MODULE 1

UNIT 1 DEFINITIONS AND EQUATIONS

CONTENTS

1.0 INTRODUCTION

What is a Partial Differential Equation, how do we classify Partial Differential Equations? How are they rendered graphically and how do we solve them? This unit addresses these questions with a tour of the basics of Partial Differential Equations; particularly on an introduction to the methods for deriving solution.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the term Partial Differential Equation
- classify First Order Equations
- investigate the methods for constructing solutions for Partial Differential Equations
- solve Quasi – Linear Equations
- explore the many definitions applied in deriving solutions
- apply the method of Lagrange in deriving solutions for Partial Differential Equations.

3.0 MAIN CONTENT

3.1 Essential Definitions

In some elementary course we encountered many physical problems that are modelled by ordinary differential equations and have learnt some of the basic solution technique for such equation. We shall now expand our view by examining Partial Differential Equations (P.D.E). Our Approach will deal with:

i) Existence and Uniqueness of solutions.
ii) Stability of solution to small perturbations.
iii) Methods for constructing solutions.

We shall focus attention largely on (iii) although it is not always possible to solve a P.D.E in closed form.

0.1 Definition

A P.D.E $G\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right) = 0$ (1)

Where $\underline{x} \in R^n$

$$\underline{x} = (x_1\ x_{2,}\ \ldots\ldots.x_n)$$

This is a relationship between a function U of several variables $\underline{x} = (x_1\ x_{2,}\ \ldots\ldots.x_n),\ n \underline{>2}$ and its partial derivatives.

0.2 Definition

By solution (0.1.0) in a domain $\Omega\, CR^n$ we mean a function U = g ($\underline{x}$) whose partial derivatives of order less than or equal to m (e m) exist in Ω and satisfy the equation. We note however that some P.D.E do not provide solution in the classical sense defined above.

Example:

$$x^2\ \frac{\partial u}{\partial x} = 1$$

Does not have a solution in any domain, Ω that contain the origin, rather than a solution in the sense of distributions or generalised functions.

0.3 Definitions

A PDE is said to be of nth – order if the order of the highest partial derivative occurring in the equation is n, and if the coefficient of the highest – order occur linearly, the equation is said to be quasi – linear.

$$\sum_{i,j=1}^{n} A_{ij}\left(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial u}{\partial x_n}\right)\frac{\partial^2 u}{\partial x \partial x_j} + g\left(x_1, u_1, \ldots, \frac{\partial u}{\partial x_n}\right) = 0$$

It is quasi – linear and of 2nd order.

If the coefficient of the highest orders derivatives are all functions of $\underline{x}$ only. The PDE is said to be Semi Linear.

Example:

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + g\left(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (0.3.1)$$

Is semi linear and of 2nd order.

The equation is linear if the coefficient of U and the coefficients of all its partial derivatives are functions of $\underline{x}$ only.

E.g. $$\sum_{ij=1}^{n} Aij\,(\underline{x})\frac{\partial^2 u}{\partial xi\, \partial xj} + \sum_{i=1}^{n} Bi\,(\underline{x})\frac{\partial u}{\partial xi}$$

$$+ c(\underline{x})U + g\,(\underline{x}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (0.3.2)$$

It is linear and of 2nd order.

An equation that is not linear is said to be non-linear. A 2nd order PDE e.g. (0.3.2) is said to be homogenous if g (x) is identically zero. Otherwise it is non – homogenous.

If (0.1.0) is a polynomial of degree k in the highest order partial derivation we say that the equation is of degree k.

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{dz}\right)^2 = 1$$

It is 1st order non-linear degree 2.

In general any equation of degree k = 1 is non – linear.

Example:

1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \cos xy$

It is 1st order, linear non-homogenous.

2) $U\frac{\partial u}{\partial y}\frac{\partial 3u}{\partial x3} + \left(\frac{\partial^2}{\partial y^2}\right)^2 \sin u$

It is 3rd order quasi – linear.

3) $\frac{\partial^2 u}{\partial xy} + \left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial u}{\partial z} + z^3$

It is 2nd order and semi – linear.

4) $\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$

It is 2nd order linear Homogenous.

