MODULE 4

Unit 1 Triple Products
Unit 2 Applications of Triple Products and Reciprocal Sets of Vectors

UNIT 1 TRIPLE PRODUCTS

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
 3.1 The dot and cross products of three vectors
 3.2 Algebraic laws of triple products
 3.3 Scalar triple products or Box products
 3.4 Volume of parallelepiped
 3.5 Box products in component form
 3.6 Vector triple products
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignments
7.0 References/Further Reading

1.0 INTRODUCTION

Having gone through scalar or dot product, vector on cross product, there is need to ask the question, what if you have more than one vector to multiply. This is the purpose of this unit. Not only will you have to learn low to get triple products, but you will have to differentiate between the dot and the cross products when it come to triple vectors.

You should take note of the `order' in which results are given.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- find the scalar triple products
- calculate correctly the volume of a parallelepiped given the sides as vectors.

3.0 MAIN CONTENT

3.1 The dot and cross product of three vectors

The dot and cross multiplication of three vector **u, v** and **w** may produce meaningful products of the form (**v . v**), **u**. (**v** x **w**) and **u** x (**v** x **w**). From examples and exercise you will learn how to evaluate these products.

3.2 Laws of triple products.

The following laws will guide you in your attempts at calculating triple products.

1. (**u . v**) **w** = **u** (**v** x **w**). This implies read the question carefully to recognize the order of the vectors in the products.

3.3 Scalar triple products or box products

2. **u**. (**v** x **w**) = **v**. (**w** x **u**) = **w**. (**u . v**) you should observe, once again the order or right **-** hand rule in these product (**uvw**), (**vwu**), (**wuv**) and note that since (**u** x **v**) = **- v** x **u**, you cannot effort to do things your own way.

These results **u**. (**v** x **w**) = **v**. (**w** x **u**) = **w**. (**u** . **v**) you will refer to as the scalar triple products or box product.

3.4 Volume of parallelepiped.

The scalar triple products or box products **u**. (**v** x **w**) = **v**. (**w** x **u**) = **w**. (**u.v**) also represents the volume of a parallelepiped having u, v and w as edges, or the negative of this volume, according as **u, v** and **w**, do or do not form a right **-** handed system.

3.5 Box products in components forms

Analytically, let **u** = **u, i** + **uj** + **uk**, **v** = **vj** + **vj** + **uk**, and **w** = **wj**+ **wk.**

Then u. (v x w) = $\begin{matrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \\ \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \\ \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{matrix}$

Determinant of the matrix formed by the coefficient of the 3 vectors **u, v** and **w** in component form.

You can continue the laws now.

3. **u** x (**v** x **w**) = (**v** x **w**) x **w**, i.e. You cannot use the associative law for cross products.

3.6 Vector triple products

u x (**u** x **w**) = (**v. w**) **v** - (**u. v**)**w** and
(**u** x **v**) x **w** = (**u.w**)**v** - (**v.w**)**u**
The product **u** . (**v** x **w**) you should refer to as the scalar triple products or box product [**u, v, w**], but the product **u** x (**v** x **w**) should be referred to as the vector triple products. You should also note that you could write **u** . (**v x w**) as **u** . (**v** x **w**). However you cannot have out the brackets in vector triple product **u** x (**v** x **w**).

Example 1

Evaluate (2**i** - 3**j**) . [(**i** + **j** - **k**) x (3**i** - **k**)]

Solution

$$= \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

= 2(-l - 0) + 3(-1 - (-3) + 0 (0-3)

= -2 + 3 (-1 + 3) + 0

= -2 + 3 (2)

= 6 – 2

=4

Note that 2**i** - 3**j** = 2**i** - 3**j** + 0**k** and 3**i** – **k** = 3**i** – **k** = 3**i** + 0**j** - **k**

Example 3

Prove that (**u** x **v**). (**v** x **w**) x (**w** x **u**) = (**u**. **v** x **w**)^2

Solution

Let a vector **p** = **v** x **w**
Then **p** x (**w** x **u**) = **w** (**p . u**) - (**p**. **v**) Substitute **p** = **v** x **w** to get,
= (**v** x **w**) x (**w** x **u**)
= **w** (**v** x **w .u**) – **u** (**u** x **w .w**)
= **w** (**u. v** x **w**) – **u** (**v. w** x **w**)
= **w** (**u. v** x **w**) And so you have,
(**u** x **v**) . (**v** x **w**) x (**w** x **u**) = (**u** x **w**). **w** (**u. v** x **w**)
= (**u** x **v .w**)(**u. v** x **w**)
= (**u. v** x **w**)^2.

