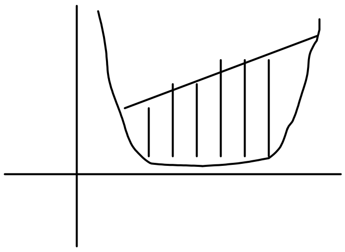
MODULE 1

UNIT 1 COMPUTATION OF AREAS BY CALCULUS

CONTENTS

1.0 INTRODUCTION

One of the early mathematicians that attempted to find the area under a curve was a Greek named Archimedes. He used ingenious methods to compute the area bounded by a parabola and a chord. See Fig (1.1).



In this unit, you will study how to develop necessary tools of calculus to compute areas under curve as a mere routine exercise. The area under a curve gave birth to the second branch of calculus known as integration. The tools that will be developed here will naturally lead to the definition of integration in the next unit – unit 2. Recall that the word to integrate connotes "whole of" which could be interpreted to mean "find the whole area of". This concept is what will be introduced in this unit and this will be fully developed in the next unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

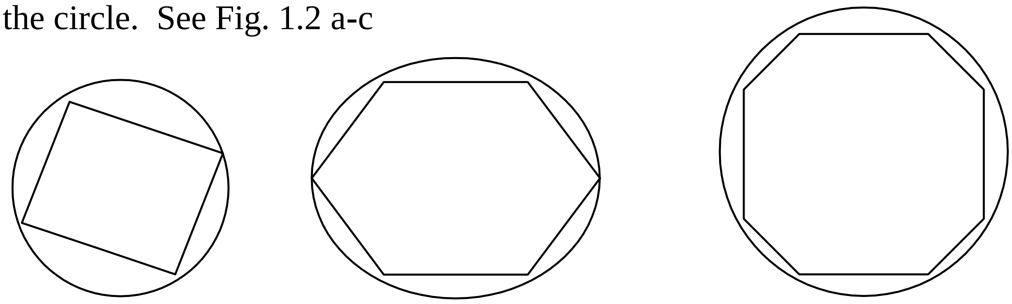
- approximate area under a curve by the sum of areas of rectangles inscribed in the curve
- approximate the area under a curve by the sum of the areas of rectangles circumscribed over the curve
- define a partition of a closed interval (a, b)
- compute the exact value of the area under a curve by the limiting process.

3.0 MAIN CONTENT

3.1 Area under a Curve

You are quite familiar with the computation of the areas of plane figures such as triangles, parallelogram trapezium, regular polygons etc. Interestingly, you studied in elementary geometry that the area of a regular polygon can be computed by cutting it into triangles and sum up the areas of the triangles.

You are also aware t hat the area of a circle is r^2. This formula was derived by the method of limit. You could recall that the limit of the areas of inscribed regular polygons as the number of sides approaches infinity is equal to the area of the circle. See Fig. 1.2 a-c



Inscribed polygons of 4 sides — **Fig. 1.2a**

Inscribed polygons of 6 sides — **Fig. 1.2b**

Inscribed polygons of 8 sides — **Fig. 1.2c**

Let y = f(x) be a continuous function (see the first course on calculus i.e. calculus I unit 4 for definition of continuous function) of x on a closed interval [a, b]. In this case for better understanding, you assume that the f(x) is positive in the closed interval i.e. $f(x) \geq 0$. for all XE [a, b]. Then the problem to be considered is to calculate the area bounded by the graph y = f(x) and the vertical lines y = f(a) and y = f(b) and below by the x – axis as shown in Fig. 1.3.

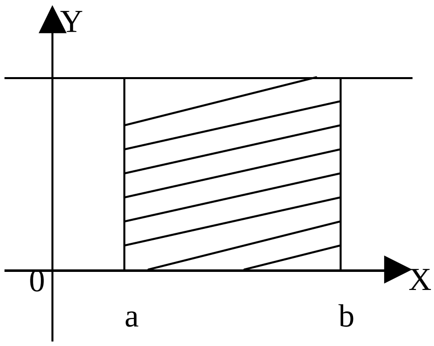


Fig. 1.3

You can start by dividing the area into n thin strips of uniform width $\Delta x = \frac{(b-a)}{n}$ by lines perpendicular to the x – axis at the end points x = a and n x = b and many intermediate points which can be numbered as X_1, X_2, Xn_{-1} (see fig 1.4).

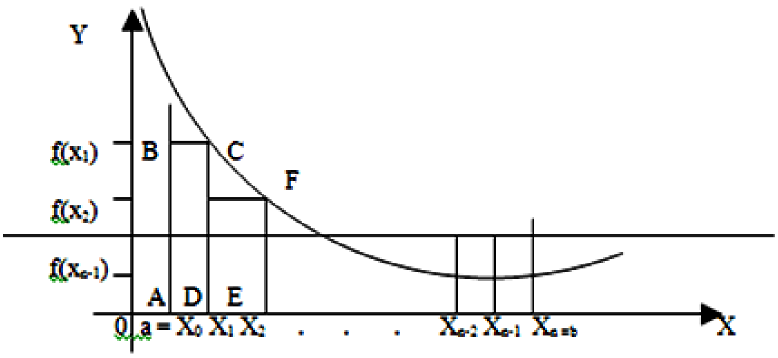


Fig. 1.4

The sum of the areas of these n rectangular strips gives an approximate value for the area under the curve. To put the above more mathematically, you can define the area of each strip in terms of f(x) and x. Given that Δx = x1 – a = x2-x1 = .. = b-xn-1. For example the area of the rectangular strip ABCD in Fig.

1.4 above is given as:

$$\text{Area of ABCD} = f(x_2)\,.\,(x_1-x_0) = f(x_2)\,\Delta x$$

Example:

Suppose $f(x) = x^2-3$ in Fig 1.5 with n = 6 were a = 2, b = 8, dx = 8-2 = 6

Therefore: dx = 1 i.e. you have 6 rectangular strips.

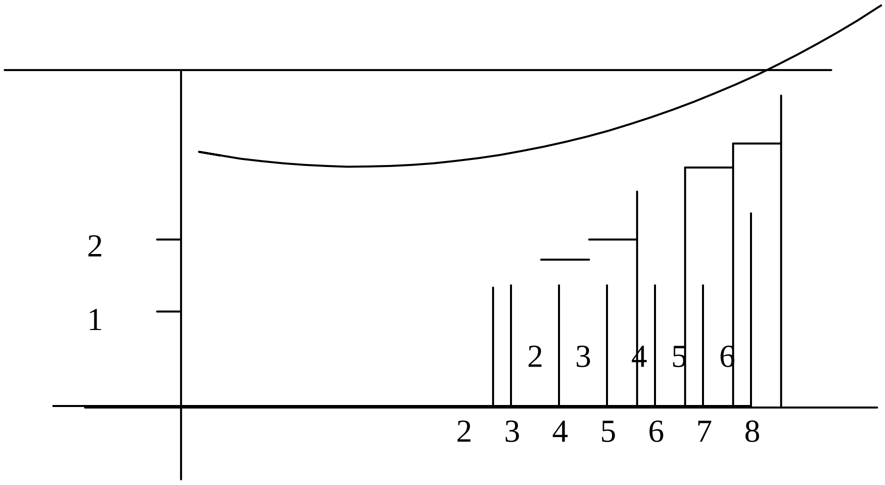


Fig. 1.5

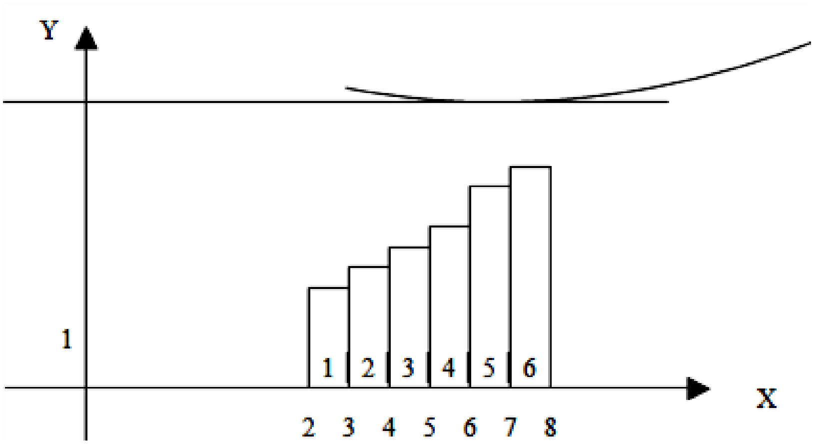
Area is given as sum of

f(2) Δx	=	1.1	=	1
f(3) Δx	=	6.1	=	6
f(4) Δx	=	13.1	=	13
f(5) Δx	=	22.1	=	22
f(6) Δx	=	33.1	=	33
f(7) Δx	=	46.1	=	46

In fig. 1.4 above the area under the curve is larger than the sum of the areas of the inscribed rectangular strips numbered 1 to 6 i.e. sum of areas of strips = 1+6+13+22+33+46 = 121 which is less than area under curve.

Example:

Using the same example Y = x^2-3 use circumscribed rectangular strips instead of inscribed ones to compute the area under the curve. See Fig. 1.6



Area is given as the sum of

f(x) . Δx	=	6.1	=	6
f(x) . Δx	=	13.1	=	13
f(x) . Δx	=	22.1	=	22
f(x) . Δx	=	33.1	=	33
f(x) . Δx	=	46.1	=	46
f(x) . Δx	=	61.1	=	61

Area = 6+13+22+33+46+61 = 181
As should be expected this area is greater than the area under the curve f(x) = x^2-3.

In the computation with the circumscribed rectangular strips the sides of the rectangles are assumed in this case to be the points of the function in their respective subintervals. In the case of the inscribed rectangles, the sides of the rectangles are the minimum values of the function in their respective subintervals.

Therefore the area under the curve lies between the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribe rectangles. This takes to the issue of limit. Therefore it will be right to say as n→ α Δx→0 this implies that the $Lim(MaxArea - MinArea) = 0\ as\Delta x \to 0$

From the foregoing, you can now define the area under curve as the limit of the sums of the areas of inscribed (circumscribed) rectangles as their common base of length dx approaches zero and the number of rectangles increases without bound. In symbols you can write the above limit as:

$$A = lim[f(x_o)\Delta x + f(x_7)\Delta x + \cdots + f(x_{n-1})]\Delta x_n \to \infty$$
$$= lim[f(x_1)\Delta x + f(x_2)f + \cdots + f(x_{n-1})]\Delta xn \to \infty$$

OR

$$A = \lim_{n \to x_1} \sum_{k=10}^{n-1} f(x_k)\,dx = \lim_{n \to x_1} \sum_{k=1}^{n} (fx_k)dx$$

SELF-ASSESSMENT EXERCISE

Repeat the above example using n = 10. Find the difference between the sum of areas of the inscribed rectangle (i.e. the minimum area) and the sum of areas of the circumscribed rectangles (i.e. the maximum area).

3.2 Partition of a Closed Interval

Let [a, b] be a bounded closed interval of real numbers. A partition of a closed interval [a, b] is a finite set of points

P = {a = x_0, x_1, x_2, .., x_{n-1}, x_{n-b}} where a

= $x_0 < x_1 < x_2 < .. < x_{n-1}$, x_n = b

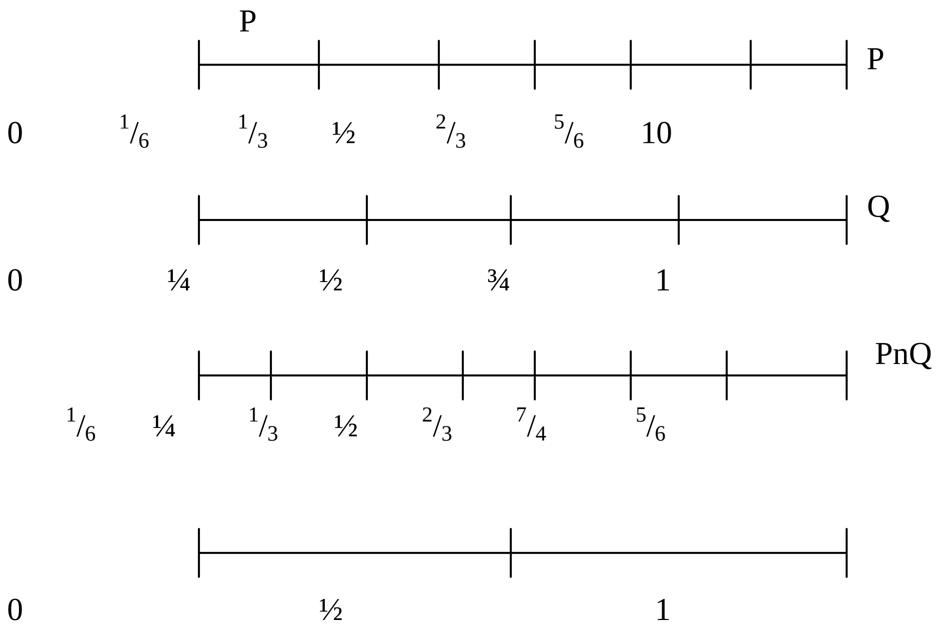
Example:

P = {0, $^1/_6$, $^1/_3$, ½, $^2/_3$, $^5/_6$, 1} and
Q = {0, ¼, ½, ¾, 13 are both partition of {0, 13}

PUQ = {0, $^1/_6$, ¼ , $^1/_3$, ½ , $^2/_3$, ¾, $^5/_6$ 1} is a partition of [0, 1] PnQ = {0, ½, 1} is

a partition of [0,1]

See fig. 10.6(a) to (c)



A partition of P = {x_0, x_1, .., x_n} of [a, b] divides [a, b] into n closed sub interval [x_0, x_1], [x_1, x_2], …, [x_{n-1}, X_n]

The closed interval

[x_{r-1}, x_r] is called the r^{th} subinterval of the partition.

