MODULE 2

UNIT 1 GENERAL FIRST ORDER EQUATION AND CAUCHY METHOD OF CHARACTERISTIC

CONTENTS

1.0 INTRODUCTION

General non – linear first order partial differential equations have a form *F(x,y,z,p,q) = 0 where p = Zx and q = zy* whose solution lead to the concept of the Monge cone and the chain curve stripe.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use General non – linear First Order Equation to solve some problems
- sketch and explain the Monge cone
- apply Cauchy Method of Characteristic equations

3.0 MAIN CONTENT

3.1 General First Order Equation

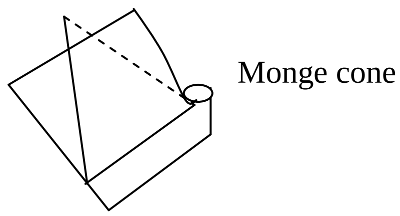
The general non – linear P.D.E of 1st order has the form

$F(x, y, z, p, q) = 0$ .... (1.9.1)

Where $p = z_x$ and $q = z_y$

At each point $p(x, y, z)$ on an integral surface $z = z\,(x, y)$ the direction number $(p, q, -1)$ of the normal to the surface are related through equation (1.9.1)

The P.D.E will restrict its solution to these surface having tangent planes belonging to a 1-parameter family $q = G\,(x, y, z, p)$. Generally this one – parameter family of planes envelope a cone called the Monge cone.

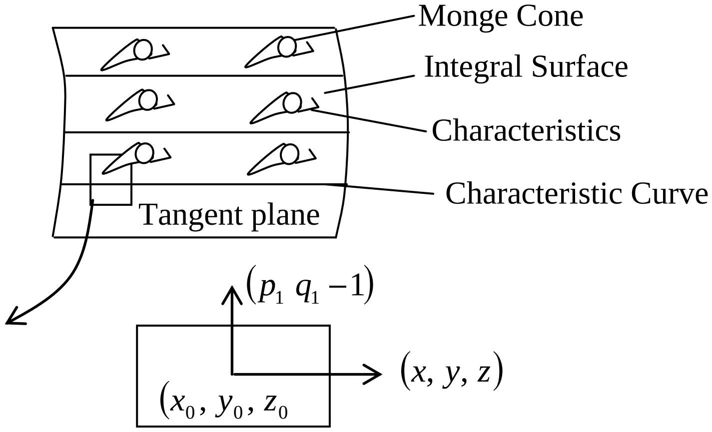


The geometrical significance of the 1st order P.D.E in (1.9.1) is that any solution surface through a point in space must be tangent to the corresponding Monge cone

3.2 Cauchy Method of Characteristic

Let $z = z\,(x, y) \in C^2$ be a given integral surface. At each point the surface will be tangent to the Monge cone

Furthermore, the lines of contacts between the tangent planes of the surfaces and the cones define a field of directions on the surface called characteristic direction. These integral curves of field define a family of characteristic curves.



The Monge cone at a fix point $(x_o\ y_o, z_o)$ in the envelope of one particular family of planes.

$$\left.\begin{aligned} z - z_o &= p\,(x - x_0) + q\,(y - yo) \\ \text{where } F\,&(x_o, y_o\, z_o, p, q) = 0 \\ \text{or } q &= q\,(x_o y_{o,} z_o, p) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.10.1)$$

It's thus given by

$$\left.\begin{aligned} z - z_0 &= p\,(x - x_0) + q\,(x_0, y_0, z_0, p)(y - y_0) \\ o &= x - x_0 + \frac{dq}{dp}(y - y_0) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.10.2)$$

Where p is adopted as the parameter using (1.10.1) we have

$$\frac{df}{dp} = Fp + Fq\frac{dq}{dp} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.10.3)$$

Eliminating $\frac{dq}{dp}$ from (1.10.2) yields the Equation

$$\frac{df}{dp} = F_p - F_q\left(\frac{x - x_0}{y - y_0}\right)$$

Equation for the Monge cone

$$\left.\begin{aligned} &F\,(x_0, y_0, Z_0, p, q) = 0 \\ &z - z_0 = p(x - x_0) + q\,(y - y_0) \\ &\frac{x - x_0}{Fp} = \frac{y - y_0}{Fq} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.10.4)$$

Eliminating p and q from (1.10.4) yield a more standard form of the equation of the cone.

If given p and q the last two equation of (1.10.4) define the line of contact between the cone and the tangent plane.

