MODULE 1

UNIT 1 INTRODUCTION TO THE NATURE OF DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

The subject of differential equation constitutes a part of mathematics that plays an important role in understanding physical sciences. In fact, it is the source of most of the ideas and theories which constitute higher analysis. In physics, engineering, chemistry and many other disciplines it has become necessary to build a mathematical model to represent certain problems. These mathematical models often involve the search for an unknown function that satisfies an equation in which derivatives of the unknown function play an important role. Such equations are called **differential equations.** The primary purpose of differential equations is to serve as a tool for studying change in the physical world.

You may recall that if $y = f(x)$ is a given function then its derivation $\frac{dy}{dx}$ can be interpreted as the rate of change of y respect to x. Sir Isaac Newton observed that certain important laws of natural sciences can be phrased in terms of equations involving rates of change. The most famous example of such a natural law is Newton's second law of motion. Newton was able to model the motion of a particle by an equation involving an unknown function and one or more of its derivatives.

As early as the 1690s, scientists such as Isaac Newton, Gottfried Leibniz, Jacques Bernoulli, Jean Bernoulli and Christian Huygens were engaged in solving differential equations. Many of the methods which they developed are in use till today. In the eighteenth century the mathematicians Leonhard Euler, Daniel Bernoulli, Joseph Lagrange and others contributed generously to the development of the subject. The pioneering work that led to the development of ordinary differential equations as a branch of modern mathematics is due to Cauchy, **Riemann, Picard, Poincare,** Lyapunov, Birkhoff and others.

Differential equations are not only applied by physicists and engineers, but they are being used more and more in certain biological problems such as the study of animal populations and the study of epidemics. Differential equations have also proved useful in economics and other social sciences. Besides its uses, the theory of differential equations involving the interplay of functions and their derivatives, is interesting in itself.

2.0 OBJECTIVES

In this unit, we introduce the basic concepts and definitions related to differential equations. We also express some of the problems of physical and engineering interest in terms of differential equations in this unit. We shall give the methods of solving differential equations of various types in Units 2 and 3. The physical problems formulated in this unit will be solved in unit 3 after we have learnt the various methods of solving the first order equations.

- Distinguish between the order and degree of a differential equation;
- Define the solution of an ordinary differential equation;
- Identify an initial value problem;
- State and use the conditions for existence and uniqueness of first order ordinary different equations;
- Derive differential equations for some physical problems.

3.0 MAIN CONTENT

3.1 Basic Concepts

In this section we shall define and explain the basic concepts in the theory of differential equations and illustrate them through examples.

In unit 1 of MTH 112 Differential calculus, you have learnt that if a relation y = y(x) involving two variables x and y exist then we call x the **independent variable** and y the **dependent variable.**

Further, suppose we are given a relation of the type $f(x,t_1, t_2, \ldots\ldots, t_n) = 0$ involving (n + 1) variables (x and $t_1, t_2, \ldots\ldots\ldots, t_n$,); where the value of x depends on the values of the variables $t_1, t_2, \ldots\ldots, t_n$ are called independent variables and x is called the dependent variable and y is dependent variable. Similarly, if $z = x^2 + y^2 +2xy$, the x and y are independent variables and z is a dependent variable.

Any equation which gives the relation between the independent and dependent variables and the derivatives of dependent variables is called a **differential equation**.

In general, we have the following definition.

Definition: An equation involving one (or more) dependent variable derivatives with respect to one or more independent variables in called a differential equation.

For example, $\frac{dy}{dx} = \cos x$ …(1)

$$Y = x\frac{dy}{dx} + \frac{a}{dy/dx} \quad \text{….(2)}$$

$$y^2 \frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = nzx \quad \text{….(3)}$$

are all differential equations.

In Eqn. (3), $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are partial derivatives of z w.r.t.x and y respectively. The partial derivatives of a function of two variables z = f(x, y) w.r.t to one of the independent variables, can be defined as

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

when the limit exist and is independent of the path of approach. $\frac{\partial z}{\partial x}$ is the first order partial derivatives of z w.r.t.x and is obtained by differentiating z w.r.t.x treating y as a constant. It is read as 'del z by del x'. Similarly, first order partial derivative of z w.r.t. y is denoted by $\frac{\partial z}{\partial y}$ (or $\frac{\partial f}{\partial y}$ or $f_y(x, y)$, so that

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Note that equations of the type

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

are not differential equations. In this equation, if you expand the left hand side then you will find that the left hand side is the same as the right hand side. Such equations are called **identities.** Moreover, a differential equation may have more than one dependent variable. For instance,

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = y$$

is a differential equation with dependent variable x and y and the independent variable t.

Differential equations are classified into various types. The most obvious classification of differential equations is based on the nature of the dependent variable and its derivatives (or derivatives) in the equation. Accordingly, we divide differential equations into three classes: ordinary, partial and total. The following definitions give these three types of equations.

Definition: A differential equation involving only ordinary derivatives (that is, derivatives with respect to a single independent variable) is called an **ordinary differential equation** (abbreviated as ODE).

Equations

$$\frac{d^2y}{dx^2} + y = x^{2,}$$

$$\left(\frac{dy}{dx}\right)^2 = [\sin(xy) + 2]^2, \text{ and}$$

$$y = x\frac{dy}{dx} + r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

are all ordinary differential equations.

The typical form of such equations is

$$g\left[x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \ldots\ldots, \frac{d^ny(x)}{dx^n}\right] = 0 \qquad \ldots(4)$$

whenever we talk of Eqn. (4) we assume that g is known real valued function and the unknown to be determined is y. secondly, in an ordinary differential equation, y and its derivatives are evaluated at x.

It may be noted that the equation

$$\left(\frac{dy}{dx}\right)_x = (y)_{x+1}$$

is not a differential equation. This is because y is evaluated at (x + 1) whereas $\frac{dy}{dx}$ is evaluated at x.

Similarly, the equation

$$\frac{dy(x)}{dx} = \int_0^x e^{xs} y(s)ds$$

is not a differential equation since the unknown y is appearing inside an integral. Also, in this case the values of y on the right hand side of the equation depends on the interval 0 to x, whereas, in a differential equation, the unknown y has to be evaluated **only at x**.

Let us now define partial equation.

Definition: Differential equation containing partial derivatives of one (or more) dependent variable with respect to two or more independent variable is called a **partial differential equation**. **(abbreviated AS PDE)**

The examples of differential equations are

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 0,$$

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2} + xtu = 0.$$

You may also note that Eqns. (1`) and (2) given earlier are ordinary differential equations, whereas, Eqn. (3) is a partial differential equation.

And now an exercise for you.

Besides ordinary and partial differential equations, namely, total differential equations. Before giving you the definition of total differential equations, we ascribe a meaning to the symbols dx and dy which permit us to manipulate the derivative $\frac{dy}{dx}$ as a quotient of two function y = f(x), we define, the differential of y, by

Dy = f' (x) dx

If u = f(x,y) be a function of two independent variables x and y, then we know that

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

and

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Let Δu be the change in u when both x and y change by the amounts Δx and Δy respectively, so that $\lim_{\Delta x, \Delta y \to 0} \Delta u = du$. Here du is called the **total differential**.

The total differential du of a function u (x, y) is defined as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad \ldots(5)$$

or

$du = u_x dx + u_y dy$

For instance,

If $u = x^2y - 3y$

then

$$Du = 2xy\, dx + (x^2 - 3)\, dy$$

Now consider the relation u (x, y, z) = c where x, y, z are variables and c is a constant. Then

$$Du = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

Here, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are known functions of x, y and z, and therefore the above equation can be put in the form

P dx + Q dy + R dz = 0

Which is called the **total differential equation** in three variables. In this equation any one of the variables x, y, z can be treated as independent and the remaining two are then the dependent variables.

Similarly, if u = u(x, y, z, t) then corresponding total differential equation will be of the form

$$P\,dx + Q\,dy + R\,dz + T\,dt = 0.$$

Remember that a total differential equation always involves three or more variables.

We now give the following definition.

Definition: A total differential equation contains two or more dependent variables together with their derivatives with respect to a single independent variable which may, or may not, exist explicitly into the equation.

For example, equations

$$yz\,(1 + 4\,xz)\,dx - xz\,(1 + 2\,xz)\,dy - xydz = 0,$$
and

$$\frac{xdx + ydy = zdz}{\sqrt{x^2 + y^2 + z^2}} + \frac{zdx - xdz}{x^2 - z^2} + 2ax^2dx + 3by^2dy + 3cz^2dz = 0.$$

are total differential equations.

We shall be dealing with only ordinary differential equations in Modules 1ansd 2 and devote Modules 3 and 4 study total and partial differential equations.

We next consider the concepts of order and degree of a differential equation on the basis of which differential equations can be further classified.

We all know that the **nth derivative** of a dependent variable with respect to one or more independent variables is called a derivative of order n, or simply an nth order derivative.
For example, $\frac{d^2y}{dx^2}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\, \partial y}$ are second order derivatives and $\frac{d^3z}{dx^3}$, $\frac{\partial^2 z}{\partial x^2 \partial y}$ are third order derivatives.

Definition: The **order** of a differential equation is the order of the highest order derivative appearing in the equation. For instance, the equation

$\frac{d^2y}{dx^2} + y = x^2$ is of **second** order …(6)

(because the highest order derivative is $\frac{d^2y}{dx^2}$, which is of second order), whereas

$(x + y)\left(\frac{dy}{dx}\right)^2 - 1$, is of **first** order …(7)

(highest order derivative is $\frac{dy}{dx}$).

Similarly, equation

$\left[\frac{d^3y}{dx^3}\right]^2 + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + x^2\left[\frac{dy}{dx}\right]^3 = 0$ is of **third** order …(8)

whereas, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} = 0$ is of **second** order. …(9)

Note that the order of a differential equation is a positive integer.

Also, if the order of a differential equation is 'n' then it is not necessary that the equation contains some or all lower order derivatives or independent variables explicitly. For instance, equation $\frac{d^4y}{dx^4} = 0$, is a fourth order differential equation.

Definition: The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and any fractional power of the derivatives or negative power. For example Equations. (6) and (9) are of **first** degree and Equations. (7) and (8) are of **second** degree.

Equation

$$y - x\frac{dy}{dx} = r\sqrt{1+\left(\frac{dy}{dx}\right)^3} \quad ….(10)$$

is of degree **three** for, in order to make the equation free from radicals, we need to square both the sides, so that

$$\left[y-x\frac{dy}{dx}\right]^2 = r^2\left[1+\left(\frac{dy}{dx}\right)^3\right]$$

since the highest exponent of the highest derivative, that is, $\frac{dy}{dx}$ is three, thus by definition the degree of Equation. (10) is three.

Similarly, Equation. (2), that is,

$y - x \ \frac{dy}{dx} + \frac{a}{dy/dx}$ is of degree two.

This is because we multiplied through by $\frac{dy}{dx}$ to remove negative power of $\frac{dy}{dx}$ and get

$$Y \ \frac{dy}{dx} = x \left[\frac{dy}{dx}\right]^2 + a.$$

You may now try the following exercise.

We now classify the differential equations depending upon the degree of dependent variables and its derivatives into two classes, namely, linear and non-linear.