Given a partition of P[a = x_0, x_2, …, x_n = b] the length of the subinterval s are the same and it is denoted by $\Delta x_r = x_r - x_{r-1}$

This equal to the length of the interval [a, b] divided by the number of subintervals n i.e. $\Delta x_r = \frac{b-a}{6}$

Example: Δx for p is $\frac{1-0}{6} = {}^1/_6$

Δx for Q is $\frac{1-0}{4} = {}^1/_4$

Not in all case you will get subintervals of the same length. Example is PUQ The length of $x_1 - X_0 = 1/6 - 0 = 1/6$
The length of $x_2 - X_1 = ¼ - 1/6 = 1/12$

Such partitions in which the subintervals are not of the same length are called irregular partition.

SELF-ASSESSMENT EXERCISE

Write down a regular partition for

(1) [2, 8], n = 12 (2) [1, 8], n = 7

Ans:

(i) [2, 5/2, 6/2, 7/2, 8/2, 9/2, 10/2, 11/2, 12/2, 13/2, 14/2, 15/2 16/2]

(ii) [1, 2, 3, 4, 5, 6, 7, 8]

3.3 Computation of Areas as Limits

In this section you will combine the results of section 3.1 and 3.2 to compute the areas under curves using the limiting process.

Example

A good starting point is to consider the area under the curve Y = X (see Fig. 10.7)

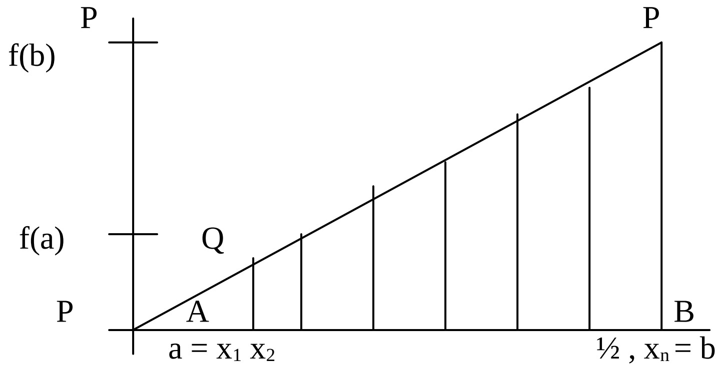


Fig 10.7

which the interval X϶ [a, b] let there be n-regular partition of [a, b] i.e.

$$\Delta x = \frac{b-a}{n}$$

P [a, x_1, x_2, .., x_{n-1}, x_n = b]

x_1 = a + xx

x_2 = a + 2Δx x_3 = a + 3Δx

x_{n-1} = a + (n-1) Δx

Areas of inscribed rectangles are

f(a) . Δx	=	a.Δx
$f(x_1)$. Δx	=	(a+Δx).Δx
$f(x_2)$. Δx	=	(a+2Δx). Δx
..		
$f(x_{n-1})$ Δx	=	(a+(n-1)Δx) . Δx

Sum of the areas of the rectangles is given as

$$S = (a.\Delta x + (a + \Delta x) + \cdots + (a + (n-1)\Delta x).\Delta x)$$

$$= [a + (1 + 2 + 3 + \cdots (n-1)dx]$$

$$= na + (\sum_{k=1}^{n-1} K)\Delta x)\Delta x$$

$$\sum^{n-1} K = \frac{(n-1)n}{2}$$

(The sum of an arithmetic 1 progression with different d = 1)

$$S = \left[na + \frac{(n-1)n}{2}dx\right]dx$$

but $dx = \frac{b-a}{n}$ therefore

$$S = \left[na + \frac{(n-1)n}{2}\frac{b-c}{n}\right]\frac{b-c}{n}$$

$$= \left[a + \frac{b-a.n-1}{2}\right]\frac{b-c}{n}$$

Taking limit as n →∞

$$LimS = \lim_{n\to\infty}\left[a + \frac{b-a.n-1}{2}\right]\frac{b-c}{n}$$

$$= \frac{(a+b-a)(b-a)}{2}\lim_{n\to\infty}\frac{n-1}{n}$$

$$= a + \frac{(b-a)(b-a)}{2}.1$$

$$= \frac{a+b}{2}.(b-a)$$

In fig. 10.8, the area of trapezium AQPB is the same as the area under the curve and as you know the area of trapezium is given as:

½ base x sum of two parallel sides

$= {}^{1}/_{2}\,(b-a) \times f(a) + f(b)$

$= {}^{1}/_{2}\,(b-c)(b+a)$

Example

Find the area under the graph Y = x + 1 $0 \leq x \leq 6$

Solution:

Let n be a positive integer that there be a partition of [a, b] into n regular partition.

Therefore; $\Delta x = \underline{b}\ n$

$x_1 = \Delta x$ $x_2 = 2\Delta x$ $x_3 =$
$3\Delta x$
..

$x_{n-1} = (n-1)\ \Delta x$

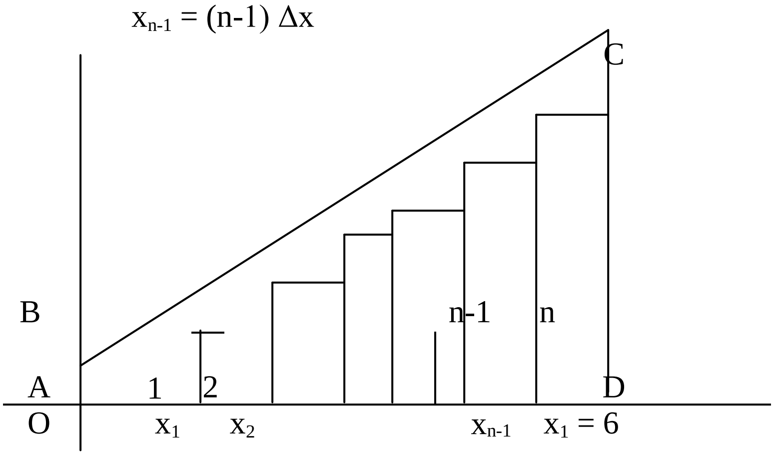


Fig. 10.8

Area of (n-1) rectangles is given as
f(0).Δx = 1.Δx
$f(x_1).\Delta x = (\Delta x + 1)\Delta x$

Sum of areas of rectangles is

$S = \Delta x + (\Delta x + 1)\Delta x + (2\Delta x + 1)\Delta x + (3\Delta r + 1)\Delta x + \cdots + (n-1)(\Delta x - 1)\Delta x$

$= (\Delta x + (\Delta x + 1) + (2\Delta x + 1) + \vdots\ (n-1)(\Delta x + 1)\Delta x)$

$= [\Delta x + n + \sum_{k=1}^{n-1} k\Delta x]\Delta x$

$S = \left[\Delta x + \frac{(n+(n-1)n)}{2}\Delta x\right]\Delta x$ let $\Delta x = \frac{1}{n}$

then

$S = \frac{b}{n} + \left[\frac{(n+(n-1)n)}{2}\right]\frac{b}{n}$

$= \frac{b}{n}\left(1 + \frac{b}{n}.\frac{n-1}{2}\right)$.b

Taking limits as n →∞

$$\lim_{n\to\infty} S = \lim_{n\to\infty}\frac{b}{n} + b\lim_{n\to\infty}\left(1 + \frac{b}{2}\frac{(n-1)}{n}\right)$$

$= 0 + b\left(1 + \frac{b}{2}\right)$

$$\lim_{n\to\infty} S = b + \frac{b^2}{2}$$

SELF-ASSESSMENT EXERCISE

Show that the area of the trapezium ABCD in Fig. 10.8 is equal to $\frac{b(b+2)}{2}$

4.0 CONCLUSION

In this unit, you have studied how to find an approximate value of the area under a curve by computing the sums of areas of rectangles inscribed under the cure and circumscribed over if you have defined a partition of a closed interval. You have studied that as the number of partitions of a closed interval [a, b] is increased without bound the value of the sum of the areas of the rectangles (inscribed or circumscribed) approaches the exact value of the area under the curve in the given interval [a, b] that is the limit of the sum of areas of the rectangles is equal to the exact area under the given curve as the number n of partition tends to infinity or the length dx of the subinterval of the partition tends to zero.

5.0 SUMMARY

In this unit you studied how to

- compute the minimum value of area under a curve i.e. sum of area of rectangles inscribed under a curve within an interval
- compute the maximum value of the area under a curve i.e. the sum of areas of rectangles circumscribed over the curve.
- define a partition of a closed interval [a, b] i.e. a= $x_1 < x_2 < \ldots < x_2$ = b = P[a, b]
- compute the exact area under a curve in a given interval [a, b] by taking the limit of the sum of the areas of the rectangles (inscribed or circumscribed) as the number n of partition of [a, b] is increased without bound i.e. $A = \lim_{n\to\infty} \int_n$ where $dx = \frac{b - a}{n}$

6.0 REFERENCES

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7.0 TUTOR-MARKED ASSIGNMENT

1. Show that the sets

   {0, 1} {0, ½, 1}, {0, ¼, ½, 1} and
   {0, 1/4 , 1/3, ½, 5/8, 1} are partition of {0, 1}

2. Which of the partition of [0, 1] in exercise (1) above are regular?

3. Find the minimum and maximum values of the area under the curve f(x) = 2x for xЄ[0, 1] and P(0, ¼, ½, 1)

2. Find the minimum value of the area under the curve f(x)= 1- x xe [0,2] P($0_1$1/3, ¾, 1, 2).

3. Find the area under the curve Y = x^2 XЄ[0, b] by taking appropriate limits.

4. Find the area under the curve Y = mx a ≤ x ≤ b by taking appropriate limits.

5. Sketch the graph of Y = X + 1. Divide the internal into n = 6 subintervals with d<u>x</u> = (b – a)/6. Sketch the inscribed rectangles.

6. Repeat ∑ x 7 but this time sketch the circumscribed rectangle.

7. Compute the sums of areas in Є x 7 and Є x 8 above.

8. Find the area under the curve Y = x + 1 a ≤ b by taking appropriate limits of results of exercise 9 above.

UNIT 2 DEFINITE INTEGRAL

CONTENTS

1.0 INTRODUCTION

In unit 1, you studied how to compute the area under a curve and showed how you could estimate it by computing sums of area of rectangles. Using the above estimate you applied the concept of limit to get the exact value of the area under a curve. These methods were applied to functions or graphs that could easily be sketched i.e. not too complicated functions. In this unit, you will be introduced to the famous path taken by Leibniz and Newton in showing how exact areas can be computed easily by using integral calculus. It is necessary you refer once more to unit 1 of this course before embarking on this one. It will help you have a proper grasp of this unit if you do so.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the definite integral of a function within an interval [a, b]
- evaluate definite integrals of function
- state the fundamental theorem of integral calculus.

3.0 MAIN CONTENT

3.1 Definition of Definite Integral

In unit 1, you studied that the sum of the areas of inscribed rectangles gives a lower (minimum) approximation of the area under the curve of the function f(x). If you list all the values of the function f(x) in a given interval [a, b] and take the least among these value you will have what is known as the infimum of f(x) for all x€ [a, b]

i.e. Inf f(x) X€[a, b].
let Inf f(x) = M_r and X€ P[a, b]
when $dx_r = x_r - x_{r-1}$. then the area is $M_r . dx_r$. The sum of such area is

$A_L = \sum M_r (x_r - x_{r-1})$ is called the Lower Sum of the function f(x).

If you take the maximum value of f(x) within [a, b] and find their areas i.e. M_r = ∫up f(x) XЄ $[x_{r-1} x_r]$ then the Upper sum for the areas is given as

$$A_u = \sum_{r=1} M_r(x_r - x_{r-1})$$

No known concept has been introduced. You are rewriting sum of areas of a rectangles inscribed under the curve f(x) as $A_L = \sum M_r (x_r - x_{r-1})$ and the sum of areas of rectangles circumscribed over f(x) as

$$A_u = \sum M_r x_r - x_{r-1})$$

Once you keep the fact you will not run into any difficulty understanding what follow next.

Definition: The unique number I which satisfies the inequality

$A_L(P) \leq I \leq A_u(P)$ for all partitions P of [a, b] is called the definite integral (or more simply the integral of f on [a, b] and is written as:

$$I = \int_a^b f(x)\, dx$$

This symbol S dates back to Leibniz and it is called the integral sign. It is an elongated S, which presents sum. The numbers in this case are called the limits of integration. This expression $\int_a^b f(x)\, dx$ read integrating from b to a with respect to x

In the above definition, it has been assumed that f(x) is continuous in the closed [a, b]. This condition guarantee the existence of a number I such that

$$A_L(p) \leq I \leq A_L(p)$$

The prove of the above theorem could be found in the text suggested for further reading given at the end of this course.