It may be written in the form

$$\left.\begin{aligned} u &= g\,(x_i\ y_i\ z) \\ v &= h\,(x_i\ y_i\ z) \end{aligned}\right\} c^i\,(\Omega)$$

The characteristic direction is

$$(Fp, Fq, pFp + qFq) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.10.6)$$

If therefore follows that the characteristic curves are determined by the O.D.E

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q}$$

$$\equiv \frac{dx}{dt} = F_p, \frac{dy}{dt} = F_q, \frac{dz}{dt} - pF_p + qF_q \qquad \text{...........} \qquad (1.10.7)$$

Assuming that the integral surface is yet unknown, the 3 equations in (1.10.9) are not sufficient to determine the characteristic curve comprising the surface.

This is because the equation contains 2 addition unknowns p and q.

However, along a characteristic curve on the given integral surface we have

$$\left.\begin{aligned} {dp}/{dt} &= p_x x_r + p_y y_t = p_x f_p + p_y f_q \\ {dq}/{dt} &= q_x x_t + q_y y_t = q_x f_p + q_y f_q \end{aligned}\right\} \qquad \text{.........................} \qquad (1.10.8)$$

Differentiating the given Partial Differential Equation in (1.9.1), we have

$$F\ x + F = p + Fp\ px + Fq\ \ qx = 0$$
$$F\ y + Fz\ \ q + Fp\ py + Fq\ \ qy = 0$$

But $\quad q_x = \dfrac{\partial^2 z}{\partial x \partial y}$

So

$$F_x + F_{zp} + \left(F_p\ P_x + F_q\ P_y\right) = 0$$
$$F_y + F_{zq} + \left(F_p\ q_x + F_q\ q_y\right) = 0$$

(1.10.8) then yields

$$\left.\begin{aligned} \frac{dp}{dt} &= -\ Fx - pFz \\ \frac{dq}{dt} &= -\ Fy - qFz \end{aligned}\right\} \qquad \text{..................................} \qquad (1.10.9)$$

The 5 equations in 1.10.7 and 1.10.9 are called the characteristics equation associated with the Partial Differential Equation. The situation is now more complicated than in (1.9.1). All together we have 6 equations.

$$F\,(x,\,y,\,z,\;p,\,q) = 0$$
$$\frac{dx}{dt} = Fp,\; \frac{dy}{dt} \;=\; Fq$$
$$\frac{dt}{dt} \;=\; -\;(Fx + pFz)$$
$$\frac{dq}{dt} \;=\; (Fy + qFz)$$
$$\equiv \quad F\,(x,\,y,\,z,\;p,\,q) = 0$$
$$\frac{dx}{Fp} \;=\; \frac{dy}{Fq} \;=\frac{dz}{pFp + qFq}$$
$$= \frac{dp}{-[Fx + pFz]} \;= \frac{dq}{-[Fy + qFz]}$$

For the 5 – unknown functions

$x(t),\ y(t),\ z(t),\ p(t),\ q(t)$

In other words if $F\,(x,\,y,\,z,\;p,q) = 0$ is satisfied at an initial point say $x_0,\, y_0,\, z_0,\; p_0,\, q_0$ for $t = 0$. The 5 characteristic equations in (1.10.7), (1.10.9) will determine a unique solution $x(t),\ y(t),\ z(t),\ p(t)\,q(t),$ passing through the x point and along which $f = 0$ will be satisfied for all t.

Theorem 1.11

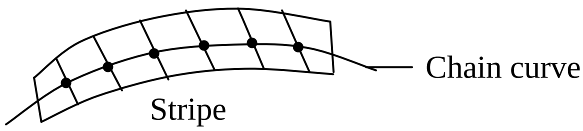
Along any solution of characteristic equation of (1.10.10) $F\,(x,\,y,\,z,\;p,\,q) = 0$

Proof

Exercises:

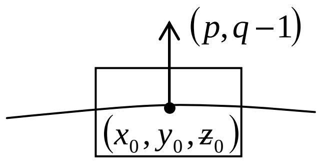
Defined 1.12

A ship is defined as a space curve $x = x(t)$ $y = y(t)$ and $z = z(t)$ in addition to the family of tangent planes with $(p,\,q,-1)$ as normal.



Defined 1.13

An element of a stripe is defined as a point on a characteristic curve including the corresponding tangent plane at that point.



An element

Remark

Note that not any set of 5 functions can be interpreted as a strip.
The planes must be tangent to the curve which is that conditions that

$$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt}$$

Theorem 1.14

If a characteristic strip $x(t), y(t), z(t), p(t), q(t)$ has $x_0\ y_0\ z_0\ p_0\ q_0$ in common with an integral surface $z = u(x, y)$ then it lies completely on that surface.