Example 4

Given u=**i** - 2**j** - 3**k**, **v** = 2**j** - **k** and **w** = **i** + 3**j** - 2**k**, find. (a) (**u** x **v**) x **w** (b) **u** x (**v** x **w**)

Solution

(a) (**u** x **v**) x **w** = (**u** . **w**) **v** - (**v** . **w**) **u**

= (1 x l) + (-2 x 3) + (-3 x -2) **v**- (2 x l) + (1 x 3) + (-1 x -2) **u**
= (1 – 6 + 6)(2**i** + **j** - **k**) - (2 + 3 + 2)(**i** – 2**j** – 3**k**)
= (2**i** + **j** - **k**) - (7**i** - 14**j** - 21**k**)
= | (2 - 7)**i** + (1 + 14)**j** + (-1 + 21)**k** |
= | -5i + 15j + 20k |
= 5| –I + 3j + 4k |
= 5 $\sqrt{(-1)^2 - 3^2 + 4^2}$
= 5 $\sqrt{26}$
(b) [u x (v x w] = [(u. w) v - (u. v) w]
u = i - 2j - 3k, v = 2i + j - k, w = i + 3j - 2k
= {(1 x 1) + (-2 x 3) + (-3 x -2)}v - {(1 x 2) + (-2 x 1) + (3 x -1)}w
= | (1 – 6 + 6)(2i + j - k) - (2 – 2 + 3)(i + 3j - 2k) |
= | (21 + j - k) - (3i + 9j - 6k) |
= | (2 - 3)i + (1 - 9)j + (-1 + 6k) |
= | (-i - 8j + 5k) |
= $\sqrt{(-1)^2 + (-8)^2 + (5)^2}$
= $\sqrt{1 + 64 + 25}$
= $\sqrt{90}$
= 3$\sqrt{10}$

4.0 CONCLUSION

Vector can be multiplied in triples.

You have the scalar Triple product, which will give you a scalar result. You can represent the scalar triple product as the determinant of the matrix formed by the coefficient of the three vectors involved.

The right-handed system must be taken into consideration when writing out the order of the scalar Triple or box product and the vector Triple product.

You, using the absolute value of scalar triple product of the vectors representing the adjacent sides can calculate the volume of a parallelepiped.

5.0 SUMMARY

- **u**.(**v** x **w**) = **v** . (**w** x **u**) = **w**. (**u** x **v**) is the scalar triple products of vectors u, v, and w and you could write it as a box product (uvw)
- It represent the determinant of the (3 x 3) matrix formed based on the coefficients of the components of the three vectors
- Its absolute value represents the volume of a parallelepiped with the adjacent sides, as the three vectors
- The vectors are coplanar when it is zero
- The vector triple products are
  u x (**v** x **w**) = (**u**. **w**) **v** - (**u**. **v**) **w** and (**u** x **v**) x **w** = (**u**. **w**)**v** - (**v**. **w**) **u.**

6.0 TUTOR- MARKED ASSIGNMENTS.

l. If **u** = **i** - 2**j** - 3**k**, **v** = 2**i** +**j** - **k** and **w** = **i** + 3**j** - 2**k**
Find
(a) 2**u**. (**v** x 3**w**) (b) (3**u** x 2**v**). (**w**)
(c) Find the volume of the parallelepiped with adjacent sides as **u**, **v**, and **w**.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. and Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.

UNIT 2 APPLICATIONS OF TRIPLE PRODUCTS AND RECIPROCAL SETS OF VECTORS

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
  3.1 Coplanar vectors
  3.2 Another look at the scalar triple products
  3.3 Properties of scalar triple products
  3.4 Equal vectors in scalar triple products
  3.5 Reciprocal sets of vectors
  3.6 Properties of the reciprocal sets of vectors
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Reading

1.0 INTRODUCTION

In this unit, you will be introduced to the second type of triple products, which are the vector triple products.

