t.

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

You can also interpret the autonomous system (4) as defining a *velocity vector field V*, where

V (x, y) = [P(x, y), Q(x, y)]

The x component of this velocity vector at a point (x, y) is given by $P(x, y)$, and the y component there is given by $Q(x, y)$. This velocity vector of a representative point R describing a path of (4) defined parametrically by a solution $x = f(t)$, $y = g(t)$. At a critical point both components of this vector velocity are zero, and hence at a critical point the point R is at rest.

In particular, you can consider the special case (3) which arises from a dynamical system described by the differential equation (1). At a critical point of (3) both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero. Since $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, you thus see that at such a point the velocity and acceleration of the dynamical system described by (1) are both zero. Thus the critical points of (3) are equilibrium points of the dynamical system described by (1).

The following are basic concept dealing with critical points and paths.

Definition 3.2 A critical point (x_0, y_0) of the system (4) is called *isolated* if there exists a circle

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

about the point (*x*0, *y*0) such that (*x*0, *y*0) is the only critical point of (4) within this circle.

In what follows, assume that every critical point 138 is isolated.

Definition 3.3 Let *C* be a path of the system (4), and let $x = f(t)$, $y = g(t)$ be a solution of (4) which represents *C* parametrically. Let (x_0, y_0) be a critical point of (4). You shall say that the path *C approaches the critical point* (x_0, y_0) *as* $t \to +\infty$ if

$$\lim_{t\to+\infty} f(t) = x_0, \ \lim_{t\to+\infty} g(t) = y_0, \qquad (12)$$

Definition 3.4 Let *C* be a path of the system (4) which approaches the critical point (x_0, y_0) as $t \to +\infty$, and let $x = f(t)$, $y = g(t)$ be a solution of (4) which represents *C* parametrically. You will say that *C enters the critical point* (x_0, y_0) *as* $t \to +\infty$ if

$$\lim_{t\to+\infty} \frac{g(t) - y_0}{f(t) - x_0} \qquad (13)$$

exists or if the quotient in (13) becomes either positively or negatively infinite as $t \to +\infty$.

3.1.2 Types of Critical Points

1. *Center:* This is a critical point that is surrounded by infinite family of closed paths which is not approached by any of the paths as $t \to +\infty$ or $t \to -\infty$.
2. *Saddle point:*
3. A critical point is called *spiral point* if such a point is approached in a spiral-like manner by an infinite family of paths as $t \to +\infty$ (or as $t \to -\infty$).
4. A critical point is called a *node* if such a point is not only approached but also entered by an infinite family of paths as $t \to +\infty$ (or as $t \to -\infty$).

3.1.3 Stability

Definition 3.5 Let $(x_0.y_0)$ be a critical point of the system (4); let C be a path of (4); and let $x = f(t)$, $y = g(t)$ be a solution of (4) represent C parametrically. Let

$$D(t) = \sqrt{[f(t) - x_0]^2 + [g(t) - y_0]^2} \qquad (14)$$

denote the distance between (x_0, y_0) and the point $R : [f(t), g(t)]$ on C. The critical point (x_0, y_0) is called *stable* if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that the following is true: Every path C for which

$$D(t_0) < \delta \text{ for some value } t_0 \qquad (15)$$

is defined for all $t \geq t0$ and is such that

$$D(t) < \epsilon \text{ for } t_0 \leq t < \infty. \qquad (16)$$

3.2 Critical Points and Paths of Linear Systems

3.2.1 Basic Theorems

Although the major interest in this unit is to classify the critical point of nonlinear systems. But you shall see that under appropriate circumstance you can replace a given nonlinear system by a related linear system and then employ this linear system to determine the nature of the critical point of the given system. Thus in this section, you shall first investigate the critical points of a linear autonomous system.

Consider the linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \qquad (17)$$

where *a, b, c* and *d* (in the right member of the second equation) are real constants. The origin (0, 0) is clearly a critical point of (15). Assume that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, \qquad (18)$$

and hence (0, 0) is the *only* critical point of (15). Note that the solutions of (15) are sought and found of the form

$$\begin{cases} x = Ae^{\lambda t} \\ y = Be^{\lambda t} \end{cases} \qquad (19)$$

and if (17) would be a solution of (15), then λ must satisfy the quadratic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \qquad (20)$$

called the *characteristic equation* of (15). Note that by condition (16), zero cannot be a root of the equation (18) in the problem under discussion. Let λ_1 and λ_2 be the roots of the characteristic equation (18). You need to prove that the nature of critical point (0, 0) of the system (15) depends upon the nature of the roots λ_1 and λ_2. You shall consider five cases according as

1. λ_1, and λ_2 are real, unequal, and of the same sign
2. λ_1, and λ_2 are real, unequal, and of opposite signs
3. λ_1, and λ_2 are real and equal
4. λ_1, and λ_2 are conjugate, complex and pure imaginary.
5. λ_1, and λ_2 are pure imaginary.

Theorem 3.1 If the roots λ_1 and λ_2 of the characteristic equation are

1. real, unequal and of the same sign, then the critical point (0, 0) of the linear system is a node.
2. are real, unequal and of opposite sign then the critical point (0, 0) of the linear system is a saddle point
3. real and equal then the critical point (0, 0) of the linear system (15) is a node

4. conjugate complex with real part not zero (that is not pure imaginary) then the critical point (0, 0) of the linear system (15) is a spiral point.

5. pure imaginary, then the critical point (0, 0) of the linear system (15) is a center

Theorem 3.2 The critical point (0, 0) of the linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad \text{where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

is stable if and only if both roots of the characteristic equation have negative or zero real parts.

