MODULE 4

UNIT 1 CAUCHY PROBLEM, CHARACTERISTICS PROBLEM AND FUNDAMENTAL EXISTENCE THEOREM

CONTENTS

1.0 INTRODUCTION

This unit zooms in on Cauchy problem and the Cauchy Kovalevsky theorem and places their significance in the solving of higher order Partial Differential Equations into context.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- solve Cauchy Problem and Characteristics Problem
- explain the strip condition
- treat the fundamental existence theorem
- explain the method of solving Cauchy Problem
- solve problems using the Cauchy Kovalevsky Theorem.

3.0 MAIN CONTENT

3.1 Cauchy Problem and Characteristics Problem

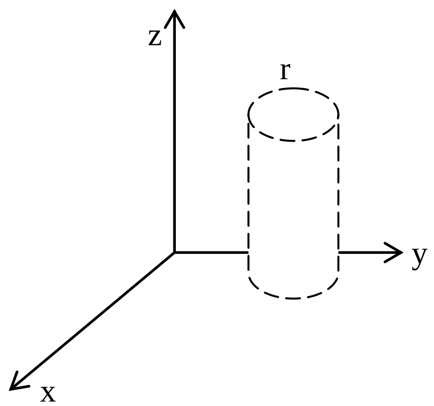
Find a solution

$U = U(x, y)$ of the equation
$aUxx + 2b\,Uxy + cUyy = \Phi(x, y, u, p, q)$ in some neighbourhood of

a space curve, set $u = z$
$r = \{(x, y, z) : x = f_1(t), \ y = f_2(t), z = h(t)\}$
$0 \le t \le 1$

Such that

$$z/n = h(x, y),$$
$$\partial z/\partial n = H(x, y)$$



If instead of prescribing

$\frac{\partial z}{\partial n}$ on r, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ are prescribed

We find that f_1, f_2, h, p, q must satisfy

$$\frac{dz}{dt} = h(i) = zx\,xt + z_y y_t \quad \text{……………………………………….(2.7.2)}$$
$$= pf : + q f_2^1$$

This is known as the strip conditions. Cauchy problem therefore becomes that of finding a solution of (2.7.1) containing the integral ship of (f_1, f_2, h, p, q) at any point of the integral strip.

$$\frac{dp}{dt} = Zxx\frac{dx}{dt} + Zxy\frac{dy}{dt} = rf_1 + sf_2^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.7.3)}$$

$$\frac{dp}{dt} = Zxy\frac{dx}{dt} + Zyy\frac{dy}{dt} = f_1 + rf_2^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.7.3b)}$$

Solving (2.7.1) and (2.7.3) above, we have

$$\frac{r}{\Delta,} = \frac{s}{\Delta_2}\frac{r}{\Delta_3} = \frac{1}{\Delta}$$

Where

$$\Delta = \begin{vmatrix} a & 2b & c \\ f_2^1 & f_2^1 & o \\ o & f_1^1 & f_2^1 \end{vmatrix}$$

Now

$$\Delta_1 = \begin{vmatrix} \Phi & 2b & c \\ p^1 & f_2^1 & o \\ q^1 & f_1^1 & f_2^1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a & \Phi & c \\ f_1^1 & p_2^1 & o \\ o & q^1 & f_2^1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a & 2b & \Phi \\ f_1^1 & f_2^1 & p^1 \\ o & f_1^1 & q^1 \end{vmatrix}$$

If $\Delta \neq 0,$ we can uniquely determine r, s, r on c differentiating (2.7.1) with respect to t and using the relations

$$\frac{dr}{dt} = \frac{\partial^3 z}{\partial x^3}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxxx\ f_1^1 + Zxxyf_2^1$$

$$\frac{ds}{dt} = \frac{\partial^3 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxxy\ f_1^1 + Zxyyf_2^1$$

$$\frac{dr}{dt} = \frac{\partial^3 z}{\partial x \partial y^2}\frac{dx}{dt} + \frac{\partial^3 z}{\partial y\,\partial x^2}\frac{dy}{dt} = Zxyy\ f_1^1 + Zyyyf_2^1$$