Theorem 1.15

Given $F\ (x, y, z, p, q) = 0$ (1.9.1)

And suppose along the initial curve

$x = x_0\ (s), y = y_0\ (s),\ 0 \le \varsigma \le 1$, the initial values $z = z_0(s)$ are assigned and $x_0, y_0, z_0 \in C^2\ [0,1]$ have been determined satisfying

$$F\ (x_0\ (s), y_0\ (s), z_0\ (s), p_0\ (s), q_0\ (s)) = 0 \text{ and}$$

$$\frac{dz_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds} \text{ with}$$

$$\frac{dx_0}{ds}\ fq\ (x_0, y_0, z_0, p_0, q_0) - \frac{dy_0}{ds}\ fp(x_0, y_0, z_0, p_0, q_0) \ne 0$$

then in the same neighbourhood of the initial curve, there exists a unique solution $z = z\ (x, y)$ of (1.9.1) containing the initial strip that is such that.

$$z\ (x_0(s), y_0\ (s)) = z_0(s)$$
$$zx\ (x_0(s), y_0\ (s)) = p_0(s)$$
$$zy\ (x_0(s), y_0\ (s)) = q_0(s)$$

4.0 CONCLUSION

Solution for general non – linear Partial Differential Equations of 1st order has a geometrical significance in relation to the Monge cone.

5.0 SUMMARY

The form of the general non – linear first order Partial Differential Equation is $F(x,y,z,p,q) = 0$ where any solution surface through a point in space must be tangent at that point to the corresponding Monge cone.

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve $$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$$

   subject to $w(x,0) = \sin x$

2. Solve the following equation using the method of characteristics

   a. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = f(x)$

   b. $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 1$ subject to $u(x,0) = f(x)$

   c. $\frac{\partial u}{\partial t} + 3t\frac{\partial u}{\partial x} = u$ subject to $u(x,0) = f(x)$

   d. $\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = \cos x$

   e. $\frac{\partial u}{\partial t} - t^2\frac{\partial u}{\partial x} = -u$ subject to $u(x,0) = 3e^{x}$

3. Show that the characteristics of

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = f(x)$$

   are straight lines

4. Take a look at the problem

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

a. Determine equations for the characteristics
b. Determine the solution $u(x, t)$
c. Sketch the characteristic curves.
d. Sketch the solution $u(x, t)$ for fixed t.

7.0 REFERENCES/FURTHER READING

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UNIT 2 TYPES OF SOLUTIONS

CONTENTS

1.0 INTRODUCTION

Partial Differential Equations can have three types of solutions; the complete solution, the general solution and the singular solution. All are treated in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- categorise the different types of solutions of partial differential equations
- decribe the methods used in deriving complete solution
- explain what a general solution is
- explain why some Partial Differential Equations are singular solutions.

3.0 MAIN CONTENT

3.1 Types of Solutions

We observed that the general solution of the 1st order P.D.E (1.9.1) is an expression involving an arbitrary function of one variable.
This naturally is the extension of the result that the general solution of first order PDE involves one arbitrary constant.

3.1.1 Complete Solution (Integral)

Any solution of the form

$$Z = \Phi(x, y, a, b) \qquad \text{.................................} \qquad (1.16.1)$$

Where a and b are arbitrary parameters represent two parameter family of surfaces. No systematic rule determining the complete integral is available. The complete integral is significant in the sense that the envelope of any family of solution of the 1st order equation (1.1.1) depending on some parameter is again a solution. Indeed equation 1.9.1 defines the tangent plane of a solution. If a surface has the same tangent plane as a solution at some point in space, then it also satisfies the equation there. The envelope of a family of solutions is also a solution since it is in contact at each of its points with one of these earlier mentioned solutions.

3.1.2 General Solution (Integral)

The general solution of 1.9.1 can thus be obtained from the complete integral if we prescribe the 2nd parameter b, say b = b(a) as an arbitrary function of a. The enveloped of the one parameter subsystem of the complete integral is then considered as follows

$$Z = \Phi(x, y, a, b(a)) \qquad \text{....................................} \qquad (1.16.1)$$

Differentiating with respect to (wrt) a, we have

$$O = \underline{\Phi}_a (x, y, a, b(a)) + \Phi_b (x, y, a, b(a)) \frac{db}{da} \qquad \text{...........................} \qquad (1.16.2)$$

Eliminating (a) between 1.16.1 and 1.16.2 yield a single expression (involving the arbitrary function b(a) which is the general solution of (1.9.1)