5) $\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = u$

It is 2nd order non linear.

$$\begin{cases} U\frac{\partial v}{\partial x} + V\frac{\partial u}{\partial y} = x + y \\ V\frac{\partial v}{\partial x} + U\frac{\partial u}{\partial y} = x - y \end{cases}$$

System of 1st order quasi linear equation

$$(ax)\ z\,(a)u_x + (a)Uz\ Za + (a)\ Vn + (av)Vz\ Zx = ay = 0$$

Example: Given that

$$\left.\begin{matrix} u = g\,(x_i\ y_i\ z) \\ v = h\,(x_i\ y_i\ z) \end{matrix}\right\} c^i\ (\Omega)$$

Determine the P.D.E of lowest order satisfied by the class of all functions defined implicitly by

$$G(u_i\ v) = 0$$

Where, $Gu\,Gv \neq 0 \text{ in } \Omega$

3.2 First Order Equation

Examples of 1st order equations are:

$$Z\!\!\!/\, x^2 + Z\!\!\!/ y^2 = 1$$

If $P = Z\!\!\!/ x \quad , q = Z\!\!\!/ y$

$$P^2 + q^2 = 1$$

$$a(z) Z\!\!\!/ x + Zy = 0$$

$$a(z)\, p + q = 0$$

$$xz_x + yz_y = z$$

3.3 Quasi-Linear Equations

This is given by

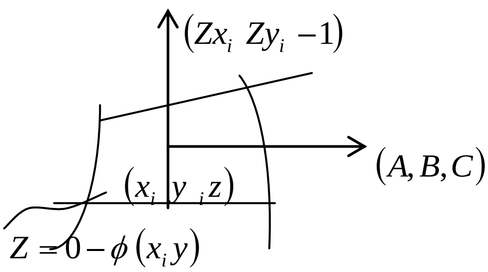
$$A(x_i\ y\ z) Zx + B(x_i\ y_i\ z) Zy = C(x_i\ y_i\ z) \qquad \text{............................} \qquad (1.1.0)$$

Where, $(x, y) \in D\ c\ R^2$ and A, B, C are

$C^0(\Omega)$, Ω being in R^3

Where projection on R^2 is 0

$(1.1.0) \Rightarrow (Zx, Zy, -1)$ is perpendicular to (A,B,C)



Implies that there exists an integral surface

$\sum = \{ (x_i \; y_i \; z) : Z \; \phi \; (x_i \; y) \}$

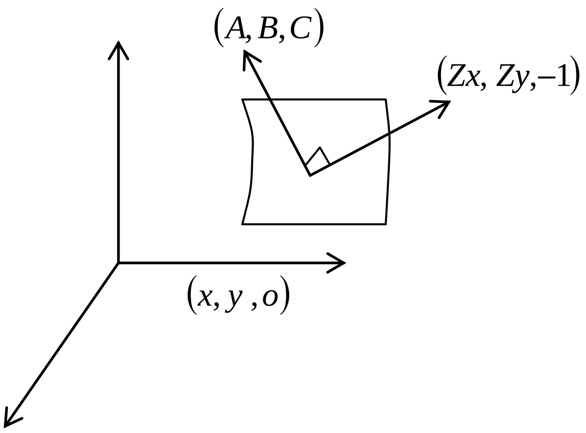
Passes thru $(x_i \; y_i \; z)$ which is target to the given vector

$A(x_i \; y_i \; z), B(x_i \; y_i \; z), \; C(x_i \; y_i \; z).$

At the given point,

$AZx + BZy = C$

can be interpreted geometrically as a requirement that any surface $Z = Z(n, y)$ thru (x, y, z) must be tangent to a prescribed vector (A, B, C).



The direction of the vector (A,B,C) is called the characteristics direction at the given point if (dx, dy, dz) lies in the tangent plane S at (x, y, z) then $(d_x \; d_y \; d_z)(Z_x, Z_y \, -1) = 0$

$\Rightarrow Zxdx + Zy\,dy = dz$

Comparing the above result with (1.1.0)

We have that

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \quad \text{..........................} \quad (1.1.1)$$

$$= \frac{dx}{dt} = A \,;\, \frac{dy}{dt} = B \,;\, \frac{dz}{dt} = C$$

$$= \frac{dy}{dx} = \frac{B}{A} ; \frac{dz}{dx} = \frac{C}{A}.$$

Define (1.1.0) $A(x, y, z)z_x + B(x, y, z)z_y = C(x, y, z)$

By the characteristic of 1.1.0 we mean the integral curves of (1.1.1)
chor $= \frac{dx}{A} = \frac{dy}{B} = \frac{dt}{C}$

