MODULE 2

UNIT 1 FURTHER TECHNIQUES OF INTEGRATION I

CONTENTS

1.0 INTRODUCTION

In continuation of development of skills in techniques for finding integrals of special functions, you will study in the unit how to evaluate integrals involving rational function with $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2 - a^2}$ and $a^2 - u^2$ as denominators. The use of inverse trigonometric functions or trigonometric identifies will be needed. In this unit the method or process used in deriving formulas for integrals of functions in the previous unit will be adopted.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recall basic differential formulas and corresponding integrals as stated in Unit 5.
- Use these basic formulas to evaluate integrals involving $\sqrt{a^2 - u^2}$, $a^2 + u^2$, u^2, $- a^2$, and $a^2 - u^2$
- evaluate integrals of rational functions with $ax^2 + b x + C$ as denominator.

3.0 MAIN CONTENT

3.1 Integrals Involving $\sqrt{a^2 \neq u^2}$ and $a^2 \neq u^2$

Recall that $\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1+u^2}\frac{du}{dx}$

therefore $\int \frac{d}{}(\text{arc tan } u) = \int \frac{1}{1+u^2} du\ d$

Therefore arc tan u + C = $\frac{1}{1+u^2}$ – (I)

Example: Find the integral of $\frac{1}{+u^2}$ i.e. $\int \frac{du}{a^2+u^2} a^2$

To evaluate the above you factor out a^2 from $a^2 + u^2$
i.e. $a^2 + u^2 = a^2 (1 + (\frac{u}{a})^2)$

let $z = \frac{u}{a}$, adz = du

then $a^2 + u^2 = a^2(1 + z^2)$

therefore: $\int \frac{du}{a^2+u^2} = \frac{1}{a^2}\int \frac{adz}{1+z^2} = \frac{1}{a}\int \frac{dz}{1+z^2}$

$= \frac{1}{} \tan^{-1} z\ a$

thus $\int \frac{du}{a^2+u^2} = \frac{1}{a}$ arc tan $\frac{u}{a}$ + C – II

Example: Find $\int \frac{du}{9+u^2}$

Solution $\int \frac{du}{a+u} = \int \frac{du}{(3)^2+u^2} = \frac{1}{3}$ arc tan $\frac{u}{3}$ + C

You have to review some trigonometric identities you studied in MATH 111.

Example:

(i) $1 - \sin^2 x = \cos^2 x$
(ii) $1 + \tan^2 x = \sec^2 x$ and
(iii) $\sec^2 x - 1 = \tan 2\ x$.

let u = a sin x then
$u^2 = a^2 \sin^2 x$
Multiplying identity (1) through by a^2 you get:

$a^2(1-\sin^2 n) = a^2 \cos^2 x.$ – (i)

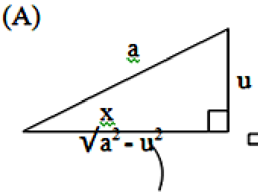
$a^2 - a^2 \sin^2 x = a^2 \cos^2 x.$ – (ii)

but $u^2 = a^2 \sin^2 x$ therefore equation (II) becomes $a^2 - u^2 = a^2 \cos^2 x$

In similar manner, if $a^2 + a^2 \tan^2 x = a^2 \sec^2 x$ and $u^2 = a^2 + ax^2 x$ then
$a^2 + a^2 \tan^2 x = a^2 + u^2 = a^2 \sec^2 x.$
If $a^2 \sec^2 x - a^2 = a^2 \tan^2 x$ and $u^2 = a^2 \sec^2 x$
then $u^2 - a^2 = a^2 \tan^2 x.$
thus:
(1) $a^2 - u^2 = a^2 \cos^2 x$,; $u = a \sin x$ (see fig. 6.1)
(2) $a^2 + u^2 = a^2 \sec^2 x$; $u = a \tan x$ (see fig. 6.2)
(3) $u^2 - a^2 = a^2 \tan^2 x$; $u = a \sec x$ (see fig. 6.3)

The above trigonometric identities you were given in equations (1) to (3) are equivalent expressions of the Pythagorean Theorem. See fig. 6.1 to 6.3

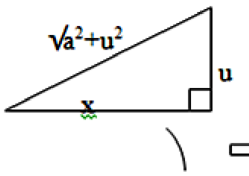
(A)



$\frac{u}{a} = \sin x$

and $\sqrt{a^2-u^2} = a \cos x$ $\Longrightarrow$ $a^2 - u^2 = a^2 \cos x$

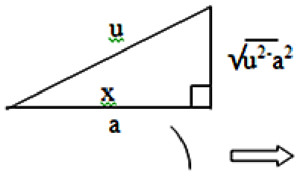
Fig. 6.1



$\frac{u}{a} = \tan x$

and $\frac{u}{\sqrt{a^2+u^2}} = \cos x\ a$
$\Longrightarrow$ $= a \sec x$
$= a^2 \sec^2 x$

Fig. 6.2



$\frac{u}{a} = \sec x$

$\frac{\sqrt{u^2-a^2}}{a} = \tan x$

$\Longrightarrow$ $u^2 - a^2 = a^2 \tan^2 x$

Fig. 6.3

Example Find $\int \frac{du}{\sqrt{a^2-u^2}}$ /a/>u

Solution: let $u = a \sin x$ $\Longrightarrow$ $du = a \cos x\, dx$
then $a^2 - u^2 = a^2 \cos^2 x$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\cos x dx}{\sqrt{a^2 \cos^2 x}} = \int dx = x + C$$

if $u = a \sin x$ $\Longrightarrow$ $\frac{u}{a} = \sin x$

and $x = \arc \sin \frac{u}{a}$

therefore $\int \frac{du}{\sqrt{a^2-u^2}} = \arc \sin \frac{u}{a} + C$ - III

(B) The usefulness of trigonometric identities in evaluating special types of integrals is numerous. Functions not involving trigonometric function can be integrated by expressing them in terms of trigonometric identities and then using standard integration formulas to evaluate them. In integration by method of completing the square is introduced in this unit.

Example: Find $\int \frac{du}{\sqrt{25-u^2}}$

here $a^2 = 25 \Rightarrow a = 5$

From above $u = 5 \sin x \Rightarrow du = 5 \cos x\, dx$

$\int \frac{du}{\sqrt{25 - u^2}} = \int \frac{5\cos x dx}{\sqrt{25 \cos^2 x}} = \int dx = x + C$

therefore $x = \arc \sin \frac{u}{5} + C$

Example: Find $\int \frac{du}{\sqrt{a^2+u^2}}$ $a > 0$

Let $u = a \tan$

$du = a \sec^2 x\, dx$

but $a^2 + u^2 = a^2 + a^2 \tan^2 x = a^2 (1 + \tan^2 x)$

$= a^2 \sec^2 x$

then $\int \frac{du}{\sqrt{a^2+u^2}} = \int \frac{a\sec^2 x dx}{\sqrt{a^2 \sec^2 x}} = \pm\int \sec x\, dx$

Recall that $\int \sec x\, dx = \ln/\sec x + \tan x/ + C^1$

i.e. $\int \frac{\sec x(\tan x+\sec x)}{\tan x + \sec x} dx = \ln/s\sec x + \tan x/ + C^1$

Hence $\pm\int \sec x\, dx = \ln/\sec + \tan x/ + C^1$

If you let $x = \arc \tan \frac{u}{} x$ $(- /_2, /_2)$

then sec x will be positive and $\int \frac{du}{\sqrt{a^2 + u^2}} = \int \sec x\, dx$

$= \ln/\sec x + \tan x/ + C$

recall that you let $a^{2+u}2 = a^2 \sec^2 x$

$\Rightarrow$ $\sec x = \frac{\sqrt{a^2+u^2}}{a}$

and $\tan x = \frac{u}{a}$

Then

$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|\frac{\sqrt{a^2+u^2}}{a} + \frac{u}{a}\right| + C^1$$

let $C = C^1 - \ln a$ you then have that

$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|\sqrt{a^2+u^2} + u\right| + C \quad \text{IV}$$

Example: Find $\int \frac{du}{\sqrt{16+u^2}}$

Solution: let a^2 16 $\Longrightarrow$ $a = 4$
then by direct substitution into IV you get

$$\int \frac{du}{\sqrt{16+u^2}} = \ln\left|\sqrt{16+u^2} + u\right| + C$$

Example: Find $\int \frac{du}{\sqrt{u^2-a^2}}$ $|u|>a>0$

Solution: You can start by trying the substitution $u = a \sec x$
then $du = a \sec x \tan x\, dx$
but $u^2 - a^2 = a^2 \sec^2 x - a = a^2(\sec^2 x - 1)$
$= a^2 \tan^2 x$

You will then have that

$$\int \frac{du}{\sqrt{u^2-a^2}} = \int \frac{a\sec x\tan x\,dx}{\sqrt{a^2\tan^2 x}} = \int \frac{a\sec x\tan x\,dx}{a\tan x}$$

$$= \pm\int \sec x\, dx$$

therefore $x = \text{arc sec } \frac{u}{a}$ $0 < x <$

but $\tan x > 0$ whenever $0 < x < /_2$.
and $\tan x < 0$ whenever $/2 < x <$

From the previous example, you know that $\pm\int \sec x\, dx = \ln|\sec x + \tan x| + C^1$

Recall tha $\sec x = \frac{u}{a}$ and $\tan x = \pm\frac{\sqrt{u^2-a2}}{a}$

If $\tan x > 0$ you get $+\ln\left|\frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a}\right| + C^1$

and $\tan x < 0$ you get $-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2 - a^2}}{a}\right| C^1$

However, $-\ln\left|\frac{u}{a} - \frac{\sqrt{u^2 - a^2}}{a}\right| = \ln\left|\frac{u + \sqrt{a^2+u^2}}{a}\right|$

therefore $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$ – (V)

(where $C = C^1 - \ln a$.)

Example Find $\int \frac{du}{\sqrt{u^2 - 64}}$

Solution: let $a^2 = 64 \Rightarrow a = 8$

Thus by direct substitution into equation (V) you get that

$$\int \frac{du}{\sqrt{u^2 - 64}} = \ln/u + \sqrt{u^2 - 64}/ + C$$

Example: Find $\int \sqrt{9 - u^2}\, du$

Solution: let $u = 3 \sin x$, $du = 3 \cos x\, dx$.
then $9 - u^2 = (9 - 9 \sin^2 x) = 9(1 - \sin^2 x)$
$= 9 \cos^2 x$

therefore: $\int \sqrt{9 - u^2}\, du = \int \sqrt{9 \cos^2 x}\ 3 \cos x\, dx$
$= \int 9 \cos^2 x\, dx$

From Unit 5 sec 3.3 you have that

$$\int 9 \cos^2 x\, dx = 9/2 \int (1 + \cos 2x)\, dx$$
$$= 9/2(x + \frac{\sin 2x}{2}) + C$$
$$= 9/2\, x + \frac{9 \sin 2x}{4} + C$$

therefore: $\int \sqrt{9 - u^2}\, du = \frac{9}{2} \arcsin \frac{u}{3} \_ \frac{u}{2} \sqrt{9 - u^2} + C$

Example: Find $\int \frac{u^2\, du}{\sqrt{4 - u^2}}$

Solution: let $u = 2 \sin x$ and $\frac{-}{2} < \theta < \frac{}{2}$

$du = 2 \cos x\, dx$

$4 - u^2 = 4 - 4 \sin^2 x = 4 \cos^2 x$

therefore $\int \frac{u^2\, du}{\sqrt{4 - u^2}} = \int \frac{4 \sin^2 x 2 \cos x dx}{\sqrt{4 \cos^2 x}} = \int 4 \sin^2 x\, dx$

From unit 5 sec 3.3 you have that

$$4 \int \sin^2 x\, dx = 4 \int \frac{(1 - \cos 2x)}{2}\, dx$$
$$= 2x - 2 \sin x \cos x + C$$
$$= 2(\arcsin \frac{u}{2} - (\frac{4 - u^2}{4}) + C$$

Example: Find $\int \frac{dx}{\sqrt{1 - 4x^2}}$

Solution: let $2x = \sin u$, $4x^2 = \sin^2 u$

$2\,dx = \cos u\,du$

$1 - 4x^2 = 1 - \sin^2 u = \cos^2 u$

therefore: $\sqrt{1-4x^2} = \sqrt{\cos^2 u} = \cos u$

hence $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{\cos u\,du}{\cos u} = \frac{1}{2}\int du$

$= \frac{1}{2}u + C$ but $u = \arcsin 2x$

$= \frac{1}{2}\arcsin 2x + C$

Example: Find $\int \frac{x\,dx}{\sqrt{4 + x^2}}$

Solution: let $x = 2\tan u$ $dx = 2\sec^2 u\,du$

$4 + x^2 = 4 + 2\tan^2 u = 4(1 + \tan^2 u) = 4\sec^2 u$

$\int \frac{x\,dx}{\sqrt{4+x^2}} = \int \frac{2\tan u . 2\sec^2 u\,du}{\sqrt{4\sec^2 u}} = \int 2\tan u \sec u\,du$

$= 2\sec u + C$

$= \sqrt{4 + x^2} + C$

Example: Find $\int \frac{dx}{\sqrt{(x-1)^2 + 4}}$

Solution: let $z = \frac{x-1}{2}$, $2dz = dx$

$4 + (x-1)^2 = 4(1 + Z^2)$

$\Rightarrow \int \frac{dx}{(x-1)^2 + 4} = \frac{1}{4}\int \frac{2\,dz}{1+Z^2} = \frac{1}{2}\int \frac{dz}{1 + Z^2}$