It should be quite easy for you to recognise the difference between the two types of triple products.

You should take advantages of the use of determinants of matrices in calculation, even if it means revising matrices.

The reciprocal sets of vectors put an end to your study of vector algebra.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find with ease the vector triple products
- calculate correctly the set of vectors reciprocal to a given set.

3.0 MAIN CONTENT

3.1 Coplanar vectors

The necessary and sufficient condition for the vectors **u v** and **w** to be coplanar is that **u . v** and **w** = 0.

Example 2

Find the constant p such that the vectors 2**i** - **j** + **k**, **i** - 2**j** - 3**k** and 3**i** + p**j** + 5**k** are coplanar.

Solution

The product **u . v** x **w** = 0 for the vectors to be coplanar.

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 0$$

.'. 2 (10 + 3p) + 1 (5 + 9) + 1 (p - 6) = 0
= 20 + 6p + 14 + p – 6 = 0
= 6p + p + 34 – 6 = 0
= 7p + 28 = 0
7p = -28
P = -4

3.2 Another look at scalar and vector triple product

You should note that the scalar triple products of three vectors **u, v** x **w** is a scalar. While the vector triple product will yield a vector.

The scalar triple product of **u , v w** could be defined as **u** . (**v** x **w**) = /**u** / **v** x **w**/ cos β where β is the angle between **u** and the vector **v** x **w**. Now you can see why it is scalar. The scalar triple product **u** . (**v** x **w**) can also be interpreted by you as the components of **u** along **v** x **w**. The geometrical meaning of the scalar triple product is that its absolute value represent the volume of a parallelepiped with **u , v** and **w** as adjacent sides.

The magnitude of the scalar triple products **u** , (**u** x **v**) is equal to the volume of the parallelepiped having sides **u , v** and **w** .

3.3 Properties of the scalar triple products

If you inter change two rows of matrices the sign is reversed for the determinate. And so you have **u**, (**v** x **w**) = **- u** . (**v** x **w**) = **- u**. (**w** x **v**) Interchanging the rows twice you get **u**. (**v** x **w**) = **v**. (**w** x **u**) = **w**. (**u** x **v**).

The geometrical significance of this result is that these three products represent the same volume. And since dot product is commutative, you can write **u**, (**v** x **w**) = (**v** x **w**). **u** = (**u** x **v**). **w** and for any constant k, [k **u** . (**v** x **w**)] = k [**u**. (**v** x **w**)]

If **u**. (**v** x **w**) = o, then the volume of the parallelepiped is zero and so **u**. (**v** x **w**) and coplanar, as already discussed i.e. they lie on the same plane.

3.4 Equal vectors in scalar triple products

Since **u** x **v** is perpendicular to u from definition of cross product, then **u**. (**u** x **v**) = 0.

In other words, if any two vectors in the scalar triple products are equal it becomes zero.

SELF-ASSESSMENT EXERCISE 1

The volume of a tetrahedron is one sixth of the volume of a parallelepiped the three sides of a tetrahedron are given by **u** = 2**i** + 3**j** - 4**k**, **v** = **i** + 2**j**-**k** and **w** = 2**i** + 3**j** + 4**k**

Find the volume of the tetrahedron.

Solution

$$\text{Volume } 1/6 \begin{vmatrix} i & j & k \\ 2 & 3 & -4 \\ 1 & 2 & -1 \\ 2 & 3 & 4 \end{vmatrix}$$

$= {}^{1}/_{6}\ |\ 2\ (8 + 3) - 3(4 + 2) - 4\ (2 - 4)\ |$

$= {}^{1}/_{6}\ |\ 2(22\text{-}18\text{+}4)\ |$

$= \dfrac{26 - 18}{6}$

$= \dfrac{8}{6}$

$= 1^{1}/_{3}$ cubic units

Example 3

Prove that (**u** x **v**). (**v** x **w**) x (**w** x **u**) = (**u**. (**v** x **w**)^2

Solution

Let a vector **p** = **v** x **w**
Then **p** x (**w** x **u**) = **w** (**p. u**) **-** (**p. w**) Substitute **p** = **v** x **w** to get,
(**u** x **w**) x (**w** x **u**) = **w** (**v** x **w. u**) – **u** (**v** x **w. w**)
= **w** (**u. v** x **w**) – **u** (**u. w** x **w**)
= **w** (**u. v** x **w**) And so you have,
(**w** x **v**). (**v** x **w**) x (**w** x **u**) = (**u** x **v**). **w** (**u. v** x **w**)
= (**u** x **v. w**) (**u. v** x **w**)
= (**u. v** x **w**)^2.