Definition: When, in an ordinary or partial differential equation, the dependent variables and its derivatives occur to the degree only, and not as higher powers or products, we call the equation **linear.**

The coefficients of a linear equation are therefore either constants or functions of the independent variable or variables. If an ordinary differential equation is not linear, we call it **non-linear.**

For example, the equation

$\frac{d^2y}{dx^2} + y = x^2$, is an ordinary linear differential equation..

However $(x + y)^2 \ \frac{dy}{dx} = 1$ is an ordinary non-linear equation, because of the presence of terms like $y^2 \ \frac{dy}{dx}$ and $2xy\frac{dy}{dx}$.

Similar, equation

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \ \partial y}\right)^2 = 0$, is a non-linear partial differential equation.

Further, if a partial differential equation is not linear, it can be **quasi-linear, semi-linear** or **non-liner.** We will discuss conditions for these classifications in the later part of this course.

You may now try this exercise.

Normally when we encounter an equation, our natural curiosity is to enquire about its solution. But, then it is natural for you to ask as to what exactly is the meaning of a solution of a differential equation. In the next section you will find an answer to this question. There, we also answer many more questions like

i) Under what conditions does the solution of a given ordinary differential equation exists?

ii) If the solution exists, then is it a unique solution?

3.2 Solution of a Differential Equation

You have seen that the general ordinary differential equation of the nth order as given by Equation (4) is

$$g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0$$

using the prime notation for derivatives ($y' = \frac{dy}{dx}$ $y'' = \frac{d^2y}{dx^2}, \ldots$ $Y^n = \frac{d^ny}{dx^n}$) we can rewrite Equation . (4) in the form

$$y^{(n)} = f(x, y, y', y'' \ldots.. y(^{(n-1)}) \qquad \ldots(11)$$

Let us assume that we can solve Eqn. (11) for $y^{(n)}$, that is, Eqn. (11) can be written in the form

$$Y^{(n)} = f(x, y, y', y'' \ldots\ldots., y^{(n-1)}) \qquad \ldots.(12)$$

It is normally a simple task to verify that a given function = $\phi(x)$ satisfies an equation like (11) or (12). All that is necessary is to compute the derivatives of y and to show that y = $\phi(x)$ and its derivatives, when substituted in the equation, reduce it to an identity in x. if such a function y exists, we call it a solution of the Eqn. (11) or (12).

However, usually we assume that

i) y = $\phi(x)$ is defined on some interval [a, b];
ii) y is n times differentiable on [a, b];
iii) We assume that y has a right derivative at point a and a left derivative at b;

iv) $y = \phi(x)$ can be real valued function or complex valued function (range is a subset of **C**) of x.

We now give the definition of the solution of an ordinary differential equation.

Definition: A real or complex valued function $y = \phi(x)$ defined on an interval I is called a solution or an integral of the differential equation $g(x, y, y', \ldots, y^{(n)}) = 0$ if $\phi(x)$ is n time differentiable and if x, $\phi(x)$, $\phi'(x)$, ….., $\phi^{(n)}(x)$ satisfy this equation for all x in I.

For example, the first-order differential equation

$$\frac{dy}{dx} = 2y - 4x$$

Note: I could represent any interval [a,b], [a, b] $[0, \infty[$, $]-\infty, \infty]$ and so on

has the solution y = 2x + 1 in the interval I = $\{x : -\infty < x < \infty\}$.
This can be checked by computing y' = 2 = 2(1 + 2x) – 4x

In the same way you can check that $y = 1 + 2x + ce^{2x}$, in the interval $-\infty < x < \infty$, is also a solution of this equation for any constant c.

In the above definition you might have noticed that a solution of (11) is real valued or complex valued. In case y is real value it is called a **real solution**. If y is complex valued, it is called **complex solution**. We are usually interested in real solution of Eqn. (11). To help you clarify what we have just said let us take some more examples.
Example 1: Show that for any constant c, the function $y(x) = ce^x$, $x \in R$ is a solution of

$$\frac{dy}{dx} = y, \qquad x \in \mathbf{R} \qquad \ldots(13)$$

Solution: Here I is R itself. For any $x \in \mathbf{R}$, we know that

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = c\frac{dy}{dx}(e^x) = ce^x = y$$

which shows that y satisfies equation(13).

Example 2: show that for real constants a and b the functions y (x) = a cos 2x and z(x) = b sin 2x are solutions of the equation below;

$$\frac{d^2y}{dx^2} + 4y = 0, \ x \in \mathbf{R} \qquad \ldots(14)$$

solution: We will first show that z (x), x $\in$**R** is a solution of Equation. (14).

Now $\frac{d}{dx}$ [z(x)] = $\frac{d}{dx}$ (b sin 2x) = 2bcos 2x.

$$\therefore \frac{d^2}{dx^2}[z(x)] = \frac{d}{dx}(2b\cos 2x) = -4b\sin 2x = -4z(x).$$

thus,

$$\frac{d^2y}{dx^2} + 4z(x) = 0,\ x\in\mathbf{R}.$$

That is, z satisfies Equation (14).

By now you must have understood the meaning of z satisfying Equation (14)._. It means that Equation (14). holds when y is replaced by z. Similarly, you can check that y(x) = a cos 2x is also a solution of Equation (14).

. You may **observe** here that the sum y(x) + z(x) that is, a cos 2x + b sin 2x is again a solution of. Equation (14).

Let us consider another example.

Example 3: Shown that y(x) = e^{ix}**,** x$\in$**R** is a solution of

$$\frac{d^2y}{dx^2} + y = 0,\ x \in\mathbf{R}$$

Solution: We have,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{ix}) = i\,e^{ix}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(ie^{ix}) = i^2 e^{ix} = -e^{ix} = -\,y(x)$$

thus, $\frac{d^2y}{dx^2} + y = 0$

In the examples taken so far, you have seen that the solution(s) differential equation exist. In Example (1) and (2) the solutions were real valued whereas, the solution in Example (3) was a complex valued function. But, there are equations for which real solution does not exist. Suppose that we are looking for real roots of the equation $x^2 + 1 = 0$. We know that it does not exist. Likewise, the equation

$$\left|\frac{dy}{dx}\right| + y^2 + 1 = 0$$

does not admit a real solution.

Similarly, the equation sin $\left(\frac{dy}{dx}\right)$ = 2 does not admit a real solution, because real value of the sin of a real function lies between – 1 and + 1.

You may now try the following exercises.
In the above discussion you must note that a differential equation may have more than one solution. It may even have infinitely many solutions. For instance, each of the functions y = sin x, y = sin x + 3, y = sin x - $\frac{4}{5}$ is a solution of the differential equation y = cosx but from your knowledge of calculus you also know that any solution of the differential equation is of the form.

$$y = \sin x + c \qquad \text{…(15)}$$

Where c is a constant. If we regard c as arbitrary then relation (15) represents the totality of all solutions of the equation. Thus, we can represents even the infinitely many solutions by a simple formula involving arbitrary constants. Accordingly, we classify various types of solutions of an ordinary differential equation as follows.

Definition: The solution of the nth order differential equation with arbitrary 'n'n constants is called its **general solution**.

Definition: Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a **particular solution**.

For example, y = a cos 2x + b sin 2x, involving two arbitrary constants a and b, is the general solution of the second order equation $\frac{d^2y}{dx^2}$ + 4y = 0 (ref. Example 2) whereas, y = 2 sin 2x + cos 2x is its particular solution (taking a = 1 and b = 2).

In some cases there may be further solutions of given equation which cannot be obtained by assigning a definite value to the constant in the general solution. Such a solution is called a **singular solution** of the equation. For example, the equation

$$y'' - xy' + y = 0 \qquad \text{…(16)}$$

has the general solution $y = cx - c^2$. A further solution of Eqn. (16) is $y = \frac{x^2}{4}$. Since this solution cannot be obtained by assigning a definite value to c in the general solution, it is a singular solution of Eqn. (16).

Thus, we have seen the various types of solution of an ordinary differential equation. We have also seen that a solution of a differential equation may or may not exist. Even if a solution exists, it may or may not be unique.

We now try to find the conditions under which the solution of a given ordinary differential equation exists and is unique. Here, we shall confine our attention to the first order ordinary differential equations only. Let us consider the general first order equation.

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(17)$$

In Eqn. (17) we assume that f is known to us. You may be surprised to know that, though this equation looks simple, it is very difficult to get its explicit solution. For clearity, let us look at the following examples.

Example 4: Does the solution y(x) of an ordinary differential equation

$$\frac{dy}{dx} = f(x), \quad \text{where} \quad f(x) = 0 \text{ for } x < 0$$
$$= 1 \text{ for } x \geq 0$$

exist $\forall\ x \in \mathbf{R}$?

Solution: The function defined by

$$y(x) = \begin{cases} c \text{for} x < 0 \\ x + c \text{ for } x \geq 0 \end{cases}$$

Satisfies this equation at the same time this function has no derivative at x = 0, because of the discontinuity of y(x) at x = 0.

Thus, this differential equation has no valid solution for x = 0.

However, y(x) defined above is the solution of the given differential equation at all points other than x = 0.

Let us look at another example.

Example 5: Does the equation $\frac{dy}{dx} = - e^{-y} x$ have a unique solution?

Solution: Rewrite the above equation in the form

$$\frac{dy}{dx}(e^y) = - x$$

Integrating, we get the solution of given equation as

$$e^y = -\frac{x^2}{2} + A,$$

or $$y = \text{In}\left(-\frac{x^2}{2} + A\right)$$

where A is an arbitrary constant.

You know that in x is defined for positive values of x only. So, the solution of the given differential equation will exists as long as $\left(-\frac{x^2}{2} + A\right) > 0$. Clearly A > 0. Also, for different values of A we get different solutions. Moreover these solutions have different intervals of existence. Thus, the solution of a given differential equation is not unique.

As regards the non-unique solutions, it is obvious that the cause for the non-uniqueness is the arbitrariness of A, (but for A > 0). Thus, we would like to impose some condition on the solution which might determine A. one such condition is to specify the value of y at some point x_0 where x_0 is in the interval of existence of y. such a condition is called **initial condition** and the problem of solving a differential equation together with the initial conditions is called the **initial value problem** (IVP). In other words, initial value problem is the problem in which we look for the solution of a given differential equation which satisfies certain conditions at a single of the independent variable. Thus, the first order initial value problem s

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x, y), \\ y(x_0) &= y_0 \end{aligned}\right\} \qquad \ldots(18)$$

From Example 4 and 5 mainly two questions arise:

1) Under what conditions does an initial value problem of the form (18) have at least one solution?

2) Under what conditions does that problem have a unique solution, that is, only one solution?

The above questions are answered by a theorem, known as **Existence Uniqueness Theorem 1.** We shall now state this theorem for the first order differential equation.

Theorem 1: (Existence – Uniqueness):
If f(x, y) is continuous at all points (x, y) in some rectangle (see Fig. 1.