$= \frac{1}{2}\arctan z = \frac{1}{2}\arctan \frac{(x-1)}{2} + C$

Exercises: Find the following integrals

(i) $\int \frac{dx}{(9 - x^2)^{3/2}}$ (ii) $\int \sqrt{16 + x^2}\,dx$

(iii) $\int \frac{dx}{x\sqrt{9x^2+4}}$ (iv) $\int \sqrt{25 - 4x^2}$

(v) $\int \frac{x^2\,dx}{\sqrt{9 - 4x^2}}$ (vi) $\int \frac{dx}{\sqrt{1-16x^2}}$

(vii) $\int \frac{\cos x\,dx}{\sqrt{2-\sin^2 x}}$ (viii) $\int \frac{dx}{\sqrt{1 - \frac{x^2}{16}}}$

(ix) $\int \frac{dx}{x\sqrt{a^2+x^2}}$ (x) $\int \frac{xdx}{\sqrt{25-4x^2}}$

Ans:

(i) x + C (ii) ½ x $\sqrt{16 + x^2} + 8\ln/x+\sqrt{16+x^2}/+C$

(iii) $\frac{1}{2}\ln/\frac{\sqrt{9x^2+4}-2}{3x}/ + C$

(iv) $\frac{x}{2}\sqrt{25 - 4x^2} + \frac{25}{4}\arcsin\frac{2x}{5} + C$

(v) $\frac{-x}{8}\sqrt{9 - 4x^2} + \frac{9}{16}\arcsin\frac{2x}{3} + C$

(vi) ¼ arc sin 4x + C (vii) $\arcsin\left(\frac{\sqrt{2}\sin x}{2}\right) + C$

(viii) $\frac{x}{8}\sqrt{16 - x^2} + 2\arcsin\frac{x}{4} + C$

(ix) $\frac{-1}{a}\ln/\frac{a + \sqrt{a^2 + x^2}}{x}/ + C$

(x) $-\frac{1}{4}\sqrt{25 - 4x^2}$

3.2 Integration by Completing the Square of $ax^2 + bx + C$

Given a quadratic function of this form $f(x) = ax^2 + bx + C$ by completing the square it can be reduced to the form $a(u^2 + A)$

i.e. $ax^2 + bx + C = a(x^2 + \frac{b}{a}x) + C$

$= a(x^2 + \frac{bx}{a} + \frac{b^2}{4}) + C - \frac{b^2}{4a}$

$= a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$

if you let $u = x + \frac{b}{2a}$ and $A = \frac{4x-b^2}{4a^2}$

then $ax^2 + bx + C = a(u^2 + A)$.

When the integral involves the square root of $ax^2 + bx + C$ then you have to consider only the case for which $\sqrt{a(u^2+A)}$ will have only real roots.

Example: Find $\int \frac{dx}{\sqrt{x^2+2x}}$

Solution: $x^2+2x = \int\sqrt{(x+1)^2 - 1}$

$= \sqrt{u^2 - 1}$, $u = x + 1$; $du = dx$

then $\int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{du}{\sqrt{u^2 - 1}}$

$= \ln/u + \sqrt{u^2-1}/ + C$

$= \ln/(x+1) + \sqrt{x^2 + 2x}/ + C$

Example: $\int \frac{dx}{\sqrt{x^2 - 8x}}$

$\sqrt{x^2 - 8x} = \sqrt{(x-4)^2 - 16}$

let $u = x - 4$ and $du = dx$ then $\sqrt{x^2} = 8x = \sqrt{u^2 - 16}$

therefore: $\int \frac{dx}{} = \ln/u \ \sqrt{u^2 - 16}/ \ _C$

$= \ln/x - 4 + \sqrt{x^2-8x}/ + C$

Example: Find $\int \frac{dx}{x^{2-}10x+29}$

Solution:

$x^2 - 10x + 29 = x^{2-}10x + 25 + 4 = (x-5)^2 + 2^2$

therefore: $\int \frac{dx}{x^2-10x + 29} = \int \frac{dx}{2^2 + (x-5)^2} = \int \frac{du}{2^2 + u^2}$

where $u = x - 5$, $du = dx$

thus $\int \frac{dx}{x^2-10x+29} = \int \frac{du}{2^2+ u^2} = \frac{1}{2} \text{arc tan } \underline{u} + C$

$= \frac{1}{2} \text{arc tan } \frac{(x-5)}{2} + C$

Example: Find $\int dx \ \sqrt{3-x^2+2x}$

Solution: $\sqrt{3-x^2+2x} = \sqrt{-(x^2 - 2x) + 3} = \sqrt{-(x^2-2x+1)+1+3}$

$= \sqrt{-(x-1)^2 + 4} = \sqrt{4 - u^2}$

where $u = x - 1$ $du = dx$ then

$\int \frac{dx}{\sqrt{3-x^2+2x}} = \int \frac{du}{\sqrt{4 - u^2}} = \text{arc sin } \underline{u} + C$

$= \text{arc sin } \frac{x- 1}{2} + C$

Example: Find $\int \frac{dx}{4x^2+4x+10}$

Solution: $4x^2 + 4x + 10 = 4(x^2 + x) + 10$

$= 4(x^2 + x + \frac{1}{4}) + 10 - 4/4$

$= 4(x + \frac{1}{2})^2 + 9$

let $u = x + \frac{1}{2}$ $du = dx$

then $\int \frac{dx}{4x^2+4x+10} = \int \frac{du}{4u^2 + 9}$

$= 1/6 \arctan \frac{2u}{3} + C$

$= 1/6 \arctan \frac{2x+1}{3} + C$

4.0 CONCLUSION

In this unit, you have studied techniques used in evaluating integrals involving $\sqrt{a^2+u^2}$, $\sqrt{a^2 - u^2}$, $\sqrt{u^2 - a^2}$, $a^2 + u^2$, $ax^2 + bx + C$, and $\sqrt{ax^2+bx+C}$. You have used the trigonometric identities and formulas studied in unit 5 to develop the techniques for solving integrals involving the expressions mentioned above. In the next unit you will study other techniques of integration.

In this unit you have reviewed important trigonometric identities such as (i) $1+ \tan^2 x = \sec^2 x$, (ii) $1 - \cos^2 x = \sin^2 x$ and (iii) $\sec^2 x - 1 = \tan^2 x$. You have used the above identities to develop techniques for evaluating integrals involving $a^2 + u^2$, $\sqrt{a^2 - u^2}$, $\sqrt{u^2 - a^2}$ and $\sqrt{a^2 + u^2}$. You have also recall the method of completing the square of a quadratic function such as $f(x) = ax^2+bx+C$. You have used the method of completing the square to evaluate the integrals involving $ax^2 + bx + C$ and $\sqrt{ax^2 + bx + C}$. You used the formulas studied in unit 5 to evaluate the above mentioned integrals. In the next unit you will study other techniques for evaluating integrals.

5.0 SUMMARY

In this unit you have studied;

- how to evaluate the following types of integrals

(i) $\int \frac{du}{a^2+u^2}$ (ii) $\int \frac{du}{\sqrt{a^2+v^2}}$

(iii) $\int \frac{du}{\sqrt{a^2-u^2}}$ (iv) $\int \frac{du}{\sqrt{u^2 - a^2}}$

- how to evaluate integrals such as:

(i) $\int \frac{dx}{ax^2+bx+C}$ (ii) $\int \frac{dx}{\sqrt{ax^2+bx+C}}$

using the method of completing the square.

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7.0 TUTOR-MARKED ASSIGNMENT

Evaluate the following integrals:

1. $\int \frac{dx}{\sqrt{2x-x^2+3}}$

2. $\int \frac{x^2\,dx}{\sqrt{25-x^2}}$

3. $\int \sqrt{4-x^2}\,dx$

4. $\int \frac{du}{\sqrt{u^2-a^2}}$

5. $\int \underline{\ du\ }$ /a/>/u/ /u/>/1/>0

6. $\int \frac{du}{u\sqrt{9u^2+4}}$

7. $\int \frac{dx}{x^2+2x+5}$

8. $\int \frac{dx}{\sqrt{x^2-8x+32}}$

9. $\frac{dx}{\sqrt{3x^2-4x+1}}$

10. $\int \frac{3x+10}{\sqrt{x^2+2x+5}}$

UNIT 2 FURTHER TECHNIQUES OF INTEGRATION II

CONTENTS

1.0 INTRODUCTION

As have been mentioned in previous section, integration is a process that involves anti-differentiation. You start by making a guess and determine whether the differentiation of your guess can give you the function you want to integrate. The techniques you have studied so far are all trying to narrow your guess to the exact solution. Therefore in this unit you will study rational functions can be integrated by first resolving the rational function into partial fractions. The method of resolving rational fraction into partial fraction before integration is called integration by method of partial fractions. Also there are functions that are formed as a product of exponential functions and trigonometric function. Finding, integrals of such product functions can only be possible by making use of the product rule studied in the first course in calculus i.e. calculus I. This type of integration by which the product rule is applied is known as integration by parts. All these techniques are studied in order to make it easier for you to evaluate integrals of function without making a wide guess. So endeavour to practice the examples given here and in the previous units.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- integrate certain types of rational functions by method of partial fractions
- evaluate integrals of product functions by method of integration by parts.

3.0 MAIN CONTENT

3.1 Integrations by Partial

In the course in algebra MTH 111 you studied how to split a rational function into a sum of fractions with simpler denomination. You were told that the process of doing this is called the method of partial fractions. It is advisable that you review this method of partial fraction in the materials given to you for the course in algebra. You could recall that

$$\frac{x+7}{6+x-x^2} = \frac{x-7}{(3-x)(2+x)} = \frac{2}{3-x} + \frac{1}{2+x}$$

Therefore if you want to integrate a rational function of this type, $\int \frac{f(x)}{g(x)} dx$

There are two things you will check before you decide to use the method of partial fractions. These are as follows:

1. The degree of f(x) should be less than the degree of g(x). If this is not the case, you must perform a long division, then resolve the remainder into partial fraction.

2. The factors of g9x) should be known by you.

Example: Find $\int \frac{x+7}{6+x-x^2} dx$

Solution: The degree of x + 7 is lower than that of $6+x-x^2$.

The factors of $6+x-x^2$ are (3-x)(2+x)

$$\text{Therefore, } \int \frac{x+7}{6+x-x^2} dx = \int \frac{2}{3-x} dx + \int \frac{dx}{2+x}$$
$$= -2\ln/3-x/+\ln/2+x/+C.$$

Example: Evaluate $\int \frac{4x^2-24x+11}{(x+2)(x-3)^2}$

Solution: The degree of $4x^2 - 24x + 11$ is less than that of $(x+2)(x-3)^2$

$$\frac{4x^2-24x+11}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

i.e. $4x^2 - 24x + 11 = (A + B)x^2 + (C - 6A - B)x + 9A - 6B + 2C$

Equating coefficient and solving the resolutions simultaneous equations yields

$$\frac{4x^2-24x+11}{x-3)^2} = \frac{3}{x+2} + \frac{1}{3-3} - \frac{5}{(x-3)^2} (x+25$$

$$\text{therefore } \int \frac{4x^2-24x+11}{(x+2)(x-3)^2} dx = \int \frac{3}{x+2} du + \int \frac{dx}{x-3} - \int \frac{5dx}{(x-3)^2}$$

$$= 3\ln/x+2/+\ln/x-3/+ \frac{5}{(x-2)} + C$$

Example: Find $\int \frac{6x+1}{3x^3+12x^2-2x-3}$

Solution: $3x^3 + 12x^2 - 2x - 3 = (4x^2 -1)(2x + 3)$.

Then $\frac{6x+1}{} = \frac{A}{} + \frac{B}{} + \frac{C}{} (4x^2-$

1)(2x+3) 2x -1 2x+1 2x+3.

6x + x = A (2x+1)(2x+3) + B(2x-1)(2x +3) + C(2x-1)(2x+1)

By equations coefficients and solving the resulting simultaneous equation you get

$$\frac{6x+1}{1)92x+3)} = \frac{1}{2(2x-1)} - \frac{1}{2(2x+1)} - \frac{1}{2x+3} (4x^2-$$

therefore: $\int \frac{6x+1}{3x^3+12x^2-2x-3} dx = \frac{1}{2}\int \frac{dx}{2x-1} + \frac{1}{2}\int \frac{dx}{2x+1} - \int \frac{dx}{2x+3}$

= ¼ ln/2x-1/+1/ ¼ ln/2x+1/- ½ ln/2x+3/ + C.