3rd order derivations of c can be calculated on $\not{z}$, similarly for fourth and higher order partial derivatives. The value of $\not{z}$ in some neighbourhood of r can be obtained by Taylor's theorem

The Cauchy problem passes a unique solution $\Delta \neq 0$

Suppose that $\Delta \neq 0$, then

$$a\left(f_2^1\right)^2 - 2bf_1^1\ f_2^1 + c\left(f_1^1\right)^2 = 0$$
$$\Rightarrow\ a\left(\frac{dy}{dt}\right)^2 - 2b\,\frac{dx}{dt}\,\frac{dy}{dt} + c\left(\frac{dx}{dt}\right)^2 = 0$$
$$\Rightarrow a\,(dy)^2 + 2b\,dx\,dy + c\,(dx)^2 = 0$$

Which is the equation for characteristics of (2.7.1) however, if $\Delta = 0$ and $\Delta_i\,(i = 1,2,3)$, a solution will exist but not unique.

3.2 Fundamental Existence Theorem

3.2.1 Cauchy Problem

Given a Partial Differential Equation, find a solution which satisfied given boundary or initial conditions if the conditions are enough to ensure existence, uniqueness and continuous dependence of the solution on the given data a Cauchy data. We say that the problem is well posed.

3.2.2 Cauchy Kovalevsky Theorem

Solutions of initial value problems may be obtained in Taylor's series. We simply compute the coefficients of the Taylor's series of the solution using initial data and the Partial Differential Equation. The method is possible if the solution is analytic. Cauchy Kovalevsky theorem gives the condition under which the initial – value problem has solution which is an analytic function.

Case 1 (1st Order Equation R^2)

$$\begin{cases} F(t),\ x,\ u,\ u_t,\ u(x) & = 0 \\ u\,(o,x) & \phi\,(x) \end{cases}$$

Assume

$$\begin{cases} ut = f\,(t,x,u,u_x) & \text{..} \quad (3.1.1) \\ u(o,x = \Phi\,(x) & \text{..} \quad (3.1.2) \end{cases}$$

Let $\Phi(x)$ be analytic in the neighbourhood of the origin x = 0, f is analytic in the neighbourhood of the point $\left(0,\,0,\ \Phi(0),\,\Phi^1(0)\right) \in \Re^4$

Then the Cauchy – problem (3.1.1) to (3.1.2) has a solution u (t, x)

Which is defined and analytic in a neighbourhood of the origin (0,0) $\in \Re^2$ and this solution is unique in the class of analytic functions.

Proof

The proof depends essentially on a specific technique. Assuming Φ is analytic in a neighbourhood of x = 0 this enables us to obtain

$$\frac{\partial^n u}{\partial x^n}(0,0)\,F\ \Phi^n(0) \quad n=1,\,2\,,3...$$

From (3.1.1) $u_t\ (0,0)= f\left(0,\,0,\,\Phi\,(0),\,\Phi^1(0)\right)$

Differentiating (3.1.1) wrt x

$$u_{tx}(t,x)=\frac{\partial f}{\partial t}\frac{\partial t}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial ux}\frac{\partial ux}{\partial x}$$

$$U_{tx}\ (t,x)= fx + fu\ ux + fux\ Uxx$$

Since f is known and U_x, U_{xx} have being determined at the origin. We can find $U_{xt}(0,0)$ to obtain U_{xxt} we differential (3.1.1) twice with respect to x and substitute $t=x=0$ and also previously determined values of $U, U_x U_{xx}, U_{xxx}$ at (0,0)

Continuing in this manner, we can determine the values of all partial derivatives

$$\frac{\partial^{n+1} u}{\partial x^n \partial t};\ \ n=0,1,2,......\text{at } (0,0)$$

Differentiating (3.1.1) w.r.t t

$$U_{tt} = f_t + f_u U_t + f_{ux} U_{xt}$$

Substituting t = x =0 and previously obtained values of $u, u_x\ u_t$ at (0.0). Continuing in this way, we obtain the values of all partial derivatives of u at (0,0.