If f(x) ≥ 0 X∀Є[a, b] then

$I = \int_a^b f(x)\, dx$ = Area under the curve f(x)

Example

Given that f(x) = K XЄ[a, b] sho∀w that $\int_a^b f(x)\, dx$ = K(b – a)

Solution: Let P = {u, x_o, x_1, x_2, .., x_n = b} be any partition of [a, b]

Since f(x) = K XЄ [a, b] the∀f(x_0) = f(x_1) = … = f(x_n)

$$\text{Let } A_L (P) = \sum m\Delta X_r = K\Delta X_1 + K + \Delta X_2 + ..K\Delta X_n$$
$$= K(\Delta X_1 + ... + \Delta X_n) = K(b\text{-}a)$$

Also
$A_u(P) = \sum M_r \Delta X_r$ = K(b-a)
But
$A_L(P) \le \int_a^b f(x)\,dx \le A_u(P)$

then K(b-a) ≤ $\int_a^b f(x)\,dx$ ≤ K(b-a)

⇨ $\int_a^b f(x)\,dx = K(b-a)$

Example

Given that f(x) = x show that $\int_a^b f(x)\,dx$ = ½ (b^2-a^2)

Solution: Let P{a$x_{o,}$ x_1, .. , x_n, = b} be an arbitrary partition of [a, b].
f(x) = x for XЄ [X_r, $X_{r2,}$] for all such subintervals.
So M_r ≤ f(x) m_r XЄ [x_r, x_{rt1}] such m_r and m_r exist for each subinterval.
Let M_r = x_r and m_r = x_{r-1}
then

$$A_u(P) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} x_r \Delta x_r$$

= $x_1(x_1-x_o)+x_2(x_2-x_1)+\ldots+ x_n(x_n-x_{r-1})$
and

$$A_L(P) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} x_{r-1} \Delta x_r$$

= $x_o(x_1-x_o) + x_1(x_2-x_1) + .. x_{2-1}(x-x_{n-1})$
For each index,

$x_{r-1} \le ½ (x_r + x_{r-1}) \le x_r$

Therefore

$A_L(P) \le ½ (x_1+x_o)(x_1-x_o) + ½ (x_2+x_1)(x_2-x_1) + \ldots + ½ (xn+xn_{-1})(x_n-x_{r-1}) \le A_u(P)$
but $½ (x_1+x_o)(x_1-x_o) + ½ (x_2+x_1)(x_2-x_1) + .. + ½ (xn+xn_{-1})(Xn+Xn_{-1})$

$$= 1/2\,(x_1^2 - x_0^2 + x_2^2 - x_1^2 + \cdots + x^2 - x_{n-1}^2) = 1/2\,(x^{2n} - x_0^2)$$

⇨ $A_L(P) \le 1/2\,(x_n^2 - x_o^2) \le A_u(P)$
⇨ $A_L(P) \le 1/2\,(b^2 - a^2)$
⇨ $\int_a^b f(x)dx = 0$

The following properties of definite integral are hereby stated without their proofs are beyond the scope this course:

1. If a < c < b then $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
2. If a < b then $-\int_a^b f(x)dx = \int_b^a f(x)dx$
3. $\int_a^a f(x)dx = 0$

Example: Given that

$$\int_0^1 f(x)dx = 6, \quad \int_1^3 f(x)dx = 5, \quad \int_3^7 f(x)dx = 2$$

Find (i) $\int_0^7 f(x)dx$ (ii) $\int_1^3 f(x)dx$ (iii) $\int_1^1 f(x)dx$ (iv) $\int_7^1 f(x)dx$

Solution: (i) $\int_0^7 f(x)dx$

$\int_0^7 f(x)dx = \int_0^t f(x)dx + \int_x^7 f(x)dx$

Let t = 3 i.e. $\int_0^7 f(x)dx = \int_0^3 f(x)dx + \int_3^7 f(x)dx = 5 + 2 = 7$

(ii) $\int_1^3 f(x)dx = \int_1^t f(x)dx + \int_t^3 f(x)dx$

Let t = 0 $\int_1^3 f(x)dx \int_1^0 f(x)dx + \int_0^3 f(x)dx = -6 + 6 = -1$

(iii) $\int_1^1 f(x)dx = 0$

(iv) $\int_7^1 f(x)dx = -\int_1^7 f(x)dx = \int_1^3 f(x)dx + \int_3^7 f(x)dx = -[-1 + 2]$

SELF-ASSESSMENT EXERCISE

Given that $\int_0^2 f(x)dx = 2, \int_0^3 f(x)dx = 4 \; and \; \int_2^4 f(x)dx = 7$

Find (i) $\int_0^4 f(x)dx$ (ii) $\int_4^3 f(x)dx$ (iii) $\int_3^2 f(x)dx$ (iv) $\int_2^3 f(x)dx$

Answer: (i) 9 (ii) – 5 (iii) – 2 (iv) 2

3.3 Fundamental Theorem of Integral Calculus

To find the value of the function F(x) = $\int_a^b f(x)dt$ for some simple function it could easily be evaluated. Either by the limiting process discussed in unit or by direct evaluation as was done in the previous section. Such process might prove very laborious for certain classes of functions. In this section you will examine the direct connection between differential calculus and integral calculus. This connection was made possible by looking at the summation process of finding areas and volumes and the differentiation process of finding the scope of a target to a curve. It is quite interesting that the process of carrying out inverse differentiation yields an easy tool of solving the summation problem.

So you will now discuss the proof of the fundamental theorem concept behind the theorem is that before you can evaluate a definite integral $\int_0^4 f(x)dx$ you will first of all find a function Fl(x) whose derivative is f(x). i.e.

$F^1(x) = f(x)$ X€ (A, B). ∀

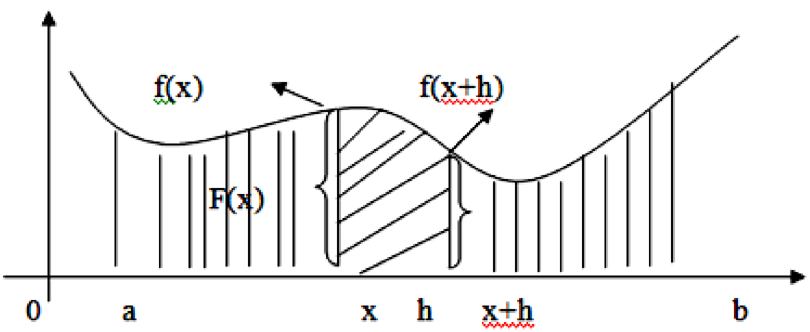
You will now as first step study the proof of the following theorem:

Theorem 1: If f(x) is a continuous function on [a, b], the function F(x) defined on [a, b] by setting F(x) = $\int_a^x f(t)dt$ is a(i) continuous function on [a, b] and (ii) satisfies $F^1(x) = f(x)$ for all x in (a, b).

Proof: You will begin with x€ [a, b] and show that:

$$\lim_{h\to 0} \frac{F(x+h) - F(x) = f(x)}{h}$$

In figure 2.1 F(x +h) = area from a to x + h



F(x) = area from a to x
F(x + h) - F(x) = area from x to x + h
(Area = base x height)

$\frac{F(x+h)-F(x)}{h}$ = $\frac{\text{areafromxtox+ h}}{h}$ ≅ f(x) if h ⟶ 0 (note f(x) = height of the area under curve in Fig. 2.1

If $x < x + h \leq b$ then

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

(since from statement of theorem F(x) = $\int_a^x f(t)dt$

It follows therefore that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^h f(t)dt$$

$$= \int_a^{x+h} f(t)dt$$

Let M_h = maximum value of f(x) on [x, x+h]
and m_h = minimum value of f(x) on [x, x+h]
since M_h (x+h – x) = M_h.h and m_h (x+h – x) = m_h.h

therefore, M_h = upper sum (see UNIT 1) and m_h = lower sum (see UNIT 1)
therefore

$$m_h.h \leq \int_x^{x+h} f(t)dt \leq M_h.h$$

$$= \quad m_h.h \leq \frac{F(x+h–F(x)}{h} \leq M_h.h$$

since f(x) is a continuous function on [x, x + h] therefore

$$\lim_{h\to 0} m_h.h = f(x) = \lim_{h\to 0} M_h.h$$

thus $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$ - I

In a similar manner you can show that if X Є (a, b), then

$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$ - - I

Now if xЄ (a, b) then equation (I) and (II) hold

Thus $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$

And $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = F^1(x)$

therefore $F^1(x) = f(x)$

since $F^1(x)$ exists then F(x) must be continuous on (a, b). Before you prove the fundamental theorem of calculus. Look at this definition.

Definition:

A function F(x) is called an anti-derivative for f(x) on (a. b) if and only if

(i) F(x) is continuous on (a, b) and
(ii) $F^1(x) = f(x)$ for all XЄ (a, b)

Using the above definition you can rewrite theorem 1 as

If f is continuous on (a, b) then

$F(x) = \int_a^x f(t)dt$

The above now says to you that you can construct or find an anti-derivative for f(x) by integration f(x). The next theorem you are going to study will tell you that you can evaluate the definite integral $\int_a^x f(x)dx$by finding an anti-derivative for f(x).

The Fundamental Theorem of Integral Calculus:

Let f(x) be continuous for all xЄ (a, b) If P(x) is an anti-derivative of f(x) for all xЄ (a, b) then

$F(x) = \int_a^b f(x)dx = P(b) – P(a)$

Proof: In theorem 1, the function $F(x) = F(x) = \int_a^x f(t)dt$

is an anti-derivative for f(x) for all XЄ (a, b).

If P(x) is another anti-derivative for f(x) for all xЄ (a, b), then it implies that both P(x) and F(x) are continuous for all XЄ (a, b) and also will satisfy that $P^1(x) = F^1(x)$ for all XЄ (a, b). There exist a constant C such that

F(x) – P(x) = C

Since $F^1(x) = P^1(x)$ and derivative of a constant is zero
i.e. F(x) – P(x) = C $\Rightarrow F^1(x) – P^1(x) = 0$

Since F(a) = 0 then P(a) + C = 0 and C = -P(a)

This implies that

F(x) = P(x) – P(a) for all XЄ (a, b) Thus

F(h) = P(b – P(a) (x = b)

Since F(b) = F(x) = $\int_a^b f(t)dt$ = P(b)-P(a)

which is the required result.

3.4 Evaluation of Definite Integral

You are now set to seek or construct anti-derivates F(x) which will evaluate the definite integral given as F(x) = $\int_a^b f(x)dx$

Example: Find $\int xdx$

Solution Let F(x) = ½x^2 as an anti-derivative

Then $\int_a^b xdx$ = ½ ($b^2 - a^2$)

Find the $\int_a^b x^n dx$ when n is a positive integral the anti-derivative to use is

$$F(x) = \frac{1}{n+1}x^{n-1}$$

⇨ $F^1(x) = x^n$ => $\int_a^b x^n = F(b) - F(a)$

$= \frac{1}{n+1}(b^{n+1} - a^{a+1})$

Notation:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

thus $\int_a^b x^4 dx \left[\frac{1}{4+1}x^{4+1}\right]_a^b = {}^1/_5\,(b^5 - a^5)$

Example:

$$\int_1^2 (6x^2 - 2x^3 - x)dx$$

Let F(x) = $\frac{x^6}{4} - \frac{2x^4}{2} - x^2$

then $\int_1^2 (6x^5 - 2x^3 - x)\,dx = \left[x^6 - \frac{1}{2}x^4 - \frac{x^2}{2}\right]_1^4 = 3960$

Example:

Evaluate the following integrals by applying the fundamental theorem.

(i) $\int_{1}^{0}(x-1(x-2)dx$

(ii) $\int_{3}^{7}\frac{dx}{(x-2)^2}$

(iii) $\int_{0}^{1}\left(x^{3/4}+{}^{1}/_{2}\,x^{1/2}\right)dx$

(iv) $\int_{a}^{9}(a^2x-x^4)$

(v) $\int_{1}^{3}\frac{2-x}{x^3}dx$

(vi) $\int_{1}^{8}\left(\sqrt{t-\frac{1}{t^2}}\right)dt$

(vii) $\int_{1}^{3}6-tdt$

(viii) $\int_{1}^{2}x^2(x-1)dx$

(ix) $\int_{1}^{4}\sqrt{x+1}\,dx$

(x) $\int_{0}^{1}(x-1)^{17}\,dx$

Solution: To evaluate $\int_{-3}^{0}(x-1)(x-1)dx$

you expand the function (x - 1) (x -1)

$= x^2 - 2x + 1$

(i) $\int_{0}^{3}(x-1)(-1)dx = \int_{0}^{3}(x^2-2x+1)dx$

let $F(x) = {}^{1}/_{3}\,x^3 - x^2 + x$ serve as anti-derivative

therefore $\int_{0}^{3}(x^2-2x+1) = \left[{}^{1}/_{3}\,x^3 - x^2 + x\right]_{0}^{3}$

(ii) $\int_{3}^{7}\frac{dx}{(x-2)}$

construct a function with derivative as $\frac{1}{(x-2)^2}$ it is not difficult to see that $\frac{d}{dx}\left[\frac{1}{x-2}\right] = \frac{1}{(x-2)^2}$

therefore: $\int_{3}^{7}\frac{dx}{(x-2)} = \left[\frac{-1}{x-2}\right]_{3}^{7} = \frac{8}{10}$

(iii) $\int_{0}^{1}\left(x^{3/4}+{}^{1}/_{2}\,x\,{}^{1}/_{2}\right)dx$

let F(x) = $4/7x^{3/4} + ½ x ½$) = 4/7 x 7/4 + 1/3 x 3/2

therefore $\int_0^1 \left(x^{3/4} + {}^1/_2\, x\, {}^1/_2\right) = \left[{}^4/_7\, x^{4/7} + {}^1/_3\, x^{3/4}\right]_0^1 = 19$

0 0

(iv) $\int_0^9 (a^2x^2 - x^4)dx$

Let F(x) = $\frac{a^2x^3}{3} - \frac{x^5}{5}$

$$\int_0^a (a^2x^2 - x^4) = \left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{a^2x^3}{3} - \frac{a^5}{5}$$

(v) $\int_1^3 \frac{2-x}{x^3} = \int_1^3 (2-1)dx$

Let F(x) = $\frac{1}{x} - \frac{1}{x^2}$

then $\int_1^3 \frac{2-x}{x^3} = \left[\frac{1}{x} - \frac{}{x^2}\right]_1^3$

(vi) $\int^8(\sqrt{E} - 1/t^2)dt$

Let F(t) = $\frac{2t^{3/2}}{3} + \frac{1}{t}$

$\int_1^8 \sqrt{t} - 1/t^2 = \left[\frac{2t^{3/2}}{3} + \frac{1}{t}\right]_1^8 = \frac{32\sqrt{2}}{3} - \frac{37}{24}$

(vii) $\int_1^3 \left(\frac{6-t}{t^4}\right)dt = \int_1^3 \left(\frac{6}{t^4} - \frac{1}{t^3}\right)dt$

F(t) = $\frac{1}{} - \frac{2}{}$

$\int_1^3 \left(\frac{6}{t^4} - \frac{1}{t^3}\right)dt = \left[\frac{1}{2t^2} - \frac{2}{t^3}\right]_1^3 = \frac{40}{27}$

(x) $\int_1^2 (x-1)\, x^2\, dx = \int_1^2 (x^3 - x^2)dx$

F(x) = $\frac{x^4}{4} - \frac{x^3}{3}$

$\int^2 (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = \frac{17}{12}$

(xi) $\int_1^4 \sqrt{x} + 1dx$

F(x) = $2/3(x + 1)^{3/2}$

then $\int_1^4 \sqrt{x} + 1 = [2/3(x + 1)^{3/2}]_1^4 = \frac{10\sqrt{5}}{3} - \frac{4\sqrt{2}}{3}$

(xii) $\int_0^1 (x-1)^7 dx$

$F(x) = \frac{1}{8}(x-1)^8$

therefore: $\int_0^1 (x-1)^7 = \left[\frac{1}{8}(x-1)^8\right]_0^1 = \frac{-1}{8}$

4.0 CONCLUSION

In this unit, you have studied how to define a definite integral. You have seen the connection between the summation process of finding the area under a curve and the differentiation of the function representing the area under the curve. You have studied that the fundamental theorem of integral calculus is the bridge between the summation process and the differentiation process i.e. you can find the area under a curve by finding an anti-derivative for the curve. You have applied the theorem in evaluation of definite integrals.