Example: Find $\int \frac{3x^2}{1+x^3} dx$

Solution: $\frac{3x^2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$

Therefore $3x^2 = A(1 - x + x^2) + (Bx + C)(1 + x)$

Equations coefficient solving the resulting simultaneous equations yields

$$\frac{3x^2}{1+x^3} = \frac{1}{1+x} + \frac{2x-1}{1-x+x^2}$$

therefore: $\int \frac{3x^2}{1+x^3} dx = \int \frac{1}{1+x} dx + \int \frac{2x-1}{1-x+x^2} dx$

Example: Find $\int \frac{x+3}{4x^3+4x^2-7x+2}$

Solution: $4x^3+4x^2 - 7x + 2 = (2x - 1)(2x -1)(x + 2)$

then $\frac{x+3}{(2x-1)^2(x+2)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+2}$

hence $x + 3 = A(2x-1)(x+2) + B(x+2) + C(2x - 1)^2$

Equating coefficients and solving the resulting simultaneous equations yields

$$\frac{x+3}{(2x-1)^2(x+7)} = \frac{-2}{25(2x-1)} + \frac{7}{5(2x-1)} + \frac{1}{25(x+2)}$$

therefore: $\int \frac{x+3}{4x^3+4x^{2-7x+2}} dx = \frac{-2}{25}\int \frac{dx}{2x-1} + \frac{7}{5}\int \frac{dx}{(2x-1)^2} + \frac{1}{25}\int \frac{dx}{x+2}$

$= \frac{-1}{25}$ ln/2x-1/ $\frac{-7}{10(2x-1)} + \frac{1}{25}$ ln/x+2/+C

Exercises: Find the following integrals

(1) $\int \frac{dx}{(x+1)(x+2)}$ c

(2) $\int \frac{3x-1}{(2+x)(x-3)} dx$

(3) $\int \frac{x-5}{(x-1)(x+4)} dx$

(4) $\int \frac{2x^2+x+6}{4x^2-4x-3} dx$

(5) $\int \frac{x-3}{x^2+2x-8} dx$

(6) $\int \frac{5}{1+3x+2x^2} dx$

(7) $\int \frac{x^2}{2x^2-3x-2} dx$

(8) $\int \frac{x^3+x+1}{x^2-1} dx$

(9) $\int \frac{2-x^2}{(2x+1)^3(x-1)} dx$

(10) $\int \frac{x^3}{(x^2+x+4)(x^2+1)} dx$

Ans:

(1) ln(x+1) – ln/x +2/ + C

(2) 7/5 ln/x+2/ + 8/5 ln/x-1/ + C

(3) 9/5 ln/x+4/ - 4/5ln/x – 1/ + C

(4) $\frac{x}{2} - \frac{3}{4}$ ln/2x + 1/ + $\frac{3}{2}$ ln2x – 3/ + C

(5) $\frac{7}{6}$ ln/x + 4/ - $\frac{1}{6}$ ln/x – 2/ + C

(6) 5ln (2x + 1) – 5ln/x + 1/ + C

(7) $\frac{x}{2}$ - $\frac{1}{20}$ ln (2x + 1) + $\frac{4}{5}$ ln/x -2/ + C

(8) ½ x^2 + 3 ln/x-1/+ ½ ln/x + 1/ + C

(9) $\frac{7}{24(2x+1)^2} + \frac{13}{36(2x+1)} - \frac{1}{27}$ ln/2x + 1/+ $\frac{1}{27}$ ln/x – 1/ + C

(10) $\frac{13}{20}$ ln/x^2 + x + 4/ - $\frac{\sqrt{15}}{30}$ arc tan $\left(\frac{2x+1}{\sqrt{15}}\right)$ $\frac{-3}{20}$ ln/x^2 +1/

- $\frac{1}{10}$ arc tan x

3.2 Integration by Parts

The method of integration by parts owns its origin to the differential of a product. That is d(uv) = u dv + v du – (i) or udv = u dv – v du - (ii)

integrating equation (ii) you get $\int u\,dv = \int d(uv) - \int v du$

$\int u\,dv = uv - \int v\,du. + C$ – III

Equation III above expresses one integral $\int u\,dv$, in terms of a second integral $\int u\,dv$. The idea behind this method is that, if by appropriate choice U and dv, the second integral is simpler than the first, you may be able to evaluate it quite simply and as such arrive at the solution.

Example: Find $\int x \sin x\,dx$

Solution: let $\int x \sin x\,dx = -\int x\,d(\cos x)$

then using the formula for integration by part given in equation III above you have that $\Longrightarrow$ v = cos x

therefore $\int x \sin x\,dx$

$= \int u\,dv = uv - \int v\,du$

$= -\cos x\,(x) + \int \cos x\,dx$

$= -x \cos x + \sin x + C$

Example: $\int xe^{-x}\,dx.$

Solution: Use integration by parts with u = x, du = dx, dv = e^{-x} dx, v =$-e^{-x}$

Therefore $\int xc^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$

$= -xe^{-x} – c^{-x} + c$

In above example, it is possible to choose u and v differently

i.e. $\int xe^{-x}\,dx = \int udv$

$u = e^{-x}$, $du = -e^{-x}\,dx$ $dv = x\,dx$ $v = \frac{x^2}{2}$

then integration by parts you get

$$\int xe^{-x}\,dx = \frac{x^2e^{-x}}{2} - \int -^{-x} \frac{x^2}{2}\,dx$$

The above is true but the resulting integral on the right is harder than the given one on the left. Therefore, you should be cautious when factoring the integrand into u and dv. With more examples, you get use to this technique of integration by parts.

Examples: Find $\int \ln x\,dx$

Solution: u = lnx, dv = dx, du = $\frac{1}{x}$ dx

therefore $\int \ln dx = x\ln x - \int x.\frac{1}{x}\,dx$.

$= x\ln x – x. + c.$

Example: Find $\int \text{arc} \cos x\,dx$

Solution: $u = \text{arc cos } x \quad du = \frac{-dx}{\sqrt{1-x^2}}$

$dv = dx, \ v = x.$

therefore

$$\int \text{arc cos } x \, dx = uv - \int vdu = x \text{ arc cos } x - \int \frac{-x \, dx}{\sqrt{1-x^2}}$$

but $\int \frac{x \, dx}{\sqrt{1-x^2}} = -\int \frac{udu}{y}$ where $y^2 = 1 - x^2$

$$-\int dy = -y + c = \frac{-ydu}{\sqrt{1-x^2} + C} = + x \, dx$$

therefore: $\int \text{arc cos } x \, dx = x \text{ arc cos } x - \sqrt{1-x^2} + C$

Example: Find $\int \frac{xe^x \, dx}{(x + 1)^2}$

Solution: After several attempts it is found that the following factors for the integrand will work.

Let $u = xe^x \quad du = \frac{1}{(x + 1)^2} dx$

$du = (xe^x + e^{x) \, dx =} e^x (x + 1) \, dx$

$v = -\frac{1}{x + 1}$

therefore: $\int \frac{xe^x}{+ 1)^2} dx = uv - \int vdu = \frac{xe^x}{(x+1)} - \int \frac{e^x(x+1)}{-(x + 1)}$ (x

$$= \frac{-xe^{-x}}{x + 1} + \int e^x \, dx$$

$$= - \frac{xe^x}{x + 1} + e^x + C$$

$$= \frac{e^x}{x + 1} + C$$

3.2.1 Repeated Integration by Parts

Some integration may require that you apply the method of integration by parts two or more times.

Example: Find $\int x^2e^x \, dx$.

Solution: Applying integration by parts you get:

$u = x^2, \ dv = e^x \, dx$

$du = 2x \, dx, \ v = e^x$

therefore

$$\int x^2 e^x\, dx = \int u dv = uv - \int v\, du$$

$$= x^2e^x - \int 2xe^x dx$$

To find $\int 2x\, e^x\, dx$, you apply integration by parts again. By letting

$u = 2x \quad dv = e^x\, dx$
$du = 2\, dx \quad v = e^x$

then $\int 2x\, e^x = \int udv = uv - \int v\, du$
$= 2x\, e^x - \int 2e^x\, dx$
$= 2x\, e^x - 2e^x + C$

hence $\int x^2 e^x\, dx = x^2 e^x - 2x\, e^x + 2e^x + C$

Example: $\int x(\ln x)^2\, dx$

Solution: let $u = (\ln x)^2 \quad du = \frac{2\ln x dx}{X}$

$dv = x\, dx \quad v = \frac{x^2}{2}$

thus $\int x(\ln x)^2\, dx = \int udv = vu - \int vdu$

$$= \frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2}.\, 2\frac{\ln x}{x}\, dx$$

$$= \frac{(x\ln x)^2}{2} - \int x\ln x\, dx$$

To evaluate $\int x \ln x\, dx$, you apply integration by parts the second time.

i.e. $u = \ln x \quad du = \frac{dx}{x} \quad dv = x\, dx \quad v = \frac{x^2}{2}$

therefore $\int x\ln x = \frac{x^2}{2}.\, \ln x - \int \frac{dx}{x}.\, \frac{x^2}{2}$
$= \frac{x^2}{2}\frac{\ln x}{} - \int \frac{x}{2}\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

thus

$$\int x(\ln x)^2 = \frac{(x\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C$$

Example: Find $\int x^3 e^x\, dx$

Solution: You will apply integration by parts three times to get the solution.

$\int x^3 e^{-x} dx$
$dv = e^{-x} dx$, $u = x^3$ $du = 3x^2 dv$ $v = e^x$.
$\int x^3 e^{-x} dx = uv - \int vdu = -x^3 e^{-x} - \int -e^{-x} 3x^2 dx$
$3\int e^{-x} x^2 dx = 3[uv - \int vdu]$
$u = x^2$ $dv = e^{-x} dx$, $du = 2x dx$, $v = -e^{-x}$
$3\int e^{-x} x^2 dx = 3 -x^2 e^{-x} -3 \int -e^{-x} 2x dx.$
$+ 6\int e^{-x} x dx = 6[uv - \int vdu]$
$u = x$, $du = dx$, $dv = e^{-x} dx$, $v = -e^{-x}$
$6\int e^{-x} x dx = -6x e^{-x} -6 \int -e^{-x} dx$
$= 6xe^{-x} + 6e^{-x} + C$

therefore: $\int x^3 e^{-x} dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6 e^{-x} + C$
$= e^{-x} (-x^3 -3x^2 - 6x -6) + C$

Example: Find $\int e^x \sin x\, dx$

Solution: let $u = e^x$, $du = e^x dx$ $du = \sin x\, dx$ $v = \cos x$

Let
$I = \int e^x \sin x\, dx$

therefore $I = uv - \int v\, du$
$= -e^x \cos x - \int - \cos x\, e^x dx.$
$\int \cos x\, e^x dx$ integrate by parts again by letting $u = e^x$ $du = e^x dx$, $dv = \cos x\, dx$.
$v = \sin x$
then $\int \cos x\, e^x dx = e^x \sin x - \int \sin x\, e^x dx.$

Therefore $I = -e^x \cos x + e^x \sin x - I$ (since $I = \int \sin x\, e^2 dx$)
$\Longrightarrow$ $2I = -e^x \cos x + e^x \sin x.$

therefore $I = \dfrac{e^x(\sin x - \cos x)}{2}$
$\Longrightarrow$ $\int \sin x\, e^x dx = \dfrac{e^x(\sin x - \cos x)}{2} + C$

Exercise: Evaluate the following integrals

(1) $\int xe^x dx$ (2) $\int x \cos x\, dx$

(3) $\int x^2 e^{2x} dx$ (4) $\int x^2 e^{-x} dx$

(5) $\int xe^{2x} dx$ (6) $\int \ln (x^2 + 1) dx$

(7) $\int x \sec^2 x\, dx$ (8) $\int x^3 e^{x2}\, dx$

(9) $\int x^2 \cos x\, dx$ (10) $\int e^{2x} \sin x\, dx$

(11) $\int x \tan^2 x\, dx$ (12) $\int e^{2x} \cos x\, dx$

Ans

(1) $xe^x – e^x + C$
(2) $\cos x + x \sin x + C$
(3) $^1e^{2x} (\frac{x^2}{2} – \frac{x}{2} + ¼) + C$
(4) $-e^{-x} (-x^2 + 2x – 2) + C$
(5) $e^{2x} (\frac{x}{2} – ¼) + C$
(6) $x\ln (x^2 + 1) – 2x + 2 \arctan x + C$
(7) $x \tan x + \ln (\cos x) + C$
(8) $–x^3 \cos x + 3x^2 \sin x – 6 \sin x + 6x \cos x + C$
(9) $x^2 \sin x – 2 \sin x + 2x \cos x + C$
(10) $e^2x (- \frac{\cos x}{5} + \frac{2}{5} \sin x) + C$
(11) $x \tan x – \frac{x^2}{2} - \frac{\ln(1+\tan^2 x)}{2} + C$
(12) $e^{2x} (\frac{2}{5} \cos x + \frac{1}{5} \sin x) + C$

4.0 CONCLUSION

You have studied two techniques of integration. The method of partial fraction requires that you factorize the denominator so that you could have simpler factors which in turn will be easier to integrate. Also you used the technique of integration to integrate product of functions that are somehow difficult to integrate. Breaking up the integration into parts by applying the product rule for differentiation yields simpler integrands that you are already familiar with in previous units of this course. In the next unit you will study a technique which is very similar to the technique of integration by parts. In this unit various solved examples have been provided for you. This is because understanding the examples will enable you to know at glance if a particular integration should be carried out by any of the techniques studie in this unit. Doing all the exercises provided in this unit will also sharpen your skills in the use of the techniques studied in the unit.

5.0 SUMMARY

In this unit you have studied

(•) the technique of integration by partial fraction.

i.e. $\int \frac{f(x)}{g(x)} dx = \int \frac{A_1}{g_1(x)} dx + \int \frac{A_2}{g_2(x)} dx + \int \frac{An}{g_n(x)} dx$
where $g(x) = g_1(x)\, g_2(x) \ldots g_n(x)$ and the integrals on the left and are simpler than the given integral on the left.

(•) how to integrate product of function such as xe^x, $e^x \sin x$ x lnx etc by the technique of integration by parts.

i.e. $\int u\, dv = uv - \int v\, du$
where the integral on the left is simpler than the given integral on the left.

(•) how to apply the method of integration by parts two or more times.

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7.0 TUTOR-MARKED ASSIGNMENT

Find the following integrals.