3.1.3 Singular Solution

This is the envelope of the full two parameter family of surfaces defined by the complete solution and is given by the 3 relation

$$Z = \Phi(x, y, a, b)$$
$$O = \Phi_a (x, y, a, b)$$
$$O = \Phi_b (x, y, a, b)$$

Examples

Type I

$F\left(p, q\right) = 0$

Solve $p^2 - q^2 = 1$

Write $f\left(p, q\right) = p^2 - q^2 - 1 = 0$

$F\left(a, h(a)\right) = a^2 - \left(h(a)\right)^2 - 1$ and

$$h(a) = \left(a^2 - 1\right)^{1/2}$$

A complete solution is

$$Z = ax + \left(a^2 - 1\right)^{1/2} y + c$$
$$Z = ax + by + c$$

Put $b = \left(a^2 - 1\right)^{1/2}$ and diff with a to get

$$O = x + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$
$$\frac{-x}{y} = \frac{a}{\left(a^2 - 1\right)^{1/2}}$$

General solution is

$$\frac{\cancel{Z}}{a} = x + \left(\frac{y}{x}\right) y + \frac{c}{a}$$
$$\alpha\ x\ z = x^2 - y^2 + \alpha x \quad \left(\alpha = 1/a\right)$$
$$\alpha\ x\left(z - 1\right) = x^2 - y^2$$

There are singular solutions since

$$z = a\ x + by + c$$
$$o = x + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$

O = 1

Example:

Consider $p^2 + q^2 = 1$

Recall $f\left(x, y, z, p, q\right) = o$

$$p^2 + q^2 - 1 = o$$

By $\frac{dx}{fp} = \frac{dy}{fq} = \frac{dz}{pf_p + qfq} = \frac{dp}{-(fx + pfz)} = \frac{dq}{-(fy + qfz)}$

$$fx = o,\ fy = o,\ fz = o,\ fp = 2p,\ fq = 2q$$

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{dz}{2(p^2 + q^2)} = \frac{dp}{o} = \frac{dq}{o}$$

$dp = o$

$$p = a\ (a \text{ is constant})$$

$$q^2 = 1 - a^2$$

$$q = (1 - a^2)^{1/2}$$

$$p = zx = \partial z/\partial y = (1 - a^2)^{1/2}$$

$$dz = \partial z/\partial x + \partial z/\partial y dy$$

$$\int dz = \int a\ d\ x + \int (1 - a^2)^{1/2} dy$$

$$z = ax + (1 - a^2)^{1/2} y + b$$

General solution is given by

$$z = ax + \sqrt{1 - a^2}\, y + \Phi(a)$$

Differentiating wrt a we have

$$o = x - \frac{a}{\sqrt{1 - a^2}}\ y + \Phi^1(a)$$

Singular solution: None

Since $z = ax + \sqrt{1 - a^2}\, y + 6$

Differentiating wrt a

$$za = o = x - \frac{a}{\sqrt{1 - a^2}}$$

$$zb = o = \sqrt{1 - a^2}$$

Examples:

Given $xp + yq = pq$, Find

1. The initial element if $x = x_o$, $y = o$ and $z = x_o/2$ $\ z(x, o) = x/2$
2. The characteristics stripe containing the initial elements
3. The integral surface which contain the initial element.

Solution

$$xp + yq = pq$$
$$xp + yq - pq = o$$
$$f\ (x, y, z, p, q) = o$$
$$\left(x_o, o, \tfrac{1}{2}x_o,\ p_o, q_o\right)\text{assume}$$
$$x_o p_o = p_o\ q_o$$
$$\Rightarrow x_o = q_o$$

According to the strip condition

$$dz\ =\ \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$
$$\frac{\partial z_o}{dx_o} = \frac{\partial z_o}{dx_o} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial x}$$
$$\tfrac{1}{2} = po$$

Initial element is $\left(x_o,\ o,\ \tfrac{1}{2}\ x_o,\ \tfrac{1}{2},\ x_o\right)$

For simplicity let us take $x_o = 1$

For the characteristic equation $\frac{dx}{fp} =, \frac{dy}{fq}\ \frac{dz}{pfp + qfq} = \frac{dp}{-(fx + pfz)} = \frac{dq}{-(fy + qfz)}$

$$\frac{dx}{dt} = x - q \qquad \frac{dq}{dt} = -q$$
$$\frac{dp}{dt} = -p \qquad \frac{dz}{dt} = -\ pq$$
$$\frac{dy}{dt} = y - p$$

Integrating we obtain

$$x\ =\ xo\cosh t$$
$$y\ =\ \tfrac{1}{2}\sinh t$$
$$z\ =\ \tfrac{1}{4}xo\left(e -^{2t}\ +1\right)$$
$$p\ =\ \tfrac{1}{2}e^{-t}$$
$$q\ =\ xoe^{-t}$$