Theorem 1.2

The integral curves of (1.1.1) generates the integral surface of (1.1.0)
$A(x,y,z)\,Zn + B\,(x,y,z)\,Zy = (x,y,z)$

Proof

Let $Z = Z(x,y)$ be an integral of (1.1.0)

Then $dz = Zx\;dx + Zy\;dy$ .... (1.2.0)

Suppose r is an integral curve of (1.1.1) then *dx* = Adt, by = Bdt *and* dz = C

Substituting into (1.2.0) we have (1.1.0)

It can be proved that exactly one characteristic passes through each point of S. The general solution of 1.1.1 is of the form

$$y = (x,\,\alpha,\,\beta)$$
$$z = (z,\,\alpha,\,\beta)$$

Where α amd β are arbitrary constant.

Solution for α and β we obtain

$\alpha = u\;(x,y,z)$
$\beta = v\;(x,y,z)$

Assuming that u and v are finally independent

i.e. $\frac{\partial\,(u,v)}{\partial\,(x,y)}, \frac{\partial(u,v)}{\partial(x,z)}, \frac{\partial(u,v)}{\partial(y,z)}$

are not all zero at any point (x, y, z) of S.

Definition 1.2

A single relation between u and v of the form

$$a(u,v)=0$$

Is called the general solution of (1.1.0)

Examples:

Find the general solution of

$$x\,Zx + yZy = Z$$

With characteristic equation

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

$$\frac{dx}{x}=\frac{dy}{y}$$

$$\Rightarrow \quad Iny = lnx + lnc$$

$$\frac{y}{x} = x = u(x,y,z)$$

Also $\frac{dz}{z} = \frac{dx}{x}$

$$lnZ = lnx + ln\beta$$

$$\frac{Z}{x} = \beta = v(x,y,z)$$

$$F(u,v)= 0$$

$$F(\alpha,\beta)=0$$

$$F\left({}^{y}\!/\!_{x}, {}^{z}\!/\!_{x}\right)=0$$

$$\frac{z}{x}=F\left({}^{y}\!/\!_{x}\right)$$

$$\Rightarrow Z = x\,F\left({}^{y}\!/\!_{x}\right)$$

3.4 Method of Lagrange

This is a useful technique for integrating first order equation from algebra we have that is

$$\frac{a}{b} = \frac{c}{d}$$

Then the following relationship is true

$$\frac{K_1 a + K_2 c}{K_1 b + K_2 d} = \frac{a}{b} = \frac{c}{d}$$

For arbitrary values of the multiples K_1 and K_2 so

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} = \frac{K_1 dx + K_2 dy + K_3 dz}{K_1 A + K_2 B + K_3 C} \quad \text{.......................... (1.2.1)}$$

Hence equation more convenient for integration maybe found by appropriate choice of K_1, K_2, K_3 in (1.2.1)

Further examples:

Find the general solution of

i) $(y + 2xz)z_x - (x + 2yz)z_y = \frac{1}{2}(x^2 - y^2)$
$x \in R$; $y > 0$

ii) $(z^2 - 2yz - y^2)z_x + (xy + xz)z_y = xy - xz$
$G\ x^2 + y^2 + z^2 = 0$

Solution

$$x^2 + 4zy + 2z^2 \text{ where } K_1 = x$$
$$K_2 = z$$
$$K_3 = 2z + y$$

Characteristic equations are

$$\frac{dx}{y + 2xz} = \frac{dy}{x + 2yz} = \frac{dz}{\frac{1}{2}(x^2 - y^2)}$$

By method of Langrage multiplier

$$\tfrac{1}{2}\, y\, d\ x + \tfrac{1}{2}\, x\, d\ y + dz = 0$$
$$\tfrac{1}{2}\, y\, x \quad + \tfrac{1}{2}\ x\, y \ + \ z = \alpha$$
$$2\ xy + \ 2\ z = 2\alpha = \beta$$
$$\tfrac{1}{2}\, x\, d\ x + \tfrac{1}{2}\, y\, d\ y - 2\ Z\ dz = 0$$
$$\frac{x^2}{4} + \frac{y^2}{4} - z^2 = \alpha$$
$$x^2 + y^2 - 4z^2 = \alpha$$
$$G\,(\alpha, \beta) = 0$$
$$G\left(x^2 + y^2 - 4z^2; xy + z\right) = 0$$