3.2.2 Examples and Applications

Example 3.2 Determine the nature of the critical point (0, 0) of the system

$$\begin{cases} \dot{x} = 2x - 7y \\ \dot{y} = 3x - 8y \end{cases} \qquad (21)$$

and determine whether or not the point is stable.

Solution

The system (19) is of the form (15) where $a = 2$, $b = -7$, $c = 3$ and $d = -8$. The characteristic equation is

$$\lambda^2 + 6\lambda + 5 = 0$$

Hence the roots of the characteristic equation are $\lambda_1 = -5$ and $\lambda_2 = -1$. Since the roots are negative, the critical point (0, 0) of (19) is a *node*. Since the roots are negative, the point is *stable*.

Example 3.3 Determine the nature of the critical point (0, 0) of the system

$$\begin{cases} \dot{x} = 2x + 4y \\ \dot{y} = 2x + 6y \end{cases} \qquad (22)$$

and determine whether or not the point is stable.

Solution

Here $a = 2$, $b = 4$, $c = -2$ and $d = 6$. The characteristic equation is

$$\lambda^2 - 8\lambda + 20 = 0$$

and its roots are $4 \pm 2i$. Since these roots are conjugate complex but not pure imaginary, conclude that the critical point (0, 0) of (20) is a *spiral point*. Since the real part of the roots is positive, the point is stable.

3.3 Critical Points and Paths of Nonlinear Systems

3.3.1 Basic Theorems on Nonlinear Systems

Consider the nonlinear real autonomous system

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \quad (23)$$

Assume that the system (21) has an isolated critical point which you shall choose to be the origin (0, 0). Assume further that the function P and Q in the right members of (21) are such that $P(x, y)$ and $Q(x, y)$ can be written in the form

$$\begin{cases} P(x, y) = ax + by + P_1(x, y) \\ Q(x, y) = cx + dy + Q_1(x, y) \end{cases} \quad (24)$$

where (i) a, b, c and d are real constants,

$$and \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

and (ii) P1 and Q1 have continuous first partial derivatives for all (x, y), and are such that

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{Q_1(x, y)}{\sqrt{x^2 + y^2}} = 0 \quad (25)$$

Thus the linear system under consideration may be written in the form

$$\begin{cases} \dot{x} = ax + by + P_1(x, y) \\ \dot{y} = cx + dy + Q_1(x, y) \end{cases} \quad (26)$$

where a, b, c, d, P1 and Q1 satisfy the requirements above.

If P(x, y) and Q(x, y) in (21) can be expanded in poyour series about (0, 0), the system (21) takes the form

$$\begin{cases} \dot{x} = \left[\frac{\partial P}{\partial x}\right]_{(0,0)} x + \left[\frac{\partial P}{\partial y}\right]_{(0,0)} y + a_{12}x^2 + a_{22}xy + a_{21}y^2 + \cdots \\ \dot{y} = \left[\frac{\partial Q}{\partial x}\right]_{(0,0)} x + \left[\frac{\partial Q}{\partial y}\right]_{(0,0)} y + b_{12}x^2 + b_{22}xy + b_{21}y^2 + \cdots \end{cases} \tag{27}$$

This system is of the form (24), where P1(x, y) and Q_1(x, y) are the terms of higher degree in the right members of the equations. The requirements above will be met, provided the Jacobian $\left.\frac{\partial(P,Q)}{\partial(x,y)}\right|_{(0,0)} \neq 0$. Observe that the constant terms are missing in the expansion in the right members of (25), since P(0, 0) = Q(0, 0) = 0.

Example 3.4 The system

$$\begin{cases} \dot{x} = x + 2y + x^2 \\ \dot{y} = -3x - 4y + 2y^2 \end{cases}$$

is of the form (24) and satisfies the requirements (i) and (ii) above. Here a = 1, b = 2, c = −3 and d = −4, and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2 \neq 0$$

Further $P_1(x, y) = x^2$, $Q1(x, y) = 2y^2$, and hence

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

and

$$\lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2y^2}{\sqrt{x^2+y^2}} = 0$$

By the requirement of (ii) the nonlinear terms P1(x, y) and Q1(x, y) in (24) tend to zero more rapidly than the linear terms ax + by and cx + dy. Hence one would suspect that the behaviour of the paths of the system (24) near (0, 0) would be similar to that of the paths of the related linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \tag{28}$$

obtained from (24) by neglecting the nonlinear terms. In other words, it would seem that the nature of the critical point (0, 0) of the nonlinear system (24) should be similar to that of the linear system (15). In general this is actually the case. It is now time to state without proof the main theorem regarding this relation.

Theorem 3.3 *Consider the nonlinear system*

$$\begin{cases} \dot{x} = ax + by + P_1(x, y) \\ \dot{y} = cx + dy + Q_1(x, y) \end{cases} \tag{29}$$

where a, b, c, d, P_1 and Q_1 satisfy the requirements (i) and (ii) above. Consider the linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \tag{30}$$

obtained from (27) by neglecting the nonlinear terms P1(x, y) and Q1(x, y). Both systems have an isolated critical point at (0, 0). Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{31}$$

of the linear system (28).

Then

(a) The critical point (0, 0) of the nonlinear system (28) in the following cases

(i) If λ_1 and λ_2 are real, unequal and of the same sign, then not only is (0, 0) a node of (28) but also (0, 0) is a node of (27).

(ii) If λ_1 and λ_2 are real, unequal, and of opposite sign, then not only is (0, 0) a saddle point of (28), but also (0, 0) is a saddle point of (27).

(iii) If λ_1 and λ_2 are real and equal and the system (28) is not such that a = d =6 0, b = c = 0. Then not only is (0, 0) a node of (28), but also (0, 0) is a node of (27).