1. $\int \frac{1}{x^2 - 4} dx$ (2) $\int \frac{5x-3}{(x+1)(x-3)} dx$

3. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}$ (4) $\int \frac{2x^2+3}{x(x-1)^2} dx$

(5) $\int \frac{x^2+1}{x^3-4x^2+x+6} dx$ (6) $\int \arctan x \, dx$

(7) $\int x^3 e^x dx$ (8) $\int x^2 \cos ax \, dx$

(9) $\int \sin(\ln x) dx$ (10) $\int_1^2 x \sin ax \, dx$

UNIT 3 FURTHER TECHNIQUES OF INTEGRATION III

CONTENTS

1.0 INTRODUCTION

In this unit you shall study two additional techniques for integration. The previous unit ended with the technique of integration by parts. In this unit you shall extend the method of integration by parts to derive what is known as reduction formulas for certain categories of product functions. The reduction is derived by applying the method of integration by parts repeatedly until the power of one of the product functions is reduced to 1 or 0. the second technique that will be studied in this unit involves using an appropriate substitution which makes it possible to integrate all rational expression of sin x and cos x, then the substitution 2 arc tan u = x transforms the integral ∫ f(x) into the integral of a rational function of u, which can be evaluated by techniques studied in previous units. The third technique that will be studied in this unit involves using appropriate substitution for rational expression containing some radicals such as $\sqrt{x}$, $x^{3/4}$, $\sqrt{1-e^x}$ etc.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive reduction formulas for integrals such as
  (i) $\int \cos^n x\, dx$, (ii) $\int \sin^n x\, dx$, (iii) $\int e^{ax} \cos bx\, dx$, etc
- evaluate integrals of rational functions of sin x and cos x using the substitution u = tan x/2.
- evaluate integrals rational functions involving radicals and fractional powers of x.
- evaluate integrals of product function of trigonometric ratios involving cos bu cos ax sin sub u cos ax and au bus u ax.

3.0 MAIN CONTENT

3.1 Reduction Formulas

Repeated application of integration by parts could reduce the power of a function from n, say + 1 or 0. Thus a formula could emerge from the above application that can be used for evaluating integrals which are similar.

Given the integral $\int x^2 e^x\, dx$ which you are quite familiar with in unit 7. The integral above requires two integration by parts. Each integration lowers the power of x by one until x disappears. In a similar way the integral $\int x^3 e^x\, dx$ requires three integration by parts and the integral ∬ x^4 e^x dx require four integration by parts. This process can continue for any power of x say n. Thus to evaluate the integral $\int x^n e^x\,dx$ requires ^{n} integration by parts.

Example: Obtain a reduction formula for the integral $J_n = \int x^n e^x\,dx$.

Solution: Integrate by parts setting $u = x^n$, $du = nx^{n-1}\,dx$, $dv = e^x$ $v = e^x$.

Then $J_n = \int x^n e^x\,dx = uv - \int v\,du$

therefore $J_n = x^n e^x - \int nx^{n-1} e^x\,dx$

$= x^n e^x - n \int_{n-1}$

Thus the reduction formula is given as $J_n = x^n e^x - n\int_{n-1}$ ---- (A)

Example: Find $\int x^{6x} e_x\,dx$

Solution: Use the reduction formula of equation (A) to evaluate J_4. By the reduction formula in equation (A) n = 6, that is $J_6 = x^6 e^x - 6\int_5$
By the reduction formula with n = 5 $J_5 = x^5 e^x - 5 J_4$
By the reduction formula with n = 4 $J_4 = x^4 e^x - 4 J_3$
Thus $J_6 = x^6 e^x - 6 (x^5 e^x - 5 J_4)$
$= x^6 e^x - 6x^5 e^x + 30 \int_4$

But by repeated use of the reduction formula

$$
\begin{aligned}
J_4 &= x^4 e^x - 4\int_3 \\
&= x^4 e^x - 4(x^3 e^x - 3\int_2) \\
&= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2\int_1) \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 (xe^x - \int_o) \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C
\end{aligned}
$$

thus $J_4 = e^x (x^4 - 4x^n + 12x^2 - 24x + 24) + C$

therefore $J_6 = e^x (x^6 - 6x^5 + 30 (x^4 - 4x^3 + 12x^2 - 24x + 24) + C$

$= e^x (x^6 - 6x^5 + 30x^4 - 120x^3 + 360x^2 - 720x + 720) + C$

Example: Obtain a reduction formula for $\int \sin^n x\,dx$.

Solution: You can write $J_n = \int \sin^n x\,dx = \int (\sin^{n-1}x)(\sin x)\,dx$
and integrate by parts with

$u = (\sin x)^{n-1}$ $du = (n-1)(\sin x)^{n-2} \cos x\,dx$
$dv = \sin x\,dx$ $v = -\cos x$

therefore $\int \sin^n x\,dx = (\sin x)^{n-1} (-\cos x) + \int (n-1) \sin x^{n-2} \cos^2 dx$

therefore $= J_n \sin^n x\,dx = (\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x (1- \sin^2 x)\,dx$

$= (\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x\,dx - \int \sin^n x\,dx$

$J_n = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2} - (n-1) J_n$

collecting like terms you get

$$(n-1) \int_n + \int_n = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2}$$

$$n \int_n = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2}$$

dividing through by n you obtain

$$Jn = \frac{(\sin x)^{n-1} \cos x}{n} + \frac{n-1}{n} \int_{n-2}$$

In the above the reduction formula lowers the power of sin x by two. Therefore, repeated application will reduce $\int_n$ to $\int_o$ or $\int_1$ accordingly as n is even or odd i.e.

$$J_n = \int \sin x \, dx = \cos x + C$$
$$J_o = \int dx = x + C$$

Example: $\int_0^{/2} (\sin x)^6 \, dx$

Solution: set $J_n = \int^{II/2} (\sin x)^6 \, dx$

Using the reduction formula of the last example, you get

$$\text{Let } J_6 = \frac{(\sin x)^5 \cos x}{6} + \frac{5}{6} J_4$$

for brevity you write

$$J_6 = I, + \frac{5}{4}[\sin^3 x\cos x + \frac{3}{4} J_2]$$

$$= I + \frac{5}{6}[I_2 + \frac{3}{4}[I_3 + ½ J_o]$$

$$I_1 = \frac{(\sin x)^5 \cos x}{6} \Big/_0^{II/2} = \frac{(\sin {}^{II/2})^5 \cos {}^{II/2}}{6} - \frac{(\sin o)^5 \cos o}{6} = 0$$

$$I_2 = 0, \quad I_3 = 0$$

hence

$$J_6 = \frac{5}{6}(¾)(1/2\, J_o) = \frac{5}{6}(¾)(½)(x)\Big/_0^{II/2}$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot ½ \cdot \frac{II}{2} \cdot = \frac{5II}{32}$$

Exercises: Find (1) $\int \sin^5 x \, dx$, (2) $\int_0^{II} \sin^7 x \, dx$

(3) $\int x^5 e^x \, dx$ (4) $\int_0^1 x^7 e^x \, dx$

(5) $\int \cos^3 x \, dx$

Ans:

(1) $e^x (x^5 - 5x^4 + 20 x^3 - 60 x^4 + 120 x - 120)$

(1) $-\frac{1}{15} \cos x (3 \sin^4 x + 4 \sin^2 x + 8) + C$

(2) $\frac{32}{35}$

(3) $e^x (x^5 - 5x^4 + 20x^3 - 60x^4 + 120x - 120) + C$

(4) $5040 - 1854e^1$

(5) $\frac{1}{3} \sin x (\cos^2 x + 2) + C$

Example: Obtain a reduction formula for $\int e^{ax} \cos bx\, dx$.

Solution: Let $U = e^{ax}$ and $dv = \cos bx\, dx$

Then $du = ae^{ax} dx$ and $v = \frac{1}{B} \sin bx$

Hence $\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{B} - \int \frac{1}{b} \sin bx.ae^{ax} dx$

(A) therefore $\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int \sin bx\, e^{ax} dx$

The integral on the right is like the first one except that it has sin bx instead of cos bx. You will apply integration by part again by letting $u = e^{ax}$ and $dv = \sin bx\, dx$. then $du = ae^{ax} dx$ and $v = - \underline{1} \cos bx\ b$

hence

(B) $\int \sin bx\, e^{ax} dx = - \frac{e^{ax} \cos bx}{b} - \int - \frac{a}{b} e^{ax} \cos bx\, dx$

$= \underline{-e^{ax} \cos bx} + \underline{a} \int e^{ax} \cos bx\, dx$

therefore: substituting equation (B) into equation (A) you get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \left(\frac{-e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx\, dx\right)$$

$$= \frac{e^{ax} \sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx\, dx$$

collecting like terms you get

$$\int e^{ax} \cos bx\, dx \left(1 + \frac{a^2}{b^2}\right) = \frac{e^{ax}}{b} \left(\sin bx + \frac{a}{b} \cos bx\right)$$

therefore $\int e^{ax} \cos bx\, dx = e^{ax} \left(\frac{\sin bx}{b} + \frac{a}{b} \cos bx\right) \left(\frac{b^2}{a^2+b^2}\right)$

$$= e^{ax} \frac{(\text{sinbx+acpsbx})}{a^2 + b^2} + C$$

Example: Find $\int e^{2x} \cos 3x \, dx$

Solution: Using the above reduction formula you have that a = 2 and b = 3 then

$$\int e^{2x} \cos 3x \, dx = e^{2x}\frac{(3\sin 3x+2\cos 3x)}{4 + 9} + C$$

$$= \frac{e^{2x}}{13} (3 \sin 3x + 2 \cos 3x) + C$$

Example: Find $\int e^{x/2} \cos 2/3 \, x \, dx$.

Solution: let a = ½ b = 2/3 then by the reduction formula

$$\int e^{ax} \cos bx \, dx = e^{ax} \frac{(b\sin bx+a\cos bx)}{a^2 + b^2} + C$$

you have that

$$\int e^{x/2} \cos 2/3 \, x \, dx = e^{x/2} \frac{(2/3x+½\cos 2/3x)}{¼ + 4/9}$$

$$= \frac{36}{25} e^{x/2} (2/3 \sin 2/3x + ½ \cos 2/3x) + C$$

3.2 Rational Expressions in SIN X AND COS X

There are certain class of trigonometric functions that techniques studied in the previous might not be able to be used to integrate them specifically, rational functions of sinx and cos x. An appropriate substitution of u = tan x/
2 might reduce the problem of integrating such class of rational functions of sinx and cox to a problem of integrating a rational function of u. This in turn can be integrated by the method of partial fraction studied in unit 7.

Example: If f(x) is a rational expression in sinx and cosx, then the substitution u = tan x/2 transforms the integral $\int f(x) \, dx$ into the integral of a rational function of u.

Solution: A typical way to explain the above is to start by expressing cosx and sinx in terms of u.

i.e. $\cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{\sec^2(\frac{x}{2})} -$

$$= \frac{2}{1 + \tan^2(\frac{x}{2})} - 1 = \frac{2}{1 + u} -$$

$$= \frac{1- u^2}{1 + u^2}$$

therefore: $\cos x = \dfrac{1-u^2}{1+u^2}$, $u = \tan \dfrac{x}{2}$

and $\sin x = 2 \sin \dfrac{x}{2} \cos \dfrac{x}{2} = 2 \dfrac{\sin x/2}{\cos(x/2)} . \cos^2 \dfrac{x}{2}$

$= 2 \tan (x/2) . \dfrac{1}{\sec^2 \frac{x}{2}}$

$= \dfrac{2\tan(x/2)}{1 + \tan^2 (x/2)}$

$= \dfrac{2u}{1+u^2}$

therefore $\sin x = \dfrac{2u}{1+u^2}$, $u = \tan x/2$

$x = 2 \text{ ac} \tan u$
$dx = \dfrac{2du}{1+u^2}$

Use the above tools to evaluate

(1) $\int \sec x \, dx$

Solution: $\int \sec x \, dx = \int \dfrac{dx}{\cos x}$

from above $\cos x = \dfrac{1-u^2}{1+u^2}$ $dx = \dfrac{2du}{1+u^{22}}$

therefore: $\int \dfrac{dx}{1+u^2} = \int \dfrac{2du}{1-u^2} . \dfrac{1+u^2}{1-u^2} = 2\int \dfrac{du}{\cos x}$

by partial fractions you arrive at $2\int \dfrac{du}{1-u^2} = \int \dfrac{Adu}{1-u} + \int \dfrac{Bdu}{1+u}$

$= \int \dfrac{du}{1-u} + \int \dfrac{du}{1+u}$

$= \ln /1-u/+\ln/1+u/+C$

$= \ln/\dfrac{1+u}{1-u}/ + C$

but $u = \tan x/2$ thus

$\int dx = \ln/ \dfrac{1+\tan x/2}{1 - \text{tax } x/2} / + C$

Example: $\int \dfrac{dx}{1 + 2\cos x}$

Solution: let $\cos x = \frac{1-u^2}{1+u^2}$

then $1 + 2\cos x = 1 + 2\frac{(1-u^2)}{1+u^2} = \frac{1+u^2+2-2u^2}{1+u^2}$

$$= \frac{3-u^2}{1+u^2}$$

$dx = \frac{2du}{1+u^2}$.

therefore: $\int \frac{dx}{1+2\cos x} = \int \frac{2du}{1+u^2} \cdot \frac{1+u^2}{3-u2} = \int \frac{2du}{3-u^2}$.

therefore: $\int \frac{2dv}{3-u^2} = 2\int \frac{du}{u^2-3} = \frac{1}{\sqrt{3}}\left(\int \frac{du}{u+\sqrt{3}} - \int \frac{du}{u-\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}(\ln|u+\sqrt{3}| - \ln|u-\sqrt{3}|) + C$$

$$\sqrt{3}\ln\left|\frac{\tan^{2/2}+\sqrt{3}}{\tan^{2/2}-\sqrt{3}}\right| + C$$

Example: Find $\int \frac{dx}{5\sec x - 3}$.