Example 4

Given **u** = **i** **-** 2**j** **-** 3**k**, **v** = **i** **-** **k** and **w** = **i** + 3**j** **-** 2k, find. (a) (**u** x **v**) x **w**
(b) **u** x (**v** x **w**)

Solution

(a) (**u** x **v**) x **w** = (**u. w**) **v** **-** (**v. w**) **u**
= (1 x l) + (-2 x 3) + (-3 x -2) v - (2 x 1) + (1 x 3) + (-1 x -2)u
= (1 – 6 + 6) (2**i** + **j** **-** **k**) **-** (2 + 3 + 2) (**i** **-** 2**j** **-** 3**k**)
= (2**i** + **j** **-** **k**) **-** (7**i** **-** 14**j** **-** 21**k**)
= [(2 **-** 7)**i** + (1 + 14)**j** + (**-** I + 21)**k**]
= [-5**i** + 15**j** + 20**k**]
= 5 [**-i**+ 3**j** + 4**k**]

$$= {}^{5}/_{(-1)^2} - 3^2 + 4^2$$
$$= {}^{5}/_{26}$$

(b) [u x (v x w] = [(u. w) v **-** (u. v)w]

u = **i** **-** 2**j** – 3**k**,**v** = 2**i** + **j** **-** **k**, **w** = **i** + 3**j** **-** 2**k**

= {(1 x 1) + (-2 x 3) + (-3 x -2)}v - 1(1 x 2) + (-2 x 1) + (3 x -1)w

= | (1 - 6 + 6) (2i + j - k) **-** (2 **-** 2 + 3) (i + 3j - 2k) |

= | (2i + j - k) **-** (3i + 9j **-** 6k) |

= | (2 - 3)i + (1 - 9)j + (-1 + 6k) |

= [-i - 8j + 5k]

= √(-1)2 + (-8)2 + (5)2

= √1 + 64 + 25

= √90

= 3√10

SELF-ASSESSMENT EXERCISE 2

If **u** = **i** - 2**j** - 3 **k**, **v** = 2**i** + **j** - **k** and **w** = **i** + 3**j** - 2**k**
Find (a) u. (v. w) (b) (u x v) w

Solution
(a) **u**. (**v** x **w**) =

i	j	k
1	-2	-3
2	1	-1
1	3	-2

= 1 (-2 + 3) – 2 (-4 + 1) - 3 (6 - 1)

= 1 (1) + 2 (-3) – 3 (5)

= 1 – 6 - 15

= -20

(b) (**u** x **v**). **w** = **w**. (**u** x **v**) (dot product is commutative.)
Therefore **w**. (**u** x **v**) =

i	j	k
l	3	-2
1	-2	-3
2	1	-1

= 1 (2 + 3) – 3 (-1 + 6) – 2 (1 + 4)

= 1 (5) – 3 (5) – 2 (5).

= 5 -15 -10

= 5 - 25

= - 20

Finally, you should note that the necessary and sufficient condition that u x (v x w) = (u x v) x w is when (u x w) x v = 0

3.5 Reciprocal sets of vectors

You refer to the sets of vectors u, v and w, and u, v and w as reciprocal sets of systems of vectors if

u. u = v. v^1= w. w^1= 1 and

u^1. v = u^1. w = v^1.u = v^1. w = w^1. u = w^1. v = 0

In other words, the dot product of reciprocal sets of vectors is 1.

You can easily remember this by recalling that the product of a number and its multiplicative inverse e.g. x $^3/_2$ is 1, the multiplicative identity.
You can also use the cross product to define the reciprocal sets of vectors. The sets u v w and u^1, v^1 and w^1 are reciprocal sets of vectors if and only if

$$\mathbf{u}' = \frac{v \times w}{u. v \times w} \quad \mathbf{v}' = \frac{w \times u}{u. v \times w} \quad \text{and } \mathbf{w}' = \frac{u \times v}{u. v \times w}$$

Where u. v x w ≠ 0.