(iv) If λ_1 and λ_2 are conjugate complex with real part not zero, then not only is (0, 0) a spiral point of (28), but also (0, 0) is a spiral point of (27).

(b) The critical point (0, 0) of the nonlinear system (27) is not necessarily of the type as that of the linear system (28) in the following cases:

(v) If λ_1 and λ_2 are real and equal and the system (28) is such that a = d =6 0, b = c = 0, then although (0, 0) is a node of (28), the point (0, 0) may be either a node, a spiral point of (28).

(vi) If λ_1 and λ_2 are pure imaginary, then although (0, 0) is a center of (28), the point may be either a center or a spiral point of (27).

Theorem 3.3 deals with the type of the critical point (0, 0) of the nonlinear system (27). Concerning the stability of this point, you have without proof the following theorem of Lyapunov. More on this is discussed in unit 8.

Theorem 3.4 With Hypothesis as exactly as in theorem 3.3,

(a) If the roots λ_1 and λ_2 of the characteristic equation (29) of the linear system (28) both have negative real parts, then not only is (0, 0) a stable critical point of (28) but also (0, 0) is a stable critical point of (27).

(b) If at least one of the roots λ_1 and λ_2 of (29) has a positive real part, then not only is (0, 0) an unstable critical point of (28), but also an unstable critical point of (27).

Example 3.5 Consider the nonlinear system

$$\begin{cases} \dot{x} = x + 4y - x^2 \\ \dot{y} = 6x - y + 2xy \end{cases} \qquad (32)$$

This is of the form (27), where $P_1(x, y) = -x^2$ and $Q_1(x, y) = 2xy$. You see at once that the hypotheses of Theorems 13.7 and 13.8 are satisfied. Hence to investigate the critical point (0,0) of (30), consider the linear system

$$\begin{cases} \dot{x} = x + 4y \\ \dot{y} = 6x - y \end{cases} \qquad (33)$$

of the form (28). The characteristic equation (29) of this system is

$$\lambda_2 - 25 = 0.$$

Hence the roots are $\lambda_1 = 5$, $\lambda_2 = -5$. Since the roots, are real, unequal, and of opposite sign, you see from conclusion (ii) of theorem 3.3, that the critical point (0, 0) of the nonlinear system (31) is a saddle point. From the conclusion (b) of theorem 3.4, you further conclude that the point is unstable.

Eliminating dt from the equation (30), you obtain the differential equation

$$\frac{dy}{dx} = \frac{6x - y + 2ry}{x + 4y - x^2} \tag{34}$$

which gives the slope of the paths in the xy-phase plane defined by the solutions of (30). The first order equation (32) is exact. Its general solution is readily found to be

$$x^2y + 3x^2 - xy - 2y^2 + c = 0 \tag{35}$$

where c is an arbitrary constant. Equation (33) is the equation of the family of paths in the *xy*-phase plane.

Example 3.6 Consider the nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = \sin x - 4y \\[2ex] \dfrac{dy}{dt} = \sin 2x - 5y \end{cases} \tag{36}$$

Using the expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

You write this system in the form

$$\begin{cases} \dot{x} = x - 4y - \frac{x^3}{6} + \frac{x^5}{120} + \cdots \\[2ex] \dot{y} = 2x - 5y - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots \end{cases} \tag{37}$$

The hypothesis of theorems 3.3 and 3.4 are satisfied. Thus to investigate the critical point (0,0) of (27) or (28), you consider the system

$$\begin{cases} \dot{x} = x - 4y \\ \dot{y} = 2x - 5y \end{cases} \tag{38}$$

The characteristic equation of this system is

$$\lambda_2 + 4\lambda + 3 = 0.$$

Thus the roots are $\lambda_1 = -3$, $\lambda_2 = -1$. Since the roots are real, unequal, and of the same sign, you see from conclusion (i) of theorem 3.3 that the critical point (0, 0) of the nonlinear system (34) is a *node*. From conclusion (a) of theorem 3.4, you can conclude that this node is *stable.*

Example 3.7 Consider the two nonlinear systems

$$\begin{cases} \dot{x} = -y - x^2 \\ \dot{y} = x \end{cases} \tag{39}$$

and

$$\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x \end{cases} \tag{40}$$

The point (0, 0) is a critical point for each of these systems. The hypotheses of Theorem 3.3 are satisfied in each case, and in each case the corresponding linear system to be investigated is

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \tag{41}$$

The characteristic equation of the system (39) is

$$\lambda^2 + 1 = 0$$

with the pure imaginary roots ±i. Thus the critical point (0, 0) of the linear system (39) is a center. However, Theorem 3.3 does not give us definite information, concerning the nature of this point for either of the nonlinear systems (37) and (38). Conclusion (vi) of theorem 3.3 tells you that in each case (0, 0) is either a center or a spiral point; but this is all that this theorem tells us concerning the two systems under consideration.

4.0 CONCLUSION

In this unit, you studied nonlinear systems. In which you learnt how to determine the critical points of a system of differential equations and discuss the nature and stability of a critical point especially (0,0) You also learnt how to linearize a non linear system.

5.0 SUMMARY

Having gone through this unit, you are now able;

- define phase plane, paths and critical points.
- describe types of critical points
- define and describe stability of a critical point.
- determine the critical points of linear system.
- describe the nature of the critical point (0, 0)
- describe the stability of the critical point (0, 0)
- linearize a nonlinear differential equation and describe the nature and stability of the critical point (0, 0)

6.0 TUTOR MARKED ASSIGNMENT

Exercise 6.1

Determine the nature of the critical point (0, 0) of each of the linear autonomous systems in the following Also determine whether or not the critical point is stable.