Solution: set $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{du}{1+u^2}$

where $u = \tan x/2$

$5\sec x - 3 = \frac{5}{x} - 3 = 5\frac{(1-u\,u^2)-3}{1+u^2}\cos$

$$= \frac{5(1-u^2)-3(1+u^2)}{1+u^2} = \frac{2-8u^2}{1+u^2}$$

therefore $\int \frac{dx}{5\sec - 3} = \int \frac{2du}{2-8u^2} = \int \frac{du}{1-4u^2}$

$$= \tfrac{1}{4}\left[\int \frac{du}{1-2u} + \int \frac{du}{1+2u}\right]$$

$$= \tfrac{1}{4}[-\ln|(2u-1) + \ln(2u+1)|$$

$$= \tfrac{1}{4}\ln\left|\frac{2u+1}{2u-1}\right| + C$$

$$= \tfrac{1}{4}\ln\left|\frac{2(\tan^{x/2})+1}{2(\tan^{x/2})-1}\right| + C$$

Example: Find $\int \frac{1}{5+4\cos x}\,dx$

Solution: set $\cos x = \frac{1-u^2}{1+u^2}$ $dx = \frac{2dx}{1+u^2}$

therefore $5 + 4\cos x = 5 + \frac{4(1-u^2)}{1+u^2}$

$$= \frac{5(1+u^2)+4(1-u^2)}{1+u^2}$$

$$= \frac{9+u^2}{1+u^2}$$

$$\int \frac{dx}{5+4\cos x} = \int \frac{2du}{1+u^2} \cdot \frac{1+u^2}{9+u^2}$$

$$= \int \frac{2du}{9+u^2} = 2\int \frac{du}{9+u^2}$$

$$= 2\int \frac{du}{3^2+u^2} = = \frac{2}{3} \arctan \frac{u}{3}$$

therefore $\int \frac{dx}{5+4\cos x} = \frac{2}{3} \arctan \frac{u}{3} = \frac{2}{3} \arctan \frac{1}{3}(\tan \frac{x}{2})$

Exercises: Evaluate the following integrals

(1) $\int_0^{\Pi/2} \frac{dx}{4+\cos x}$

(2) $\int \frac{\sin x}{3-\sin x} dx$

(3) $\int \frac{1+\sin x}{1+\cos x} dx$

(4) $\int \frac{1-\sin x}{1+\cos x} dx$

(5) $\int \frac{1+\sin x}{1+\cos x} dx$

Ans:

(1) $\frac{2}{15} \sqrt{15} \arctan (\frac{1}{5} \tan (\frac{x}{2}) \sqrt{15}) + C$

(2) $\frac{1}{3} \ln (3 + 3(\tan^2 \frac{x}{2}) - 2\tan \frac{x}{2}) + \frac{\sqrt{2}}{6} \arctan (\frac{\sqrt{2}}{8}(6\tan \frac{x}{2} - 2)$

(3) $\frac{2}{7} \sqrt{7} \arctan (\frac{7}{7} \tan \frac{x}{2})$

(4) $\tan \frac{x}{2} - \ln (1 + \tan^2 \frac{x}{2})$

(5) $\tan \frac{x}{2} + \ln (1 + \tan^2 \frac{x}{2})$

3.3 Further Substitutions

You are quite familiar with the following trigonometric identities;

(1) $\sin px \sin tx = ½ [\cos (p - t) x - \cos (p + t) x]$

(2) $\sin px \cos t x = ½ [\sin (p – t) x + \sin (p + t) x]$

(3) $\cos px \cos t x = ½ [\cos (p – t) x + \cos (p + n) x]$

using the above evaluate

(i) $\int \sin 3x \cos 7x \, dx = ½ \int \sin (3 – 7)x + \sin(3 + 7)x \, dx$
$= ½ \int(\sin 10x – \sin 4x) \, dx$
$= - \frac{\cos10x}{20} + \frac{\cos4x}{8} + C$

(ii) $\int \cos 2x \cos 3x \, dx = ½ \int\cos(3 – 2)x + \cos(2 + 3)x \, dx$
$= ½ \int(\cos x + \cos 5x) \, dx$
$= \frac{\sin x}{2} + \frac{\sin 5x}{5} + C$

(iv) $\int \sin 7x \sin x \, dx = ½ \int \cos (7 – 1) u – \cos (t + 1) x$
$= ½ \int (\cos 6x – \cos 8x)$

$½ [\frac{\sin6x}{6} - \frac{\sin8x}{8}] = \frac{\sin6x}{12} - \frac{\sin16x}{2} + C$

SELF-ASSESSMENT EXERCISES

Evaluate the following integrals:

(i) $\int_{II}^{II} \sin 2x \sin 5x \, dx$ (ii) $\int \cos 5x \cos 6x \, dx$

(iii) $\int_{o}^{II} \cos 5x \sin 6x \, dx$

3.4 Other Rationalising Substitutions

You will study how to integrate rational functions f(x) involving fractional powers of x. This you will carry out by using the substitution $x = u^n$.

Example: Find $\int \frac{dx}{1 + \sqrt{x}}$

Solution: set $u^2 = x$ $2u\,du = dx$ where $u = \sqrt{x}$ then
$\int \frac{dxmm}{1 + \sqrt{x}} = \int\frac{2udu}{1 + u} - 2\int \frac{udu}{1 + u}$

$= \quad 2\int\left(1 - \frac{1}{1+u}\right) du$

$= \quad 2(u - \ln 11 + u/ + C$

$= \quad 2u - 2\ln/1 + u/ + C$

$= \quad 2\sqrt{x} - 2\ln/1 + \sqrt{x}/ + C$

Example: Find $\int \frac{dx}{1 + \sqrt[4]{x}}$

Solution: Set $u^4 = x \;\; 4u^3\, du = dx$

Then $\int \frac{dx}{1 + \sqrt[4]{x}} = \int \frac{4u^3\, du}{1 + u} = 4\int \frac{u^3\, du}{1 + u}$

$= \frac{4}{3}\; u^3 - 2u^2 + 4u - 4\ln/1 + u/ + C$

$= \; \underline{4}\; x^{3/4} - 2x^{1/2} + 4x^{1/4} - 4/1 + x^{1/4}/ + C$

Example: $\int x^3 \sqrt{x^2 + 4}$

Solution: set $u^2 = x^2 + 4$

$x^2 = u^2 - 4$

$2x\, dx = 2u\, du$

therefore $\int x^3 \sqrt{x^2 + 4}\, dx = \int x^2 \sqrt{x^2 + 4}\, .x\, dx$

$= \int (u^2 - 4)\, u.\, u\, dv = \int (u^2 - 4)\, u^2\, du$

$= \int (u^4 - 4\, u^2)\, du$

$= \int \frac{u^5}{5} - \frac{4\, u^3}{3} + C$

$= \frac{(x^2 + 4)^{5/2}}{5} - 4\,\frac{(x^2 + 4)^{3/2}}{3} + C$

$= \; \underline{1}\;\; (x^2 + 4)^{3/2}\, (3x^2 - 8) + C$

4.0 CONCLUSION

In this unit you have studied how to obtain reduction formula of certain product functions. Also you have studied how to use appropriate substitution to integrate rational function involving expressions such as sinx, cosx, tanx etc. You have used trigonometric ideitities to integrate product function involving expression like sinbx cosux, sinbx, sincx and cosbx cosax. You have also studied how to integrate rational functions involving fractional powers of the variable x. This unit deals mainly with integration of functions emanating from problems of alternating correct theory, heat transfer, bending of beams, cable stress analysis in suspension bridges, and many other places where trigonometric series is involved.

5.0 SUMMARY

In this unit you have studied how to:

(1) obtain reduction formula of special integrals such as

(i) $\int \cos^n x\, dx$ (ii) $\int \sin^n x\, dx$

(iii) $\int e^{ax} \cos bx\, dx$ (iv) $\int e^x x^n\, dx$

(2) integrate rational functions involving expressions such as sinx, cosx, secx tanx and cosecx.

(3) integrate product functions of trigonometric functions such as $\int$ cosbx cosax dx, $\int$ cosbx sinax dx and $\int$sinax sinbx dx

(4) evaluate integrals of rational functions involving fractional powers of the variable x.

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7.0 TUTOR-MARKED ASSIGNMENT

1. $\int_0^1 x^4 e^x \, dx$
2. $\int \sin x \, dx$
3. $\int \cos^5 x \, dx$
4. $\int e^3 \cos 2x \, dx$
5. $\int e^{x/2} \cos x \, dx$

obtain a reduction formula for

6. $\int \cos^n x \, dx$
7. $\int e^{ax} \sin bx \, dx$
8. $\int \frac{dx}{1 + \cos x}$
9. $\int \frac{\sqrt{x} \, dx}{1 + x^{1/4}}$
10. $\int \frac{\sec x}{2 \tan n + \sec 2\text{-}1} dx$
11. $\int \frac{\sqrt{x} \, dx \quad 5}{4(1 + x^{3/4})}$
12. $\int \cos 3x \sin 2x \, du$
13. $\int \cos 5 x \cos 3x \, dx.$
14. $\int \cos 4x \cos 7x \, dx$
15. $\int \sin 7n \sin 2x \, d$

UNIT 4 APPLICATION OF DEFINITE INTEGRATION I

CONTENTS

1.0 INTRODUCTION

In unit 1, you studied the connection between sums of thin rectangles of area f(x). Δxr and the integration of f(x). You discovered that when f(x) is continuous on $a \leq x \leq b$ then the limit lim $\sum f(x)\ \Delta x = F(b)$ as $\Delta x \to 0$. In unit 1 you applied the above when finding the area under the curve of f(x) within the interval [a,b]. In this unit you shall extend the concept of area under a curve to the following, area between two curves. Distance traveled can be calculated by integrating the velocity $v_1 = f(t)$ of the body. Here $v = f(t) \geq 0$ and continuous function of t within a specified interval of time t. In the next unit calculation of volumes of a solid of revolution will be treated as well as computing the work done by a body.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- evaluate area between two curves
- calculate the distance traveled by a body moving with a velocity v = f(t)
- calculate volumes of any solid of revolution

3.0 MAIN CONTENT

3.1 Area between two Curves

Suppose you consider two continuum functions f(x) and f(x) in a closed interval [a, b]. Suppose that $f_1(x) \geq f_2(x)$ for all X€ [a, b]. Then the curve of $f_1(x)$ is always above the curve of $f_2(x)$. (see fig. 9,1).

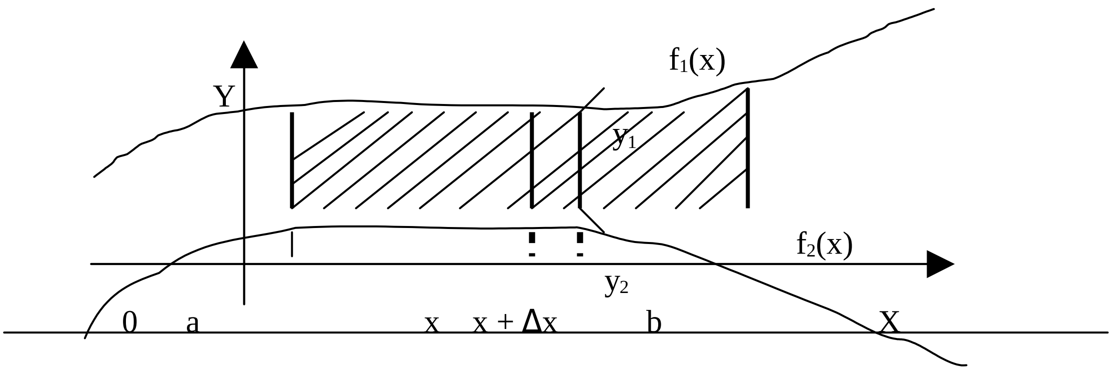

Fig. 9.1

To calculate the area between the two curves you will consider the area under curve f(x) and the vertical lines x = a and x = b then the area above curve $f_2(x)$ and the vertical lines x = a and x = b. To understand this cut out a rectangular STRIP OF WIDTH ΔX. You will not find it difficult to know that the length of the rectangular strip is $y_1 - y_2$. Therefore, the area of rectangular strip is $(y_1 - y_2)\,.\Delta = [f_1(x) - f_2(x)].\Delta x$. Using the concept studied in unit 1, the area under the two curves will be given by the sum of the areas of the rectangular strip i.e.

$$A \approx \sum_a^b [f_1(x) - f_2(x)]\,\Delta x = \int (f_1(x) - f_2(x)\,dx$$

If you allow $\Delta x \to 0$ then you can obtain the exact area as

$$A = \lim_{\Delta x \to 0} \sum_a^b [f_1(x) - f_2(x)]\,\Delta x = \int (f_1(x) - f_2(x)\,)\,dx$$

Example: Find the area bounded by the parabole y = -2x.

Solution: The first step is to know which curve is the upper boundary and which is the lower boundary. This can be achieved by plotting the curves on the same rectangular axes. See fig. 9.2).

