MODULE 3

UNIT 1 SECOND ORDER P.D.E. CLASSIFICATIONS

CONTENTS

1.0 INTRODUCTION

With specific reference to Second Order Partial Differential Equations, the various classifications are discussed with reference to the Tricomi's equation, characteristics, and case of two independent variables.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- classify Second Order Partial Differential Equations
- explain the importance of the Eigen-values of the matrix of coefficients
- state Tricomi's Equation
- work with Laplace, heat and Wave Equations
- describe the special case of two Independent Variables.

3.0 MAIN CONTENT

3.1 Second Order P.D.E. Classifications

A second order semi – linear equation

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2 u}{\partial x \partial x_j} = g\left(x, u, \frac{\partial u}{\partial x_i}, \ldots, \frac{\partial u}{\partial x_n}\right) \quad \text{.........................} \quad (2.1.1)$$

Will be classified according to the properties of the Eigen-values of the matrix$(Aij) = (Aji)$ of the coefficients of the highest order P.D.E. Derivations at any point x of a domain $\Omega\, C\, R^n$

a. If all the Eigen-values are different from zero and all are of the same sign at x, the equation is said to be elliptic at the point x.

$$\begin{pmatrix} +++ & +++ \\ --- & --- \end{pmatrix}$$

b. If all the Eigen-value are different from zero and all but one have the same sign at x. It means that the Partial Differential Equation is normal hyperbolic at x.

$$\begin{pmatrix} -+++ & +++ \\ +--- & --- \end{pmatrix}$$

c. If all the Eigen-values are different from zero and there are at least two of each sign at x. The Partial Differential Equation is ultra hyperbaric at that pod.

$$\begin{pmatrix} --+ & ++++ \\ ++------ & \end{pmatrix}$$

d If one Eigen-value is zero and the rest are of one sign at x. The Partial Differential Equation is parabolic at that point. If at least two Eigen-values are zero and the rest are of one sign at x. The equation is elliptic parabolic at that point.

$$\begin{pmatrix} 00 & ++++++ \\ 00 & ------ \end{pmatrix}$$

If an Eigen-value is zero and there is at least one positive and one negative at x, the equation is hyperbolic parabolic at x $\begin{pmatrix} 0-++++ \\ 0+---- \end{pmatrix}$ $\begin{pmatrix} 0--1-1 \\ 0+1--- \end{pmatrix}$. The Partial Differential Equation

(2.1.1) is said to be of one of the above types in a domain Ω in R^n if it is so at every point x in Ω. Otherwise the equation is said to be of the mixed type in Ω. The above classification is applicable to quasi – linear second order equations.

$$\sum_{i,j=1}^{n} A\left(\underline{x}, u, u_{x_1}, \cdots\cdots u_{x_n}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} = g\left(\underline{x}, u, u_{x_1}, \cdots, u_{x_n}\right)$$

However, the sign of the solution $u(\underline{x})$ and the signs of the 1st order P.D.E derivation $ux, r........, uxn$ may have to be known.

Examples:

a) A Laplace equation $\Delta u = 0$ is elliptic since

$$\Delta u = \frac{\partial^2 u}{\partial x_1{}^2} + \cdots + \frac{\partial^2 u}{\partial x_n{}^2} = 0 \text{ and}$$

$$Aij = \begin{pmatrix} 1\,0\,0---0 \\ 0\,1\,0---0 \\ \vdots \quad 1 \qquad \vdots \\ 0 \quad \ddots \qquad \vdots \end{pmatrix}$$

b) Heat Equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u = o \text{ is parabolic}$$

Since $\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_n^2} + \frac{o\partial^2 u}{\partial t^2} - \frac{\partial 4}{\partial t} = o$ and

$$Aij = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & & & & & & \end{pmatrix}$$

$$\lambda = 1, 1, 1, 1, 0$$

c) Wave Equation

$$\boxed{u} = \left(\Delta - \frac{\partial^2}{\partial t^2}\right)u = 0$$

e.g. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{dy^2} - \frac{d^2 y}{dz^2} = 0$$

$$Aij \quad \neq \begin{pmatrix} 1 & 0 & - & - & - & 0 & 0 \\ 0 & 1 & - & - & - & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & & & \\ 0 & \vdots & & - & 1 & & \end{pmatrix} \lambda = 1,1,1,---1$$