5.0 SUMMARY

You have studied the following in this unit:

- How to define a definite integral
- How to evaluate definite, integral using the following properties:

  (i) $\int_a^a f = 0$, (ii) $\int_b^b f + \int_a^c f = \int_a^c f$ and (iii) $\int^b f = -\int^a f$

- How to apply the fundamental theorem of integral calculus in evaluating the definite integral of rational functions.

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7.0 TUTOR-MARKED ASSIGNMENT

Evaluate the following integrals by applying the fundamental theorem of integral calculus.

(1) $\int_0^1 (4x - 3)dx$

(2) $\int_1^0 5x - 3dx$

(3) $\int_0^1 (3x + 2)dx$

(4) $f^5\sqrt{x}$

(5) $\int_{-c}^{a} (x - a)^2 dx$

(11) $\int_1^2 (\sqrt{x} - \frac{1}{\sqrt{x}})dx$

(12) $\int_1^2 (3t + 4t^2)dt$

(13) $\int_1^3 (\frac{x^2 + 1}{x^2})dx$

(14) $\int_0^1 x^2(x - 1)dx$

(15) $\int_1^4 (t^3 - t)dt$

(6) $\int_{1}^{2}(\frac{5}{t} + t^x)dx$

(7) $\int_{0}^{2}(1 - x)dx$

(8) $\int_{-4}^{-t}(\frac{1}{x^2} + x)\, dx$

(9) $\int_{-2}^{-1}\frac{1}{x^4}$

(10) $\int_{-2}^{2}(3 + 2x - x^2)dx$

(16) $\int_{-2}^{1}(x + 1)\,(x - 2)dx$

(17) $\int_{1}^{2}x^{-1/2}dx$

(18) $\int_{1}^{2}\frac{2(x+3)}{x^3}dx$

(19) $\int_{1}^{3}(\sqrt{x} + \frac{1}{\sqrt{x}})^2$

(20) $\int_{2}^{3}(2v - 3\sqrt{v})dv$

UNIT 3 INDEFINITE INTEGRAL

CONTENTS

1.0 INTRODUCTION

You have studied rules for differentiation of various functions such as polynomids functions, rational functions, trigonometric function of sines, cosines, tangent etc. hyperbolic functions and then inverses, exponent and logarithm functions. All these you studied in the first course in calculus. However, the reverse process i.e. anti-differentiation is some how not as straight forward process as the differentiation. The reasons being that there are no systematic rules or procedures for anti-differentiation. Rather success on techniques of anti-differentiation depends much more on your familiarity with differentiation itself. So before embarking on the study of this unit, it might be worth the time to practice some of the differentiation in calculus I. Do not be discouraged when you come across functions whose derivatives are not very common. In this unit and subsequent ones you will study some basic methods of anti-differentiation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- evaluate indefinite integral as anti-differentiation
- recall notations for integration and
- recall properties of indefinite integration
- evaluate indefinite integrals using the properties of indefinite integration.
- integrate differential equations that are separable.

3.0 MAIN CONTENT

3.1 Indefinite Integration

In this section an informal definition of what is anti-differentiation will be given. Suppose that the derivative of the function is given as:

$$\frac{dy}{dx} = f(x)$$

and you were asked to find the function y = F(x). For example you are given the differential equation. $\frac{dy}{dx} = 2.$

From your experience with differentiation you can easily know that $y = x^2$ since $\frac{dy}{dx} = 2x$

Interestingly, it is not only $y = x^2$ that can be differentiated to give $\frac{dy}{dx} = 2x$.

Other function like $y = x^2 - 1$, $y = x^2 + 2$, $y = x^2 + a$, $y = x^2 + 4$ can be differentiated to yield $\frac{dy}{dx} = 2x$

In general any function of this form $y = x^2 + c$, where C is any constant will yield a differential equation of this type $\frac{dy}{dx} = 2x$

You are now ready to take this definition.

Definition 1: An equation such as $\frac{dy}{dx} = f(x)$ which specifies the derivative as a function of x (or as a function of x and y) is called a differential equation. For example $\frac{dy}{dx} = \text{sine}$ is differential equation

Definition 2: A function y = F(x) is called a solution of the differential equation $\frac{dy}{dx} = f(x)$ if over domain a<xcb F(x) is differentiable and

$$\frac{d}{dx} F(x) = F^1(x) = f(x)$$

in this case F(x) is called an integral of f(x) with respect to x.

Definition 3: If F(x) is an integral of the function f(x) with respect to x so is the function F(x) + C an integral of f(x) with respect to x, where c is an arbitrary constant. If $\frac{d}{dx} F(x) = f(x)$ so also is F(x) + C i.e. $\frac{d}{dx} [F(x) + C] = \frac{d}{dx} F(x) + C \frac{dc}{dx} = \frac{df(x)}{dx} + 0 = F^1(x) = f(x)$

From the above if y = F(x) is any solution of $\frac{dy}{dx} = f(x)$ then all other solutions are contained in the formula y= F(x)+C where C is an arbitrary constant this gives rise to the symbol. $\int f(x)\, dx = F(x) + C$ – (1) where the symbol *S* is called an integral sign (see unit 2). Equation 1 is read the integral of f(x)dx is equal to F(x) plus C since $\frac{dy}{dx} = 2x$ and a typical

solution is $F(x) = x^2 + C$. then $\frac{dF(x)}{dx} = 2x = \frac{d(x^2+C)}{dx}$

$= y = x^2 + C$

and $\frac{dy}{dx} = \frac{d}{dx}(x^2 + C) = 2x\ dx$

Example: If $y = x$ $\frac{dy}{dx} = 1\ dx$

$= \int \frac{dy}{dx} dx = \int 1dx = x + C$

In other words, when you integrate the differential of a function you get that function plus an arbitrary constant.

Example: Solve the differential equation $\frac{dy}{dx} = 4x^3$

Solution: let $\frac{dy}{dx} = 4x^3$

then $dy = 4x^3dx$ integrate both side you get $\int dy = \int 4x^3dx$ but $\frac{d}{dx}(x^2)=4x^3dx$.

therefore $y = \int 4x^3dx = \int d(x^4) = x^4$ C.

Example: Solve the differential equation $\frac{dy}{Dx} = 2x + 1$

$= dy = (2x + 1)\ dx$

but $\frac{d}{dx}(x^2 + x) = 2x + 1\ dx$

therefore $\int dy = \int (2x + 1)dx$ becomes $y = \int dx^2 + x) = x^2 + x + C$. compare $\int d(F(x)) = F(x)$ with the result of UNIT 2.

Example: Solve the following differential equation:

(1) $\frac{dy}{dx} = x^2 - 1$ (2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

(3) $\frac{dy}{dx} = \frac{x}{y}$ (4) $\frac{dy}{dx} = 2x + 3$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})dx$ (6) $\frac{dy}{dx} = 3x^2-2x+3$

(7) $\frac{ds}{dt} = 3t^2-2t-6$ (8) $\frac{dv}{du} = 5u^4-3xu^2-1$

(9) $\frac{dx}{dt} = 8\sqrt{x}$ (10) $\frac{dy}{dx} = (2x^2- \frac{1}{x^2})$

Solution: $\frac{dy}{dx} = x^2- 1$

$= dy\ (x^2 - 1)dx$

$\int dy = S(x^2-1)dx$

but $d(\frac{x^3}{3} - x) = (x^2-1)dx$

therefore: $y = \int d(\frac{x^3}{3}-x) = \frac{x^3}{3} - x + C$

(2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

$\int dy = S(\frac{1}{x^2} + x)dx$

$d(\frac{-1}{x} + \frac{x^2}{2}) = (\frac{1}{x^2} + x)\, dx$

$y = \int d(\frac{x^2}{2} - \frac{1}{x}) = \frac{x^2}{2} - \frac{1}{x} + c$

(3) $\frac{dy}{dx} = \frac{x}{y}$

$\int ydy = \int x\, dx$

$d(\frac{y^2}{2}) = ydy$ and $d(\frac{x^2}{2}) = xdx$

therefore: $\int ydy = \int d(y\frac{2}{2})$

$\int d(\frac{y^2}{2}) = \int d(\frac{x^2}{2})$

$= \frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 = x^2 + 2C_1$

$y^2 = x^2 + C$

(4) $\underline{dy} = 2x + 3\, dx$

$\int dy = \int (2 + 3)\, dx$

$y = \int d(x^2 + 3x) = x^2 + 3x + C$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})$

$dy = (x^2 + \sqrt{x})\, dx$
$\int dy = \int (x^2 + \sqrt{x})dx$
$y = \int d(\frac{x^3}{3} + \frac{2}{3}x^{3/2}) = \frac{x^3}{3} + \frac{2}{3}x^{3/2} + C$

(6) $\underline{dy} = 3x^2 - 2x - 5\ \underline{dx}$

$\underline{dy} = 3x^2 - 2x - 5$

$\int dy = \int (3x^3 - x^2 - 5x) = x^3 - x^2 - 5x + C$

(7) $\frac{ds}{dt} = 3t^2 - 2t - 6$

$\int ds = \int (3t^2 - 2t - 6)dt$

$\int = \int d(t^3 - t^{2-} 6t) = t^3 - t^2 - 6t + C$

(8) $\frac{dv}{du} = 5u^4 - 3u^2 - 1$

$\int dv = \int (5u^4 - 3u^2 - 1)du$
$V = \int d(u^5 - U^{3-} U) = U^5 - U^3 - U + C$

(9) $\frac{dx}{dt} = 8\sqrt{x}\ dt$

$dx = 8\sqrt{x}\ dt$

$= \int \frac{dx}{\sqrt{x}} = \int 8\ dt$

$\int d(2\sqrt{x}) = \int d(8t)$
$2\sqrt{x} + C_x = 8t + C_t$
$2\sqrt{x} = 8t + C_x + C_t$
$2\sqrt{x} = 8t + C$, where $C = C_x + C_t$

(10) $\frac{dy}{dx} = (4x^2 - \frac{1}{x^2})$

$\int dy = \int (4x^2 - \underline{1})dx\ x$
$y = \int d(\frac{4x^3}{3} - \frac{1}{x}) = \frac{4x^3}{3} - \frac{1}{x} + C$

SELF-ASSESSMENT EXERCISE

Evaluate the following:

(1) $\int \frac{dx}{x^5}$ (2) $\int (x + 1)^3 dx$

(3) $\int (ax^2 + b)dx$ (4) $\int (\frac{x^3 + 1}{x^6})\ dx$

(5) $\int (\frac{x^3 - 1}{x^2})dx$ (6) $\int (t^2 - a)(t^2 - b)dt$

(7) $\int (\sqrt{x} - \frac{1}{x^{1/3}})dx$ (8) $\int \frac{(5x)^4}{x^5}dx$

(9) $\int \frac{dx}{\sqrt{1+x}}$ (10) $\int (x-1)^2 + \frac{1}{(x+2)^2})dx$

Ans:

(1) $\frac{-1}{4x^4} + C$ (2) $\frac{1}{4}(x+3)^4 + C$

(3) $1/3ax^3 + bx + C$ (4) $\frac{-1}{10}\frac{5x^3}{x^5} \frac{+2}{} + C$

(5) $\frac{1}{2}\frac{(x^3+2)}{x}$ (6) $\frac{1}{5}t^5 + \frac{1}{3}(b-a)t^3 - abt + C$

(7) $2/3\, x^{3/2} - 3/2\, x^{2/3} + C$ (8) $\frac{125}{3x^{15}} + C$

(9) $2\sqrt{x} + 1 + C$ (10) $1/3(x-1)^3 - \underline{1} + C\ (x+1)$

3.2 Properties of Indefinite Integral

So far, you would have been doing much of guess work to find an appropriate anti-derivative that will fit the answers above you will now be given some properties of indefinite integral. It would help reduce the amount of guesswork when evaluating integrals.