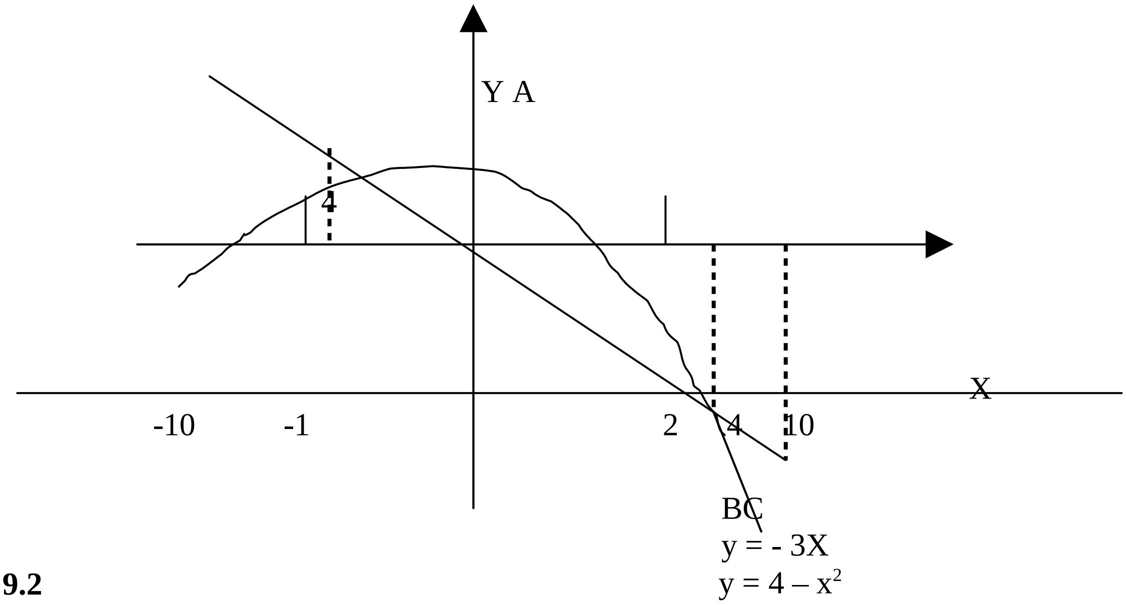

Fig 9.2

Second step is to know where the curves intersect. This you can do by finding points that satisfy both equations simultaneously. That is you solve

$4 - x^2 = -3x$

$x^2 - 3x - 4 = 0$

$(x - 4 Xx + 1) = 0$

$x = 4$ or -1

Thus the curves intersect at A(-1, 3) and B(4, - 12).

For values of $X \in [-1, 4]$ the curve $y = 4 - x^2$ is above the line $y = -3x$ by an amount

$(4 - x^2) - (-3x) = 4 + 3x - x^2$

therefore the area between the two curves is given as

$$\int_{-1}^{4} (4 + 3x - x^2)\, dx = 4x + \frac{3x^2}{2} - \frac{x^3}{3} \Big/_{-1}^{4}$$

$$= \frac{125}{6} \text{ sq units}$$

Example: Find the area bounded the parabola $y = x^2 - 2$ and the strainght line
$y = 2x + 1$.

Step1.Plotting the curves on the same graph gives you what you find in Fig. 3

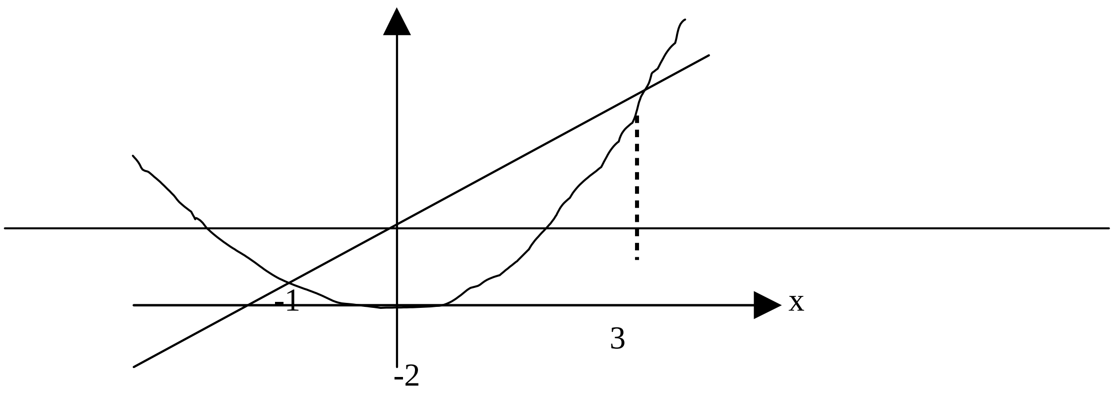

Fig 9.3

Step 2: Point of intersection is given as $x^2 - 2 = 2x + 1 \Rightarrow x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$
$x = 3$ or -1

Step 3: The line $y = 2x + 1$ is above the parabola $y = x^2 - 2$ by an amount
$2x + 1 - (x^2 - 2) = 2x + 3 - x^2$

Therefore the area is given as $\int_{-1}^{3}(2x + 3 - x^2)\,dx = \frac{32}{3}$ sq units

You will now extend the above to finding areas between curves that are crossed. Consider two f(x) and g(x) shown in Fig 9.4

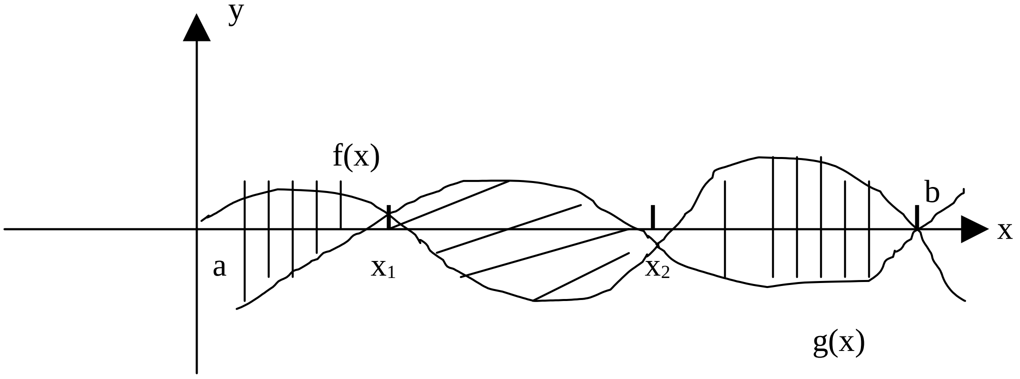

Fig 9.4

In fig 9.4 neither f(x) or g(x) remains positive, i.e. $f(x) > g(x)$ $X \in [a_1\ x_1]$ and $X \in [x_2, b]$ while $g(x) > f(x)$ for $X \in [x_1, x_2]$. Then the area is given as $\int_{a}^{x_1}(f(x) -$

$g(x))\,dx + \int_{x_1}^{x_2}(g(x) - f(x))\,dx + \int_{x_2}^{b}(f(x) - g(x)\,dx = \int_{a}^{x}/f(x) - g(x)/dx + \int_{x_1}^{x_2}/f(x) - g(x)/dx$

Under each integral sign, the upper curve is written first. If you compute just

$$\int^{b} (f(x) - g(x)\,dx$$

the areas will be counted with opposite signs and may cancel out to give you a zero value.

Example: Find the area between y = sinx and y = cosx x [, 2]

Solution: The region covered by the area is given in Fig 9.5. The area is given by the integral

$\int_{0}^{2II} /\sin x - \cos x/\ dx$

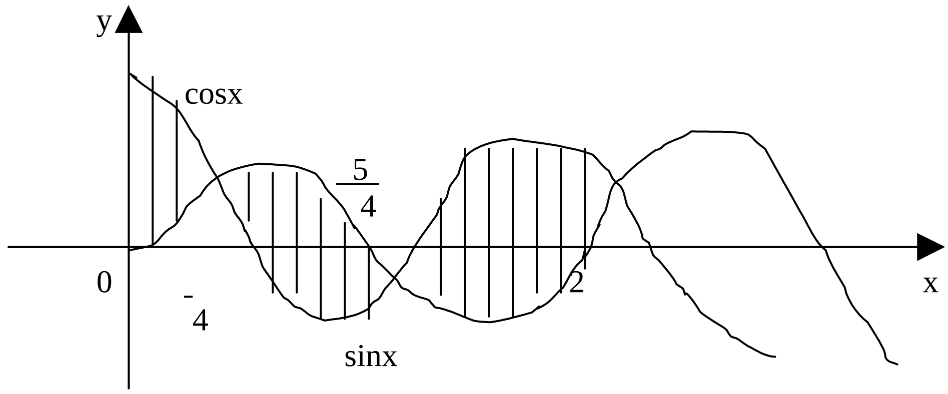

Fig 9.5

We solve for x simultaneously i.e. sinx = cosx ⇨ sinx – cosx = 0.

here from the g raph above,

sinx – cosx ≤ 0 x [0, ___]
4 sinx
– cosx ≤0 x [, 5] ___
4 4
sinx – cosx ≤ 0 x [5 , 2]
4

thus

area $\int_{0}^{II/4} /\sin x - \cos x\ /dx + \int_{II/4}^{5II/4} /\sin x - \cos x/dx$

$+ \int_{5II/4}^{2II} /\sin x - \cos x/dx$

$= \int_{0}^{II/4} (\cos x - \sin x)\ dx + \int_{II/4}^{5II/4} (\sin x - \cos x)\ dx$

$+ \int_{5II/4}^{2II} (\cos x - \sin x)\ dx$

$= \sin x + \cos x/_{0}^{II/4} + -\cos x - \sin x/_{II/4}^{5II/4}$

$+ \sin x + \cos x\ /_{5II/4}^{2II}$

$= \sqrt{2} - 1 + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2}$ sq units

You could also compute areas in which the boundary curves are not functions of x but functions of y (see Fig 9.6). In such cases the area is given by the integral.

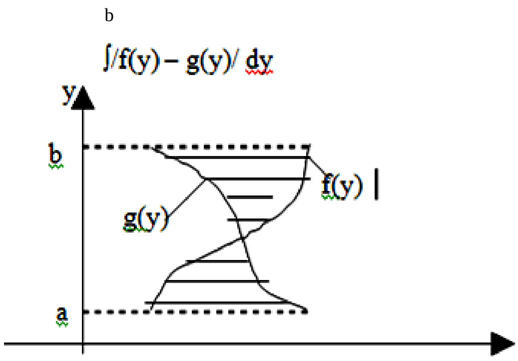

Fig. 9.6

Example: Find the area between the parabola $x = -3y^2 + 4$ and $x = y^2$.

Solution: The region covered by area is displayed in Fig. 9.7. The two parabolas intersect at $y = -1$ and $y = \underline{1} \Rightarrow -3y^2 + 4 = y^2 \qquad -4y^2 = -4$

$y = \pm 1$

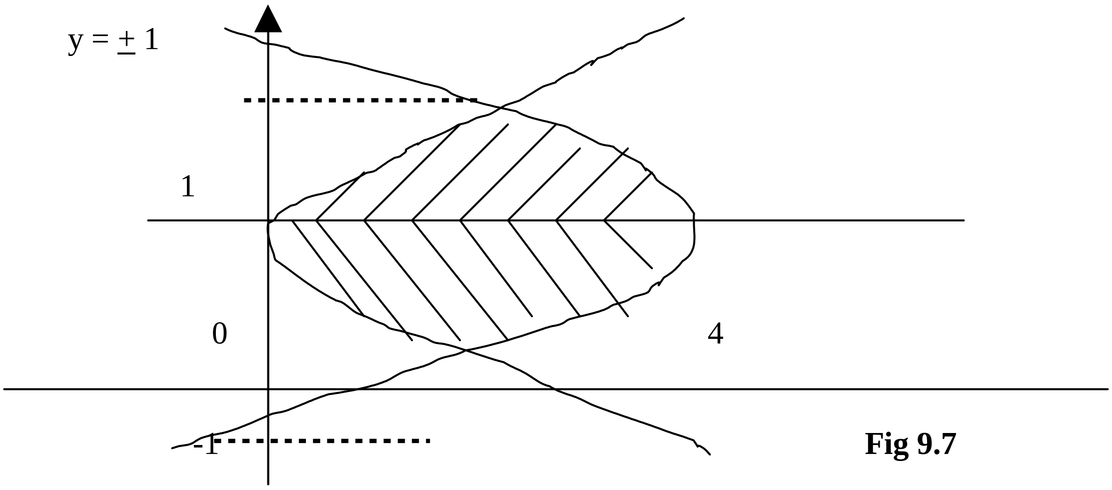

Fig 9.7

The area is given by the definite integral $A = \int_{-1} (-3y^2 + 4 - y^2)\,dy$

$$= \int_{-1}^{1} (4 - 4y^2)\,dy = \frac{16}{3}$$

Example: Find the area between

$x = \frac{1}{4} y^3 \quad y \in [-2, 1]$ and
$x = \frac{1}{2} y^3 + \frac{1}{4} y^2 - \frac{1}{2} y \quad y \in [-2, 1]$

Solution: The region bounded by the two curves is disploayed in Fig. 9.8.