It is normal hyperbolic

From $(x\ y\ z)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$a_{11}x^2 + 2a_{21}xy + 2a_{31}xy$$
$$+ 2\ a_{32}yz + a_{22}y^2 + a_{33}z^2$$

Exercise:

$$z\ \frac{\partial^2 u}{\partial x \partial y} + 2x\ \frac{\partial^2 u}{\partial y\ \partial z} = 0$$

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x dz} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{dy^2 z} & \frac{\partial^2 y}{\partial z^2} \end{pmatrix}$$

Is hyperbolic – parabolic in R^3

$$Aij = \begin{pmatrix} o & z & o \\ z & o & x \\ o & x & o \end{pmatrix}$$

Using $|A - \lambda I| = o$

$$= \begin{vmatrix} o-\lambda & z & o \\ z & o-\lambda & o \\ o & x & o-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & z & o \\ z & -\lambda & o \\ o & x & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & x \\ x & -\lambda \end{vmatrix} - z \begin{vmatrix} z & x \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda\left(\lambda^2 - x^2\right) - z^2(-\lambda)$$

$$= -\lambda\left(\lambda^2 - x^2 - z^2\right) = 0$$

$$\lambda\left(x^2 + z^2 - \lambda^2\right) = 0$$

$$\lambda = 0 \text{ or } \pm\sqrt{x^2 + z^2}$$

3.2 The Tricomi's Equation

Given by $y\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$Aij = \begin{pmatrix} y & 0 \\ o & 1 \end{pmatrix}$$

This equation is of the mixed type elliptic for y > 0, parabolic for y = 0 and hyperbolic y < 0

3.3 Characteristics

Consider $\displaystyle\sum_{ij=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = g\left(x, u, u_{x_1}, ...u_{x_n}\right)$ ………………. (2.5.1)

Definition:

By the characteristic of (2.5.1) we mean the solution of the 1st order equation

$$\sum_{ij=1}^{n} Aij \frac{\partial^2 u}{\partial x_i \partial xj} = 0 \text{ ………………………………….. (2.5.2)}$$

3.4 Case of Two Independent Variables

$a(x,y)\, z_{xx} + 2b(x,y) z_{xy} + c(x,y) z_{yy} = \Phi\left(x, y, z, z_x, z_y\right)$……………2.5.3

x and y are the independent variable and z the depend variable

$$Aij = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Eigen-values are given by

$$|A - \lambda \pm| = 0 \quad \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1}{2}\left\{(a+c) \pm \sqrt{\{(a-c)^2 + 4b^2\}}\right\} \ldots\ldots\ldots\ldots\ldots \text{(2.5.4)}$$

Eigen-values are of different signs if

$a\,c - b^2 < o$ and one is zero if

$a\,c - b^2 = 0$

Therefore (2.5.3) is elliptic at x, y if b2 - ac < o

Hyperbolic $at \;\; x, y$ if $b^2 - ac > o$

Parabolic $at \;\; x, y$ if $\mathrm{b}^2 - ac = 0$ ………………………………… (2.5.5)

Equation 2.5.1 is one of these types in a domain Ω of xy plane if it is at any point of Ω.

No other type is possible since 2.5.4 can only admit two roots.