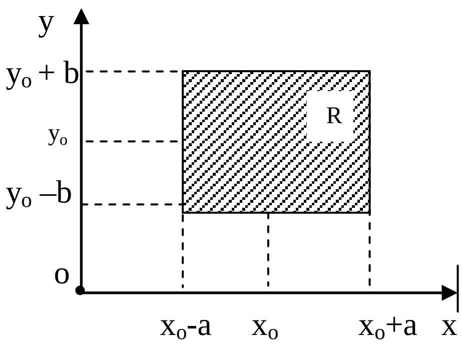

Fig. 1: Rectangle R

R: $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R, say
$|f(x, y)| \leq k \ \forall (x, y)$ in R. …(19)

then the IVP (18) has **at least one solution** y(x) defined for all x the interval
$|x - x_0| < \mathbf{h}$,

further, if $\frac{\partial f}{\partial y}$ is continuous for all (x, y) in R and bounded say,

$$\left|\frac{\partial f}{\partial y}\right| \leq \mathbf{M}, \ \forall (x, y) \text{ in R} \qquad \ldots(20)$$

then the solution y(x) is the **unique solution** for all x in that interval $|x - x_0| < h$,

Note: A function f(x, y) is aid to be **bounded** when (x, y) varies in a region in the xy-plane and if there is a number k such that $|f| \leq k$ when (x, y) is in that region. For example.

$F = x^2 + y^2$ is bounded, with K = 2 if $|x| < 1$ and $|y| < 1$.

We shall not be proving this theorem. The proof of this theorem requires familiarity with many other concepts which are beyond the scope of this course. However, in which, we give some remarks which may be helpful for a good understanding of the theorem.

Remark: Since y' = f(x,, y), the condition (19) implies that $|y'| \leq k$, that is, the slope of any solution curve y(x) in R is at least – k and at most k. Hence a solution curve which passes through the point (x_0, y) must lie in the shaded region in Fig. 2 bounded by the lines 1_1, and 1_2 whose slopes are – k and k, respectively

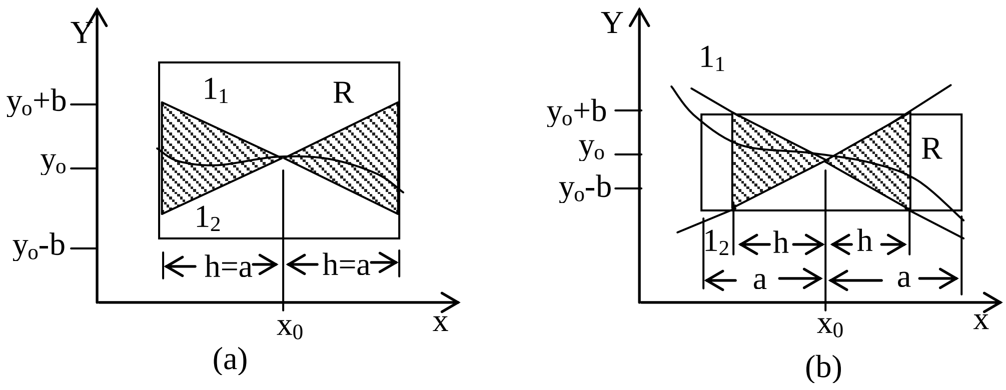

Fig. 2

Now two different cases may arise, depending on the form of R.

i) We may have $\frac{b}{k} \geq a$. Therefore, h = a, which asserts that the solution exists for all x between $x_0 - a$ and $x_0 + a$ (see Fig. 2 (a)).

ii) We may have $\frac{b}{k} < a$. Where, $h = \frac{b}{k}$, and we concluded that the solution exists for all x between $x_0 - \frac{b}{k}$ and $x_0 + \frac{b}{k}$. In this case, for larger or smaller values of x; the solution curve may leave the rectangle R (see fig. 2 (b)). Since we have not assumed anything about f outside R, nothing can be concluded about the solution for those corresponding value of x.

The condition stated in Theorem 1 are **sufficient** but not necessary and can be relaxed. For example, by the mean value theorem of differential calculus, we have (ref. Theorem 1).

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\frac{\partial f}{\partial y}$$

where (x, y_1) and (x, y_2) are assumed to be in R. From condition (20) then it follows that

$$|f(x, y_2) - f(x, y_1)| \leq M|(y_2 - y_1)|$$

condition (20) may be replaced by the condition (21) which is known as a Lipschitx condition, named after the German mathematician, Rudolf Lipschitz (1831 – 1903).

Thus, we can say that for the existence of the solution of the IVP (18), we must have

i) f continuous in T.
ii) f bounded in T.

Further the solution is unique if in addition to (i) and (ii), we have

iii) $\frac{\partial f}{\partial y}$ continuous in T.
iv) $\frac{\partial f}{\partial y}$ bounded in T (or, Lipschitx condition)

However, if the above conditions do not hold, then the IVP (18) may still have either (a) no solution (b) more than one solution (c) a unique solution.

This is because theorem provides only sufficient conditions and not necessary. For instance, consider

$$\frac{dy}{dx} = 3y^{2/3},\ y\ (0) = 0.$$

Here, $f(x, y) = 3y^{2/3}$, $\frac{\partial f}{\partial y} = 3.\frac{2}{3}\ y^{-1/3}$ for $y \neq 0$. $\frac{\partial f}{\partial y}$ does not exists at y = 0. so $\frac{\partial f}{\partial y}$ is not bounded but, the solutions $y = x^3$ and y = 0 exist.

Let us examine conditions (i) – (iv) for a few differential equations through examples.

Example 6: Examine $\frac{dy}{dx} = y$, with y (0) = 1 for existence and uniqueness of the solution.

Solution: Here f(x, y) =y, $f_y(x, y) = 1$. Also $x_0 = 0$ and $y_0 = 1$.

In this case consider a rectangle T defined by

T: $|x – 0| < a$, $|y – 1| < b$
Where a and b are positive numbers.

In any rectangle T (containing the point (0, 1)) the function f(x, y) is continuous and bounded. Hence the solution exists. Further $f_y(x, y)$ is also continuous and bounded in any such rectangle T. Therefore, the solution is unique.

You may verify that $y = e^x$ is a solution of the given equation satisfying the initial condition y (0) = 1. Hence, it is the unique solution.

However, if the initial condition is changed to y(0) = 0 then rectangle T will be of the form

T: |x – 0| < a, |y – 0| <b
And in that case y = 0 will be the unique solution for all x and y in any rectangle T containing (0, 0).

Example 7: Examine $\frac{dy}{dx} = \sqrt{|y|}$ when y (0) = 0, for existence and uniqueness of solutions.

Solution: Here f(x, y) = $\sqrt{|y|}$, $x_0 = 0$ and $y_0 = 0$. In this case consider the region T with |x| < a, |y| < b, a and b positive numbers. Function f(x,y) is continuous and bounded in any rectangle T, containing the point (0,0).

Hence solution exists. In order to test the uniqueness of the solution, consider the Lipschitz condition.

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{|\sqrt{|y_2|} - \sqrt{|y_1|}|}{|y_2 - y_1|}$$

for any region containing the line y = 0, Lipschitz condition is violated. Because for $y_1 = 0$ and $y_2 > 0$, we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, (\sqrt{y_2} > 0)$$

and this can be made as large as we please by choosing y_2 sufficiently small, whereas condition (21) requires that the quotient on the left-hand side of (21) does not exceed a fixed constant M.

therefore, the solution is not unique.

Further, it can be checked that the given problem has the following solutions

i) $y = 0 \ \forall x$

ii) $y = \begin{cases} \frac{1}{4}x^2 \text{for } x \geq 0 \\ -\frac{1}{4}x^2 \text{for } x \leq 0 \end{cases}$

Example 8: Examine $\frac{dy}{dx} = f(x, y) = \begin{cases} y(1-2x) \text{ for } x > 0 \\ y(2x-1) \text{ for } x < 0 \end{cases}$ with y(1) = 1 for existence and uniqueness of solution.

Solution: Here x_0 = 1 and y_0 = 1. Rectangle T can be any rectangle containing point (1, 1). You may note that the function is not defined at x = 0. It is discontinuous at x = 0. Thus, at x = 0 the solution does not exist. At all other points the function

$$f(x, y) = \begin{cases} y(1-2x) \text{ for } x > 0 \\ y(2x-1) \text{ for } x < 0 \end{cases}$$

is continuous and bounded in T with (x) = 1. Hence, the solution, exists and is unique for all x other than x = 0. Further, you may verify that

$y = x^{x-x^2}$ for x > = 0

and $y = e^{x^2-x}$ for x < 0

is the unique solution of the given problem for all x other than x = 0

you may now try the following exercise.

From the definitions given in page 14, you may have realized that the general solution of a first order differential equation normally contains one arbitrary constant which is called a **parameter**. When this parameter is vary in values, we obtain a one parameter family of curves. Each of these curves is a particular solution or integral curve of the given differential equation, and all of them together constitute its general solution. On the other hand, we expect that the curve of any one-parameter family are integral curves of some first order differential equation. In general we pose a problem: given an n-parameter family of curves, can thus say that differential equations arise from a family of curves. In the next section we shall take up this.

3.3 **Family of Curves and Differential Equations**

Let us consider a family of straight lines

$Y = mx + c$ …(22)

Which is a two-parameter family of curves, parameters being m and c.

It is clear from Eqn. (22) that y can be treated as a function of x, $x \in R$. Differentiating Eqn. (22) w.r.t.x, we have

$y' = m$ …(23)

Again, differentiating (23) we get

$y'' = 0$ …(24)

Equation (23) and (24) are differential equations of order one and two respectively. The way in which we have arrived at Eqn. (23) or (24) is clear. We have actually tried to eliminate the parameters, or constants, m and c and the result is Eqn. (23) or (24).

In general, we represent one-parameter family of curves by an equation

$$F(x, y, a) = 0 \qquad \ldots(25)$$

where a is a constant.

In Eqn. (25), let us regard y as a function of x and differentiate if w.r.t.x. Suppose we get

$$G(x, y, y', a) = 0 \qquad \ldots(26)$$

In case, we are able to eliminate the constant a between Eqn. (25) and (26), then we have a relation connecting x, y and y', say

$$h(x, y, y',) = 0 \qquad \ldots(27)$$

Equation (27) is an ODE of order one. In particular, if Eqn. (25) has the form

$$\psi(x, y) = a \qquad \ldots(28)$$

then the elimination of the constant a from Eqn. (28) leads us to the differential equation

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0 \qquad \ldots(29)$$

for example, $x^2 + y^2 = a^2$ …(30)

is the equation of the family of all **concentric circles** with centre at the origin (fig. 3)

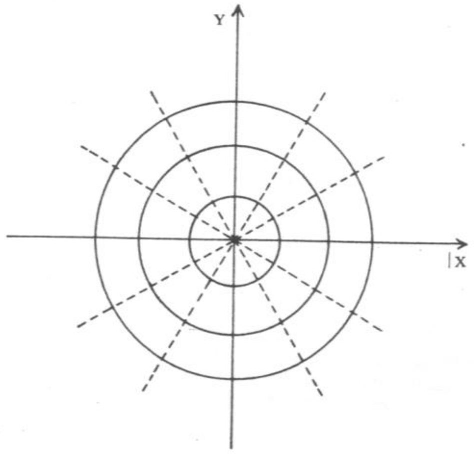

Fig. 3

For different values of a , we get different circles of the family. Differentiating Eqn. (30) with respect to x, we get

$$2x + 2y \frac{dy}{dx} = 0.$$

Or $\quad x + y\dfrac{dy}{dx} = 0$, as the differential equation of the given family of circles.