(1) The integral of the differential of a function U is U plus an arbitrary constant. $\int du = u + c$

(2) A constant may be moved across the integral sign $Sadu_1 = aSdu$

(3) The integral of the sum of two differentials is the sum of their integrals $\int (du + dv) = \int du + \int dv$

(4) The integral of difference of two differential is the difference of their integrals $\int (du - dv) = \int du - \int dv$

(5) As a consequent of 2, 3 and 4 above, you have that
$\int a(du \pm dv) = a\int du \pm a \int \underline{dv}$

$\int du_1 \pm du_2 \pm du_3 \ldots du_h = \int du_1 \pm \int du_2 \pm .. \pm \int du_h$

(7) If n is not equal to minus 1, the integral of $U^n du$ is obtained by adding one to the exponent dwindling by the new exponent and adding an arbitrary constant $\int u^n du = \frac{U^{n+1}}{n+1} = C$

Find the following

Example (1) $\int (5x^{10} - x^8 + 2x)dx = \int 5x^{10}dx - \int x^8 dxt \int 2xdx$

$= \frac{5x^{10+1}}{10+1} - \frac{x^{8+1}}{8+1} + \frac{2x^{1+1}}{1+1}$

$= \frac{5x^{11}}{11} - \frac{x^9}{9} + x^2 + C$

(2) $\int x^{3/2}dx = \frac{x^{3/2+1}}{3/2+1} = \frac{x^{5/2}}{5/2}$

$= 2/5\ x^{5/2}$

(3) $\int$ 3x + 1dx

Let u = 3x+1

then $\frac{du}{dx} = 3 = du = 3dx$

Therefore $\int\sqrt{3x+1}dx = \int u^{1/2} \frac{du}{3}$

here $dx = \frac{du}{3}$

therefore $1/3 S u^{1/2} du = 1/3 \frac{u^{1/2+1}}{½+1}$

$= \frac{1}{3} \frac{2}{3} u^{3/2} = \frac{2}{9} U^{3/2}$

$= \frac{2}{9}(3x+1)$

(4) $\int\sqrt{-}$ 1 dx

Let U = 4x – 1 $\frac{du}{dx} = 4 = dx = \frac{du}{4}$

then $\int\sqrt{4x-1}dx = \int u^{1/2} \frac{du}{4}$

$¼ \int u^{1/2} du = \frac{U^{1/2+1}}{½+1} = \frac{2}{12}(4x-1)^{3/2} + C$

Examples: Evaluate the following integrals

(i) $\int\sqrt{1=4x}$ (ii) $\int \sqrt[3]{1+x}\,dx$

(iii) $\int 5\sqrt{2x+1}$ (iv) $\int 4\sqrt{4x - 2}$

(v) $\int \sqrt[6]{6x + 4}$

Solution:

(i) $\int \sqrt{1-4x}\,dx$ let U = 1- 4x then $\frac{du}{dx} = 4$, $dx = \frac{-du}{4}$

therefore $\int\sqrt{1-4x}\,dx = \int u^{1/2}\left(\frac{-du}{4}\right) = -\frac{1}{4}\int u^{1/2}du$

$= -\frac{1}{4}\left[\frac{2u^{3/2}}{3}\right] = -\frac{1}{6}(1-4x)^{3/2} + C$

(ii) $\int \sqrt[3]{1+x}\,dx$

then du = 1 du = dx dx

therefore $\int\sqrt[3]{1 + x}\,dx = \int u^{1/3}du = \frac{3}{4}u^{4/3} = \frac{3}{4}(1 + x)^{4/3} + C$

(iii) $\int \sqrt[5]{2x\ 1}\,dx$ let U = 2x + 1

then du/dx = 2 ⟹ = dx = du/2

therefore $\int \sqrt[5]{2x+1}\,dx = \int U^{1/5}\frac{du}{2} = \frac{1}{2}\left[\frac{5U^{6/5}}{6}\right] = \frac{5(2x+1)^{6/5}}{}C$

$= \frac{5}{12}(2x + 1)^{6/5} + C$

(iv) $\int \sqrt[4]{4x-2}\,dx$ let U = 4x - 2

then $\underline{du}$ = 4dx ⟹ dx = du/4

$\int\ 4\ 4x -2dx = \frac{1}{4}\int U^{1/4}du = \frac{1}{4}\,[4/5(4x-2)^{5/4}]$

$= 1/5\ (4x-2)^{5/4}$

(v) $\int \sqrt[6]{6x+4}\,dx$ let U = 6x + 4 then dx = du/6

$\int \sqrt[6]{6x+4}\,dx = 1/6\int U^{1/6}du = 1/6.6/7\ (6x+4)^{7/6}+C$

$= 1/7\ (6x + 4)^{7/6} + C$

Exercise: Evaluate the integrals

(1) $\int (8x^7 – 6x^5 – x^4 + 3x^3 + 2)\,dx$

(2) $\int (6x + 1)^{1/6}dx$

(3) $\int (1 – 4x)^{1/4}dx$

(4) $\int (4 – 10x)^{1/10}dx$

(5) $\int (x-1)^{1/3}dx$

Ans:

(1) $x^8 - x^6 - \frac{x^5}{5} + \frac{3x^4}{4} + 2x + C$

(2) $\frac{1}{7}(6x+1)^{7/6} + C$

(3) $\frac{-1}{5}(1-4x)^{5/4} + C$

(4) $\frac{-1}{11}(4-10x)^{11/10} + C$

(5) $\frac{3}{4}(x-1)^{4/3} + C$

3.3 Application of Indefinite Integration

Most elementary differential equation could be solved by integrating them.

Example: Solve the differential equation given as $\frac{dy}{dx} = f(x)$

$dy = f(x)\,dx \Rightarrow$

$\int dy = \int f(x)\,dx \Rightarrow$

$y = \int f(x)\,dx \Rightarrow$

Such class of differential equation is used to solve various types of problems arising from Biology, all branches of engineering, physics, chemistry and economics.

In application of indefinite integral the value of the arbitrary constant must be found by applying the initial conditions of the problem that is being solved. Therefore before continuing it is important that you know more about this arbitrary constant C.

Example: Let $\frac{dy}{dx} = 2x$

then $y = x^2 + C$

The graph of $y = x^2$ for C = 0 is given in Fig. 3.1

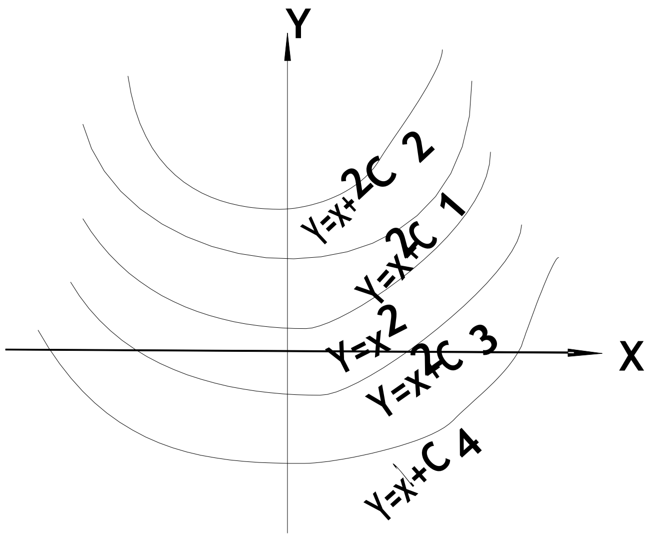

Fig. 3.1

Any other integral curve $y^2 + C$ can be obtained by shifting this curve $y = x^2$ through a vertical displacement C. In Fig.3.1 such vertical displacements give rise to a family of parallel curves. They are parallel since the slope of each curve is equal to 2x. This family of curves has the property that for any given part (x_o, y_o) where x_o ^{1}D (i.e. D is the domain of definition) there is only one and only one curve from the family of curves that passes through the part (x_o, y_o). Hence the part (x_o, y_o) must satisfy the equation

$$Y_o = x_o^2 + C$$

i.e. $C = y_o - x_o^2$ so for any particular point $(x_o\ y_o)$ C can be uniquely be determined.

This condition that $y = y_o$ and $x = x_o$ imposed on the differential equation du/dx = 2x is referred to as initial condition. You will use this method to solve problems on application of integration.

Example: Total profit P(x) from selling X units of a product can be determined by integrating the differential equation of the marginal profit dp/dt and using some initial conditions based on the market forces to obtain the constant of integration. Given that

$$dp/dt = 2 + 3/(2x-1)^3$$

Find P(x) for $0 \leq x \leq C$ if P(1) = 1.

Solution:

$$\int dp = \int \frac{(2+2)}{(2x-1)^2}\,dx = \int \left(\frac{2+2}{(2x-1)^3}\right)dx$$

$$\int dp = \int \left(\frac{2+2}{(2x-1)^3}\right)dx$$

$$P = \frac{2x-1}{2(2x-1)^2} + C$$

Since $P_o = 1$ $X_o = 1$

$$1 = 2.1 - \frac{1}{2(2-1)^2} + C$$

⇨ $C = -{}^{1}/_{2}$

$$\text{Therefore } P(x) = 2x - \frac{1}{2(2x-1)} - {}^{1}/_{2}$$

Example: Given that $dy/dx = 8x^7$

Find y when y = -1 and x = 1

Solution:

$\int dy = \int 8x^7\, dx$

$y = x^8 + C$

$-1 = 1 + C$

$C = x^8 - 2$

SELF-ASSESSMENT EXERCISES

Solve the following equations subject to the prescribed initial conditions: (1) $dy/dx = 4x^2 - 2x - 5$ x = -1, y = 0

(2) $dy/dx = 4(x-5)^3$ x = 0, y = 2

(3) $dy/dx = \frac{x^2+1}{x^4} dx$ x = 1, y = 1

(4) $\frac{dy}{dx} = \sqrt[x]{1+x^2}$ x = 0, y = 0

(5) $\frac{dy}{dx} = x^{1/2} + x^{1/5}$ x = 0, y = 2

You will study more on application of indefinite integration in the last unit in the course.

Ans: (1) $y = 4/3x^3 - x^2 + 5x + 22/3$ (ii) $y = (x-5)^4 - 623$

(iii) $y = -1/x - 1/3x^3 + 7/3$ (iv) $y = 1/3(x^2+1)^{3/2} - 1/3$

(v) $y = 2/3x^{3/2}$

4.0 CONCLUSION

In this unit emphasis has been on techniques of finding anti-derivative. Therefore, you have studied numerous solved examples on method of finding anti-derivatives of functions. You have known the notation for indefinite integration as $\int f(x)dx = f(x) + C$. You have studied properties of indefinite integration and how to use them to evaluate integrals. You have studied how to integrate simple differential equations.

5.0 SUMMARY

You have studied:

- the definition of indefinite integral
- Properties and notation of indefinite integration
- To evaluate integrals using both the notation and properties of indefinite integration.
- To integrate differential equation that are separable.

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7.0 TUTOR-MARKED ASSIGNMENT

Evaluate the following integrals:

(1) $\int \sqrt{} dx$ (2) $\int \sqrt{4x}\ 1$

(3) $\int (7x^6 - 4x^3 + 4x^6 - 2x)$ (4) $\int dx/x^7\ dx$

(5) $\int \frac{x^4 - 1}{X^6} dx$ (6) $\int \left(\sqrt{x} + \frac{1}{\sqrt{1+x}} \right) dx$

(7) $\int \frac{(5x - 1)^2}{x^3} dx$ (8) $\left(\int \frac{4x^3 - 1}{x^6} dx \right)$

(9) $\int (\sqrt{4x} + 1 - \sqrt{3x})\ dx$ (10) $\int\ x^2 + \frac{x^2 + 2x}{}\ \ dx\ x2$

(11) $\int (1 - 8x)^{1/8} dx$ (12) $\int (5x - 2)^{1/5} dx$

Solved the differential equation at the specified points:

(13) $dy/dx = \frac{x^2 - 1}{x^4}$ $y = 0\ x = 1$

(14) $dy/dx = \frac{1}{\sqrt{1+7x}}$ $y = 2,\ x = 1$

(15) $dy/dx = (1 - 4x)^{1/4}$ $y = 1, x = -3$

(16) $dy/dx = 6c\sqrt{1 - x^2})dx$ $y = 0, x = 1$

(17) Find the total profit of a product if the marginal profit is given as $dp/dx = x^4 + x^2 (\sqrt{1 - x^3})$ where $P(0) = 0$

(22) Solve $dy/dx = \frac{2\sqrt{1+y^2}}{y}$ if $x = 1, y = 1$

(19) Solve $dy/dx = x^2/y^3$ if $x = 0, y = 1$

(20) Solve $ds/dt = (t^2+1)^2$ when $S = 0, t = 0$

UNIT 4 INTEGRATION OF TRANSCENDENTAL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In the previous unit, you studied the integration of polynomial function and simple rational function. However, there are some functions whose derivatives are not very common. Integration of such functions uncommon derivatives can only be possible by using derivatives of known functions to do the evaluation. In this unit integration of transcendental and rational function are discussed. These integration will form part of the basic tools that will be needed in applying techniques of integration that will be studied in the next unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive the formula for integrating rational functions, exponential function and trigonometric functions
- evaluate definite and indefinite integrals of sinx, cos x, e^x and any combination of them
- to evaluate integrals by using the derivatives of inverse trigonometric functions of sinx and tan x.

3.0 MAIN CONTENT

3.1 Integration of Rational and Exponental Function

3.1.1 The integral $\int du/u$ =, u ≠0 Recall that d/dx 1nu = du/u - I (see unit 8 of calculus I) then the integral counterpart of equation I above is that $\int du/u$ = ln1u1+C

In the above u is a differentiable function of x and u>0 for all values of x in the specified domain.