1. $\begin{cases} \dot{x} = x + 3y \\ \dot{y} = 3x + y \end{cases}$

2. $\begin{cases} \dot{x} = 3x + 4y \\ \dot{y} = 3x + 2y \end{cases}$

3. $\begin{cases} \dot{x} = 2x - 4y \\ \dot{y} = 2x - 2y \end{cases}$

4. $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + 5y \end{cases}$

Determine the type and stability of the critical point (0, 0) of each of the nonlinear autonomous systems

5. $\begin{cases} \dot{x} = x + x^2 - 3xy \\ \dot{y} = -2x + y + 3y^2 \end{cases}$

6. $\begin{cases} \dot{x} = x + y - x^2 y \\ \dot{y} = 3x - y + 2xy^3 \end{cases}$

7. $\begin{cases} \dot{x} = (y+1)^2 - \cos x \\ \dot{y} = \sin(x+y) \end{cases}$

8. Consider the autonomous system $\begin{cases} \dot{x} = -ye^x \\ \dot{y} = e^x - 1 \end{cases}$

(a) What type of critical point is (0, 0)?
(b) Obtain the differential equation of the paths and find its general solution.

UNIT 3 THEOREMS AND SOLUTIONS OF LYAPUNOV EQUATION

CONTENTS

1.0 INTRODUCTION

In this unit, you shall be introduced to stability theory. This will aid you to determine the stability of a system of ordinary differential equation.

2.0 OBJECTIVES

At the end of this section, you should be able to:

- say whether a system of an ODE is stable or not;
- determine whether a critical point is stable or not; and
- use Lyapunov's theory to determine the stability of a critical point.

3.0 MAIN CONTENT

Stability

The term stability is an expression that almost tells its own story. Suppose a device of some sort operates under general conditions, and these conditions are slightly changed or modified. The question now is, "Does this change or modification have little or considerable effect on the device? In your thought, if the first instance is stable, then the second is unstable.

How does this apply to physical systems in particular? The system will depend upon certain number of physical parameters $x_1, x_2, \ldots, x_n$ which define position and also velocity. These will be represented in some space $\mathbb{R}^n$ by a vector point x. The state of the system at time t will be $x = x(t)$. As will be produce a trajectory g, in $\mathbb{R}^n$ space. The question again is, how do trajectories g which start near g behave with respect to g? Do they as time goes on remain near g which is stability or do they shift away from g which is instability.

Suppose at time t, the state of a physical system is described by

$$x_i = x_i(t) \qquad i = 1, 2, ..., n$$

and suppose the conditions of motion of the system require variable to satisfy

$$\dot{x}i = xi(x_1, x_2, x_3, ..., x_n) \qquad (1)$$

Suppose $x_i = \eta_i(t)$ is some particular state of the system i.e., $x_i = \eta_i(t)$ is a solution of (1). To study the properties of solutions of (1) in the neighbourhood of $\eta_i(t)$, you make

$$y_i = x_i - \eta_i$$

where $yi = 0$ *or* $xi = \eta i$ is the unperturbed motion or trajectory and $xi(t)$ describes another solution or state of the system. The new variable $yi(t)$, now satisfies an equation of the form

$$\frac{dy_i}{dt} = y_i(y_1, y_2, ..., y_n, t) \qquad (2)$$

Where

$$yi(y_1, y_2, ..., y_n, t) = xi(y_1 + \eta_1,..., y_n - \eta_n, t) - x_i(\eta_1, \eta_2,..., \eta_n, t). \qquad (3)$$

Here the curve $\{\eta_i(t)\}$ is denoted by

$$y_i = 0$$

is called the null-solution (or trivial solution) as can be seen from (2) and (3). Thus the stability of the solution $\eta_i(t)$ of (1) is reduced to that of the trivial solution of y_i 0 of (2).

Given that equation

$$\dot{x} = f(t, x), \qquad df(t, 0) = 0 \qquad (4)$$

in which $f : I \times D \rightarrow \mathbb{R}^n$ is assumed continuous and satisfy conditions for uniqueness and continuous dependence of solutions on initial data. Then the following definitions hold.

3.1 Stability in the sense of Lyapunov

Definition 3.1 The trivial solution $x(t) = 0$ of (4) is said to be stable (in the sense of ***Lyapunov***) if $\epsilon > 0$ is given and $t \in I$, there exists, for any $y(t)$ a solution of (4), a positive number

$\delta = \delta(t_0,\epsilon)$ such that $\|y(t_0)\| < \delta$ implies that $\|y(t)\| < \epsilon$ or $\|y(t_0) - x(t_0)\| < \delta$ implies that $\|y(t) - x(t)\| < \epsilon$ fort $> t_0$.

If δ can be chosen independent of t0, then the x(t) is said to be uniformly stable.

Definition 3.2 The solution *x(t)* is said to be asymptotically stable if it is stable and for any given $\delta > 0$ and a solution y(t) of (4),

$$\lim_{t\to\infty}\left\|x(t) - y(t)\right\| = 0 \qquad \text{for } \|x(t0) - y(t0)\| < \delta$$

Note that the definition of stability given are local in nature in the sense that you are concerned with solutions where the initial values are sufficiently close.

Example 3.1 Show that the differential equation

$$\ddot{x} + x = 0$$

is stable in the sense of Lyapunov but not as symptotically stable.