Initial value problem (or Cauchy problem in R^2 consists of a determination of an integral surface S of (1.1.0) which passes through a pre-assigned space curve 6. We noticed that those are the following possibilities:

i) Unique surface.
ii) Infinitely many surface.
iii) No surface depending on the pre assigned curve ξ

Examples:

2) Consider the ivp

$$\begin{cases} yZx - xZy = 0 \\ \xi\ ;\ Z\,(x,0) = x4 \end{cases}$$

The characteristic equations are

$$\frac{dx}{y}\ \frac{dy}{x} = \frac{dz}{0}$$

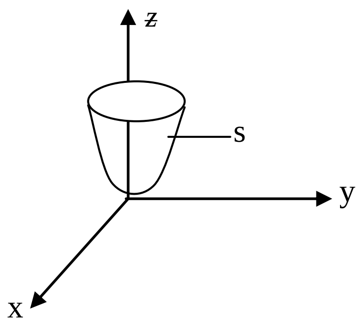
$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\, d\ x = y\, d\ y$$
$$\frac{x^2}{2} + \frac{y^2}{2} = \alpha$$
$$x^2 + y^2 = \alpha$$

and

$$\frac{dx}{y} = \frac{dz}{0}$$
$$dZ = 0 \Rightarrow Z = \beta$$
$$F(\lambda, \beta) = 0$$
$$\beta = F(\lambda)$$
$$Z = F(x^2 + y^2)$$

The general solution is any surface of revolution about z-axis



Given the curve $z = Z(x, 0) = F(x^2) = x^4$

$$\Rightarrow F(x)\, x^2$$
$$\Rightarrow Z(x, y) = (x^2 + y^2)^2$$

2) Consider the ivp

$$\begin{cases} y\ z\ x - xZy = 0 \\ \xi \ \text{: circle} \begin{cases} x^2 + y^2 = 1 \\ z = 1 \end{cases} \end{cases}$$

$$Z = F(x^2 + y^2)$$
$$1 = F(x^2 + y^2) = 1$$
$$\Rightarrow z = F1(x^2 + y^2) \text{ where f1 is any function which satisfier fi}(1) = 1$$

The solution exist but not unique.

These are certainly infinitely many such surfaces.

In this cape ξ itself is a characteristic.

3) Consider the ivp

$$\begin{cases} yZx - xZy = 0 \\ \xi : \text{ellipse} \begin{cases} x^2 + y^2 = 1 \\ Z = y \end{cases} \end{cases}$$

$$Z = F\left(x^2 + y^2\right)$$
$$\Rightarrow \quad y = F\left(x^2 + y^2\right) = F(1)$$
$Z = y$, which is impossible.

$\therefore$ No such integral surface exists.

Theorem1.8:

Let $AZx + BZy = C, \quad (x, y, z) \in \Omega;$ .... (1.8.0)

$$\left.\begin{array}{ll} A, B, C \in C^0\ (\Omega) \text{ and } \xi : & x = x_0(s) \\ & y = y_0(s) \\ 0 \le s \le 1 & z = Z_0(s) \end{array}\right\}$$ .... (1.8.1)

A given space in $\Omega \ \exists$

$$x_0, y_0, z_0 \in C^1[0_1 1]$$

Let $Ay_0{}^1 - Bx_0{}^1 \neq 0$ .... (1.8.2)

Then $\exists$ a unique solution $z = z(x, y)$ of (1.8.0) defined in some neighbourhood of the given curve ξ and which satisfiers the initial condition $Z\left(x_0^{(s)}, y_0^{(s)}\right)$

$Z\left(x_0(s), y_0(s)\right) = Z_0\ (s)$ .... (1.8.3)

Proof:

Consider the characteristic system

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$$
$$\left.\begin{array}{l} \equiv \dfrac{dx}{dt} = A \\ \dfrac{dy}{dt} = B \\ \dfrac{dz}{dt} = C \end{array}\right\}$$ .... (1.8.4)

From the existence and uniqueness theorem for P.D.E we many solve (1.8.4) for a uniquely family of characteristics

$$\left.\begin{aligned} x &= x\left(x_0(s), y_0(s), z_0(s), t\right) \\ y &= y\left(x_0(s), y_0(s), z_0(s), t\right) \\ z &= z\left(x_0(s), y_0\ (s), z_0(s), t\right) \end{aligned}\right\} \in C^1[0,1]\, x \quad \text{......................} \quad (1.8.5)$$