You should take note of the denominators, which are all the same and are the box product of the vectors.

3.6 Properties of the reciprocal sets of vectors

Given the vectors $u = \frac{v \times w,}{u.v \times w}$, $v= \frac{w \times u}{u.v \times w}$ and

$$w = \frac{u \times v}{u.v \times w}$$

With u. v x w ≠ 0
Then
(a) u’ u = v. v = w. w = 1

(b) u'. v = v. w = 0, v. u = v. w = 0 w. u = w x v = 0. (c) If u. v x w = a, then u'. v 'x w' = $^{I}/_{a}$
(d) u', v and w are non-coplanar. Or if u.v x w ≠ 0, then u.v x w ≠ 0.

SELF-ASSESSMENT EXERCISE 3

Find a set of vector reciprocal to the set
2i + 3j - k, i – j - 2k, -i + 2j + 2k.

Solution:

Let u = 2i + 3j - k, v = i – j - 2k, w = -i + 2j + 2k

$$U, = \frac{v \times w,}{u.\ v \times w} \qquad v, = \frac{u \times w,}{u.v \times w} \qquad w, = \frac{u \times y}{u.\ v \times w}$$

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

= (-6 - 1)i - (-4 + 1)j + (-2 - 3)k
= -7i + 3j - 5k

$$v \ \ \times w = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

= (-2 + 4)i - (2 - 2)j + (2 -1)k
= 2i + k

$$u \times w = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ -1 & 2 & 2 \end{vmatrix}$$

= (6 + 2)i - (4(-1)j + (4 + 3)k

=8i -3j +7k

$$u.\ v \times w = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

= 2 (-2 + 4) – 3 (2 - 2) – 1 (2 +1)
= 4 – 0 – 3
= 1

u' = 2**i** + **k**, w = -7**i** + 3**j** - 5**k**. **v** = 8**i** - 3**j** + 7**k**

Or

u' = i + k, v' = $^8/_3$i - 5 + $^7/_3$k w' = $^7/_3$i + j - $^5/_3$k.

4.0 CONCLUSION

Vector can be multiplied in triples.

You should, however be very careful which product you are involved with, and so what formula to use.

The necessary and sufficient condition for the vectors u, v and w to be coplanar is that u.v x w = 0.

The scalar triple products u.v x w can be interpreted by you as the components of u along v x w.

To complete the vector algebra, you were given the sets of vectors reciprocal to a given set of vectors, just like the inverse of a number in the Real number system.

5.0 SUMMARY

- u (u x v) = 0. i.e. the Scalar triple products is zero if any two of the vectors are equal.
- (u x v). (v x w) x (w x v) = $(u.v \times w)^2$
- The necessary and sufficient condition that (**u** x **v**) x **w** = **u** x (**v** x **w**) is (**u** x **w**) x **v** = 0.
- The reciprocal Sets of vector **u**.**v**, **w** are

$$\mathbf{u} = \frac{\mathbf{v} \times \mathbf{w},}{\mathbf{u}.(\mathbf{v} \times \mathbf{w})} \qquad \mathbf{v} = \frac{\mathbf{u} \times \mathbf{w},}{\mathbf{u}.(\mathbf{v} \times \mathbf{w})} \qquad \mathbf{w} = \frac{\mathbf{u} \times \mathbf{v}}{\mathbf{u}.(\mathbf{v} \times \mathbf{w})}$$

6.0 TUTOR- MARKED ASSIGNMENTS

1. If **u** = 3**i** - **j** - 2**k** and **v** = **i** - 2**j** - 4**k** and **w** = 2**i** + **j** - 3**k**.
   Find (**u** x **v**) x **w**.

7.0 REFERENCES/FURTHER READING

Keisler, H.J. (2005). Elementary Calculus. An Infinitesimal Approach, 559 Nathan Abbott, Stanford, Califonia, USA.

Wrede, R.C. & Spegel M. (2002). Schaum's and Problems of Advanced Calculus, McGraw – Hill N. Y.