Solution

The scalar equation

$\ddot{x} + x = 0$

is equivalent to the system

$$\begin{matrix} \dot{x} = y \\ \dot{y} = -x \end{matrix} \quad or\ x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with solution as

$x = A\cos t + B\sin t$ *and* $y = -A\sin t + B\cos t$

Define the a norm on $X = \binom{x}{y} \equiv \binom{x_1}{y_2}$ by

$$\|X\| = |x_1| + |x_2|$$

$$\|X(0)\| = |A| + |B|$$

$$\begin{aligned} \|X(t)\| &= |A\cos t + B\sin t| + |-A\sin t + B\cos t| \\ &\leq 2(|A| + |B|) < \epsilon \end{aligned}$$

Choose $\delta = \frac{\epsilon}{2}$
Then from the definition, the trivial solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is stable in the sense of Lyapunov. However, the trivial solution is NOT asymptotically stable, why?

Example 3.2 Show that the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0$$

is both stable and assymptotically stable in the sense of Lyapunov.

Solution

The scalar equation

$$\ddot{x} + 3\dot{x} + 2x = 0$$

is equivalent to the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_2 - 2x_1 \end{aligned} \quad \text{or } X = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X \text{ with } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The auxiliary equation is

$$\lambda^2 + 3\lambda + 2 = 0 \text{ i.e., } \lambda_1 = -_1, \text{ and } \lambda_2 = -2$$

The general solution is given by

$$X = c_1 e^{-t} + c_2 e^{-2t} \text{ and } X^{\cdot} = -c_1 e^{-t} - 2c_2 e^{-2t}$$

$$||X(0)|| = |x(0)| + |\dot{x}(0)| \leq 2c_1 + 3c_2$$

$$||X(t)|| \leq 2|c_1|e^{-t} + 3|c_2|e^{-2t}$$

$$\leq (2|c_1| + 3|c_2|)e^{-t}$$

Since $e^{-t} \leq 1$. Choose $\delta = \frac{\in}{2|c_1|+3|c_2|}$ and conclude that the trivial solution is stable.

$$\lim_{t\to\infty} \|X(t)\| \leq \lim_{t\to\infty}\{2|c_1| + 3|c_2|\}e^{-t} \to 0 \; as \; t \; \to 0$$

Therefore, $\lim_{t\to\infty}\|X(t)\| = 0$. Thus, the trivial solution of the system

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X$$

is assymptotically stable in the sense of Lyapunov.

Example 3.3 Consider the homogeneous system

$$\dot{x} = Ax$$

where A is a constant $n \times n$ matrix all of whose eigenvalues have negative real parts. Then you can conclude both stability and asymptotic stability.

Proof. Suppose all the roots of a are distinct, recall that Ty_i, $i = 1, 2,..., n$ are the n linearly independent solution, where

$$y_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_i t} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1, 2, ..., n$$

$$Ty_1 = \begin{pmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Any solution $x(t)$ of the system is of the form

$$x(t) = (Ty_1, Ty_2,..., Ty_n)x^n$$

$$||x(t)|| \leq Ae^{-\mu t}||x_0||$$

where µ is the smallest of

$$\alpha_i < 0 \lambda_i = \alpha_i + j\beta_i \alpha_1 < \alpha_2 < \ldots < \alpha_j < 0$$

you can conclude both stability and asymptotic stability.

Suppose the roots are not necessarily distinct. Then the general solution is given by

$$x(t) = \sum_{j=1}^{k}\sum_{i=1}^{m_j} t^{i-1}e^{\lambda_j t}c_{ij} \qquad m_1 + m_2 + \cdots + m_k = n \qquad (5)$$

Fix ***j*** = 1**,** Then the expression (5) becomes

$$x(t) = c_{11}e^{\lambda_1 t} + c_{21}te^{\lambda_1 t} + c_{31}t^2e^{\lambda_1 t} + \cdots + c_{m_1 1}t^{m_1-1}e^{\lambda_1 t}$$

$$\begin{aligned} \|t^{i-1}e^{\lambda_j t}c_{i,j}\| &\leq m_{ij}|t^{i-1}|\left|\frac{e^{(\alpha_j+\mu)t}}{e^{\mu t}}\cdot e^{\beta_j t}\right| \qquad \alpha_j + \mu < 0,\ \mu > 0 \\ &\leq m_{ij}|t^{i-1}|\frac{e^{-(\alpha_j+\mu)t}}{e^{\mu t}} \\ &\leq Mm_{ij}e^{-\mu t} \to 0\ as\ t \to \infty \end{aligned}$$

Therefore asymptotically stable.

Note that given the system

$$\dot{x} = Ax$$

A is a constant $n \times n$ matrix all of whose eigenvalues have negative real parts then there exists $\alpha > 0, \beta > 0$ such that

$$||e^{At}|| \leq \beta e^{-\alpha t}, \qquad t > 0 \qquad (6)$$

(i) If all the roots of A are distinct then each column of the fundamental matrix solution of Φ is of the form

$$e^{\lambda kt}P^k \qquad k = 1, 2, ..., n$$

where the P^k's are some constants n–vectors, Clearly, you can write

$$\lambda_k = \mu_k + \alpha k, \qquad u_k < 0$$

Assume that $u_k < -\alpha$ for all k. Then

$$\Phi(t) \leq n \max |P_j^k|e^{-\alpha t}, \qquad t > 0, k, j = 1, 2, ..., n$$

If λi is a repeated root of A, then the corresponding column of Φ involve terms of the form

$$\Phi^r(t) = e^{t\lambda_i} \sum_{i=1}^{n} \frac{t^{i-1}}{(i-1)!} P^i \qquad (7)$$

where P_i*'s* are constant n–vectors. Since

$$\frac{t^r e^{t\lambda_s}}{e^{-\alpha t}} = t^r|e^{(\mu_s+\alpha_i)t} \qquad \mu_s < -\alpha_s < 0$$