Continuing with equation y = mx + c, if we regard only c as an arbitrary constant to be eliminated, then y' = m, represents the required differential equation. Geometrically, for a fixed m, y' = m represents a family or **straight lines** (in the plane) **whose slope is m** (see fig. 4).

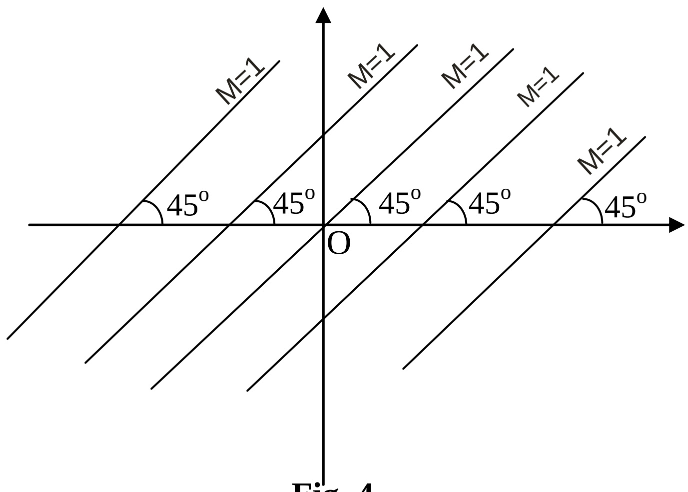

Fig. 4

On the other hand, if we assume that in equation y = mx + c both m and c are constants to be eliminated, then equation y'' = 0 represents the required differential equation. Geometrically, it is the family of the **straight lines in the plane** (see Fig. 5)

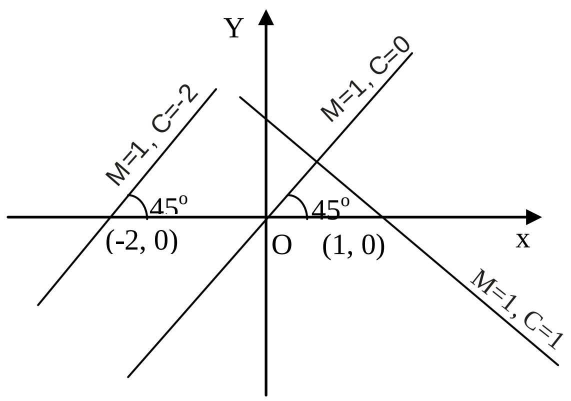

Fig. 5

You may now try the following exercise.

In the introduction of this unit we mentioned that there are many problems of physical and engineering interest which give rise to differential equations. In other words, we can say that some problems have representations through the use of differential equations. In the next section we shall take up such problems.

3.4 Differential Equations Arising From Physical Situations

In this section we shall show that differential equations arise not only out of consideration of families of geometric curves, but an attempt to describe physical problems, in mathematical terms, also result in differential equations.

The initial-value problem

$$\left.\begin{aligned} \frac{dy}{dx} &= ky \\ y(t_0) &= t_0 \end{aligned}\right\} \qquad \ldots(31)$$

where k is a constant of proportionality, occurs in many physical theories involving either **grow** or **decay.** For example, in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacterial present at any time. In physics an IVP such as Eqn. (31) provides a model for approximating the remaining amount of a substance that is disintegrating, or decaying, through radioactivity. The differential Eqn. (31) could also determine the temperature of a cooling body. In chemistry, the amount of a substance remaining during certain reactions is also described by Eqn. (31).

Let us now see the formulation of some of these problems.

I: Population Model

Let N(t) = denote the number or amount of a certain species at time t. then the growth of N(t) is given by its derivative $\frac{d}{dt}$ N(t). Thus, if N(t) is growing at a constant rate, $\frac{d}{dt}N(t) = \beta$, a constant. It is sometimes more appropriate to consider the relative rate of growth defined by

$$\text{relative rate of growth} = \frac{\text{actual rate of growth}}{\text{size of } N(t)} = \frac{N'(t)}{N(t)} = \frac{dN(t)/dt}{N(t)}$$

The relative rate of growth indicates the percentage increase in N(t) or decrease in N(t). For example, an increase of 100 individuals for a species with a population size of 500 would probably have a significant impact being an increase of 20 percent. On the other hand, if the population were 1,000,000 then the addition of 100 would hardly be noticed, being an increase of 0.01 percent. If we assume that the rate of change of N at time t is proportional to population N(t), present at the time t then,

$$\frac{d}{dt}N(t) \propto N(t)$$

which is written as

$$\frac{d}{dt} N(t) = k N(t), \qquad \ldots(32)$$

where k is a constant

if N increases with t, then $k > 0$ in Eqn. (32)

If N decreases with t, $k \leq 0$ in Eqn. (32).

Normally, we have the knowledge of the population, say N_0, at some initial time t_0. so along with Eqn. (32) we have

$$N(t_0) = N_0. \qquad \ldots(33)$$

Thus, the population N(t) at time t can be found by solving Eqn. (32) with condition (33). We shall reconsider this problem with some modifications later in unit 3.

II: Newton's Law of Cooling

Here we deal with the temperature variations of a hot object kept in a surrounding which is kept at a constant temperature, say T_0. Under certain conditions, a good approximation to the temperature of an object can be obtained by using Newton's law of cooling. Let the temperature of the object be T. If $T \geq T_0$, we know that the object radiates heat to the surrounding resulting in the reduction of its (object's) temperature. Newton's law of cooling states that the rate at which the temperature T(t) changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature T_0 of the surrounding medium. That is

$$\frac{d}{dt} T(t) \propto T(t) - T_0$$

or $$\frac{d}{dt} T(t) = k\,(T(t) - T_0), \qquad \ldots(34)$$

where k is a constant of proportionality.

Constant $k < 0$, because the temperature of the body is reducing (we have assume that $T(t) \geq T_0$). We observe that the Eqn. (34) is a differential equation of order one.

III: Radioactive Decay

Many substances are radioactive. This means that the atoms of such a substance break down into atoms of other substances. In Physic, is has been noticed that the radioactive material, at time t, decays at rate proportional to its amount y(t). In other words,

$$\frac{d}{dt} y(t) = ky(t) \qquad \ldots(35)$$

where $k < 0$, is a constant. If the mass of the substance at some initial time, say $t = 0$, is A, then $y(t)$ also satisfies the initial condition

$y(0) = A$.

Thus, the physical problem of radioactive decay is modeled by the IVP.

$$\frac{d}{dt} y(t) = ky(t), \; y(0) = A \qquad \ldots(36)$$

where k is a constant.

Remark: I, II and III above indicate situations where differential equations occur naturally. In unit 3 we shall give the methods of solving these equations.

You may now try the following exercise.

4.0 CONCLUSION

We now conclude this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit, we have covered the following points:

1) a) An equation involving one (or more) dependent variables and its derivative w.r. to one or more independent variables is called a **differential equations**.

b) A differential equation involving only ordinary derivatives is called an **Ordinary Differential Equation (ODE).**

c) A differential equation involving partial derivatives is called a **partial differential equation (PDE)**.

d) The order of a differential equation is the order of the highest order derivative appearing in the equation.

e) The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and fractions of the derivatives.

f) In a differential equation, when the dependent variable and its derivatives occur in the first degree only, and not as higher powers or products, the equation is said to be **linear**.

g) If an ordinary differential equation is not linear, it is said to be **non-linear**.

2) a) A real or complex value function = $\phi(x)$ defined on an interval I is called a **solution** of equation g(x, y, y', y'', ….., $y^{(n)}$) = 0

if $\phi(x)$ is differentiable n times and if $\phi(x)$, $\phi'(x)$, $\phi''(x)$, ….,(x) satisfy the above equation for all x in I.

b) The solution of the nth order differential equation which contains n arbitrary constants is called its **general solution**.

c) Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called **a particular solution** of the differential equation.

d) A solution of a differential equation, which cannot be obtained by assigning definite values to the arbitrary constants in the general solution is called its **singular solution**.

3) a) Conditions on the value of the dependent variable, and its derivatives, at a single value of the independent variable in the interval of existence of the solution are called the **initial conditions**.

b) The problem of solving a differential equation together with the initial conditions is called the **initial value problem**.

4) The **sufficient** conditions for the existence of solution of the first order equation

$$\frac{dy}{dx} = f(x, y), \text{ with } y(x_0) = y_0,$$

in a region T defined by $|x - x_0| < a$ and $|y - y_0| < b$ are

i) f is continuous in T

and

ii) f is bounded in T.

further if the solution exists, then it is unique if, in addition to (i) and (ii), we have

(iii) $\frac{\partial f}{\partial y}$ is continuous in T.

iv) $\frac{\partial f}{\partial y}$ is bounded in T (or, Lipschitz condition is satisfied).

5) The general solution of a first order (nth order) differential equation represents one-parameter (nm-parameter) family of curves.

6) Many physical situation such as population model, Newton's law of cooling, radioactive decay, can be represented by first order differential equations.

6.0 TUTOR MARKED ASSIGNMENT

1. Which of the following are differential equations? Which of the differential equations are ordinary and which are partial?

a) $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y^3 = 5x + 2$

b) $\frac{dy}{dx} = \int_x^x \sin[xy(s)]ds$

c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

d) $\frac{dy(x)}{dx} = 5x\, y\, (x + 1)$

e) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \int_x^x \sin[xy(s)]ds$

f) $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$

2. Find the order and degree of the following differential equations.

a) $\left(\frac{d^2y}{dx^2}\right)^{2/3} = 1 + 2\frac{dy}{dx}$

b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = x$

c) $\sin\left(\frac{d^2y}{dx^2}\right) + x^2y^2 = 0$

d) $\frac{dy}{dx} + y^3 = 0$

e) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = r\,\frac{d^2y}{dx^2}$

f) $\frac{\partial^4 z}{\partial x^4} + \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = x$

g) $x^2\,(dx)^2 + 2xy\,dx\,dy + y^2(dy)^2 - z^2(dz)^2 = 0$

3. Classify the following differential equations into linear and non-linear.

a) $x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} - z = 0$

b) $\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = 0$

c) $\frac{dy}{dx} = (x + y)^2$

d) $(1 - x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + n(n + 1)\,y = 0$

e) $(x^2 + y^2)^{3/2}\,\frac{d^2y}{dx^2} + \mu x = 0$

4. Verify that $y = \cos^{-1}\left(-\frac{x^2}{2}\right)$, and $2\cos y = -x^2$ are solutions of the equation $\sin y\,\frac{dy}{dx} = x$. Can you state the interval on which y is defined?

5. Verify that $y = \frac{1}{x}(\text{In } y + c)$ is a solution of the equation $\frac{dy}{dx} = \frac{y^2}{1 - xy}$ for every value of the constant c.

6. Verify that $y = e^{2x}$ and $y = e^{3x}$ are both solutions of the second order equation $y'' - 5y' + 6y = 0$
Can you find any other solution?