The characteristic equation is

$$a(x,y)(w_n)^2 + 2b\,(x,y)w\,x\,y + c(x,y)(wy)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.5.6)}$$

$\Rightarrow a\;p^2 + 2\,b\;p\,q + cq^2 = 0$

$\Rightarrow \quad p = q = o$ is a solution

$\Rightarrow \quad z(x,y)$ is a constant along a characterisation

$$\frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = 0 \;(Z = w)$$

Equation 2.5.6 is homogenous wrt $wx : wy$. If we substitute for this with the proportional quantities dy and $-dx$, we then get

$$a\,(dy)^2 - 2b\;d\;x\;d\;y + c(dx)^2 = 0$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.7)$$

Which coincide with (2.5.3) a parabolic.

i Parabolic case $b^2 - ac = o$

Let $\mathrm{E}(x, y)$ be constant and the general solution of (2.5.7). Introduce a regular transformation.

$$\mathrm{E} = \mathrm{E}(x, y)$$
$$y = y(x, y)$$

Where $y \in C^2\ (\Omega)$ is any function independent of E. The transformed characteristic equation

$$\bar{a}\left(\bar{z}_{\mathrm{E}}\right)^2 + 2\,\bar{b}\,\bar{z}_{\mathrm{E}}\ \bar{z}y + \bar{c}\left(\bar{z}y\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.8)$$

has the solution $\bar{z} = \mathrm{E}$

So $\bar{a} = 0$

Since a regular transformation does not alter the type of an equation

$$\bar{b}^2 = \bar{a}\,\bar{c}, \quad \bar{a} = 0$$

Divide the transformed P.D.E by $\bar{C}$ to get the canonical form

$$\frac{\partial^2 z}{\partial^2 y} = \frac{\partial^2 u}{\partial^2 y} = \Phi\left(\mathrm{E}, y, u, u_{\mathrm{E}}, u_y\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5.9)$$

1) Characteristics are invariant under regular transformation.
2) Type is not altered by a regular transformation.

4.0 CONCLUSION

Second order semi – linear Partial Differential Equations are classified according to the properties of their Eigen-values of the matrix of coefficients of the highest order.

5.0 SUMMARY

In this unit, we have been able to work with the classification of second order Partial Differential Equation with a special focus on the Tricomi's equation, characteristics and the case of two independent variables.

6.0 TUTOR-MARKED ASSIGNMENT

1. Reduce to canonical form and find the general solution from
$y^2u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y$
2. Find the characteristic of
$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y, u, u_x, u_y)$

7.0 REFERENCES/FURTHER READING

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UNIT 2 TRANSFORMATION OF INDEPENDENT VARIABLES

CONTENTS

1.0 INTRODUCTION

Sometimes it can be expedient to transform the independent variables in solving Partial Differential Equations with Regular case, Hyperbolic case and Elliptic case. Proof of theorem is presented that regular transformation of independent variable does not alter the type of Partial Differential Equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- transform independent variables
- apply the theorems to the regular case
- solve equations of the hyperbolic types by transformations
- explain why Elliptic equations have no real characteristics.

3.0 MAIN CONTENT

3.1 Transformation of Independent Variables

Let $\mathrm{E} = \Sigma(x, y), \quad \gamma = \gamma(x, y)$ be a regular transformation $\Leftrightarrow 1-1$

$$\overline{JJ} = \frac{\partial(\mathrm{E}, \gamma)}{\partial(x, y)}$$

$$= \begin{vmatrix} \mathrm{E}_x & \mathrm{E}_y \\ \gamma_x & \gamma_y \end{vmatrix} \neq 0$$

$$\Leftrightarrow \quad x = x(\mathrm{E}, \gamma)$$
$$y = y(\mathrm{E}, \gamma)$$
$$Ux \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \mathrm{E}}\frac{\partial \mathrm{E}}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x}$$
$$= U_{\mathrm{E}}\,\mathrm{E}_x + U_\gamma \gamma_x$$

Similarly

$$U_y = U_{\mathrm{E}}\mathrm{E}_y + U_\gamma \gamma_y$$
$$U_{xx} = U_{\mathrm{EE}}\,\mathrm{E}_x^2 + 2U_{\mathrm{E}\gamma}\gamma_x\,\mathrm{E}_x + U_{yy}\,\gamma_x^2$$
$$+ U_{\mathrm{E}}\mathrm{E}_{xx} + U_\gamma\,\gamma_{xx}$$
$$U_{yy} = U_{\mathrm{EE}}\mathrm{E}_y^2 + 2\,U_{\mathrm{E}y}\,\gamma_y\mathrm{E}_y + U_{yy}\gamma_y^2$$
$$+ U_{\mathrm{E}}\mathrm{E}_{yy} + U_\gamma\,\gamma_{yy}$$
$$U_{xy} = U_{\mathrm{EE}}\,\mathrm{E}_x\mathrm{E}_y + U_{\mathrm{E}\gamma}\left(\gamma_x\mathrm{E}_y + \mathrm{E}_x\gamma_y\right) + U_{\gamma\gamma}\,\gamma_x\,\gamma_y$$
$$+ U_{\mathrm{E}}\,\mathrm{E}_{xy} + U_\gamma\gamma_{yx}$$