and $\mu_s + \alpha_s < 0$**,** it follows that

$$t^r e^{(\mu s+\alpha)t} \to 0 \text{ as } t \to \infty$$

Thus

$$t^r e(\mu_s+\alpha)t \leq M_s, \qquad M_s > 0$$

is a constant. Combining with (7) it follows readily that there exist constants $M > 0$ and $\alpha > 0$ such that

$$||\Phi(t)|| \leq Me^{-\alpha t}$$

Remark 3.1

$$\dot{x} = A(t)x$$

with A(t + ω) = A(t), *the solution*

$$\Phi(t) = P(t)e^{tR}$$

If all the roots of R *have negative real parts then*

$$||\Phi(t)|| \leq Me^{-\alpha t} \qquad t \geq 0,\ \alpha > 0,\ M > 0$$

3.2 Quasilinear System

Theorem 3.1 *Given the differential equation*

$$\dot{x} = Ax + f(t, x) \qquad x(0) = x0 \qquad (8)$$

where A is a constant $n \times n$ matrix all whose eigenvalues have negative real parts, f is a continuous function of x and t. Suppose that

$$||f(t, x)|| \leq K||x|| \tag{9}$$

for t and x. Then the solution of the system (8) is asymptotically stable provided K is small enough

Proof: Any solution of (8) can be written in the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} f(\tau, x(\tau))d\tau$$

$$\|x(t)\| \leq \|e^{At}\|\|x_0\| + \int_0^t \|e^{A(t-\tau)}\|\|f(\tau, x(\tau))\|d\tau$$

Clearly there exists $\alpha > 0$, $\beta > 0$ such that

$$||e^{At}|| \leq \beta e^{-\alpha t}, \qquad t \geq 0$$

and so

$$\|x(t)\| < \beta\|x_0\|e^{-\alpha t} + \beta K \int_0^t e^{\alpha\tau}\|x(\tau)\|d\tau$$

Thus

$$\frac{\|x(t)e^{\alpha t\|}}{\beta\|x_0\| + \int_0^t \beta k e^{\alpha\tau}\|x(t)\|d\tau} \leq 1 \tag{10}$$

Multiplying both sides by βk

$$\frac{\beta k\|x(t)e^{\alpha t}\|}{\beta\|x_0\| + \int_0^t \beta k e^{\alpha\tau}\|x(\tau)\|d\tau} \leq \beta k$$

Integrating between 0 and t you obtain

$$\log\left[\beta\|x_0\| + \int_0^t \beta k e^{\alpha\tau}\|x(\tau)\|d\tau\right]_0^t \leq \beta kt$$

$$\frac{\beta\|x_0\| + \int_0^t \beta k e^{\alpha\tau}\|x(\tau)\|d\tau}{\beta\|x_0\|} \leq e^{\beta kt}$$

Thus

$$\|x(t)\|e^{\alpha t} \leq \beta\|x_0\| + \int_0^t \beta k e^{\alpha\tau}\|x(\tau)\|d\tau \leq \beta\|x_0\|e^{\beta kt}$$

and so

$$||x(t)|| \leq \beta||x0||e^{-(\alpha-\beta k)t}$$

Fix $k > 0$ such that $\alpha - k\beta > 0$, *then*

$$e^{-(\alpha-k\beta)t} \leq 1 \qquad \text{for all } t$$

Therefore

$$\|x(t)\| \leq \beta\|x_0\| < \epsilon \qquad \text{provided } \|x_0\| < \delta = \tfrac{\epsilon}{\beta}$$

The solutions

$$x(t) \leq \beta\|x_0\|e^{-(\alpha-\beta k)t}$$

are assymptotically stable since

$$\lim_{t\to\infty}\|x(t)\| \leq \lim_{t\to\infty}\beta\|x_0\|e^{-(\alpha-\beta)t} = 0$$

3.3 Lyapunov Second Method

The examples you have considered so far presume a knowledge of solutions before you can conclude stability. There are only very few equations whose solutions can be determined in closed form (i.e., in terms of elementary functions).

An alternative method initiated by a Russian Mathematician A.M. Lyapunov is a generalization based on the youll known observation that near the equilibrium point of a physical system, the total energy of a given system is either constant or decreasing. Therefore idea here is introduction of some functions now known as Lyapunov functions, which generalize the total energy in a system.

Consider the differential equation

$$\dot{x} = f(x, t), \qquad f(0) = 0 \qquad (11)$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the solutions are unique and very continuously with the initial data.

Let $V: \mathbb{R}^n \to \mathbb{R}$ be defined and continuous together with the first partial derivatives $\frac{\partial V}{\partial x_i}$ ($i = 1, 2, ..., n$) on some open let $\Omega \subset \mathbb{R}^n$

$$\Omega = \{x : x \in \mathbb{R}^n, \|x\| < h\}$$

The following are some definitions that would be of help you as you proceed in the understanding of this topic.

Definitions

1. A function $V: \Omega \to \mathbb{R}$ is said to be positive definite (negative definite) if $V(0) = 0$ and V assume positive (negative) values on Ω.
2. A function $V: \Omega \to \mathbb{R} \Omega \subset \mathrm{R}$ is said to be positive(negative) semi-definite if $V(0) = 0$ and $V(x) \geq 0$ ($V(x) \leq 0$) on Ω. If the functions assume arbitrary values then it is said to be indefinite.

Example 3.4

(a) $V = x^2 + y^2$ is positive definite.
(b) $(x + y)^2 + z^2$ is positive semi-definite.
(c) $V = x^2 + y^2 - z^2$ is indefinite.