Such that

$$\left.\begin{aligned} x\left(s,o\right) &= x_o\ (s) \\ y\left(s,o\right) &= y_o(s) \\ z\left(s,o\right) &= z_0\ (s) \end{aligned}\right\} \quad \text{.........................} \quad (1.8.6)$$

By hypothesis the Jacobian (*J)*

$$J \;=\; \frac{\partial(x,y)}{\partial(sit)}\bigg|_{t=0} = \begin{vmatrix} x_s & xt \\ y_s & y_t \end{vmatrix} = x_s\ y_t - x_s\ y_s\big|$$

$$= \left(B\ X_s - A\ y_s\right)\big|t = 0$$

$$= BX_0\ \ Ay_0^{\ 1} \neq 0$$

$\therefore$ We can solve (1.8.5) uniquely for s and t in terms of x and y in the neighbourhood of the given curve

$$\begin{aligned} \xi :\ & t = 0 \\ & s = s\ (x,y) \\ & t = t\ \ (x,y) \end{aligned}$$

Substituting into (1.8.5) we have

$$\begin{aligned} Z = \left(s(x,y), t(x,y)\right) &= z\ (x,y) \\ &= \Phi\ (x,y) \end{aligned}$$

That $Z = \Phi(x,y)$ satisfiers the initial conditions follows from
$\Phi(x,y)\big|t = 0 = Z(s,o) = Z_0(s)$

Φ satisfies the Partial Differential Equation for

$$\begin{aligned} & A\,\Phi x \quad + \quad B\Phi y \\ &= A\left(Z_s\ S_x\ + Z_t\ \ t_x\right) + \ B\left(Z_s\ S_y\ + Z_t\ t_y\right) \\ &= Z_s\ \left(AS_x + BS_y\right)\ \ +\ \ Z_t\left(At_x\ +\ \ Bt_y\right) \\ &= Z_s\ \left(s_x\ x_t\ + s_y\ \ y_t\right) + Z_t\ \left(t_x\ x_t\ + t_y\ \ y_t\right) \\ &= Z_s\ \frac{ds}{dt} + z_t\ \frac{dt}{dt} = Zs\ (0) + Z_t(1) \\ &= Z_t\ = c \end{aligned}$$

Uniqueness follows from theorem (1.2). $AZ_n + BZ_y = C$

The integral curves of $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$ generates the integral surface.

Summary: - Cauchy problem has a unique solution provided the initial curve is not characteristic.

Exercises:

1) Solve the following:

$$ZZx \;+\; Zy = 1$$
$$x = \; s$$
$$y = s$$
$$z = \tfrac{1}{2}\, s,\; 0 \le s \le 1$$

2)

$$Zy + \;(Zx \;= 0, \quad x \in R, \;\; y$$

$$Z\,(x,o) = \begin{cases} 1 - x^2, \; |x| \ge 1 \\ \\ 0\,, \quad\; |x| > 1 \end{cases}$$

$$Z\,(x,y) = F\,(x - cy) = 1 - (x - cy)^2 \;\; - \text{unique solution}$$
$$\mathrm{Z}\,(\mathrm{x},\, \mathrm{y}) = F\,(n - cy) = 0 \qquad\qquad - \text{intrnitty many}$$

Solution 2

We observed that $Ay_0^1 - Bx_0^1 =$

$$A = Z, \;\; B = 1$$
$$y_0^1\,(s) = 1,\, x_0^1 \;= 1$$
$$Ay_0^1 - Bx_0^1 \;= Z - 1 \ne 0 \quad \text{for } \mathrm{Z} \ne 1$$
$$0 < \le \in 1$$