Example: Find $\int \frac{8x}{2^{x-1}} dx$

Solution: let $u = x^{2-1}$, $du = 2x\,dx$
then

$$\frac{du}{2} = x\,dx$$

Therefore

$$\int \frac{8x\ dx}{x^{2-1}} = \frac{8\int du}{2}$$

$$= \frac{4\int du}{x} = 4/u/u\int + C$$

Example: Find $\int \frac{x^2 - dx}{1+3x^3}$

let $u = 1+3x^3$, $du = 9x^2dx$
⇨ $x^2dx = du/9$

$$\int \frac{x^2dx}{1_{3x}{}^3} = \int \frac{du}{\frac{9}{u}} = \frac{1}{9}\int \frac{du}{u}$$

$$= \frac{1}{9}\ln/u/ + C = \frac{1}{9}\ln/1+3x^3/+C$$

Example: Find

$$\int \frac{8x^3 - 2}{x^4-x+1}\,dx$$

let $u = x^4 - x + 1$, $du = (4x^3 - 1)\,dv$

but $(8x^3 - 2)dx = 2(4x^3-1)\,dx$

therefore: $\int \frac{(8x^3 - 2)dx}{x^4-x+1} = \int \frac{2(4x^3-1)dx}{x^4-x+1} = \int \frac{2du}{u}$

$= 2\ln/u/+C$

$= 2\ln/x^4 - x + 1/ + C$

Example: Find $\int \left(\frac{1}{(x+1} - \frac{1}{x+2)}\right) dx$

let $u = x + 1$ and $v = x + 2$
$du = dx$ $du = dx$

$$\int \left(\frac{1}{(x+1} - \frac{1}{x+2)}\right) dx = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \int \frac{du}{u} - \int \frac{dv}{v}$$

$= \ln/u/ - \ln/v/ + C$

$= \ln|x+1| - \ln|x+2| + C$

Example: Find $\int \frac{\log(x+1)}{x+1}dx$

let $u = \log(x+1)$ $du = \frac{1}{x+1}dx$

therefore: $(x+1)\,du = dx \implies \int \frac{\log(x+1)}{x+1}dx = \int \frac{u}{x+1}.(x+1)du$

$= \int u\,du = \frac{1}{2}u^2 + C$

$= \frac{1}{2}\log(x+1)^2 + C$

Exercise: Evaluate the following integrals

(1) $\int \frac{dx}{3-4x}$ (2) $\int \frac{3}{x-5}dx$

(3) $\int \frac{x}{x^2-2}dx$ (4) $\int \frac{\log x}{x}du$

(5) $\int \frac{4x-2}{x^2-x+1}dx$

Ans: (1) $-\frac{1}{4}\ln|(3-4x)|+C$ (2) $3\ln|(x-5)| + C$

(3) $\frac{1}{2}\ln|x^2-2| + C$ (4) $\frac{1}{2}\log x^2 + C$

(5) $2\ln|x^2 - x + 1|$

The method adopted above is to differentiate the denominator and check if it is a factor of the numerator; if so with appropriate algebraic manipulation, the derivative of the denominator will be made to look like the numerator. This method was used in UNIT 3.

i.e. $\int \frac{g(x)}{P(x)}dx$ let $u = P(x)$

and $du = P^1(x)dx = g(x)dx$ then $\int \frac{g(x)}{P(v)} = \int \frac{du}{u} = \ln|u| + C$

$\implies \ln|P(x)| + C$

3.1.2 The Integral $\int e^x du$

Recall that $\frac{de^u}{dx} = \frac{de^u}{du} . \frac{du}{dx} = e^u \frac{du}{dx}$

then $\frac{de^u}{dx} = e^u\frac{du}{dx}$

⇨ $de^u = e^u du$ then $\int de^u = \int e^u du$ therefore $\int e^u du = e^u + C$

Example: Find $Se^{-x}dx$.
Let $u = -x$, $du = 1dx$

⇨ dx $-du$

therefore $\int e^{-x}dx = \int e(-du) = -\int e^u du$

$= -e^u + C = e^{-x}$ C.

Example: Find $\int e^{2x}dx$. Let $u = 2x \Rightarrow du = 2dx$

$dx = \frac{du}{2}$

therefore $\int e^{2x}dx = \int e^u(\frac{du}{2}) = ½ \int e^u du$

$= ½ e^{2x} + C$

Example: Find $\int e^{x/3}dx$ let $u = \frac{x}{3}$, $du = \frac{dx}{3}$

$dx = 3du$, $Se^{x/3}$ $\int e^{x/3}du = \int e^{u\cdot}(3du)$

$\int e^{x/3}dx = 3Se^u du = 3e^u$ C
$= 3e^{x/3} + C$

Example: $\int 4e^{2x}dx$ Let $U = e^{2x}$ $du = 2e^{2x}dx$.
$\int 4e^{2x}du = 2\int 2e^{2x}dx = 2\int du = 2u + C$
$= 2e^{2x} + C$

Example: $\int (e^x\ x)^2 (e^x\ 1)dx$
Let $u = e^x + x$ = $du = (e^x\ _1)dx$
$\int (e^x + x)^2 (e^x + 1)\,dx = \int u^2 du$
$= \frac{U^3}{3} + C = \frac{(e^x +x)^3}{3} + C$

Example: $\int xe^{x2}dx$
Let $u = x^2$ $du = 2x\,dx$ $\frac{du}{2} = x\,dx$

⟹ then $\int xe^{x2}\,dx = ½\, Se^u\,du$
$= ½\, e^u + C = ½\, e^{x2} + C$

SELF-ASSESSMENT EXERCISE
Evaluate the following integrals

(1) $\int e^{3x}dx$ (2) $\int e^{\frac{sx}{2}}$

(3) $\int 8e^{4x}\,dx$ (4) $\int (e^x – x)^2 (e^x – 1)\,dx$

(5) $\int 3x^2 e^{x3}$

Ans: (1) $1/3e^{3x} + C$ (2) $\frac{2}{5}e^{\frac{5x}{2}} + C$ (3) $2e^{4x} + C$

(5) $e^{x3} + C$

3.2 Integration of Trigonometric Functions

Recall from UNIT 8 of the first course on calculus that for any differentiable function U of X that

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = \sin^2 U\frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = -\operatorname{cosec}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$$

$$\frac{d}{dx}(\text{asec} u) = -\text{wsecucotu}\frac{du}{dx}$$

Using the above you will integrate the following trigonometric function as

(1) $\int \sin u du = -\int -\sin u du = -\int \frac{d}{dx}(\cos u)$

$= - \cos u \; C$

therefore

$\int \sin u \, du = - \text{Cos}u + C$

(ii) $\int \text{Cos}u \, du = \int d\sin u) = \sin u \; C$

therefore

$\int \cos u du = \sin u + C$

Given that $\int \frac{1}{f(x)} \cdot \frac{d}{dx} [f(x)]\, dx = \log/f(x) + C$

then

(iii) $\int tanudu = \int \frac{sinu}{cosu} du = -\int \frac{1}{cosu}.d\ cosu)$

$= -\int \frac{dv}{v} = lnv + C$, where v = cosu v

$= -lnkosul + C = ln(\frac{1}{cosu}) = ln/secU/+C$

therefore $\int tan\ u\ du = ln/secu/ + C$

(iv) $\int secudu = \int secu\frac{(secu+tanu)}{(sec\ u + tan\ u)}du$

$= \int \frac{sec^2 + secutanu}{sec\ u + tan\ u} du$

Let V = tan u + sec u, $dv = sec^{2\,u} + tan\ u\ sec\ u\ du$

therefore: $\int \frac{sec^{2u} + tan\ u\ secu\ du}{tan\ u + tan\ u} = \int \frac{dv}{v}$

(v) $\mathcal{S}cotudu = \int \frac{cosu}{sin\ u} du = \int \frac{1}{sin\ u} d(sin\ u)$

$= ln/sinu/ + C$

(vi) $\int cosec\ u\ du = \int cosec\ u\ \frac{cosecu - cot\ u}{cosecU - cotU}$

$= \int \frac{cosec^{2\,u} - cot\ ucosec\ u}{cosec\ u - cot\ u} du$

$= \int \frac{dv}{v}$, v = cosec u – cot u

$dv = cosec^{2\,u} - cot\ cosec\ u\ du$

⇨ $ln/v/ + C = ln/cosec\ u - cot\ u/ + C.$

Example: Find $\int sec^2 u\ du = \int d(tan\ u)\ C$

$= tan\ u + C$

Example: Find $\int cosec^2\ u\ du = -\int -cosec^2 u\ du\ \ -\int (d(cot\ u) = -cotu + C$

Example: Find $\int sec\ u\ tan\ u\ du\ \int sec\ u\ tan\ u = \int d(sec\ u) = sec\ u + C$

Example: Find $\int cos\ x\ sin\ x\ dx$

Let u = sinx du = cos x dx

therefore $\int sin\ x\ cos\ x\ dx = \int u\ du$

$= \frac{u^2}{2} + C = \frac{sin^2 x}{2} + C$

Example: Find $\int \sec^3 x \tan x \, dx$

Let $u = \sec x$ $du = \sec x \tan x \, dx$

therefore $\int \sec^3 x \tan x \, dx = \int \sec^2 x \sec x \tan x \, dx$

$$= \int u^2 du = \frac{u^3}{3} + C$$

$$= \frac{\sec^3 x}{3} + C$$

Example: Find $\int \text{cosec}^3 x \cot x \, dx$. Let $u = \text{cosec } x$ $du = -\text{cosec } x \cot x \, dx$

Therefore $\int \text{cosec}^3 x \cot x \, dx = \int \text{cosec}^2 x \, \text{cosec } x \cot x \, dx$

$$= -\int u^2 du = \frac{-u^3}{3} + C$$

$$= -\frac{\text{cosec}^3 x}{3} + C$$

Example: Find $\int x \cos ax^2 \, dx$

Let $U = ax^2$ $du = 2axdx$

$\int x \cos ax^2 dx = \int ½ a (\cos ax^2)(2ax) \, dx$

$= ½ a \int \cos U.du = ½ a (\sin U + C)$

$= ½ a \sin ax^2 + C$

Example: Find $\int \frac{\sec^2 x}{1+\tan x} dx$

let $U = 1 + \tan x$ $du = \sec^2 x \, dx$

therefore $\int \frac{\sec^2 x dx}{1+\tan x} = \int \frac{du}{u} = \ln/u/+C$

$= \ln/1+ \tan x1 + C$

Exercises: Find the following integrals

(i)	$\int \sin(2x-1) \, dx$	(ii)	$\int \sin ½ ax \, dx$
(iii)	$\int 2\cos^2 x \sin x \, dx$	(iv)	$\int \sin^4 x \cos x \, dx$
(v)	$\int x \tan x^2 \, dx$	(vi)	$\int \frac{dx}{\cos^{2x}}$
(vii)	$\int \frac{\sin x}{1 + \cos x} dx$	(viii)	$\int \cot ax \, dx$
(ix)	$\int \cos^6 ax \sin ax \, dx$	(x)	$\int (1+\tan x) \sec^2 x \, dx$

Ans: (i) $-\frac{1}{2}\cos(2x - 1) + C$

(ii) $\frac{-2}{a}\cos\frac{1}{2}ax + C$

(iii) $\frac{-2}{3}\cos^3 x + C$

(iv) $\frac{1}{5}\sin^5 x + C$

(v) $\frac{1}{2}\ln|\sec x^2| + C.$

(vi) $\tan x + C$

(vii) $-\ln(\cos x + 1)$

(viii) $\ln|\sin ax| + C$

(ix) $\frac{-1}{7}\frac{\cos^7 ax}{a}$

(x) $\underline{(+\tan x)}^2 + C$

3.3 Integration of Inverse Trigonometric Function

Recall that $\frac{d}{dx}(\text{arc sin u}) = \frac{1}{\sqrt{1-x^2}dx} \quad \underline{du}$

to evaluate $\int$arc sin u du you have to know how to integrate by part which is one of the techniques of integration that you will study next unit. For now

$\int \text{arc sin u du} = \text{u arc sin u} + \sqrt{1 - u^2} + C$ and

$\int \text{arc tan u du} = \text{u ar tan u} = -\frac{1}{2}\ln|1+u^2| + C$

You can proceed to make use of the derivative of arc tan x to evaluate special integrals.

Recall $\frac{d}{du}(\text{arc tan u}) = \frac{1}{1+u^2}$

$u^2 = a^2v^2$

therefore: $\int\frac{du}{\sqrt{a^2-u^2}} = \int\frac{adv}{\sqrt{a^2-a^2v^2}} = \int\frac{adv}{a\sqrt{1-v^2}}$

$= \int\frac{dv}{\sqrt{1-v^2}} = \text{arc sin v} + C$

$= \text{arc sin}\frac{u}{a} + C$

Example: Find $\int\frac{du}{\sqrt{4-u^2}}$

Solution: $\int\frac{du}{\sqrt{4-u^2}} = \int\frac{du}{\sqrt{(2)^2 - u^2}} = \text{arc sin}\frac{u}{2} + C$ = arc sin $\underline{u}$ + C

Example: Find (1) $\int \frac{dx}{a^2+(x+2)^2}$

Solution let u = (x + 2), du = dx

therefore $\frac{dx}{a^2 + (x+2)^2} = \int \frac{du}{a^2 + u^2}$

$= \frac{1}{a} \text{arc tan} \frac{u}{a} + C$

$= \frac{1}{a} \text{ arc tan } \frac{(x+2)}{a} + C$

Example: Find $\int \frac{dx}{\sqrt{a^2 + (x-1)^2}}$

Let u = x-1 du = dx therefore

$\int \frac{dx}{\sqrt{a^2+u^2}} = \text{arc sin} \frac{u}{a} + C$

$= \text{arc sin} \frac{x-1}{a} + C$

Exercises: Find the following integrals:

(i) $\int \frac{dx}{16 + 4x^2}$ (ii) $\int \frac{dx}{\sqrt{9 - 64x^2}}$

(iii) $\int \frac{dx}{49 + (x+2)^2}$ (iv) $\int \frac{dx}{\sqrt{25 - 9x^2}}$

(v) $\int_0^5 \frac{dx}{25+x^2}$

Ans: (i) ¼ arc tan $\frac{x}{8}$ (ii) arc sin $\frac{8x}{3}$

(iii) 1/7 art tan $\frac{x+2}{7}$ (iv) arc sin $\frac{3x}{5}$

(v) ∏/20

4.0 CONCLUSION

In this unit you have derived the formula for common rational functions and how to find their integrals. You studied how to derive the integration formula of trigonometric functions. Evaluation and trigonometric functions were treated. You also find the integrals of special functions using the inverse functions of sin x and tan x. The formulas derived in this unit will be used to study methods and techniques of integration which will be studied in the next unit of this course.