Substitute U_{xx}, U_{xy}, U_{yy} in

$$a\,(x,y)U_{xx} + 2\,b\,U\,(x,y)U_{xy} + C(x,y)U_{yy} = \Phi(x,y,u,ux,uy)$$

We get

$$\tilde{a}\,(\mathrm{E},\gamma)U_{\mathrm{EE}} + \widehat{2}b(\mathrm{E},\gamma)U_{\mathrm{E}\gamma} + \widehat{C}\,(\mathrm{E},\gamma)U_{\gamma\gamma}$$
$$= \Phi\left(\mathrm{E}\,\gamma, U, U_{\mathrm{E}}, U_\gamma\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.5.9)$$

Where

$$\hat{a} = a\,\mathrm{E}^2 + 2b\mathrm{E}_x\mathrm{E}_y + C\mathrm{E}_y^2$$
$$\hat{b} = a\,\mathrm{E}_x\gamma_x + b\left(\mathrm{E}_x\,\gamma_y + \mathrm{E}_y\,\gamma_x\right) + C\mathrm{E}_y\,\gamma_y$$
$$\hat{c} = a\,\gamma_x^2 + 2b\,\gamma_x\gamma_y + C\,\gamma_y^2$$
$$b^2 - a = \Delta\left(\Delta \text{ is criminant}\right)\left(\mathrm{E}_x\gamma_y - \gamma_x\mathrm{E}_y\right)^2$$
$$\hat{b}^2 - \hat{a}\,\hat{c} = \left(b^2 - ac\right)^2$$
$$= \left(b^2 - ac\right)\overline{JJ}^2$$

The steps that led to the result is a proof of the theorem that regular transformation of independent variable does not alter the type of P.D.E.

3.1.1 Regular Case

Theorem

Characteristics are invariant under regular transformation.

Proof

Equation of characteristics

$$a\left(\frac{\partial w}{\partial x}\right)^2 + 2b\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} + c\left(\frac{\partial w}{\partial y}\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.5.10)$$

$$wx = w_{\mathrm{E}}\mathrm{E}_x + w_\gamma\,\gamma_x$$

$$wy = w_{\mathrm{E}}\mathrm{E}_y + w_\gamma\,\gamma_y$$

Substituting into (2.5.10) It will be shown that it is the same as

$$\hat{a}\,w_w^2 + 2\,\hat{b}\,w_{\mathrm{E}}\,w_\gamma + \hat{c}\,w\gamma^2 = 0$$

Where

$$\hat{a} = a\mathrm{E}_x^2 + 2b\mathrm{E}_x\mathrm{E}_y + c\mathrm{E}_y$$

$$\hat{b} = a\mathrm{E}_x\gamma_x + b\left(\mathrm{E}_x\gamma_y + \gamma_x\mathrm{E}_y\right) + c\mathrm{E}_y\gamma_y$$

$$\hat{c} = a\gamma_x^2 + 2b\gamma_x\gamma_y + c\gamma_y^2$$

3.1.2 Hyperbolic Case $\left(b^2 - ac > 0\right)$

Let E (x,y) = constant, γ(x,y) = constant from the general solution of (2.5.7). Then (2.5.10) has two independent solutions.

$$w = \mathrm{E},\ w = \gamma$$

$$\Rightarrow \hat{a} = \hat{c} = 0$$

Divide the transformed equations by $2\hat{b}$ to obtain

$$\frac{\partial^2 u}{\partial \mathrm{E}\partial y} = \Phi\left(\mathrm{E}, \gamma, U, U_{\mathrm{E}}, U_\gamma\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.5.11)$$