7. Examine $\frac{dy}{dx}$ = f(x, y) = $\begin{cases} \frac{4x^3y}{(x^4+y^2)}, \text{when x and y are not both zero} \\ 0, \text{when } x = y = 0 \end{cases}$

   With y (0) = 0
   For existence and uniqueness of the solution.

8. Assuming y to be a function of x, determine the differential equations by Eliminating the arbitrary constant (or constants) indicated in the following problems.

   a) xy = c (arbitrary constant is c)
   b) y = cos (ax) (arbitrary constant is a).
   c) y = A cos (ax) (arbitrary constants are A and a).

9. In the following problems derive the differential equation describing the given physical situations.

   a) A culture initially has P_0 number of bacteria. Growth of the bacteria is proportional to the number of bacteria present. What is the number p of bacteria at any time t.

   b) A quantity of a radioactive substance originally weighing x_0 gms decomposes at a rate proportional to the amount present and half the original quantity is left after 2 years. Find the amount x of the substance remaining after t years.

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 METHODS OF SOLVING DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

CONTENTS

1.0 INTRODUCTION

In unit 1, we introduced the basic concepts and definitions involved in the study of differential equations. We discussed various types of solutions of an ordinary differential equation. We also stated the conditions for the existence and uniqueness of the solution of the first order ordinary differential equation. However, we do not seem to have paid any attention to the methods of finding these solutions. Accordingly, in this unit we shall confine our attention to the same.

In general, it may not be feasible to solve even the apparently simple equation $\frac{dy}{dx} = f(x, y)$ or $g(x, y, \frac{dy}{dx}) = 0$ where f and g are arbitrary functions. This is because no systematic; procedure exists for obtaining its solution for arbitrary forms of f and g. However, there are certain standard types of first order equations for which methods of solution are available. In this unit we shall discuss a few of them with special reference to their applications.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define separable equations and solve them
- define homogeneous equations and solve them
- obtain the solution of equations which are reducible to homogenous equations
- identify exact equations
- obtain an integrating factor which may reduce a given differential equation into an exact one and eventually provide its solution.

3.0 MAIN CONTENT

3.1 Separation of Variables

You know that the problem of finding the tangent to a given curve at a point was solved by Leibniz. The search for the solution to the inverse problem of tangents, that is, given the equation of the tangent to a curve at any point to find the equation of the curve led Leibniz to many important developments. A particular mention may be made of the method of separation of variables which was discovered by Leibniz in 1691 by providing that a differential equation of the form

$$\frac{dy}{dx} = X(x)\, Y(y)$$

is integrable quadratures. However, it is John Bernoulli (1694) who is credited with the introduction of the terminology and the process of separation of variables.

In short, it is a method for solving a class of differential equations that arises quite frequently and is defined as follows:

Note: The process of finding the areas of plane regions if called quadrature

Definition: An equation of the form

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

is called a **separable equation** or **equation in variable separable form** if f(x, y) can be put in the form

$$f(x, y) = X(x)\, Y(y), \qquad \ldots(2)$$

where X and Y are given functions of x and y respectively.

In other words, Eqn. (1) is a separable equation if f is a product of two functions, one of which is a function of x and the other is a function of y. Here X (x) and Y(y) are real value functions of x and y respectively.

For instance, equation $\frac{dy}{dx} = e^{x+y}$ is a separable equation, since $e^{x+y} = e^x.e^y$ (here X(x) $= e^x$ and Y(y) $= e^y$). The equation $\frac{dy}{dx} = x^2(y^2 + y^3)$ is also a separable equation. But the equation $\frac{dy}{dx} = e^{xy}$ is not a separable equation, because it is not possible to express e^{xy} as a product of two functions in which one is a function of x only and the other is a function of only. Similarly, equation $\frac{dy}{dx} = x + y$ is not a separable equation.

In order to solve Eqn. (1), when it is in variable separable form, we write it as

$$a(y)\ \frac{dy}{dx} + b\,(x) = 0 \qquad \ldots(3)$$

where a(y) and b(x) are each functions of only one variable

let us assume that there exist functions A and B such that A' (y) = a(y) and B' (x) = b(x). With this hypothesis, Eqn. (3) can be rewritten as

$$\frac{d}{dx}\ A\,(y(x)) + B'\,(x) = 0 \qquad \ldots(4)$$

[by chain rule $\frac{d}{dx}\ A\,(y(x)) = A'\,(y(x))\ \frac{dy}{dx} = a\,(y(x))\ \frac{dy}{dx}$]

Integrating Eqn. (4) with respect to x, we get

$$A\,(y(x)) + B(x) = c \qquad \ldots(5)$$

Where c is a constant.

Thus, any solution y of (3) is implicitly given by (5).

We now take up a few examples to illustrate this method.

Example 1: Solve $\frac{dy}{dx} = e^{x-y}$

Solution: This equation may be written as

$$\frac{dy}{dx} = e^{x}\, e^{-y}$$

or $e^{y}\ \frac{dy}{dx} = e^{x}$

or $\frac{d}{dx}\ (e^{y}) = e^{x}$

which, on integration, gives $e^{y} = e^{x} + c$, where c is a constant.

In case $e^{x} + c \geq 0$, then $y(x) = \text{In}\ (e^{x} + c)$.

Example 2: Solve the equation

$(1 + y^2)\ dx + (1 + x^2)\ dy = 0$ with $y\,(0) = -\,1$.

Solution: The given equation can be rewritten as

$$\frac{dx}{1+x^2} + \frac{dy}{1+x^2} = 0.$$

Integrating, we get

$\tan^{-1}x + \tan^{-1}y = c$.

The initial condition that y = -1 when x = 0 permits us to determine the value of c that must be used to obtain the particular solution desired here. Since $\tan^{-1}0 = 0$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$, $c = 0 - \frac{\pi}{4}$. Thus, the solution of the initial value problem is $\tan^{-1}x + \tan^{-1}y = -\frac{\pi}{4}$.

Let us look at another example.

Example 3: Solve $\frac{dy}{dx} = ky - my^2$, where k > 0 and m > 0 are real constants.

Solution: Let us write the given equation as

$$\frac{1}{ky - my^2}\frac{dy}{dx} = 1.$$

Now we will try do decompose the coefficient of $\frac{dy}{dx}$ into partial fractions.

Let $$\frac{1}{y(k - my)} = \frac{A}{y} + \frac{B}{k - my} \qquad \ldots(6)$$
Where A and B are constants to be determined.

From (6), we get
1 = A (k – my) + By
which gives $1 = Ak$ and $1 = \frac{Bk}{m}$

or $A = \frac{1}{k}$ and $B = \frac{m}{k}$

Hence $$\frac{1}{y(k - my)} = \frac{1}{ky} + \frac{m}{k}\frac{1}{(k - my)}$$

Thus the given differential equation can be rewritten as
$$\left(\frac{1}{ky} + \frac{m}{k}.\frac{1}{k - my}\right)\frac{dy}{dx} = 1, \qquad \ldots(7)$$
for $y \neq 0$ and $k - my \neq 0$.

In the integration of Eqn. (7) the sign of y and k –my play a important role. We now discuss the following possible cases:

Case 1: y > 0 and k-my > 0 (0 < y < $\frac{k}{m}$).

For the case under consideration, Eqn. (7) can be expressed as

$$\frac{d}{dx}\left[\left(\frac{1}{k}\text{In}y\right)-\frac{1}{k}\text{In}(k-my)\right] = 1$$

which on integrating, yields

$$\frac{1}{k}\ \text{In}y - \frac{1}{k}\text{In}\ (k\text{-}my) = x + c,$$

where c is a constant of integration. The above equation can be further express as

$$\text{In}\ (y)^{1/k} - \text{In}\ (k - my)^{1/k} = x + c$$

or $\left[\frac{y}{k-my}\right]^{1/k} = e^{x + c}$

Case II: y < 0

When y < 0 then k – my > 0 because m > 0,. In this case, Eqn. (7) on integration can be written as

$$-\frac{1}{k}\ \text{In}\ (-\ y)\ \pm \frac{1}{k}\ \text{In}\ (k - my) = x + c$$

or $\frac{1}{(-y)^{1/k}(k-my)^{\pm 1/k}} = e^{x+c}$

Cases III: y> 0 and k – my < 0 (y > $\frac{k}{m}$).

In this case Eqn. (7) after integration gives

$$\frac{1}{k}\ \text{In}(y) - \frac{1}{k}\ \text{In}(-k + my) = x + c$$

or $\left[\frac{y}{-k+my}\right]^{1/k} = e^{x+c}$

You may now try the following exercises.

Many differential equations that are not separable can be reduced to the separable form by a suitable substitution. In the next section we shall study one class of such equations.

3.2 Homogeneous Equations

In this section we shall study equations like

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{3x^2 + y^2}$$

This is an example of homogenous differential equations. In 1691 Leibniz made known to the world the method of solving homogeneous equation differential equations of the first order.

Before we discuss the method of solving a homogeneous equation, we define homogeneous functions of two variables x and y.

Definition: A real-value function h(x, y) of two variables x and y is called a homogeneous function of degree n, where n is a real number, if we have

$$h(\lambda x, \lambda y) = \lambda^n h(x, y)$$

for $\forall$ x, y and any constant $\lambda > 0$.

For example, $h(x,y) = x^3 + 2x^2y + 3xy^2 + 4y^3$ is homogeneous of degree three because $h(\lambda x, \lambda y) = \lambda^3 h(x, y)$

Also, $h(x,y) = x^2 \cos\left(\frac{y}{x}\right) + (\text{In}|x| - \text{In}|y|)\, xy$ is homogeneous function of degree 2 and $\frac{x^2}{x^2 + 2xy + y^2}$ is homogeneous of degree 0.

But, the function $h(x, y) = x^2 + 2xy + 4$ is not homogeneous because $h(\lambda x, \lambda y) \neq \lambda^n (x^2 + xy + 4)$ for any value of n.

if h(x, y) is a homogeneous function of degree n, that is, $h(\lambda x, \lambda y) = \lambda^n h(x,y)$, then a useful relation is obtained by letting $\lambda = \frac{1}{x}$. This gives $\frac{1}{x^n} h(x,y) = h\left(1, \frac{y}{x}\right) = \varnothing\left(\frac{y}{x}\right)$ (say) or , $h(x,y) = x^n \varnothing\left(\frac{y}{x}\right)$.

We shall be particularly interested in the case where h(x,y) is **homogeneous of degree 0** that is, if $h(\lambda x, \lambda y) = \lambda^0 h(x,y) = h(x,y)$. We now give the following definition.

Definition: A differential equation

$$y' = f(x,y) \qquad \ldots(8)$$

is called a **homogenous differential equation** when f is a homogeneous function of degree 0.

For instance, the following equations are homogeneous differential equations:

i) $\frac{dy}{dx} = \frac{2y}{x}$,

ii) $\frac{dy}{dx} = \frac{2x + 3y}{4x} = \frac{2 + 3(y/x)}{4}$

iii) $\frac{dy}{dx} = \frac{x^3 + x^2y + y^3}{3x^2y + y^3} = \frac{1 + (y/x) + (y/x)^3}{3(y/x) + (y/x)^3}$

from the above equations, you may have noticed that if an equation can be put in the form

$\frac{dy}{dx} = f(x, y) = \frac{f_1(x, y)}{f_2(x, y)}$,

where f_1 and f_2 are homogeneous expressions of the same degree in x and y, then f is a homogeneous function of degree 0.