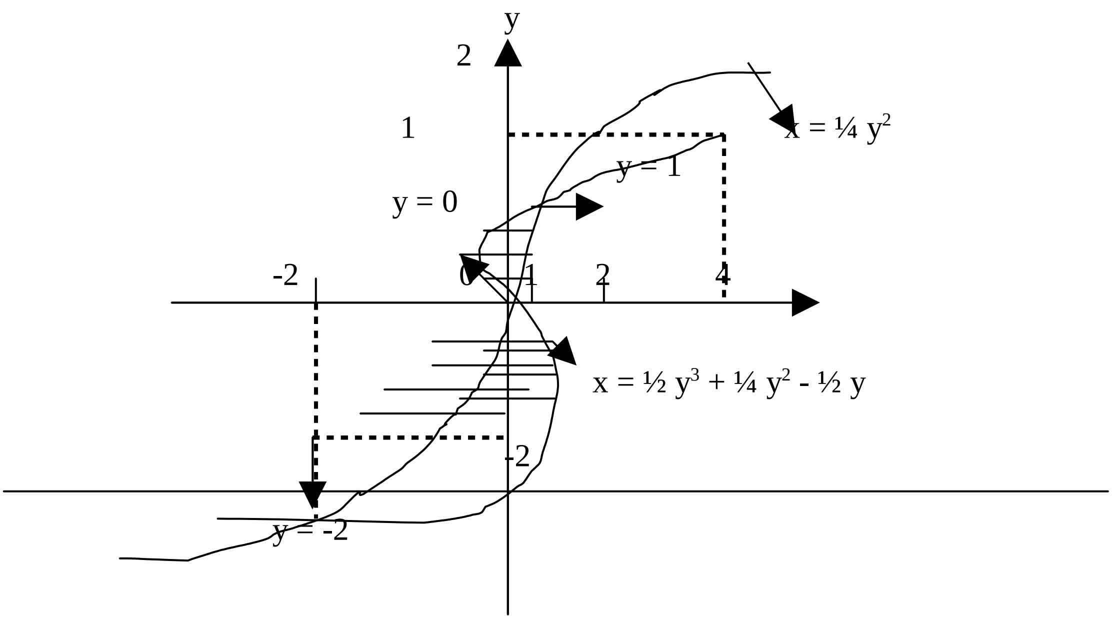

Fig 9.8

The two curves meet at where $\frac{1}{4} y^3 = \frac{1}{2} y^3 + \frac{1}{4} y^2 - \frac{1}{2} y$ x 4

$y^3 = 2y^3 + y^2 - 2y$

$\Rightarrow y^3 + y^2 - 2y = 0$

$y(y^2 + y - 2) = 0$

$y(y - 1)(y + 2) = 0$

$y = 0, y = 1, y = -2$

therefore area is given as

$$\int_{2}^{1} / \tfrac{1}{2} y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} y - (1/4\, y^3)/\, dx$$

$$= \int_{-2}^{0} / \tfrac{1}{2} y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} y - (1/4\, y^3)/\, dx$$

$$+ \int_{0}^{1} / \tfrac{1}{2} y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} - (\tfrac{1}{4} y^3)/\, dx$$

$$= \int_{-2}^{0} (\tfrac{1}{4} y^3 - \tfrac{1}{4} y\, y^2 + \tfrac{1}{2} y)\, dx$$

$$+ \int_{1}^{1} (-\tfrac{1}{4} y^3 - \tfrac{1}{4} y^2 + \tfrac{1}{2} y)\, dx$$

$$= \frac{2}{3} + \frac{5}{48} = \frac{37}{48}$$

Exercises: Sketch the region that is bounded by the following curves and find the area.

1. $y = x^2$, $y = 4x - 3$
2. $\frac{1}{2} y^2 = x$, $x = 4 + y$
3. $x + y^2 - 4 = 0$, $x - 2y = 0$

Ans: (1) $\frac{4}{3}$, (2) 36 (3) $\frac{49}{6}$

3.2 Distance

Let the distance traveled by a body moving with velocity v = f(t) be denoted by the letter S. If f(t) ≥ 0 and continuous in a closed interval t€[a, b]. Then the distance traveled is given as:

$$\int_a^b ds = \int_a^b f(t)\, dt$$

$$S = \int_a^b f(t)\, dt$$

So if you integrate the velocity function you can get the distance traveled by a body.

Example: A boy enters a car at time t = 0. After tsecs, the velocity of the car is $10t^3$m/s. How far does the car move during the first 1 sec?

Solution: Think of the time t = 0 and t = 1 sec. The integral

$$\int_0^1 10t^3\, dt$$

$$= \int_0^1 10t^3\, dt = \frac{10t^{43}}{4}\Big/_0^1 = \frac{5}{2}$$

Example: Find the total distance traveled by a moving body as a function of time. If f(t) = 2t + 1 0 ≤ 2.

Solution: $\int_0^2 f(t)\, dt = \int_0^2 92t + 1)\, dt$

$$= t^2 + t \Big/_0^2$$

$$= 6m$$

Example: A ball is thrown up from the ground at t = 0. At time t its velocity is 30 – 20t m/s. After 3secs, the ball hhits the ground. How far has it traveled?

Solution: In this example, you will encounter a negative velocity which will result to negative distance. This is simply because of the fact that at the maximum height attained the ball falls back towards the ground. The movement backwards is measured as negative velocity and negative distance (see Fig 9.10).

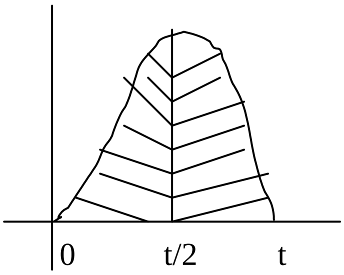

Fig. 10

$v = f(t) \geq 0 \quad t€[0, t/2]$

$v = f(t) \leq 0 \quad t€[\, t/2, t]$

Therefore $S = \int_0^t /f(t)/\, dt = \int_0^{t/5} /f(t)/\, dt + \int_{t/2}^{t} /f(t)/\, dt$

Apply the above to the problem you set $t = 3$, $v = 30 - 20t$

then $\int_0^7 /30 - 20t/\, dt = \int_0^{3/2} (30 - 20)\, dt + \int_{3/2}^{3} (20t - 30)\, dt$

$= 30t - 10t^2 /_0^{3/2} + \; 10t^2 - 30t/_{3/2}^{3}$

$= \frac{45}{2} + \frac{45}{2} = 45m$

In the above example, $\int_0^3 (30 - 20t)dt = 0$. This is because the ball moves

45/2m with positive velocity (upwards) and 45/2m with negative velocity (downwards). The integral, however, counts these distances with opposite signs so they cancel out.

Exercises: A body has velocity $v = f(t)$. Find the distance covered between $t = a$ and $t = b$.

1. $f(t) = (3t - 1)$m/s $a = 0, b = 3$
2. $f(t) = (3t - t^2)$m/s $a = 0, b = 2$
3. $f(t) = (4t^2 + 3t + 1)$m/s $a = 0$ $b = 3$

Ans: (1) $\frac{21}{2m}$ (2) $\frac{9}{2m}$ (3) $\frac{105}{2m}$

4.0 CONCLUSION

You have studied how to use integration to find the areas between two curves. You observed that when the curves crossed one another, the sum of the areas cancel out. As such caution is applied when finding areas bounded by two crossing curves. That is the curve that is above within a given region is ued first in the integral. You have studied how distance traveled by a body with a constant. Velocity can be calculated by integrating the velocity function within a given interval of time. In the next unit, you will apply the same method to finding the work done when an object is moved along a straight line. Also the volumes of a solid of revolution.

5.0 SUMMARY

In this unit you have studied how to:

- find the area bounded by two curves
- find the area bounded by two crossing curves
- find the distance traveled by an object with a velocity v = f(t)

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7.0 TUTOR-MARKED ASSIGNMENT

Sketch the area of the region bounded by the following curves and find the area

1. $y = x^2 - 2$ and $y = 2x + 1$
2. $y = 4 - x^2$ and $y = ½ x + 1$
3. y = cosx and y = sinx XЄ [0, 3∏ and t = b
4. $f(t) = 2t - 1$ and a = 0, b = 3
5. $f(t) = t^2 + 1$ and a = 0, b = 2
6. $f(t) = t^2 + \frac{2}{3t + 1}$, a = 0 b = 4
7. $f(t) = 2t - t^2$ a = 0, b = 28 $f(t) = 1 - t^3$ a = 0, b = 1
8. Find the area between

   $x = \frac{1}{8} y^3 + \frac{1}{16} y^2 - \frac{1}{8} y$ yЄ [-1, 1]

   and $x = \frac{1}{16}$ y3 yЄ [-2, 1]

9. A particle is put inside an accelerator at time t = 0. After t sec its velocity is $10^5 t^2$m/s. How far does the particle move during the first 10^{-2} sec?

UNIT 5 APPLICATION OF INTEGRATION II

CONTENTS

1.0 INTRODUCTION

In continuation the application of definite integration to specific problems or situation, will in this unit consider further application namely: (1) computing the work done by a force applied along a line. And (ii) volume of a solid. The method that will be adopted in finding the volume of a solid by integration is to slice the solid into numerous thin pieces, each of which is approximately a familiar shape of a known volume. The above could be executed in four steps (1) choose a method of slicing the solid (2) choose a variable x which locates the typical slice and find the range of values of x that applies to the problem. (3) compute the volume f(x) dx of a typical slice and finally (4) find anti-derivative (integration) of f(x) and compute $\int_a^b f(x)dx$ where x€[a, b].

The four steps enumerated above will be useful in finding volumes of any solid by integration.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- compute the work done by applying a force on an object alng a line.
- compute the volume of a solid.

3.0 MAIN CONTENT

3.1 Work

When a constant force F(N) is applied along a distances in the work done, (Nm) is the product of force and distance. i.e. Work = W = F.s suppose an object is moved along a straight line fro x = a to x = b by a force of magnitude f(x). Dividing the interval [a, b] into sub-intervals of length Δx then the work done moving the object from $xe i_{-1}$ to xi is approximately f(xl).Δxi since Δx0 this force is constant. The total work done will then be the sum of work done in each subinterval. This is given as:

$$\text{Total Work} = \int_a^b f(x)\,dx$$

Example: Suppose at each point of the x – axis there is a force of $(3x^2 - x + 1)$N pulling an object. Find the work done in moving it from x = 1 to x = 3.

Solution: Work = $\int_1^3 (3x^2 - x + 1)\, dx = x^3 - U\, x\, U + x /^3$

$= 24$

An interesting example of this can be derived from Newton's Law of Gravitation.

Example: Given that two bodies pull at each other with a force $F = k \frac{Mm}{x^2}$ where M and m afd their masses respectively and x is distance between

Them. If one of the bodies is fixed at origin, how much work is done in moving the other body from x = 1 to x = 3? (Assume k, M, and m are known)?

Solution:

$$W = \int f(x)\, dx = \int_1^3 \frac{kMn}{x^2}\, dx$$

$$= kMm\int_1^3 \frac{1}{x^2}\, dx = -kMm \left.\frac{1}{x}\right/_1^3 = \frac{2}{3}\, kMn.$$

For most springs, there is a law governing their functions. According to the law known as Hooke's law when a spring is stretched a short distance there is a compressing or restoring force proportional to the amount of stretching. This force is given as F = cx where x is the amont the spring has been displaced from its natural or unstressed length and c is a spring constant. Beyond this range, the spring will become unreliable and unpredictable.

Example: A spring has a natural length of L = 0.20m. A force of 1N streches the spring to a length of 0.21m. Find the spring constant. Find the amount of work required to stretch the spring from its natural length to a length of 0.22m. Calculate the amount of work done in stretching the spring from 0.21m to 0.22m.

Solution: F = 1, extra length = .20 – 21 = 0.1m
then F = cx ⇨ 1 = c(0.1)
c = 1/0.1 = 10.

To calculate the work done in stretching the spring 0.02m beyond it natural length, you have

$$W = \int_0^{0.02} 10x\, dx = x^2 /_0^{.02} = 2 + 10^{-3}N$$

To find the work done in stretching the spring from a length of 0.21m to a length of 0.22m you compute

$$W = \int_{0.01}^{0.02} 10x\, dx = 1.5 \times 10^{-3}N.$$

Example: Find the work done by a force f(x) = 2x + 3 N in moving an object from x = 1m to x = 5m.

Solution: $W = \int_1^5 f(x)\,dx = \int_1^5 (2x + 3)dx = 36$

Exercises:

1. The force in Newton's required to stretch a certain spring xm is given as F = 5x. How much work is required to stretch the spring (1) 0.1m (ii) 0.15m and (iii) 0.025m?

2. Find the work done by a force f9x) = 7x + 5 Newton in pushing an object from x = 1m to x = 2m.

3.2 Volumes

In this section you will study how to find the volumes of solids by integration. Before doing this, you will compute the area of circle and then extend the same methd to that of computing the volumes of solids.

Examples: Find the area of a circle of radius r. Where the circumference is given as C = 2Лr.

Solution: Step 1. Cut the circle into this concentric rings (Fig 10.0)

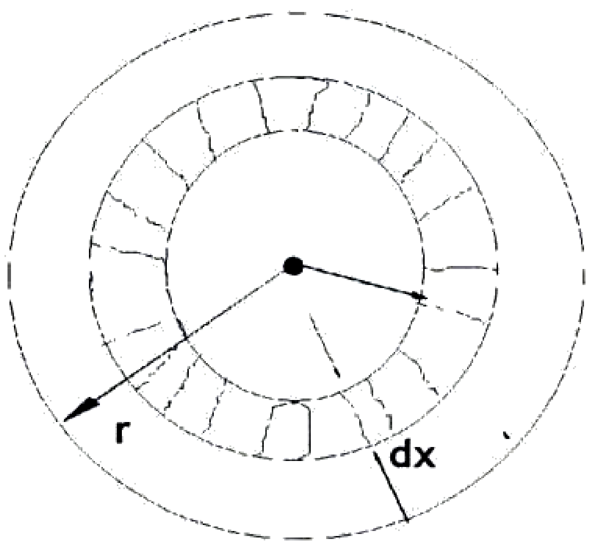


Fig. 10.0

Step 2: Let x denote the distance of a ring from the centre as shown in Fig 10.1 here $0 \leq x \leq r$.