Characteristic equation is

$$\frac{dx}{Z} = \frac{dy}{i} \;= \frac{dZ}{i}$$

So that we now have

$$\frac{dx}{y} = Z,\; \frac{dy}{-x} = \;1, \frac{dz}{dt} = 1$$

$$\frac{dz}{dt} = 1 => Z = t + \lambda$$

$$Z(so) = \frac{s}{2} = \lambda$$

$$Z = t + \lambda = t + s/2$$

Similarly

$$y = t + \beta$$

$$y(so) = s = \beta$$

$$\Rightarrow y = t + s$$

$$\frac{dx}{dt} = Z = t + 1/2\, s$$

$$x = \frac{t^2}{2} + \frac{1}{2}st + \alpha$$

$$S \; s = \alpha$$

$$\Rightarrow x = \frac{t^2}{2} + \frac{1}{2}\, st + s$$

$$= 1/2\, t\,(t + s) + s$$

$$= 1/2\, yt + s$$

Write t = y –s and substitute into x so that

$$X = 1/2\, y(y - s) + s$$

$$= \frac{1}{2}y^2 - 1/2\, ys + s$$

$$s = x - 1/2\, y^2 / 1 - y/2$$

$$s - \frac{x - 1/2\, y^2}{1 - y/2} \text{ ....}(i)$$

Also write s = y – t and substitute into

$$x = 1/2\, yt + y - t$$

$$t = \frac{y - x}{1 - 1/2} \text{ ....}(ii)$$

Substitute (i) and (ii) in

$$Z = t + \tfrac{1}{2} S$$

$$Z = \frac{y - x}{1 - y/2} + \frac{1}{2}\left(\frac{x - y^2/2}{1 - \frac{y}{2}}\right)$$

$$= \frac{\frac{y^2}{2} - 2y - x}{y - 2.}$$

4.0 CONCLUSION

In this unit we have studied some basic and essential definitions of Partial Differential Equations; specifically those properties and general characteristics of First Order Equation, Quasi – Linear Equations and the utilisation of the Method of Lagrange in solving Partial Differential Equations.

We examined Partial Differential Equations from the perspectives of existence and uniqueness of solutions, stability of solution to small perturbations around the solution as well as the different methods for constructing solutions.

5.0 SUMMARY

Partial Differential Equations can be generically classified into families and methods of solution for classes categories based on their properties.

6.0 TUTOR-MARKED ASSIGNMENT

1. Which of the following Partial Differential Equations is linear, quasi-linear or non-linear?

   If P.D.E. is linear, state whether it is homogeneous equation or not.

   a. $u_{xx} + u_{yy} - 2u = x^2$
   b. $u_{xy} = u$
   c. $u\,u_x + x\,u_y = 0$
   d. $u_x^2 + \log u = 2xy$
   e. $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
   f. $u_x(1 + u_y) = u_{xx}$
   g. $(\sin u_x)u_x + u_y = e^x$
   h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   i. $u_x + u_x u_y - u_{xy} = 0$

2. Give the order of each of the following:

a. $u_{xx} + u_{yy} = 0$
b. $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x, y)$
c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x, y)$
d. $u\, u_{xx} + u_{yy}^2 + e^u = 0$
e. $u_x + cu_y = d$

3. Find the general solution of

$$u_{xy} + u_y = 0$$

4. Show that $u = F(xy) + x\,G(\frac{y}{x})$

is a general solution of $x^2u_{xx} - y^2u_{yy} = 0$

7.0 REFERENCES/FURTHER READING

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UNIT 2 APPLICATION OF IVP CONSERVATION LAW, DEVELOPMENT OF SHOCK

CONTENTS

1.0 INTRODUCTION

The characteristic equation for the single conservation law is derived and solved with the assumption of implicit function, discontinuity which implies shock is also demonstrated.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state conservation law
- explain the concept of shock.

3.0 MAIN CONTENT

3.1 Application of IVP Conservation Law, Development of Shock

The conservation law states that rate of change of total substance contained in a fixed (arbitrary) domain Ω is equal to the flux of that substance across the boundary $\partial\Omega$.

Let U be the density of the substance and F = flux, then the conservation law is given by rate of flow $\frac{d}{dt}\int_{\Omega} udx = -\int_{\partial\Omega} \underline{F}.\,\underline{n}\,ds$

$\Rightarrow \int \frac{\partial}{\partial t} u\,d\,x = \int_{\Omega} Ut\ \ dx + \int_{\partial\Omega} \underline{F}.\,\underline{n}\,ds = 0$, $n-$normal vector to the surface.

$$\int_{\Omega} utdx - \int_{\Omega} div\,Fdx = 0$$

$ut - div\, F = 0$

Single conservation law

$$ut \ + \underline{f}x = 0$$
$$\Rightarrow \quad ut \ + a(u)\, ux = 0 \quad . \underline{\qquad\qquad\qquad\qquad} (ii)$$

Characteristic equation

$$\frac{dt}{1} = \frac{dx}{a(u)} = \frac{du}{o}$$
$$\frac{du}{0} = \frac{dt}{1}$$
$$\Rightarrow U = constant\ along\ \frac{du}{dt} = a\ (u)$$

i.e. on the characteristic $x = x(t)$ which propagates with speed a
u is a constant
a = signal of the speed.