Further, if in Eqn. (8) we let $\lambda = \frac{1}{x}$, then we have

$$y' = f\left(1, \frac{y}{x}\right) = f\left(\frac{y}{x}\right) \qquad \ldots(9)$$

This suggests making the substitution $v = \frac{y}{x}$ to solve this equation. Since we seek y as a function of x this substitution means

$$V(x) = \frac{y(x)}{x} \text{ or } y(x) = xv(x)$$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

with this substitution Eqn, (9) reduces to

$$v + x\frac{dv}{dx} = f(v)$$

or $$\frac{dv}{dx} = \frac{F(v) - v}{x} \qquad \ldots(10)$$

which shows that Eqn. (10) is a separable equation in v and x. if we can solve Eqn. (10) for v in term of x, using the technique of Sec. 2.2 then the solution of Eqn. (9) is y = vx and hence we can solve equations of the type (9).

We now illustrate this method with the help of the following examples.

Example 4: Solve $\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2}$

Solution: You can easily check that the given equation is homogeneous of degree 0. it can be rewritten as

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) \qquad \dots(11)$$

By making the substitution, $v = \frac{y}{x}$, Eqn. (11) reduces to

$x\frac{dv}{dx} + v = 2v^2 + 3v$

or $x\frac{dv}{dx} = 2v^2 + 2v = 2v(v + 1)$

or $\frac{dv}{v(v+1)} = \frac{2dx}{x}$

which is in variable separable form.

Resolving $\frac{1}{v(v+1)}$ into partial fractions, we have

$\left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \frac{2}{x}\, dx$

which on integration, gives

$$\text{In } |v| - \text{In}|v+1| = \text{In}x^2 + \text{In } |c| \qquad \dots(12)$$

Where c is an arbitrary constant.

From Eqn. (12), we have

$\frac{v}{v+1} = cx^2$

replacing v by $\frac{y}{x}$, we get

$cx^2 = \frac{y/x}{(y/x)+1} = \frac{y}{x + y}$

or $y = \frac{cx^3}{1-cx^2}$,

which is the general solution of the given equation.

Example 5: Solve $\frac{dy}{dx} = \frac{y^3}{x^3} + \frac{y}{x}$, x > 0.

Solution: With the substitution y = vx, we have

$$V + x\,\frac{dv}{dx} = v^3 + v$$

$$\text{or } \frac{1}{v^3}\,\frac{dv}{dx} = \frac{1}{x} \qquad \ldots(13)$$

Integration of Eqn. (13) yields

$$-\frac{1}{2v^2} = \text{In}x + \text{In}\,|c|,$$

where c is a real constant. On replacing $v = \frac{y}{x}$, the general solution of the given equation can be expressed as

$$y^2 = -\frac{x^2}{2[\text{In}x + \text{In}\,|\,c\,|]} \text{ or } y^2 = -\frac{x^2}{2}\,\frac{1}{\text{In}(x\,|\,c\,|)}$$

Let us consider another example.

Example 6: Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad \ldots(14)$$

Putting y = vx, in Eqn. (14), we get

$$V + x\,\frac{dv}{dx} = \frac{1+v^2}{2v},$$

$$\text{or } x\,\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\text{or } \frac{2v}{1-v^2}\,\frac{dv}{dx} = \frac{1}{x}$$

Integrating, we get,

In |x| |(1 – v^2)| = In |c|, where c is a constant of integration or
X (1- v^2) = c

On substituting for v, we can write the solution of Eqn. (14) in the form;

$$x^2 - y^2 = cx.$$

How about trying some exercise now?

Sometimes it may happen that a given equation is not homogeneous but can be reduced to a homogeneous form by considering a transformation of the variables. We now consider such equations.

Equations reducible to homogeneous form

Consider the equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \qquad \ldots(15)$$

where a, b, c, a', b' and c' are all constants.

Eqn. (15) can be reduced to a homogeneous form by using the substitution
X = x' + h and y = y' + k,

Where h and k are constants to be so chosen as to make the given equation homogeneous. In terms of these new variables, Eqn. (15) becomes

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \frac{ax' + by' + (ah + bk + c)}{a'\times' + b'y' + (a'h + b'k + c')}, \qquad \ldots(16)$$

which will be homogeneous provided h and k are so chosen that

$$\left.\begin{aligned} ah + bk + c = 0 \\ a'h + b'k + c' = 0 \end{aligned}\right\} \qquad \ldots(17)$$

Consequently Eqn. (16) reduces to

$$\frac{dy'}{dx'} = \frac{ax' + by'}{a'x' + b'y'} \qquad \ldots(18)$$

which can be solved by means of the substitution y' = vx'.

If the solution of the Eqn. (18) is of the form
g(x', y') = 0,

then the solution of Eqn. (15) is

g(x – h, y – k) = 0,

where h and k are obtained by solving the simultaneous Eqns. (17)
Solving Eqns. (17) for h and k, we get,

$h = \dfrac{bc' - bc'}{ab' - a'b}$, $k = \dfrac{a'c - ac'}{ab' - a'b}$

which are defined except when
ab' – a'b = 0 that is, when $\dfrac{a}{a'} = \dfrac{b}{b'}$.

If $\dfrac{a}{a'} = \dfrac{b}{b'}$, then h and k have either infinite values or are indeterminate. But then the question is what happens if $\dfrac{a}{a'} = \dfrac{b}{b'}$?

In such cases, we let $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{m}$ (say)

Then Eqn. (15) can be written as

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c'} \qquad \ldots(19)$$

On putting ax to by = v, Eqn (19) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = \frac{v+c}{mv+c'}$$

so that the variables are separated and hence the equation can be solved by the method given in the Sec 2.2.

We now take up some examples to illustrate the above discussion.

Example 7: Solve $\dfrac{dy}{dx} = \dfrac{y-x+1}{y+x+5}$ …(20)

Solution: Comparing the given equation with Eqn. (15), we have

a = 1, b = -1, a' = 1, b' = 1.

$\therefore \dfrac{a}{a'} = 1, \dfrac{b}{b'} = -$ and $\dfrac{a}{a'} \neq \dfrac{b}{b'}$

Putting x = x' + h and y = y' + k in Eqn. (20), we get

$$\frac{dy'}{dx'} = \frac{y' - x' + k - h + 1}{y'x' + k + h + 5} \qquad \ldots(21)$$

we choose h and k such that

$$\left.\begin{array}{l} k - h + 1 = 0 \\ k + h + 5 = 0 \end{array}\right\} \qquad \ldots(22)$$

on solving Eqn. (22), we get h = - 2 and k = -3 . with these values of h and k, Eqn. (21) reduces to

$$\frac{dy'}{dx'} = \frac{y' - x'}{y' + x'} \qquad \ldots(23)$$

which is a homogeneous equation.

on putting y' = vx' in Eqn. (23) and simplifying the resulting equation, we get

$$-\frac{1+v}{1+v^2}\,\frac{dv}{dx'} = \frac{1}{x'}$$

$$\text{or } \left(\frac{1}{1+v^2} + \frac{v}{1+v^2}\right)\frac{dv}{dx'} = -\frac{1}{x'} \qquad \ldots(24)$$

Integration of Eqn. (24) yields

$\tan^{-1}v + \frac{1}{2}$ In $(1+ v^2) = -$ Inx' + c, where c is a constant.

or, $\frac{1}{2}$ In $(1 + v^2)\, x'^2 + \tan^{-1}v = c$.

Replacing v by $\frac{y'}{x'}$, we have

$\frac{1}{2}$ In $(x'^2 + y'^2) + \tan^{-1} \frac{y'}{x'} = c$.

substituting x' = x + 2 and y' = y + 3, solution of Eqn. (20) is given by

$\frac{1}{2}$ In $\left[(x+2)^2 + (y+3)^2\right] + \tan^{-1}\left(\frac{y+3}{x+2}\right) = c$.

Example 8: Solve the differential equation (4x + 6y + 5) dy = (3y + 2x + 5) dx.
Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{3y+2x+5}{4x+6y+5}$$

$$= \frac{(2x+3y)+5}{2(2x+3y)+5} \qquad \ldots(25)$$

In this case a = 2, b= 3, a' = 4, b' = 6. Thus,

$\frac{a}{a'} = \frac{b}{b'}$. Therefore, we put 2x + 3y = v, and Eqn. (25) reduces to,

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right) = \frac{v+5}{2v+5} \quad \left(\text{here } 2+3\frac{dy}{dx} = \frac{dv}{dx}\right)$$

or $\frac{dv}{dx} = \frac{3(v+5)}{2v+5} + 2 = \frac{3v+15+4v+10}{2v+5} = \frac{7v+25}{2v+5}$

Now variables are separated and we get

$$\frac{2v+5}{7v+25}\ \frac{dv}{dx} = 1$$

or $\left[\frac{2}{7} - \frac{15}{7(7v+25)}\right] \frac{dv}{dx} = 1.$

Integrating, we get

$\frac{2}{7}$ v - $\frac{15}{49}$ In $\left(v+\frac{25}{7}\right)$ **= x + c,** where c is a constant of integration, substituting v = 2x+ 3y, we get

$\frac{2}{7}$ (2x + 3y) - $\frac{15}{49}$ In $\left(2x+3x+\frac{25}{7}\right)$ = x + c,

or, 14(2x + 3y) – 15 In (2x + 3y + $\frac{25}{7}$) = 49 (x + c)

or, 42y – 21x – 15 In (14x + 21y + 25) = 49c – 15 In 7 = c_1, say, which is the required solution.

You may now try the following exercise. In each of the equations in this exercise you should first see whether $\frac{a}{a'} = \frac{b}{b'}$ and then decide on the method.

In Unit 1, we defined the total differential of a given function. In the next section we shall make use of this to define and solve exact differential equations.

3.3 Exact Equations

Let us start with a family of curves h(x, y) = c. Then its differential equation can be written in terms of its total differential as

dh = 0, or

$$\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy = 0.$$

For example, the family $x^2y^3 = c$ has $2xy^3dx + 3x^2y^2dy = 0$ as its differential equation. suppose we now consider the reverse situation and begin with the differential equation

a(x, y)dy + b(x,y)dx = 0

If there exists a function h(x,y) such that

$$\frac{\partial h}{\partial x} = b(x, y) \text{ and } \frac{\partial h}{\partial y} = a(x, y),$$

then Eqn. (26) can be written in the form

$$\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}\,dy = 0 \text{ or } dh = 0$$

that is, h(x, y) = constant represents a solution of Eqn. (26).

In this case we call the expression a(x, y)dy + b(x, y)dx an **exact differential** and (26) is called an **exact differential equation**. For instance, equation $x^2y^3\,dx + x{+}3^{y+2}\,dy = 0$ is exact, since we have $d\left(\frac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2dy$.

Thus, an exact differential equation is formed by equating an exact differential to zero.