Let

$$\begin{aligned} \mathrm{E}_i &= \mathrm{E} + \gamma \quad \Leftrightarrow \quad \mathrm{E} = \tfrac{1}{2}\left(\mathrm{E}_i + \gamma_i\right) \\ \gamma_i &= \mathrm{E} - \gamma \qquad\quad\ \gamma = \tfrac{1}{2}\left(\mathrm{E}_i - \gamma_i\right) \end{aligned} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.5.12)$$

be a linear transformation

$$\frac{\partial}{\partial \mathrm{E}} = \left(\frac{\partial}{\partial \mathrm{E}} + \frac{\partial}{\partial \gamma}\right), \quad \frac{\partial}{\partial \gamma} = \frac{\partial}{\partial \mathrm{E}_i} - \frac{\partial}{\partial \gamma_i}$$

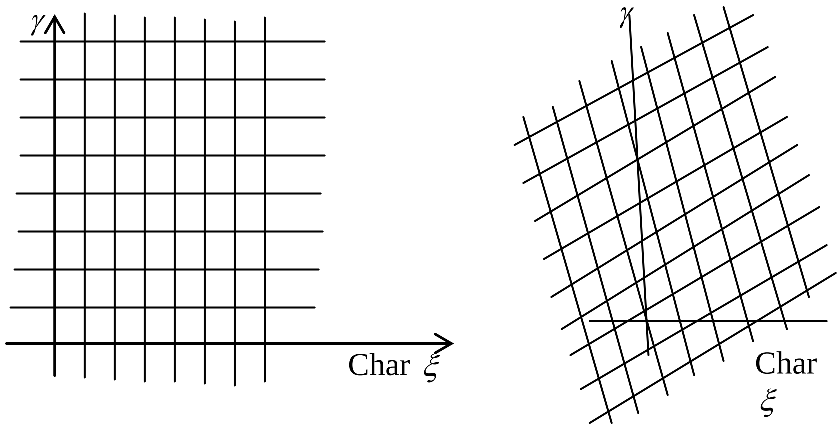
Substituting into (2.5.11) yields

$$= \frac{\partial}{\partial \mathrm{E}}\,\frac{\partial}{\partial \gamma}\,(u)$$

$$= \left(\frac{\partial}{\partial E_i} + \frac{\partial}{\partial \gamma}\right)\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial \gamma}\right) u$$

$$= \frac{\partial^2 u}{\partial E^2} - \frac{\partial^2 u}{\partial \gamma} = \Phi 2\left(E_i, \gamma_i, U, U_E, U_\gamma\right) \text{ ....(2.5.13)}$$

In the $E\,\gamma$ – plane, the characteristic are lines $\nearrow\nearrow$ to the coordinate areas while in $E_i\ \gamma_i$ - plane they are lines of slopes $\pm$ 1



3.1.3 Elliptic Case $\left(b^2 - ac < 0\right)$

Theorem

An Elliptic equation has no real characteristics

Proof

The characteristics are $\left(\frac{dy}{dx}\right)_{1,2} = \frac{b \pm \sqrt{b^2 - ac}}{a}$

But $b^2 - ac < 0$ this $\Rightarrow$ that $\nexists$ no real curve in the real x-y plane. We assume that a,b,c admits complex values. Equation (2.5.7) becomes P.D.Es in a complex values domain suppose – $w(x, y) \neq$ constant is a general solution of the 1st order equation, then $w(x, y) \neq$ satisfying (2.5.10)

Suppose $w(x, y) = E(x, y) + \gamma(x, y)$
Where $x,\ y,\ E, \gamma$ are real

$$\overline{JJ} = \frac{\partial(E, \gamma)}{\partial(x, y)} \neq 0$$

For if $\overline{JJ} = 0$ at (x, y)

$$\frac{\partial E}{\partial x} = \lambda \frac{\partial E}{\partial y}$$

$$\frac{\partial E}{\partial x} = \lambda \frac{\partial \gamma}{\partial y}$$
$$\frac{\partial w}{\partial x} = \lambda \frac{\partial w}{\partial y} \text{ for some } \lambda \in R$$