$$U(t,x)=\sum_{\alpha=0}\sum_{\beta=0}\frac{D_t^{\alpha}\ D_x^{\beta}u(0,0)\,t^{\alpha}x^{\beta}}{\alpha!\,\beta!} \qquad ....(3.1.3)$$

Cauchy Kovalevsky's theorem assert that this series converges for all (t,x) in some neighbourhood of the origin (0,0) and defines the solution of (3.1.1) in this neighbourhood. Uniqueness follows from the fact that any two analytic functions having the same Taylor's series coefficients are identical.

Examples:

$Ut = Uux$
$U(0,x) = 1 + x^2$

Taylor's series expansion

$U(t,x) = U(0,0) + Ut\,(0,0)t + Ux(0,0)x +$
$\frac{1}{2!}\{Utt\,(0,0)\,t^2 + 2Utx\,(0,0)\,tx + Uxx\,(0,0)\,x^2\}$
$+\frac{1}{3}\{Uott(0,0)\,t^3 + ...$

$\Phi\,(x) = 1 + x^2$ is analytic in the neighbourhood of $x = 0$ (analytic in x)
$f(t, x, u, ux) = uux$ is analytic in the neighbourhood $(0,0,1,0) \in \Re^4$

$u(o,x) = \Phi(x) = 1 + x^2$
$u(0,0) = \Phi(0) = 1$
$u_x(0,0) = 2x$ at x = 0 = 0

$uux = \int(t, x, u, ux)$ is analytic in the neighbourhood of point (0 ,0, 1, 0) $\in \Re^4$

$u\,(o,x) = 11x^2 \qquad u\,(0,0) = 1$
$u_x\,(0,x) = 2x \qquad u_x\,(0,0) = 0$
$u_{xx}\,(0,x) = 2 \qquad u_{xx}\,(0,0) = 2$

$D_x^n\; u(0,x) = 0, \quad n \geq 3$
$D_x^n\; u(0,0) = 0, \quad n \geq 3$
$u_t = uu_x \;\; u_t\,(0,0) = u\,(0,0)\,u_x\,(0,0) = 1 \times 0 = 0$
$u_t = u_x{}^2 + uu_{xx}\; u_{tx}\,(0,0) = 0 + 1 \times 2 = 2$
$u_{tt} = u_t u_x + uu_{xt},\, u_{tt}\,(0,0) = 2$
$u_{txx} = 3u_x u_{xx} + uu_{xxx};\, u_{txx}\,(0,0) = 0$
$u_{tt} \quad = u_t u_{xx} + 2u_x u_{tx} - uu_{txx};\, u_{ttx}\,(0,0) = 0$
$u_{tt}\,(0,0) = 0$

Neglecting terms of order ≥ 4

$$u\,(x,t) = \sum_{\alpha=0} \sum_{\beta=0} \frac{D_t^\alpha D_x^\beta\; u(0,0)\,t^\alpha x^\beta}{\alpha!\,\beta!}$$
$$= 1 + 0 + 0 + t^2 + 2tx + x^2$$
$$= 1 + t^2 + 2tx + x^2 + ...$$

Exercise:

Let $u(x,y)$ satisfy $u_x^{\,2} + u_y^{\,2} = 1$ and let $u(o,y) = \Phi(y) \nleftarrow y$. Determine the Taylor's series expansion of $u(x,y)$ and sum the senses to show that

$$u(x,y) = \Phi_1(y)x + \Phi(y)$$

2nd Order Equation in R^2

We want to consider Partial Differential Equation of the form

$$F(t,\ x, u,\ \ u_t, u_x, u_{xx}, u_{tt}, u_{tx}) = 0$$

Assume $u_{tt} = F\,(t,\ x, u, u_t, u_x, u_{tx}, u_{xx})$

The Cauchy problem in this case will be

$$\begin{cases} u_{tt} = F\,(t,\ x, u, u_t, u_x, u_{tx}, u_{xx}) \\ u\,(0,x) = \Phi\,(x) \\ u_t\,(0,x) = \Psi\,(x) \end{cases}$$