5.0 SUMMARY

In this unit you have studied how to;

1. derive formula such as: (i) $\int \frac{1}{u} du = \ln/u/+C$
   (ii) $\int \sin u\, du = -\cos u + C$ (iii) $\int \cos du = \sin u + C$
   (iv) $\int \tan u\, du = \ln/\sec u/+ C$ (v) $\int \cot u\, du = \ln/\sin u/+C$
   (vi) $\int \sec u\, du = \ln/\tan u + \sec u/$
   (vii) $\int \operatorname{cosec} u\, du = \ln/\operatorname{cosec} u - \cot u/ + C$ (viii) $\int e^u\, du = e^u + C$

2. evaluate integral of this form $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$
   and $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. how to use the formula in (i) above to evaluate integrals.

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7.0 TUTOR-MARKED ASSIGNMENT

Find the following integrals

(1) $\int \frac{dx}{5-7x}$ (2) $\int \frac{1}{x-6}\, dx$ (3) $\int \frac{x\ dx}{x^2-4}$

(4) $\int \frac{10x+5}{5x^2+5x+1}\, dx$ (5) $\int e^4\, dx$ (6) $\int \sin(4x-1)\, dx$

(7) $\int \sin^c x \text{ covx } du$ (8) $\int \frac{du}{\sin^2 x}$ (9) $\int \sin^4 ax \cos ax\, dx$

(10) $\int x \cot(x)^2 dx$ (11) $\int \frac{du}{16+x^2}$ (12) $\int \frac{dx}{\sqrt{90^2-4x^2}}$

(13) $\int 4x^3 e^{x4}\, dx$ (14) $\int (e^x + x)^2 (e^x + 1)\, dx$

(15) $\int \cos 2x \sin 2x\, dx$ (16) $\int \frac{dx}{\sqrt{36-(x+3)^2}}$ (17) $\int 3\tan (x+1)^2\, dx$

(18) $\int x e^{x2} dx$ (19) $\int \cos^8 x \sin x\, dx$ (20) $\int \frac{3x^2}{x^3-8} dx$

UNIT 5 INTEGRATION OF POWERS OF TRIGONOMETRIC FUNCTIONS

CONTENTS

1.0 INTRODUCTION

So far what you have studied in the last two units is to find the function whose derivative gives you the integral of another function. This process is summed up in the fundamental theorem of integral calculus. For a review, consider evaluating the integral ∫f(x)dx what you have studied in unit 2 and 3 is to find a function F(x) such that d/dxF(x) = f(x) – 1 then F(x)+C = ∫f(x)dx.

The process of finding F(x) that satisfies equation 1 above is the difficult aspect and that is why differentiation is taught before integration. So far, all you have been doing is making a good guess for the function F(x) which is dependent on how familiar you are with differentials of functions. In this unit you will study how to make the guesswork a lot easier. This will be done by introducing firstly the use of differentiation formulas alongside their integration formulas, second, by applying some techniques that will be developed here based on the knowledge of function as well as their respective derivative. Since it is the anti-derivative that gives the solution to the integral it is necessary once again you review basic rules and formulas for derivatives of function in the course calculus I.

The emphasis in this unit would be on developing skills rather than finding specific answer to any given problem. Therefore as was done in the previous units a particular example might be solved several times with different methods. Therefore the examples in this unit have been kept fairly simple so that you would be able to develop the necessary skills expected of you.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recall differential formulas and their corresponding integrals

- evaluate integrals involving powers of trigonometric functions
- evaluate integrals involving products of even powers of sines and cosines
- to develop techniques and methods for evaluating integrals of any function formed by functions of the trigonometric functions.

3.0 MAIN CONTENT

3.1 Basic Formula

The first requirement for skill in integration is a thorough mastery of the formulas for differentiation. Therefore, a good starting point for you to develop the skill required of you in this course is for you to build your own table of integral. You may make your own note in which the various sections are headed by standard form like $SU^{n}du$ and then under each heading include several examples to illustrate the range of application of the particular formula. Therefore, what will be done in this unit is to list formulas for differentiation together with their integration counterparts.

Summary of Differential Formulas and Corresponding Integrals

1.	$du = \underline{du}\ dx$	1	$\int du = u\ C$
2.	$d(au) = a\ du$	2	$\int a\ du = a \int du$
3.	$d(u + v) = du + dv$	3.	$\int (du + dv) = \int du + \int dv$
4..	$d(u^n) = nu^{n-1}du$	4.	$\int u^n\ du = \frac{u^{n+1}}{n+1} + C,\ n \neq 1$
5.	$d(\ln u) = \underline{du}$	5.	$\int \underline{du} = \ln /u/ + C$
6.	a) $d(e^u) = e^u\ du$	6.	a) $\int e^u\ du = e^u + C$
	b) $d(a^u) = a^u \ln a\ du$		b) $\int a^u\ du = \frac{a^u}{\ln a} + C$
7.	$d(\sin u) = -\sin u\ du$	7.	$\int \cos u\ du = \sin u + C$
8.	$d(\cos u) = -\sin u\ du$	8.	$\int \sin u\ du = \tan u + C$
9.	$d(\tan u) = \sec u \tan u\ du$	9.	$\int \sec^3 u\ du = \tan u + C$
10.	$d(\cot u) = -\csc^2 u\ du$	10.	$\int \csc^2 u\ du = -\cot u + C$
11.	$d(\sec u) = \sec u \tan u\ du$	11.	$\int \sec u \tan u\ du = \sec u + C$
12.	$d(\csc u) = -\csc u \cot u\ du$	12.	$\int \csc u \cot u\ du = -\csc u + C$
13.	$d(\sin^{-1} u) = \frac{du}{\sqrt{1 - u^2}}$	13. and	$\int \frac{du}{1 - u^2} = \{\sin^{-1} u + C$ $\{-\cos^{-1}\sqrt{u} + C$
14.	$d(\cos^{-1} u) = \frac{du}{\sqrt{1 - u^2}}$	14.	

15. $d(\tan^{-1} u) = \frac{du}{1 + u^2}$ 15. $\int \frac{du}{1 + u^2} = \{\tan^{-1} u + C$ and $\{-\cot^{-1} u + C$

16. $d(\cot^{-1} u) = \frac{-du}{1+ u^2}$ 16.

17. $d(\sec^{-1} u) = \frac{du}{/u/ \sqrt{u^2 - 1}}$ 17. and $\int \frac{du}{u\sqrt{u^2 - 1}} = \{\sec^{-1/u/} + C$ $\{ -\csc^{-1}/u/ + C$

18. $d(\csc^{-1} u) = \frac{-du}{/u/\sqrt{u^2 - 1}}$

3.2 Integration Involving Powers of Trigonometric Functions

From the above basic formula you have that:

(1) $\int u^n du = \frac{u^{n+1} + C}{n + 1}$ for n -1

and

(2) $\int u^n du = \ln/u/+C$ n = 1

This could be used to evaluate integrals involving powers of trigonometric functions.

Example: Find $\int \sin^n ax \cos ax\, dx$

Let u = sin ax du = acos ax dx

then $\frac{du}{a} = \cos ax\, dx$, $u^n = \sin^n ax$

therefore: $\int \sin^n ax \cos ax\, dx = \int u^n \frac{du}{a}$

$= \frac{n^{n \pm 1}}{a(n + 1)} + C$

using equation (1) above you get

(3) $\int \sin^n ax \cos ax\, dx = \frac{\sin^{n+1} ax + }{a(n + 1)} C$

with equation (2) you get n + 1

(4) $\int \frac{\cos ax dx}{\sin ax} = \frac{1}{a} \ln /\sin ax/+C$

Interestingly this is the same result arrive at when you derive the formula for

$\int \cot u\, du = \int \frac{\cos ax}{\sin ax} dx = \ln/\sin u/ + C$

In a similar manner you can find $\int \cos^n ax \sin ax\, dx$

Let u = cos ax du = -a sin ax

$U^n = \cos^n ax$ then

$\int \cos^n ax \sin ax\, dx = \int u^u(-\underline{du}) = -\frac{U^{n+1}C}{n + 1}$

for $n \neq 1$

therefore $\int \cos^n ax \sin ax\, dx = -\frac{\cos^{n+1} ax}{(n+1)a}$ C

for n = 1 $\int \frac{\sin ax}{\cos ax} dx$ $\frac{-1}{a}\ln/\cos ax/ + C$

this is the same as $\int \tan ax\, dx$

i.e. $\int \tan ax\, dx = \frac{-1.}{}$ ln/cos ax/ + C

$= \frac{1}{a}$ ln/sec ax/ + C

(see 3.2 of Unit 4)

Example: Try finding $\int \sin^3 x\, dx$ you find out that the above method does not work because there is cos x side of it to give d (sin x)/ therefore, another method has to be tried.

Recall that $\sin^3 x = \sin^2 x \sin x$

$= (1 - \cos^2 x) \sin x$ $\sin^3 x = \sin x - \cos^2 x \sin x$

then let u = cos x du = -sin x

$\int \sin^3 dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx$

$= -\cos x + \frac{\cos^3 x}{3} + C$

The above give rise to a formula or technique for integrating odd powers of sin x or cos x

i.e. $\cos^{2n+1} x = \cos^{2n} x \cos x$

but $\cos^{2n} x = (\cos^2 x)^n = (1 - \sin^2 x)$ therefore $\cos^{2n+1} x = (1-\sin^2 x)^n$. cos x let u = sin x du = cos x dx

therefore $\int \cos^{2n+1} x\, dx = \int (1 - \sin^2 x)^n \cos x\, dx$

$= \int (1 - u^2)\, du.$

What follows next is to expand the expression $(1-u^2)$ du where u = cos x smf

$\int \cos^{2n+1} x\, dx = -\int (1-u^2)^n du$ where u = sin x

Example: Find (1) $\int \cos^3 x\, dx$ ii $\int \sin^5 x\, dx$

Solution: $\int \cos^{2n+1} x\, dx = \int (1 - u^2)^n du$ 2n + 1 = 3 n = 1, u = sin x

therefore: $\int \cos^3 x\, dx = -\int (1 - u^2)\, du = u - \frac{u^3}{3}$

$= \sin x - \frac{\sin^3 x}{3} + C$

(optimal) $= \sin x - \frac{\sin x}{3} + \frac{\sin^2 x \cos x}{3}$

$= \frac{\sin^2 x\cos x}{3} - \frac{2}{3} \sin x$

(ii) $\int \sin^5 x \, dx$

$2n + 1 = 5 \Longrightarrow n\ 2, \ u = \cos x$

therefore $\int \sin^5 x \, dx = \int (1 - u^2)^2 \, du$

$= \int (1 - 2u^2 + u^4) \, du$

$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$

therefore $\int \sin^5 x \, dx = \cos x - \frac{2}{3} \cos^3 + \frac{1}{5} \cos^5 x + C$

(optimal) $= \frac{1}{5} \cos^4 x \sin x + \frac{4}{15} \cos^2 x \sin x + \frac{8}{15} \sin x + C$

Example: Find $\int \sec x \tan x \, dx$

Solution: $\int \sec x \tan x \, dx = \int \frac{1}{\text{Cos } x} \frac{\sin x}{\cos x} = \int \frac{\sin x}{\cos^2 x}$

then $\int \sec x \tan x \, dx = \int \cos^{-2} x \sin x \, dx$

$\Longrightarrow$ therefore $\int \cos^{-2} x \sin x = \frac{-\cos^{-2+1}}{-2 + 1} + C$

$= \frac{\cos^{-1} x}{-1} + C$

$= \frac{-1}{\cos^x} + C$

$= \sec x + C$

Example: Find $\int \tan^4 x \, dx$

recall that $\sin^2 + \cos^2 x = 1$

therefore $\tan^2 x = \sec^2 x - 1$ then

$\int \tan^4 x \, dx = \int \tan^2 x . \tan^2 x \, dx$

$= \int \tan^2 x (\sec^2 x - 1) \, dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx$

$= \int \tan^2 x \sec^2 x - \int (\sec^2 x - 1) \, dx$

$= \int (\tan^2 x \sec^2 x) \, dx - \int \sec^2 x \, dx + \int dx$ let $u = \tan x$

$du = \sec^2 x \, dx$

therefore $\int \tan^4 x \, dx = \int u^2 \, du - \int du - \int dx$

$= \frac{u^3}{3} - u - x$

$= \frac{1}{3} \tan^3 x - \tan x + x + C$

Therefore, for n = even you can derive the formula using the technique above.

$\int \tan^n x \, dx = \int \tan^{n-1} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x) \, dx - \int \tan^{n-2} x \, dx$

$= \int (\tan^{n-2} x \sec^2 x) \, dx - \int (\sec^{n-2} x - 1) \, dx$

$= \int (\tan^{n-2} x \sec^2 x) \, dx - \int \sec^{n-2} + dx$

$$= \int \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x \, dx$$

Example: Find $\int \tan^2 x \, dx$

n= 2, therefore n – 1 $\Rightarrow$ 2 – 1 = 1

therefore $\int \tan^2 x \, dx = \frac{\tan x}{1} - \int \tan^0 x \, dx$

$= \tan x - x$

The above formula also works for the case n is odd. Let n = 2m + 1 then after m steps it will be reduced by 2m leaving $\int \tan x = -\ln/\cos x/ + C$.

From the two examples above, you can see the usefulness of the two trigonometric identities.

$\sin^2 x + \cos^2 x = 1$ and $\tan^2 x + 1 = \sec^2 x$

in evaluating integrals involving powers of trigonometric functions such as

(a) odd powers of sin x or cos x
(b) any integral powers of tan x (or cot x) and
(c) even powers of sec x (cos x u)

To get the integral C of even powers of sec x all you need do is to express $\sec^2 x$ in terms of $\tan^2 x$ and then use the reduction process above to get the integral.