Substitute in 2.5.10

$$a(wx)^2 + 2bw_\lambda w_y + c(wy)^2 = 0$$

We have

$$a\lambda^2 + 2b\lambda + c = 0$$

Since $wx = \lambda wy$

$$a\,(\lambda wy)^2 + 2b\lambda\; wy\; wy + c(wy)^2 = 0$$
$$a\lambda^2\,(wy)^2 + 2b\lambda + c \quad wy = 1$$
$$a\lambda^2 + 2b\lambda + c$$

Which has real root this is impossible since $b^2 - ac < 0$, therefore $\overline{JJ} \neq o$
Take $E = E(x, y)$ and $\gamma\ \gamma(x, y)$ as the new independent variables equation 2.5.10 must be satisfied by $w = E + 1\gamma$

$$\Rightarrow \qquad \hat{a}\,(wE)^2 + 2\hat{b}\; wE\; w\gamma + \hat{c}\,(w\gamma)^2 = 0$$
$$\hat{a} + 2\hat{b}i + \hat{c}i^2 = 0$$
$$\hat{a} - \hat{c} + 2\hat{b}i = 0$$
$$\Rightarrow \qquad \hat{a} - \hat{c} = 0 \quad \Rightarrow \hat{a} = \hat{c}$$
$$2\hat{b} = 0 \qquad \Rightarrow \hat{b} = 0$$

Dividing the transformation equation by $\hat{a}$, we arrived at the canonical form

$$\frac{\partial^2 u}{\partial E^2} + \frac{\partial^2 u}{\partial \gamma^2} = \Phi\left(E, \gamma, U, U_E, U_\gamma\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.5.14)$$

Examples:

Solve the Partial Differential Equation by the method of characteristics

$$yu_{xx} + (x + y)uxy + xu_{yy} = 0$$

Sketch the characteristics curves

Have $a = y,\ b = \frac{1}{2}(x + y), c = \lambda$

So $b^2 - ac = \frac{1}{4}(x - y)^2$

Which implies that the equation a hyperbolic for $x \neq y$, parabolic for $x \neq y$ the characteristics equations

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+y) + \frac{1}{2}(x-y)}{y} = \frac{x}{y} \text{ both are separable equation}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(x+y) - \frac{1}{2}(x-y)}{y} = 1$$

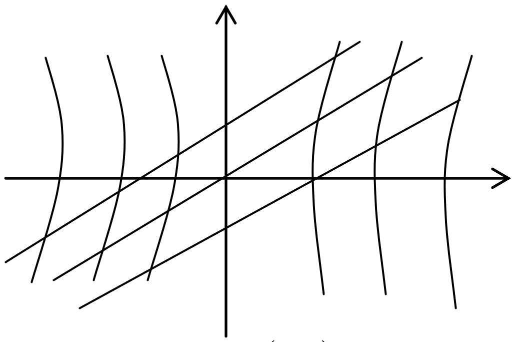
then solving it $\Rightarrow$

$$y^2 - x^2 = \alpha : r. \quad y\,dy = xdx$$

$$y - x = \beta : r_2 \quad \frac{y^2}{2} - \frac{x^2}{2} = 8$$

$$y^2 - x^2 = \beta = 28$$

where r_1, r_2 are the characteristics curves
r_1 is defined as rectangular hyperboles
r_2 is defined as straight line with slope



Now write $\begin{matrix} y^2 - x^2 = \mathrm{E}(x,y) \\ y \quad x = \gamma(x,y) \end{matrix}$