It is sometimes possible to determine exactness and find the function h by merer inspection. Consider, for example, the equations.

$3x^2y^4dx + 4x^3y^3\,dy = 0$
and $xe^{xy}\,dy + (ye^{xy} – 2x)\,dx = 0$.

These two equations can be alternatively written as $d(x^3y^4) = 0$ and $d(e^{xy} – x^2) = 0$, respectively. Thus, the general solution of these equations are give by $x^3y^4 = c$ and $e^{xy} = x^2 + c$, where c is constant.

However except for some cases, this technique of “solution by insight” is clearly impractical. Consequently we seek an answer to the following question: when does a function h(x, y) exist such that Eqn. (26) is exact? An answer to this question is given by the following theorem.

Theorem 1: If the functions a(x, y), b(x, y), $a_x\ \frac{\partial a}{\partial x}$ and $b_y = \frac{\partial b}{\partial y}$ are continuous functions of x and y, then Eqn. (26), namely,

a(x, y) dy + b(x, y) dx = 0 is exact if and only if

$$\frac{\partial}{\partial y} b(x, y) = \frac{\partial}{\partial x} a(x, y) \qquad \ldots(27)$$

Indeed condition (27) is a necessary and sufficient condition for a function h(x, y) to be such that

$$\frac{\partial}{\partial x} h(x, y) = b(x, y) \text{ and } \frac{\partial}{\partial y} h(x, y) = a(x, y) \qquad \ldots(28)$$

you may note here that if relation (28) is satisfied, then

$$d[h(x, y(x))] = \frac{\partial}{\partial x} h(x,y(x))\, dx + \frac{\partial}{\partial y} h(x,y(x))\, dy$$
$$= b(x, y(x))dx + a(x, y(x))\, dy$$

and hence Eqn. (26) can be rewritten as

d[h(x, y(x))] = 0

or that the solution of Eqn. (26) is given by

h (x, y) = c,

where c is a constant.

We now give the proof of Theorem 1.

Proof: The condition is necessary

Let the equation

$$a(x, y)\frac{dy}{dx} + b(x, y) = 0$$

Or, a(x, y) dy + b(x, y) dx = 0

Be exact.

Then there exists a function h(x, y) such that

dh = b(x, y) dx + a(x, y) dy

But $dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$.

$$\therefore \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy = b(x, y)\, dx + a(x, y)\, dy$$

Thus, necessarily,

$$\frac{\partial h}{\partial x} = b(x, y) \text{ and } \frac{\partial h}{\partial y} = a(x, y) \qquad \ldots(29)$$

Since a(x, y) and b(x, y) have continuous first order partial derivatives, h possess continuous second order partial derivatives namely, $\frac{\partial^2 h}{\partial y \partial x}$ and $\frac{\partial^2 h}{\partial y \partial y}$. Refer Unit 6, Block 2 of MTE-07 for second and higher order partial derivatives.
Now,

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial b}{\partial y} \text{ (x, y)} \qquad \ldots(30)$$

and $$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial a}{\partial x} \text{ (x, y)} \qquad \ldots(31)$$

Since $\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial x}$ are continuous,

$\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y}$ (ref. Unit 6, Block 2 of MTE-07).

There, from Eqn. (30) and (31), we get

$\frac{\partial a}{\partial x}$(x, y) = $\frac{\partial b}{\partial y}$(x, y)

The condition is sufficient: Now suppose that

$\frac{\partial a}{\partial x}$(x, y) = $\frac{\partial b}{\partial y}$(x, y)

and we shall show that a(x, y) dy + b(x, y) dx is an exact differential

Let $\int$ b(x, y) dx = V, then $\frac{\partial V}{\partial x}$ = b(x, y)

and

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial b}{\partial y} \text{ (x, y)}$$

$$= \frac{\partial a}{\partial x} \text{ (x, y) (using given condition)}$$

$$\therefore \frac{\partial a}{\partial x} \text{ (x, y)} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) \qquad \ldots(32)$$

Integration of Eqn. (32) with respect to x, holding y fixed, yields

a(x, y) = $\frac{\partial V}{\partial y}$ + ø (y)

where ø, a function of y only, is therefore a constant of integration, when y is held fixed.

Thus,

$$a(x, y)\,dy + b(x, y)\,dx = \frac{\partial V}{\partial y}dy + ø(y)\,dy + \frac{\partial V}{\partial x}dx$$
$$= d[V(x, y) + ø(y)\,dy]$$

which establishes that a(x, y) dy + b(x, y) dx is an exact differential implying thereby that, a(x, y) dy + b(x, y)dx = 0 is an exact differential equation. This completes the proof of Theorem 1.

We shall now illustrate this theorem with the help of the following examples.

Example 9: Solve the differential equation
Sin(y) + x cos (y)y' = 0.

Solution: For the case under consideration, a(x, y) = xcos(y) and b(x, y) = sin(y). also

$$\frac{\partial}{\partial x}\, a(x, y) = \cos(y) = \frac{\partial}{\partial y}\, b(x, y)$$

which shows that the given equation is an exact equation.

Therefore, there exists a function h(x, y) = constant such that, $\frac{\partial h}{\partial x}$ = b(x, y) and $\frac{\partial h}{\partial y}$ = a(x, y)

Then we have

$$\frac{\partial h}{\partial x} = \sin y \qquad \ldots(33)$$

and

$$\frac{\partial h}{\partial y} = x \cos y \qquad \ldots(34)$$

Integrating Eqn. (33) with respect to x, treating y as a constant, we get

$$h(x, y) = x \sin y + ø\,(y) \qquad \ldots(35)$$

Where ø(y) is a constant of integration. Differentiating Eqn. (35) partially w.r.t.y, we get

$$\frac{\partial}{\partial y}\, h(x, y) = x \cos y + ø'(y) \qquad \ldots(36)$$

from Eqns. (34) and (36), we get
x cos y = x cos y + ø'(y)
which shows that ø'(y) = 0 $\Rightarrow$ø(y) = constant = c_1. Hence from Eqn. (35), we can write

$h(x, y) = x\sin y + c_1$
so the required solution, h(x, y) = constant, is
$x \sin y + c_1 = c_2$, where c_2 is a constant or,
$x \sin y = c$,

where $c = c_2 – c_1$ is a constant.

Example 10: Solve $e^x \sin y + e^x \cos y\, y' + 2x = 0$.

Solution: Comparing with Eqn. (26), we have $a(x, y) = e^x\cos y$ and $b(x, y) = e^x\sin y + 2x$. Therefore,

$$\frac{\partial}{\partial x} a(x, y) = e^x\cos y$$

and $\frac{\partial}{\partial y} b(x, y) = e^x\cos y = \frac{\partial}{\partial x} a(x, y)$.

Hence the given equation is exact and can be written in the form dh(x, y(x) = 0 where

$$\frac{\partial}{\partial x} h(x, y) = e^x\sin y + 2x \qquad \ldots(37)$$

and $\frac{\partial}{\partial y} h(x, y) = e^x \cos y$ …(38)

Integrating Eqn. (37) w.r.t.x, we get

$$h(x, y) = e^x \sin y + x^2 + ø(y) \qquad \ldots(39)$$

Where ø, a function of y only, is a constant of integration
From Eqns. (38) and (39), we get

$$\frac{\partial}{\partial y} h(x, y) = e^x\cos y + ø'(y) = e^x \cos y$$

So we have $ø'(y) = 0$ or $ø(y) = c_1$ where c_1 is a constant.
Hence from Eqn. (39) we have the required solution as

$$h(x, y) = e^x \sin y + x^2 = - c_1 = c$$

where c is a constant.

On the basis of Theorem 1 and Example (9) and (10) we can say that various steps involved in solving an exact differential equation b(x, y) dx + a(x, y) dy = 0 are as follows:

Step 1: Integrate b(x, y) w.r.t.x, regarding y as a constant.

Step 2: Integrate, with respect to y, those terms in a(x, y) which do not involve x.

Step 3: The sum of the two expressions obtained in steps 1 and 2 equated to a constant is the required solution.

We now illustrate these various steps with the help of an example.

Example 11: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dx = 0$.

Solution: Here $a(x, y) = y^2 - 4xy - 2x^2$ and $b(x, y) = x^2 - 4xy - 2y^2$

$\therefore \frac{\partial a}{\partial x} = -4y - 4x$ and $\frac{\partial b}{\partial y} = -4x - 4y$

$\therefore \frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$; hence it is an exact equation.

Step1: Integrating b(x, y) w.r.t.x. regarding y as a constant, we have

$\int (x^2 - 4xy - 2y^2)\,dx = \frac{x^3}{3} - 2x^x y - 2xy^2.$

Step 2: We integrate those terms in a(x, y) w.r.t.y, which do not involve x. There is only one such term namely, y^2.

$\therefore \int y^2\,dy = \frac{y^3}{3}.$

Step 3: The required solution is the sum of expressions obtained from Steps 1 and 2 equated to a constant, that is.

$\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c_1,$

or $x^3 - 6x^2y - 6xy^2 + y^3 = c.$

where c and c_1 are constants.

Note that the test for an exact differential equation and the general procedure for finding the solution can sometimes be simplified. We can pick out those terms of a(x, y) dy + b(x, y) dx = 0 that obviously form an exact differential or can take the form f(u) du. The remaining, expression which is less cumbersome than the original can then be tested and integrated. This is illustrated by the following example.

Example 12: Solve $x\,dx + y\,dy + \frac{x^2 d(^y/_x)}{x^2 + y^2} = 0.$

Solution: Note that the first two terms on the left hand side of the given equation are exact differentials and hence need not be touched. Dividing the numerator and denominator of the last term by x^2, we get

$$xdx + ydy + \frac{d(y/x)}{1+(y/x)^2} = 0$$

Now each term of the above equation is an exact differential. Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}\frac{y}{x} = c$$

as the required solution with c as a constant.

You may now try the following exercises.

In practice the differential equations of the form a(x, y) dy + b(x, y) dx = 0 are rarely exact, since the condition in Theorem 1 requires a precise balance of the functions a(x, y) and b(x, y). But they can often be transformed into exact equations on multiplication by a suitable function F(x, y) $\neq$ 0. This function is then called an integrating factor. The question we, now, must ask is: if

a(x, y) dy + b(x, y) dx = 0

is not exact, then how to find a function

F(x, y) $\neq$ 0 so that
F(x, y) [a dy + b dx] = 0

Is exact? In the next section we shall give an answer to this question.

3.4 Integrating Factor

We begin with a very simple equation, namely,

$y' + y = 0$ …(40)

In this case a(x, y) = 1 and b(x, y) = y. Here $\frac{\partial}{\partial x}$ a(x, y) = 0 and $\frac{\partial}{\partial y}$ b(x, y) = 1 and hence the given equation is not exact. Let us multiply Eqn. (40) by $e^{x \text{ to get}}$

$e^x y' + e^x y = 0$ …(41)

you may now check that Eqn. (41) is an exact equation. Thus Eqn. (40) is not exact whereas when we multiply EQn. (40) by e^x the resulting equation becomes an exact one.

Here e^x is termed as **integrating factor** for Eqn. (40).

We now give the following definition.