Example: Find $\int \sec^4 x \, dx$

$= \int \sec^2 dx \sec^2 x = \int \sec^2 x (1 + \tan^2 x) \, dx$
$= \int \sec^2 dx + \int \tan^2 x \sec^2 x \, dx$
$= \int /+ \tan^2 x \, dx + \int u^2 \, du$
where $u = \tan x$ and $du = \sec^2 x \, dx$.

$\Rightarrow \int u^2 \, du = \frac{u^3}{3} + C$

but $\int \tan^2 x \, dx = \tan x - x$

$\int \sec^u x \, dx = \int dx + \int \tan^2 x \, dx + \frac{\tan^3 x}{3} + C$

You can now derive the integral for any even powers of sec x

Example: Find $\int \sec^n x \, dx$

Solution: let $\int \sec^2 x \, dx = \int (\sec^{2n-2} x)(\sec^2 x)$

$= \int (\sec x)^{2(n-1)} \sec^2 x \, dx$

$= \int (\sec^2 x)^{n-1} \sec^2 x \, dx$

$= \int (1 + \tan^2 x)^{n-1} \sec^2 x \, dx$

$= \int (1 + u^2)^{n-1} du$ (where $u = \tan x$ and $du = \sec^2 x \, dx$) where $(1 + u^2)^{n-1}$ can be expanded by the binomial theorem and then the result will be integrated term by term as;

SELF-ASSESSMENT EXERCISE

(i) $\int \sin^3 x\, dx$ (ii) $\int \tan^2 4x\, dx$ (iii) $\int \cos^5 x\, dx$

(iv) $\int \cot^3 x\, dx$ (v) $\int \cos^3 x \sin^2 x\, dx$ (vi) $\int \sec^u x \tan u\, du$

(vii) $\int \frac{dx}{\sin x}$ (viii) $\int \cos^n x \sin x\, dx$ (ix) $\int \cos^2 x \sin 2x\, dx$

(x) $\int \cos x^4\, 3x\, dx$

Ans:

(i) $1/3 \cos^3 x - \cos x + C$ (ii) $\tan^4 x - 4x + C$

(iii) $\sin x - \frac{2}{3}\sin^3 x + 1/5 \sin^5 x + C$ (iv) $-\frac{-\cot^2 x}{2} - \ln/\sin x / +$

(v) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$ (vi) $\ln(\text{cosec } x - \cot x) +$

(vii) $\ln(\text{cosec } x - \cot x) + C$ $_2$ (viii) $-\frac{\cos^{n+1} x}{n+1} + C$

(ix) $\frac{-\cos^3 2x}{6} + C$ (x) $\frac{-1}{9}\text{cosec}^2 3x \cot 3x \frac{-2}{9} \cot 3x$

3.3 Integration of Even Powers of Sines and Cosines

In the previous section you have studied how to integrate odd powers of sin x and cos x. You will attempt to evaluate integrals of even powers of sines and cosines by applying the same technique used above for odd powers i.e.

$\int \sin^n x \cos^m x\, dx$ where m or n is an even numbers. that $\int \cos^{½} x \sin^3 dx$

evaluate the integral.

Recall that 3 is odd as such $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$. therefore: $\int$

$\cos^{½} x \sin^3 x\, dx = \int \cos^{½} x\ 91 - \cos^2 x) \sin x\, dx$

for $u = \cos x\ du = \sin x\, dx$.

therefore: $\int \cos ½ x (1 - \cos^{12} x) \sin dx = \int u^{½} (1 - u^2)\, du$
$= \int (u^{½} - u)\, du = \frac{2u^{3/2}}{3} - \frac{u^2}{2}$
$= \frac{2\cos^{3/2} x}{3} = \frac{\cos^2 x}{2} + C$

If in the above you have $\sin^4 x$ instead of $\sin^3 x$ then you have to evaluate

$\int \cos^{½} x \sin^4 x\, dx$

Then using the above method will fail because $\sin^4 x = (1 – \cos^2 x)^2$ which give

$$\int \cos^{½} x \sin^4 x\, dx = \int \cos^{½} x (1-\cos^2 x)^2\, dx$$

missing above is $–\sin x\, dx = du$ that goes with the cos x. Therefore, there is a need to use another trigonometric identity. The one that will be used is given as $\sin^2 x = ½ (1 – \cos 2x)$ and $\cos^2 x = ½ (1 + \cos 2x)$.

Note: The above identities are derived by adding or subtracting the equations $\cos^2 x + \sin^2 x = 1$ and $\cos^2 x – \sin^2 x = \cos 2x$

Recall

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx$$
$$= \int [½ (1 – \cos 2x]^2\, dx$$
$$= \int ¼ (1 – 2 \cos 2x + \cos^2 2x)\, dx$$
$$= ¼ \int (1 – 2 \cos 2 x + ½ (1 – \cos 4x)\, dx$$
$$= ¼ [x – \sin 2x + \frac{x}{2} – \frac{1}{8} \sin 4 x]$$
$$= \frac{3}{8} x – ¼ \sin 2x = \frac{1}{32} \sin 4x + C$$

Example: Find $\int \sin^2 x \cos^2 x\, dx$

Here both powers are even. Let $\sin^2 x = (1 – \cos^2 x)$

Therefore $\sin^2 x \cos^2 x = (1 – \cos^2 x) \cos^2 x$

$\Rightarrow \int \sin^2 x \cos^2 x\, dx = \int (\cos^1 x – \cos^4 x)\, dx$

$\int \cos^2 x\, dx - \int \cos^4 x\, dx$

$\int \cos^2 x\, dx = \int ½ (1 + \cos 2x)\, dx = \frac{x}{2} + \frac{\sin 2x}{4}$

$$\int \cos^4 x\, dx = \int (\cos^2 x)^2\, dx = \int [½ (1 + \cos 2x)]^2\, dx$$

$$= \int ¼ [1 + 2 \cos 2x + \cos^2 2x]\, dx$$

$$= ¼ [1 + 2\cos 2x + ½ (1 + \cos 4x)]\, dx$$

$$= \frac{3x}{8} + ¼ \sin 2 x + \frac{1}{32} \sin 4x$$

$$\int \sin^2 x \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + \frac{3x}{8} + ¼ \sin 2x + \frac{1}{32} \sin 4x$$

$$= \frac{7}{8} x \;\; ½ \sin 2x + \frac{1}{32} \sin 4x + C$$

Example: Find $\int \cos^6 x\, dx$

$\Rightarrow \int \cos^6 x\,dx = \int(\cos^2 x)^3\,dx = \int \frac{1}{8}(1 + \cos 2x)^3\,dx$

$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\,dx$

$= \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.$

SELF-ASSESSMENT EXERCISES

Find the following integrals:

(i) $\int \sin^2 x \cos^4 x\,dx$ (ii) $\int \sin^2 4t\,dt$

(iii) $\int \cos^2 6x\,dx$ (iv) $\int \sin^6 x\,dx$

(vi) $\int \cos^4 ax\,dx$

Ans:

(i) $\frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$ (ii) $\frac{x}{2} - \frac{\sin 8x}{16} + C$

(iii) $\frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$

(iv) $\frac{5}{16}x - \frac{1}{4}\sin 2x - \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

(v) $\frac{3}{8}x + \frac{1}{4}\sin 2ax + \frac{1}{32}\sin 4ax + C$

3.4 Powers and Products of Other Trigonometric Functions

In this section, you shall evaluate two types of integrals

(1) $\int \tan^m x \sec^n x\,dx$ and

(2) $\int \cot^m x\, \mathrm{cosex}^n\,dx$

Example: When n is even you write $\tan^m x \sec^n x = \tan^m x \sec^{n-2} x \sec^2 x$ and then express $\sec^{n-2}$ in terms of $\tan^2 x$ using $\sec^2 x + 1 = \tan^2 x$.

Example: $\int \tan^3 x \sec^2 x\,dx$

let $u = \tan x$ $du = \sec^2 x\,dx$. then $\int \tan^3 x \sec^2 x\,dx = \int u^3\,du$

$= \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$

When n and m are both odd you write

$\tan x \sec^n x = \tan^{m-1} x \sec^{n-1} \sec x$ 1an u

and express $\tan^{m-1} x$ in terms of $\sec^2 x$ using $\tan^2 x = \sec^2 x - 1$

Example: $\int \tan^3 x \sec^3 x \, dx$

$\tan^3 x \sec^3 x = \tan^2 x \sec^2 x \tan x \sec x$ and $\tan^2 x = (\sec^2 x - 1)$

therefore: $\int \tan^3 x \sec^3 x \, dx = \int (\sec 3x - 1) \sec^2 x \sec x \tan x \, dx$

$= \int (\sec^4 x - \sec^2 x) \sec x \tan x \, dx$

(but u = sec x, du = se x 1 an x du)

therefore $\int \tan^3 x \sec^3 x \, dx = \int (u^4 - u^2) \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^3 x}{5} - \frac{\sec^3 x}{3} + C$

you can do the same for $\cot^m x \operatorname{cosec}^n x$ in a similar manner. That is for $\int \cot^m x \operatorname{cosec}^n x \, dx$ when n is even you write out $\cot^m x \operatorname{cosec}^n x = \cot^m x \operatorname{cosec}^{n-2} \operatorname{cosec} x$ and express $\operatorname{cosec}^{n-2}$ in terms of $\cot^2 x$ using $\operatorname{cosec}^2 x = \cot^2 x + 1$

Example: $\int \cot^5 x \operatorname{cosec}^4 x \, dx$

$= \int \cot^5 x \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx$
$= \int \cot^5 x (\cot^2 x + 1) \operatorname{cosec}2 x \, dx$
$= \int \cot^7 \operatorname{cosec}^2 x \, dx + \int \cot^5 x \operatorname{cosec}^2 x \, dx$
$(u = \cot x \; du = -\operatorname{cosec}^2 x \, dx)$.
$= \frac{-\cot^8 x}{8} - \frac{-\cot^6 x}{6} + C$

In similar manner when m and n are both odd you have $\cot^m x \operatorname{cosex}^n x$
$= \cot^{m-1} x \operatorname{cosec}^{n-1} x \operatorname{cosec} x \cot x$ and then express $\cot^{m-1} x$ in terms of $\operatorname{cosec}^2 x$ using $\cot^2 x = \operatorname{cosec}^2 x - 1$

Example: $\int \cot^5 x \operatorname{cosec}^3 x \, dx$

$= \int \cot^4 x \operatorname{cosec}^2 x \operatorname{cosec} x \cot x \, dx$
$= \int (\operatorname{cosec}^2 x - 1)^2 \operatorname{cosec}^2 x \operatorname{cosec} x \cot x \, dx$
$= \int (\operatorname{cosec}^6 x - 2 \operatorname{cosec}^4 x + \operatorname{cosec}^2 x) \operatorname{cosec} x \cot x \, dx$ $u = \operatorname{cosec} x$
$du = -\operatorname{cosec} x \cot x \, dx$

$= \int (u^6 - 2u^4 + u^2) (-du)$
$= \frac{-u^7}{7} \pm \frac{2u^5}{5} - \frac{u^3}{3} \pm C$

$$= \frac{-\operatorname{cosec}^7 x}{7} + \frac{2\operatorname{cosec}^5 x}{5} - \frac{\operatorname{cosec}^3 x}{3} + C$$

SELF-ASSESSMENT EXERCISES

Find

1. $\int \cot^3 x \operatorname{cosec}^3 x \, dx$
2. $\int \cot^3 x \operatorname{cosec}^2 x \, dx$
3. $\int \tan^5 x \sec^2 x \, dx$

Ans:

1. $\frac{-\operatorname{cosec}^5 x}{5} + \frac{\operatorname{cosec}^3 x}{3} + C$
2. $\frac{-\cot^4 x}{4} + C$
3. $\frac{\tan^6 x}{6} + C$

4.0 CONCLUSION

In this unit, you have reviewed differential formulas and their corresponding integrals. These basic formulas will be used throughout the remaining part of the course. You have developed techniques of finding integrals of powers of trigonometric functions by using the trigonometric identities;

(i) $\cos^2 x + \sin^2 x = 1$ and
(ii) $1 + \tan^2 x = \sec^2 x$ etc.

You have also studied how to evaluate the products of even powers of sines and cosines functions. These integrals will be used when developing other techniques of integration in the next unit of this course.

5.0 SUMMARY

You have studied in the unit how to

- recall basic differential formulas and corresponding integrals
- use these basic formulas to develop techniques of integration of powers of trigonometric function
- evaluate the integrals of odd powers of trigonometric function such as $\int \sin^n x \, dx$, $\int \cos^n x \, dx$
- evaluate the integrals of trigonometric function such as $\int \tan^n x \, dx$, $\int \cot^n x \, dx$ where n is odd or even
- evaluate the integrals of even powers of sec x and cosec x
- evaluate the integrals of products of even powers of sin x and cos x such as $\int \cos^n x \, dx$, $\int \sin^n x \, dx$, $\int \cos^n x \, dx \sin^m x \, dx$ where n or m is even or both are even.

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7.0 TUTOR-MARKED ASSIGNMENT

1. Find $\int \sin^2 x \cos^2 x\, dx$
2. Show that $\int \tan ax\, dx = \frac{1}{a} \ln|\cos ax| + C$
3. Find $\int \sin^3 4x\, dx$
4. Find $\int \tan^5 x \sec^3 x\, dx$
5. Show that $\int \sec^{2n} x\, dx = \int (1+u^2)^{n-1}\, du$ where $u = \tan x$
6. Find $\int \cos^{2/3} x \sin^5 x\, dx$
7. Find $\int \sin^2 x \cos^5 x\, dx$
8. Find $\int \sin 4x \cos^2 x\, dx$
9. Find $\int \tan^6 x\, dx$
10. Find $\int \tan^5 x \sec^4 x\, dx$