3. The derivatives (Euler's) of V along solution paths of (11) is given by

$$\dot{V} = \frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x_i}\dot{x}_i = \frac{\partial V}{\partial x_1}f(x_i)$$

Note: On investigation of stability or instability Lyapunov pioneered the work which appeared in France 1907. The definitions (1), (2) and (3) are very essential in this study. These will be tied up ultimately in the context of systems.

Stability or Instability can be assumed directly using the following theorems accordingly.

Theorem 3.2 Given the differential equation (4) that is

$$\dot{x} = f(t, x), \qquad f(t, 0) = 0$$

Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and is such that the time derivative of V along the solution paths of (4), that is, $F \cdot \text{grad}V$ is negative semi-definite. Then the trivial solution $x \equiv 0$ is stablen in the sense of Lyapunov.

Theorem 3.3 Lyapunov Given the system (4). Suppose that there exists a C^1 function $V : \mathbb{R}^2 \to \mathbb{R}$, with the following *properties;*

(i) V is positive definite.
(ii) The time derivative U (x1, x2) of V (x1, x2) along the solution paths of (4) is negative definite.
(iii) Then the trivial solution x = 0 of (4) is asymptotically stable.

Theorem 3.4 Given the system (4). Suppose that there exists a C^1 function $V: \mathbb{R}^2 \to \mathbb{R}$, with the following properties

(i) V is positive definite.
(ii) The time derivatives $U(x_1, x_2)$ of $V(x_1, x_2)$ along the solution paths of (4) is positive definite.

Then the trivial solution of (4) is unstable in the sense of Lyapunov.

Theorem 3.5 (Cêtaev) On Instability Consider the system (11) i.e.,

$$\dot{x} = f(x); \qquad f(0) = 0$$

which f is sufficiently smooth in the domain G (in $\mathbb{R}^n$) containing the origin. Let $D \subset G$ be a domain in $\mathbb{R}^n$ with the boundary of D which lies inside G passing through the origin.
Suppose there exists a C^1 function $V: \mathbb{R}^n \to \mathbb{R}$ such that

(i) V (x) = 0 on that part of the boundary D lying inside G and V (x) > 0 elsewhere.
(ii) The time derivative elsewhere of V along the solution paths of (11) that is $f \cdot \text{grad}V > 0$ in D.

Then the trivial solution $x \equiv 0$ of (11) is unstable

The following are some applications of these theorems to some specific cases.

Example 3.5 Consider the section equation

$$\ddot{x} + x = 0$$

or rather the equivalent 2-system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 \qquad (12)$$

with the function V defined by

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

Show that the system (12) is stable in the sense of Lyapunov

Solution

Clearly V given by

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

is positive definite. Along the solution paths of (12)

$$\dot{V}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 - x_1x_2 = 0$$

Thus $\dot{V}$ (x1, x2) is negative semi-definite. Hence V is a suitable function to which theorem 1.2 can be applied to give that the trivial solution x ≡ 0 is stable in the sense of Lyapunov.

Example 3.6 Consider the system

$$\left.\begin{aligned} \dot{x}_1 &= -x_1^3 - 2x_1x_2^2 \\ \dot{x}_2 &= x_1^2x_2 - x_2^2 \end{aligned}\right\} \qquad (13)$$

with the function V (x_1, x_2) defined by

$$V(x_1, x_2) = x_1^2 + x_1^2x_2^2 + x_2^4$$

prove that the system (13) is asymptotically stable

Solution

Note that V (x_1, x_2) is positive definite since

$$V(x_1, x_2) = x_1^2 + x_1^2x_2^2 + x_2^4 = x_1^2(1 + x_2^2) + x_2^4 > 0$$

Along the solution paths of (13)

$$V(x_1, x_2) = 2x_1\dot{x}_1 + 2x_1x_2^2\dot{x}_1 2x_2x_1^2\dot{x}_2 + 4x_2^3\dot{x}_2$$

$$= -2x_1(x_1^3 + 2x_1x_2^2) - 2x_1x_2^2\,(x_1^3 + 2x_1x_2^2)$$

$$+2x_1^2x_2(2x_1^2x_2 - x_2^2) + 4x_2^3\,(x_1^2x_2 - x_2^2)$$

$$= -2x_1^4 - 4x_1^2x_2^2 - 2x_1^2x_2^4 - 4x_2^6$$

$$\dot{V}(x_1, x_2) = -2x_1^2(x_1^2 + 2x_2^2 + x_2^4) - 4x_2^6$$

This is negative definite. Thus by theorem 1.3, the system (13) is asymptotically stable in the sense of Lyapunov.

Example 3.7 Consider the 2-system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_2 - bx_1 \qquad (14)$$

a, b are constants, with the function $V(x_1, x_2)$ defined by $V(x_1, x_2) = ax_2^2 + bx_1^2$. Discuss the conditions for

(i) assymptotic stability
(ii) instability on the system (14)

Solution

Our $V(x_1, x_2)$ defined by

$$V(x_1, x_2) = ax_2^2 + bx_1^2$$

is positive semi definite if a $\geq$ 0, b $\geq$ 0. Along the solution paths of (14)

$$\begin{aligned}\dot{V}(x_1, x_2) &= \dot{x}_1\frac{\partial V}{\partial x_1} + \dot{x}_2\frac{\partial V}{\partial x_2} \\ &= \dot{x}_2(2bx_1) + (-ax_2 - bx_1)2x_2 + 2bx_1x_2 - 2ax_2^2 - 2bx_1x_2 \\ &= -2ax_2^2\end{aligned}$$

Note: Our V is positive definite if a > 0, b > 0, also that $\dot{V}$, considered as a function x_1 and x_2 satisfies the following