Eliminating x_o and t from above, we obtain

$$8xyz + x^2 = 4z^2$$

Exercise

Solve

1) $pq = u$ with $u\left(o,s\right) = s^2$

2) determine the integral surface of $xpq + yq^2 = 1$ which contain the curve $z = x_o y = o$

Earlier Example

$$p^2 - q^2 = 1$$
$$f\left(x,y,z,p,q\right) = p^2 - q^2 - 1$$
$$fx = 0,\ fy = 0,\ fz = 0,\ fp = 2p,\ fq = 2q$$
$$\frac{dx}{fp}, = \frac{dy}{fq} = \frac{dz}{pfp + qfq} = \frac{dp}{-\left(fx + pfz\right)} = \frac{dq}{-\left(fy + qfz\right)}$$
$$\frac{dx}{2p} = \frac{dy}{-2p} = \frac{dz}{2p^2 - 2q^2} = \frac{dp}{o} = \frac{dq}{o}$$
$$p = a$$
$$q^2 = p^2 - 1$$
$$q^2 = \left(a^2 - 1\right)$$
$$q = \sqrt{q^2 - 1}$$
$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$
$$dz = adx + \sqrt{a^2 - 1}dy$$
$$z = ax + \sqrt{a^2 - 1}\ y + c$$

Put

$$b = \left(a^2 - 1\right)^{1/2} \text{ and diff wrt a}$$
$$\text{o} = \text{x} + \frac{ay}{\left(a^2 - 1\right)^{1/2}}$$
$$\frac{-x}{y} = \frac{a}{\left(a^2 - 1\right)^{1/2}}$$

General solution is

$$\frac{z}{a} = x + \left(-y/x\right) y + c/a$$

We can rewrite it as

$$\alpha\ x\ z\ = x^2 - y^2 + \alpha cx \quad \left(\alpha = 1/a\right)$$
$$\alpha\ x\ z - \alpha cx = x^2 - y^2$$
$$\alpha\ x\left(z - c\right) - x^2 - y^2$$

There is no singular solution

$$z = ax + by + c$$
$$\left.\begin{array}{ll} o = x + \dfrac{a}{\left(a^2 - 1\right)^{1/2}}\, y & \text{wrt a} \\ \text{o} = \text{y} & \text{wrt b} \end{array}\right\}\text{are not consistent}$$

Exercise:

Find the complete and singular solution of

$$p^2 + q^2 = 9$$

TYPE II

Consider $z = p_x + q_y + f\left(p, q\right)$

Solution

Using the characteristic equation

$$i.e\ \frac{dx}{fp} = \frac{dy}{fq} = \frac{dz}{pfp + qfq} = \frac{dp}{-\left(fx + pfz\right)} = \frac{dq}{-\left(fy + qf\right)}$$

Then

$$f\left(x, y, z, p, q\right) = z - px - qy - f\left(p, q\right) = 0$$
$$fx = -p \quad fy = -q$$
$$fz = 1 \qquad fp = -x \ \ fp = -\left(x + fp\right)$$
$$fq = -\left(y + fq\right)$$

Then

$$\frac{dx}{-\left(x + fp\right)} = \frac{dy}{-\left(y + fp\right)} = \frac{dz}{-p\left(x + fp\right) - q\left(y + fq\right)}$$
$$= -\frac{dp}{o} = -\frac{dq}{o}$$
$$\left.\begin{array}{l} dp = o \Rightarrow\ p = a \\ dq = o \Rightarrow\ q = b \end{array}\right\}\text{constant}$$

Complete solution b

$$z = ax + by + f\left(a, b\right)$$

Exercise:

Solve $(p+q)(z-xp-yq)=1$

Find the complete solution

$$z = xp + yp - \frac{1}{(prq)}$$

Solve 4 $(1+z^3) = 9z^4 pq$

$$\frac{4+4z^3}{qz^4} = pq$$

$$\frac{4}{q}\ z^{-4} + \frac{4}{q} z^{-1} - pq = 0$$

4.0 CONCLUSION

General solution of First Order Partial Differential Equations results in an expression involving an arbitrary function of one variable.

5.0 SUMMARY

The different types of solution of Partial Differential Equations are categorised into complete solution, general solution and singular solution.

6.0 TUTOR-MARKED ASSIGNMENT

1. Determine the general solution of

   a. $u_{xx} - \frac{1}{c^2}u_{yy} = 0 \qquad c = \text{constant}$
   b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
   c. $u_{xx} + u_{xy} = 0$
   d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

2. Show that
   is parabolic for *a*, *b*, *d* constants.

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