(t = 0 is not characteristic)

Theorem:

Let $\Phi\,(x), \Psi(x)$ be analytic in a nbd of the origin x = 0 in R and suppose f is analytic in a nbd of the point $(0,\ 0,\ \Phi(0),\ \Phi^1(0),\ \Psi^3(0)\,\Phi^4) \in \Re^n$. The Cauchy problem (3.14) – (3.16) has a solution u (t,x) which is defined and analytic in a nbd of the origin t = 0, x, 0 of R^2 and this solution is unique in the clan of analytic functions.

Outline of technique:

From (3.15) and (3.16) we obtain

$$\frac{\partial^n u}{\partial x^n}\,(0,0) = \Phi^n(0)$$
$$\frac{\partial^{n+1} u}{\partial t\ \partial x^n}(0,0) = \Psi^n(0)$$

Differentiating (3.14) successively and using the calculated values, we find all the coefficients in the Taylor's series solution as before.

Examples:

1) Let $u(x,y)$ satisfy $u_{xx}+u_{yy}=0$ and let $u(0,y)=\sin y$

$$u(o,y)=y \ \forall\ y$$

Determine the Taylor's series expansion for the solution $u\,(x,y)$ of Partial Differential Equation and sum the series to show that

$$u(x,y)=\sin y\cosh x+xy$$

2) Find the former series solution for the Partial Differential Equation

$$y^2u_{xx}=x^2u_{yy}+2\left(x^2-y^2\right)u$$
$$u(o,y)=e^{-y2}$$
$$u_x\left(o,y\right)=0$$
$$u_{xx}-u_{yy}=f\left(x,y,u,u_x,u_y,u_{xy},u_{yy}\right)$$
$$u\left(o,y\right)=\sin y$$
$$u_x\left(o,y\right)=y$$
$$u\left(x,y\right)=u\left(0,0\right)+xu_y\left(0,0\right)+yu_y\left(0,0\right)+$$
$$\frac{1}{2}\xi u_{xx}\,10dx^2+..........$$
$$u(0,0)=\sin\!-0$$
$$u_x\left(0,0\right)=0$$
$$u_y\left(0,0\right)=1$$
$$u_y\left(0,0\right)=\cos y$$
$$u_{yy}=-\sin y$$

3) $y^2u_{xx}=x^2u_{yy}+2\left(x^2-y^2\right)u$

$$u\left(0,\,0\right)=1$$
$$u_x\left(0.0\right)=0$$
$$u_y\left(o,y\right)=2y\,e^{-y2}$$
$$u_{xx}=0$$
$$u_{xy}=0$$
$$u_{yy}=-2y\left(-2ye^{-y2}\right)-2e^{-y2}$$
$$=-\underline{2}$$

4.0 CONCLUSION

Cauchy problem can be summed up as the problem of finding a solution containing the integral strip of functions at any point of the integral strip while Cauchy Kovalevsky theorem simply states the condition under which an initial – value problem has an analytic function solution.

5.0 SUMMARY

We have seen in this unit that Cauchy problem, Characteristics problem and Cauchy Kovalevsky theorem are useful in addressing certain types of partial differential equations

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve the Cauchy problem

$$\begin{aligned} u_t - xuu_x &= 0 \qquad -\infty < x < \infty,\ t \geq 0 \\ u(x,0) &= f(x) \qquad -\infty < x < \infty. \end{aligned}$$

and find a class of initial data such that this problem has a global solution for all t. and then, compute the critical time for the existence of a smooth solution for initial data, f, which is not in the above class.

2. Find an implicit formula for the solution u of the initial-value problem

$$u_t = (2x-1)tu_x + \sin(\pi x) - t,$$

$$u(x, t=0) = 0.$$

Evaluate u explicitly at the point (x = 0.5, t = 2).

7.0 REFERENCES/FURTHER READING

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