(i) $\dot{V}$ is negative definite if a > 0
(ii) $\dot{V}$ is positive definite if a < 0

and conclusions follows accordingly That is

(i) Assymptotically stable if $\dot{V}$ is negative definite.
(ii) Instability if $\dot{V}$ is positive definite

Example 3.8 Given the scalar equation

$$\ddot{x} + a\,\dot{x} + h(x) = 0 \qquad (15)$$

where $a > 0$ is a constant and the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is such that solutions exist and are unique and very continuously with initial data. By considering with initial data. By considering an appropriate equivalent system and considering the function

$$2V(x, y) = (y + ax)^2 + y^2 + 4H(x)$$

where $H(x) = \int_0^x h(s)ds$. Determine the conditions on h which ensure (i) stability (ii) assymptotic stability of the trivial solution of scalar equation (15)

Solution

1. The scalar equation (15) is equivalent to the system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay - h(x) \end{aligned}\right\} \qquad (16)$$

Along the solution paths of (16)

$$\dot{V} = (y + ax)\,\dot{y} + a(y + ax)\,\dot{x} + y\dot{y} + 2h(x)\,\dot{x}$$

$$-(y + ax)(ay + h(x)) + ay^2 + a^2xy - ay^2 - yh(x) + 2h(x)y$$

$$= -ay^2 - yh(x) - a^2xy - ah(x)x + ay^2 + ax^2y - ay^2 - yh(x) + 2h(x)y$$

$$= -(ay^2 + ah(x)x) = -a(y^2 + h(x)x)$$

If $xh(x) < 0$ for all $x \neq 0$ when you conclude stability b if $xh(x) > 0$ for all $x \neq 0$ then you conclude assymptotic stability.

4.0 CONCLUSION

Lyapunov and other theorems listed earlier depend heavily on the construction of suitable Lyapunov functions. There is no fixed standard technique for constructing such Lyapunov functions for a given ordinary differential equation. This remains the main problems in the application of the theorems.

If a suitable function V can be found then stability or instability follows and if not, you cannot proceed.

5.0 SUMMARY

Having gone through this unit, you are now able to;

- determine the stability of the trivial solution and a critical point of a system of ODE.
- use the Lyapunov's theorem to determine the stability of a solution of a linear system.

6.0 TUTOR MARKED ASSIGNMENT

Exercise 6.1

1. Given the function V defined by

$$V(x, y) = \frac{1}{2}y^2 + G(x)$$

where $G(x) = \int_0^x g(s)ds$ Determine the conditions on g which ensure stability of the trivial solution of

$$\ddot{x} + g(x) = 0, \qquad g(0) = 0$$

Note: Any relevant theorems used must be stated.

2. By considering the function

$$V = \frac{1}{2}(x^2 + y^2)$$

Prove that the trivial solution of the system

$$\dot{x} = -x - x^3 - x\sin y$$

$$\dot{y} = -y - \frac{y^3}{3}$$

is assymptotically stable in the sense of Lyapunov.

3. The scalar equation

$$\dddot{x} + f(\dot{x})\ddot{x}a\dot{x} + bx = 0$$

$a > 0$, $b > 0$ are constants, f is a continuous function such that solutions exist and are uniquely determined by the initial conditions. Furthermore the function

$$f(y) \geq C \ \geq \frac{b}{a}$$

where C is a constant by considering the function V defined by

$$V = \frac{1}{2}(az^2) + \frac{1}{2}(b^2x^2 + a^2y^2) + byz + abxy + b\int_0^y uf(u)du$$

Prove that the scalar equation

$\dddot{x} + f(\dot{x})\ddot{x} + a\dot{x} + bx = 0$

is stable and assymptotically stable in the sense of Lyapunov.

4. Consider the scalar equation

$\ddot{x} - x^3 = 0$

and $V(x_1, x_2)$ defined by

$V(x_1, x_2) = x_1x_2$

Show that the scalar equation is unstable in the sense of Lyapunov. Any relevant theorem used must be clearly stated

5. Consider the 2-system

$\dot{x}_1 = x_2 + x^2{}_1x_2, \dot{x}_2 = x_1 + 1_0x_2{}^4$

and by using V (x_1, x_2) defined by

$V(x_1, x_2) = x_1x_2$

Show that the trivial solution $x = 0$ is unstable.

6. (a) State and prove a theorem due Lyapunov used in establishing stability of a trivial solution $x = 0$ of the scalar equation

$$\dot{x} = f(x), \qquad f(0) = 0$$

(b) Hence or otherwise show that the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - x_1 - 2x^3_1$$

and the function $V_3(x_1, x_2)$ is positive definite.

(c) Show that the zero solution $\mathbf{x} = 0$ is assymptotically stable in the sense of Lyapunov.

7. By considering the system

$$\dot{x}_1 = x_2 + 2x_2, \qquad \dot{x}_2 = 3x_1 + x_2$$

and the function $V_4(x_1, x_2) = x_1x_2$

(a) Show that the zero solution $x \equiv 0$ is unstable in the sense of Lyapunov

(b) Any theorem used in the above must be stated (No Proof).

8. By considering the function V**:** $\mathbb{R}^2 \to \mathbb{R}$ defined by

$$V(x_1, x_2) = x^2_2 + 9x^2_1$$

Show that the zero solution $\mathbf{x} \equiv 0$ of the trivial solution of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -9x_1$$

is stable in the sense of Lyapunov

9. Given that the function $V: \mathbb{R}^2 \to \mathbb{R}$ defined by $V(x_1, x_2) = x_1x_2$**.** Prove that the trivial solution, $x = 0$ of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x^3_1$$

is unstable.