$U(x,y)$ can be transformed into

$$Ux = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \mathrm{E}}\frac{\partial \mathrm{E}}{\partial x} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial x}$$

$$Ux = -2xU_{\mathrm{E}} - U\gamma$$

$$U_y = \frac{\partial u}{\partial \mathrm{E}}\frac{\partial \mathrm{E}}{\partial y} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial y} \quad Uy = 2y\,U_{\mathrm{E}} + U\gamma$$

$$Uxx = 4x^2U_{\mathrm{E}\gamma} + 4xU_{\mathrm{E}\gamma} + U_{\gamma\gamma} - 2U_{\mathrm{E}}$$

$$Uxy = -4xyU_{\mathrm{EE}} - 2U_{\mathrm{E}\gamma}(y+x) - U_{\gamma\gamma}$$

$$Uyy = 4y^2U_{\mathrm{EE}} + 4yU_{\mathrm{E}\gamma} + U_{\gamma\gamma} + 2U_{\mathrm{E}}$$

The Partial Differential Equation (x) becomes

$$-2\,U_{\mathrm{E}}\gamma\left(y^2+x^2-2xy\right)-2U_{\mathrm{E}}\left(y-x\right)=0$$

So that we now have

$$-2\gamma^2\,U_{\mathrm{E}}\gamma-2\gamma U_{\mathrm{E}}=0$$

We now have

$$2\,\gamma\left(\gamma\,U_{\mathrm{E}\gamma}+U_{\mathrm{E}}\right)\ =0$$

$$\Rightarrow\ \gamma\,U_{\mathrm{E}\gamma}+U_{\mathrm{E}}\ =0$$

$$\gamma\frac{\partial}{\partial\gamma}\left(\frac{\partial u}{\partial \mathrm{E}}\right)+\frac{\partial u}{\partial \mathrm{E}}=0$$

Let $w=U_{\mathrm{E}}$

$$\gamma\,W\gamma\ +w\ =0$$

$$\frac{\partial}{\partial\gamma}\left(\gamma\,\frac{\partial u}{\partial \mathrm{E}}\right)=0$$

$$\gamma\,\frac{\partial u}{\partial \mathrm{E}}\ =F\,(\mathrm{E})$$

$$\frac{\partial u}{\partial \mathrm{E}}=\frac{1}{\gamma}\,F(\mathrm{E})$$

$$U\,(\mathrm{E},\gamma)=\frac{1}{\gamma}\int F(\mathrm{E})d\mathrm{E}\ +\ G(\gamma)$$

Where

$$F\,(\mathrm{E})\ =\ \int F(\mathrm{E})\,d\mathrm{E}$$

$$\Phi\,(x,y)U(x,y)\ =\frac{1}{y-x}\,F\,\left(y^2-x^2\right)+G\,(y-x)$$

Where F and G are arbitrary differentiable functions

Exercise:

Solve the following 2nd order P.D.E by the method of characteristics

$$Uxx+2Uxy+Uyy+Ux+\ Uy=0$$

Sketch the characteristics

$$a=1\ \ b=1\ \ c=1$$

$$b^2-ac=0$$

Further Exercise

Classify and solve the following P.D.Es by the method of characteristics and sketch the characteristics

i) $y^2Uxx - 2y\,Uxy + Uyy - Ux - 6y = 0$

ii) $Uxx + \; x^2Uyy = 0$

iii) $uxx + xUyy = 0$

$\Rightarrow F\,\left(y + \frac{1}{4}\,i\right) + x\;g\;\left(y + \frac{1}{4}i\right)$

$F\,\left(2y + x^2 i\right) + G\,\left(2y - x^2\right)$

4.0 CONCLUSION

Regular transformation of independent variables does not alter the type of a Partial Differential Equation.

5.0 SUMMARY

The potential of transformations to ease the arrival at solution of Partial Differential Equation never be understated and this was adequately demonstrated in the transformation of independent variables with three specific scenarios of the regular, hyperbolic and the elliptic cases visited.

6.0 TUTOR-MARKED ASSIGNMENT

1. Let a, b be real numbers. The Partial Differential Equation

   $u_y + au_{xx} + bu_{yy} = 0$

   is to be solved in the box Ω =[0, 1]2.

2. Find data, given on an appropriate part of ∂Ω, that will make this a well-posed problem and cover all cases according to the possible values of a and b.

   Justify your answer.

7.0 REFERENCES/FURTHER READING

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