Solve the following IVP

i) $ut + a(u)\, ux = 0$

$u\,(x,o) = f(x)$

$$\frac{dt}{1} = \frac{dx}{a(u)} = \frac{du}{0}$$
$$\Rightarrow u = \alpha$$
$$\frac{dx}{dt} = a(u) = a(\alpha)$$
$$x = at + \beta$$

General solution $F\ (\alpha,\, \beta) = 0$

$F\ (u,\, x - a(u)\, t) = 0$

$\Rightarrow \ u = F\ (x - a(u)\, t)$

$\Rightarrow$ Solution is implicitly defined by

$u = F\ (x - a(u)\, t)$

$Ut \quad Fu\, Ut \qquad u = x - a(u)t$

$= F^1(-a(u)) - \ u = x - a(u\ t)$

$(1 + F^1 a^1 t)\, ut = -f^1 a(u)$

$$\Rightarrow Ut = \frac{a(u)\ f^1}{1 + a^1 f^1 t}$$

$U\ x \ = F^1(1 - a^1 u\ x\ t)$

$$U\ x \ = \frac{f^1}{1 + a^1 f^{1t}} \qquad \text{Assuming implicit function theorem}$$

Therefore U given implicitly satisfies the P.D.E provided

$1+a^1 f^1 t \neq 0.\text{if } 1+\text{a}^1 f^1 t = 0$

$Ut,\ Ux$ will become infinite and shock is said to be developed i.e.

a discontinuity exist in Ω. If:

1) a is constant, no shock $\forall\ t \geq 0$

2) F is constant, no shock $\forall\ t \geq 0$

3) a, f, both non – deterring or non-increasing

For non – decreasing $f^1 \geq 0$

For non – increasing $f^1 \leq 0$

$a^1 f^1 \geq 0, \text{no - shock}\ \lambda \in\geq 0$

Exercises:

1. Find a solution of $Z\ Z\ x + Zy = 0$

$Z\,(x,o) = x$

Draw the lines in the x – y plane, along where solution is constant. Do shocks ever developed for $y \geq 0$?

2. $Z^2\ x + Zy = 0$

$Z\,(x,o) = x$

Derive the solution $Z(x,y) = \begin{cases} x \ \ \text{when y} = 0 \\ \sqrt{\dfrac{1+4xy}{2y}} \ -1\ y \neq 0 \\ \qquad\qquad 1+4xy > 0 \end{cases}$

Do shocks ever developed? Show that

$\text{Lim}\ \text{Z}(\text{x, y}) = x$

$y \to 0$

$\dfrac{dx}{z} = \dfrac{dy}{1} = \dfrac{dz}{0}$

$\Rightarrow Z = \alpha \quad d\ x = z\ d\ y$

$x = zy + \beta$

$z\,(x,o) = \beta = x$

$= z\ y + x = z$

$z\ = \dfrac{x}{1-y}$

The solution is constant along the lines

$y = 0,\ y >^1_0,\ y \leq -1$

$zx\ = \dfrac{x}{(1-y)^2}$

Shock develop for y = 1.

4.0 CONCLUSION

This unit has practically exposed us to the real world application of Partial Differential Equations through a scenario involving conservation law where we determine shock.

5.0 SUMMARY

The law of conservation states that the rate of change of total substance contained in a fixed domain is equal to the flux of that substance across the domain boundary.

6.0 TUTOR-MARKED ASSIGNMENT

1. Derive the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}$$

by considering the vibration of a string under a damping force proportional to the velocity and a restoring force proportional to the displacement.

2. Use Kirchhoff's law to show that the current and potential in a wire satisfy

$$\begin{aligned} i_x + C\,v_t + Gv &= 0 \\ v_x + L\,i_t + Ri &= 0 \end{aligned}$$

where *i* = current, *v* = *L* = inductance potential, *C* = capacitance, *G* = leakage conductance, *R* = resistance

7.0 REFERENCES/FURTHER READING

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