Step 3: The length of any ring is given as 2Лx and the width is dx. Thus the area of a typical ring is given as 2лx dx. The area of the circle is therefore given as the sum of areas of the concentric rings.

$$\text{i.e. } A = \int_0^r 2\text{л}x\, dx.$$

$$= \text{Л}x^2/^r = \text{Л}r^2 \quad \text{which is the area of a circle with radius r.}$$

Volume of Revolution: The volumes of many solids can be found by the method of slicing describd above. Before you continue it is necessary to define what is meant by solid of revolution A solid which has a central axis of symmetric is called a solid of revolution. Most solid that will be consider in this unit are solid of revolution. For example, a cone, a cylinder, a bucket etc. To find the volume of such solid displayed in fig. 10.1 you irst consider the area under the region AB of the curve y = f(x) revolved about the x - axis

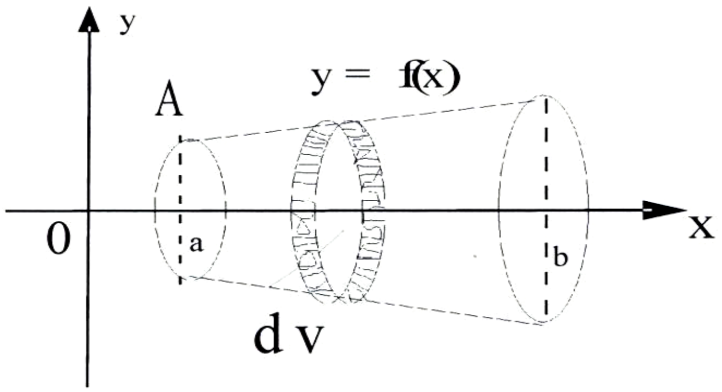


Fig 10.1

through 360^0. Each point on the curve represents a circle with centre on the x-axis. A solid of revolution as displayed in Fig 10.1 hs two circular plane ends cutting the x-axis at x = a at x = b. As ws done in the example of the area of a circle. Let v be the volume of the solid of revolution from x = a up to any value XЄ (a, b) see Fig 10.2

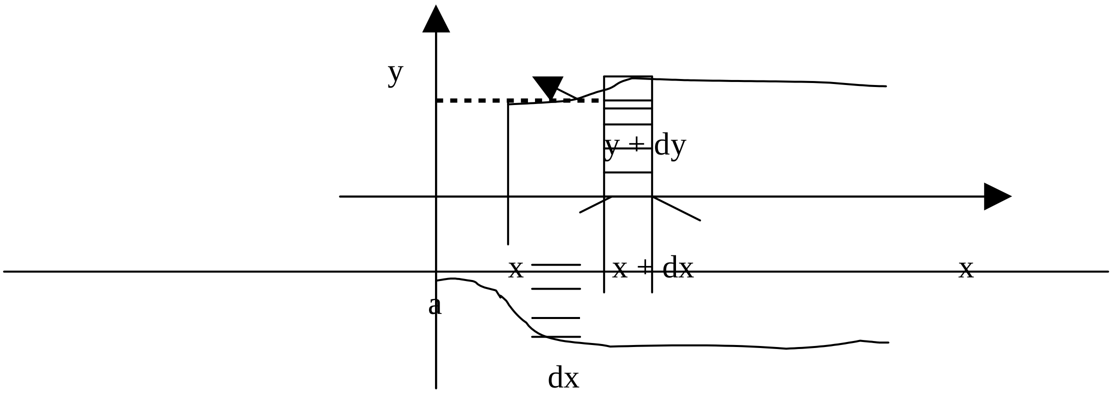


Fig 10.2 Showing a cross section of solid of revolution

Suppose thee is an increment in x i.e. dx with a corresponding increment in y, dy and increment in V, dv. In fig 10.2 he volume of the slice with thickness dx is given by dv and is enclosed between two cylinders of outer radius y = dy and inner radius y.

Thus $Лy^2 sx = sv \leq Л(y + sy)^2 dx$

$\Rightarrow Лy^2 \leq \frac{sv}{sx} \leq Л(y + sy)^2$ — (A)

hence as $sx \to 0$, $sx \to 0$ and $\frac{sv}{sx} \to \frac{dv}{dx}$

thus (A) becomes

$y^2 \leq \frac{dv}{dx} \leq Лy^2$

$\Rightarrow \frac{dv}{} = Лy^2$ integrating both sides you have

$V = \int Лy^2 dx$

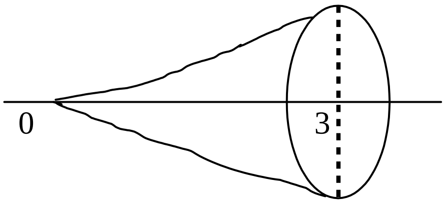
where y = f(x) a continuous function and V is the volume of solid generated when the curve y = f(x) is rotated through 360^0 around the x-axix between the limits x = a and x = b.

Example: The region in the x-y plane bounded by the curve $y = x^2$, the line x = 0 and x = 3 and the x = axis is revolved about x – axis. What is the resulting volume?

Solution: See Fig (10.3)

$V = \int_0^3 Лy^2 dx$

$y = x^2$



$$V = \int_0^3 \text{Л}\, x^4 dx$$

$$= \frac{\text{Л}x^5}{5}\Big/_0^3 = \frac{243\text{Л}}{5}$$

Fig 10.3

Example: Let the region be revolved around they –axis from x = 0 to x = 2. What is its volume?

Solution: See Fig 10.4

$$V = \int^2 \text{Л}x^2\, dy$$

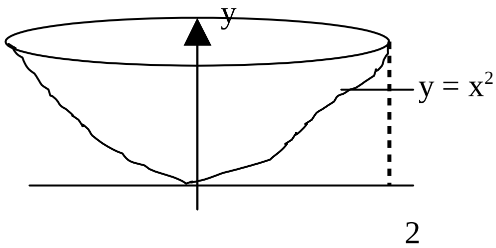


Fig. 10.4

Since $y = x^2$ thus implies that for x = 0, y = 0 and x = 2, y = 4 then

$$V = \int_0^4 \text{Л}y\, dy = \text{Л}\, \frac{y^2}{2}\Big/_0^4 = 8\, \text{Л}$$

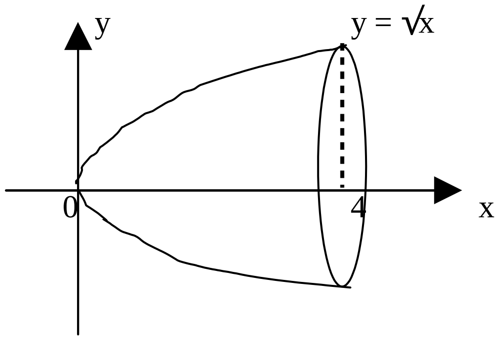
Example: The portion of the curve $y = \sqrt{x}$ between x = 0 and x = 4 is rotated completely round the x-axis. Find the volume of the solid generated.

Solution: See Fig. 10.5

$$V = \int_0^4 \text{Л}y^2\, dx$$

$$= \int_0^4 \text{Л}x\, dx$$

$$= \text{Л}\, \frac{x^2}{2}\Big/_0^4 = 8\text{л}$$



Example: Find the volume of a cone with base radius r and height h.

Solution: The axis of the cone is a line that passes through the vertex of the cone and the centre of the base.

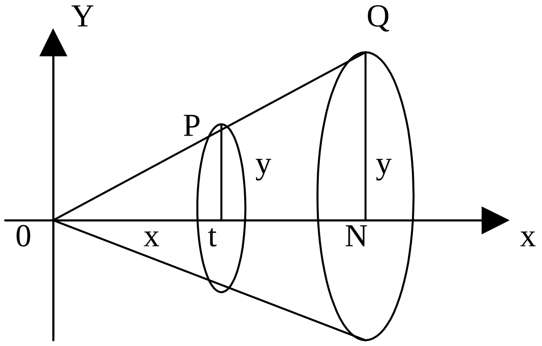


Fig. 10.5

$$V = \int_0^h Л\, y^2\, dx$$

To get y^2 you consider Fig. 10.5 where a cross section gives you a picture of two similar triangles $\Delta 0QN$ and Δ OPT. Thus

$$\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{rx}{h} \Rightarrow y^2 = \frac{r^2\, x^2}{h^2}\,h$$

therefore $V = Л \int_0^h \frac{r^2}{h^2}\frac{x^2}{}\, dx = \frac{Лr^2}{h^2}\left[\frac{x^3}{}\right]_0^h = \frac{1}{3}Лr^2h$

which is te volume a circular cone.

Example: Find the volume of a sphere by rotating the circle x^2 t$y^2 = r^2$ about the x-axis.

Solution: Using the formula

$$V = \int_0^b лy^2\, dx\ .\ Y$$

since the centre of the circle is at origin $y = \sqrt{r^2 - x^2}$ $-r \leq x$ r

then a = r , b = r

$$V = \int_{-r}^{r} Л(r^2 - x^2)\, dx = Л\left[r^2x^2 - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}Лr^3$$

SELF-ASSESSMENT EXERCISES

Sketch the graph and find the volume generated by revolving the region below it about he x-axis.

1. $y = 3x^2$ XЄ (0, 1)
2. $y = e^{-x}$ XЄ (0, 2)
3. $y = 4x^3$ XЄ (0, 1)
4. $y = 1/x$ XЄ (1, 2)
5. Sketch the graphs and find the volume generated by revolving the region between them about the x-axis $y = 4-x^2$ $y = ½ x + 1$

Ans:

1. $\frac{9}{5Л}$ (2) $\frac{Л(1 – e^{-4})}{2}$ (3) $\frac{16Л}{7}$

4. $\frac{Л}{2}$ (5) $\frac{2401}{80}$

3.3 Average value of a Function

You are quite familiar with how to find average value of a finite number of data. For example, if y_1, y_2,…, y_n are scores obtained in a class test by n number of students, then the class average score will be given as

$$y_{av} = \underline{y_1, + y_2 + y_n} – 1 \; n$$

When the number is infinite then it will not be possible to use the above formula (1). The above is also possible for a discrete case. If data is continuously used in a given interval, formula 1 will be difficult or meaningless. In such situation another method is needed to be able to calculate the average value of the data y.

Let y = f(x) a ≤ XЄ b then $Yav = \frac{1}{b-a} \int_a^b f(x)\, dx$

Example: Find the average value of [0_1 a] if $f(x) = x^3$

Solution:

$$Yav = \frac{1}{a} \int_a^b f(x)\, dx, \; a = 0, \; b = a \; b-$$

$$= \frac{1}{a} \int_0^a x^3\, dx = \frac{1}{a} \left[\frac{x^4}{4} \right]_0^a$$

$$= \frac{1}{a} \; \frac{a^4}{4} – = \frac{a^3}{4}$$

Example: Find the average value of $f(x) = \sqrt{r^2 - x^2}$ XЄ [-r, r]

Solution: a = r, b = r, $f(x) = \sqrt{r^2 - x^2}$ r
Therefore: Average = $\frac{1}{} \int (r^2 - x^2)^{½} dx$

$$= ½ r [½ x \sqrt{r^2 - x^2} + ½ r^2 \arcsin \frac{x}{r}]^{r}$$

$$= ½ r [½ r^2 Л] = \frac{rЛ}{4}$$

4.0 CONCLUSION

In this unit you have studied how to find the work done when a force is applied on an object along a straight line. You have studied how to compute the work done when a spring is compressed or stretched. You have studied how to compute volumes of a solid generated by revolving a region along the axis of symmetry of the solid. You have also studied how to find the average value of a set of continuous data in a gien interval.

5.0 SUMMARY

In this unit you have studied how to:

- compute the work done when a spring is compressed or stretched

  i.e. $W = \int_a^b f(x)\, dx$

- compute the volumes of solid of revolution

  i.e. $V = \int_{x=a}^{x=b} Л y^2 dx$ or $V = \int_{y=a}^{y=b} Лx^2 dy$

- compute the average value of a function f(2) i.e.

  Average $f(2) = \frac{1}{-a} \int_a^b f(x) dx$ $a \le x \le b$

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7.0 TUTOR-MARKED ASSIGNMENT

1. A certain spring exerts a force of 0.5N when stretched .33m beyond its natural length. What is the work done in stretching the spring 0.1m beyond its natural length? What is the work done in stretching it an additional 0.1m?

2. A hemispherical oil tank of radius 10m is being pumped out. Find the work done in lowering the oil level from 2m below the top of the tank to 4m below the top of the tank. Given that the pump is placed right on top of the tank. Take the weight of water wkg.

3. The base of a solid is the region between the curves $\sqrt{x} + \sqrt{y} = 1$ and $y = 1 - x$. Sketch the graphs and find the volume of the solid generated by revolving the region about the x-axis.

4. Find the volume generated when the plane figure bounded by $y = 5 \sin 2x$, the x-axis and the ordinates $x = 0$ and $x = \pi/4$ rotates about the x-axis through a complete revolution.

5. Suppose a supermarket receives a consignment of 1400 satchets of pure water every 30 days. The pure water is sold to retailers at a steady rate; and x days after the consignment arrives, the inventory I(x) of satchets still on hand is $I(x) = 1400 - 14x$. Find the average daily inventory.