Definition: A factor, which when multiplied with a non-exact differential equation makes it exact, is known as an integrating factor (abbreviated as I.F.).

The term I.F., to solve a differential equation, was first introduced by Fatio de Duillier in 1687.

For a given equation, there may not be a unique integrating factor.

Consider, for example, the equation

$$ydx - xdy = 0 \qquad \ldots(42)$$

You can check that Eqn. (42) is not exact, but when multiplied by $\frac{1}{y^2}$, it becomes

$$\frac{ydx - xdy}{y^2} = 0$$

which is exact. This can now be written as d $\left(\frac{x}{y}\right)$ = 0 and thus has for its solution $\frac{x}{y}$ = c with c being an arbitrary constant.

Further, when Eqn. (42) is multiplied by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

which is given exact and has its solution as Inx –Iny = c.

you may notice that this solution can be transformed into the earlier solution obtained through the I.F. $\frac{1}{y^2}$. Also Eqn. (42) when multiplied by $\frac{1}{x^2}$ reduces to an exact equation $\frac{y}{x^2}$ dx - $\frac{dy}{x}$ = 0 or, - d$\left(\frac{y}{x}\right)$ = 0 with | - $\frac{y}{x}$ = c as its solution.

Thus, we have seen that some of the integrating factors for Eqn. (42) are $\frac{1}{y^2}$, $\frac{1}{xy}$ and $\frac{1}{x^2}$

Now the question arises: Is this the case only with Eqn. (42) or, in general, does an equation of the form a(x, y) dy + b(x, y) dx = 0 have infinitely many integrating factors?

An answer to this question is given in Theorem 2.

Before we give you this theorem, here is an exercise for you.

Theorem 2: The number of integrating factors for the equation
A(x, y) dy + b)x, y) dx = 0
is infinite

proof: Let g(x, y) be an integrating factor of the given equation. Then, by definition

$$g(x, y)\left[a(x, y)\frac{dy}{dx} + b(x, y)\right] = 0 \qquad \ldots(43)$$

is an exact differential equation.

Therefore, there exist a function h(x, y) such that

$$dh = g(x, y)\left[a(x, y)\frac{dy}{dx} + b(x, y)\right] = f(h)dh = d\,[\int f(h)\ dh] \qquad \ldots(44)$$

Since the term on the right hand side of Eqn (44) is an exact differential, the term in the left must also be an exact differential. Therefore, g(x, y).f(h) is an integrating factors of the given differential equation.

Since f(h) is an arbitrary function of h, hence the number of integrating factors for equation a(x, y) dy + b(x, y) dx = 0 is infinite.

This fact is, however, of no special assistance in solving the differential equations.

So far, in our discussion we have not paid any attention to the problem of finding the integrating factors. In general, it is quite difficult to obtain an integrating factor for a given equation. However, rules for finding the integrating factors do exist, we shall now take up these rules one by one.

Rules for finding integrating factors

Rule 1: Integrating factors of obtainable inspection: Sometime integrating factors of a differential equation can be seen at a glance, as in the case of EQn. (42) above. We give below some more examples in this regard.

Example 13: Solve (1 + xy) ydx + (1 – xy)xdy = 0, x>0, y > 0 …(45)

Solution: Rearranging the terms of Eqn. (45), we get

$$ydx + xdy + xy^2dx - x^2ydy = 0$$

$$\Rightarrow d(xy) + xy^2dx - x^2ydy = 0 \qquad \ldots(46)$$

It is immediately seen that multiplication by $\frac{1}{x^2y^2}$ makes Eqn. (46) exact and the equation becomes

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

integrating, we get

$$- \frac{1}{xy} + \text{In}x – \text{In}y = \text{In}c,$$

or $x = cy\, e^{1/(xy)}$ (where c is a constant).

Example 14: Solve $(x^4e^x – 2my^2x)\, dx + 2mx^2y\, dy = 0$.

Solution: We can write the given equation as

$$x^4e^x\, dx + 2m\, (x^2ydy - xy^2dx) = 0$$

$$\Rightarrow x^4e^x\, dx + 2mx^3y\, d\left(\frac{y}{x}\right) = 0$$

Dividing by x^4, we get

$$dx + 2m\, \frac{y}{x}\, d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow d\left[e^x + m\left(\frac{y}{x}\right)^2\right] = 0$$

thus $\frac{1}{x^4}$ has served the role of an integrating factor in this case.

The required solution is, then, given by

$e^x + m\left(\frac{y}{x}\right)^2 = c$ with c as a constant.

We would like to mention that determination of an integrating factor by inspection is a skill and can be developed through practice only.

At this stage you may try the following exercises by finding an integrating factor through inspection.

Rule II: For a homogeneous equation a(x, y)dy + b(x, y) dx -= 0, when bx + ay $\neq$ 0, then $\frac{1}{bx+ay}$ is an integrating factor.

Proof: Consider an equation

$a(x, y)\, dy + b(x, y)\, dx = 0$

$$\text{Now } ady + bdx = \frac{1}{2}\left[(bx+ay)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (bx-ay)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\therefore \frac{ady + bdx}{ay + bx} = \frac{1}{2}\left[\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{bx - ay}{x + ay}\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

since the given equation is homogeneous, therefore a and b are of the same degree in x and y and therefore $\frac{bx - ay}{bx + ay}$ can be written as a function of $\frac{x}{y}$, say $f\left(\frac{x}{y}\right)$.

$$\therefore \frac{ady + bdx}{ay + bx} = \frac{1}{2}\left[\left(\frac{dx}{x} + \frac{dy}{y}\right) + f\left(\frac{x}{y}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[d(\text{In } xy) + f(e^{\text{In} x/y})\ d\left(\text{In}\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[d(\text{In } xy) + df\left(\text{In}\frac{x}{y}\right)\right]$$

$$= d\left[\frac{1}{2}\ \text{In} xy + \frac{1}{2} f'\left(\text{In}\frac{x}{y}\right)\right] \qquad \ldots(47)$$

where $d\, f\left(\frac{x}{y}\right) = f(e^{\text{In} x/y})\ d\left(\text{In}\frac{x}{y}\right)$.

Since right hand side of Eqn. (47) is an exact differential, it shows that $\frac{1}{ay + bx}$ is an integrating factor for the homogeneous equation a(x, y)dy + b(x, y)dx = 0.

We illustrate this rule by the following example.

Example 15: Solve $(x^2y - 2xy^2)\ dx - (x^2 - 3x^2 + y)\ dy = 0$.

Solution: Here the given equation is homogeneous and
$a(x, y) = x^3 + 3x^2y$ and $b(x, y) = x^2y - 2xy^2$
$\therefore bx + ay = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y) = x^2y^2 \neq 0$,
$\therefore \frac{1}{x^2y^2}$ is an integrating factor.

Multiplying the given differential equation by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0,$$

or $\frac{dx}{y} + 3\frac{dy}{y} - 2\frac{dx}{x} - \frac{x}{y^2}\ dy = 0,$

or $\left(\frac{dx}{y} - \frac{x}{y^2}dy\right) + 3\frac{dy}{y} - 2\frac{dx}{x} = 0,$

or $d\left(\frac{x}{y}\right) + d(3 \text{ Iny} + 2 \text{ Inx}) = 0.$

Therefore, the solution is

$\frac{x}{y} + 2 \text{ Iny} - 2 \text{ Inx} = c_1$ and c are constants.

Note: In case bx + ay = 0, then $\frac{a}{b} = -\frac{y}{x}$ and the given equation reduces to $\frac{dy}{dx} = \frac{y}{x}$, Whose solution is straightaway obtained as x = cy.

You may now try this exercise.

Rule III: when bx – ay $\neq$ 0 and the differential equation a(x, y) dy + b(x, y) dx = 0 can be written in the form yf_1(x y) dx + xf_2(xy) dy = 0 then $\frac{1}{bx - ay}$ is an integrating factor.

Proof: If equation a(x, y) dy + b(x, y)dx = 0 can be written in the form $yf_1(xy)dx + xf_2(xy)dy = 0$

Then evidently,

$$\frac{a}{xf_2(xy)} = \frac{b}{yf_1(xy)} = \lambda, \text{ say}$$

$\therefore a = \lambda xf_2(xy)$ and $b = \lambda yf_1(xy)$

Also $ady + bdx = \frac{1}{2}\left[(bx+ay)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (bx-ay)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$

$$\therefore \frac{ady+bdx}{bx+ay} = \frac{1}{2}\left[\frac{bx+ay}{bx-ay}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f_1+f_2}{f_1-f_2} d(\text{Inxy}) + d\left(\text{In}\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[f(xy)d(\text{In } xy) + d\left(\text{In}\frac{x}{y}\right)\right]$$

where $f(xy) = \frac{f_1+f_2}{f_1-f_2}$

$$= \frac{1}{2}\left[df(\text{Inxy}) + d\left(\text{In}\left(\frac{x}{y}\right)\right)\right]$$

$$= d\left[\frac{1}{2}\text{In}\left(\frac{x}{y}\right)+\frac{1}{2}f(\text{In}(xy))\right]$$

where dF (In xy) = f(x, y) d(In xy),
which is an exact differential.

Hence, $\frac{1}{bx-ay}$ is an integrating factor.

We now illustrate this through the following example.

Example 16: Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)dy = 0$,

Solution: Here $a = x(xy - x^2y^2)$ and $b = y(xy + 2x^2y^2)$

$\therefore bx - ay = xy[xy + x^2y^2 - xy + 2x^2y^2]$

$= 3x^3y^3 \neq 0.$

$\therefore \frac{1}{3x^3y^3}$ is an I.F.

Multiplying the given equation by $\frac{1}{3x^3y^3}$, we get

$$\frac{1}{3x^3y^2}(xy + 2x^2y^2)\,dx + \frac{1}{3x^2y^3}(xy - x^2y^2)\,dy = 0$$

$$\text{or } \frac{dx}{3x^2y} + \frac{3dx}{3x} + \frac{dy}{3xy^2} - \frac{dy}{3y} = 0$$

$$\text{or}\left[\frac{dx}{3x^2y}+\frac{dy}{3xy^2}\right]+\frac{2}{3}\frac{dx}{x}-\frac{1}{3}\frac{dy}{y} = 0$$

$$\text{or } d\left(-\frac{1}{3}\frac{1}{xy}+\frac{2}{3}\text{ Inx}-\frac{1}{3}\text{ Iny}\right) = 0.$$

Therefore, the solution is

$-\frac{1}{3xy}+\frac{2}{3}\text{Inx}-\frac{1}{3}\text{Iny} = c_1$ where c_1 is a constant.

or $-\frac{1}{xy}+\text{Inx}^2-\text{Iny} = 3c_1 = c$ for c being an arbitrary constant.

or $\text{In}\left(\frac{x^2}{y}\right) = c + \frac{1}{xy}$.

Note: If bx – ay = 0, i.e., $\frac{a}{b}=\frac{y}{x}$, then given equation will be of the form $\frac{dy}{dx}=-\frac{y}{x}$ and have a solution xy = c.

Before we go to the next rule here is an exercise for you.

Rule IV: When $\frac{1}{a}\left(\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right)$ **is a function of x alone, say f(x), then** $e^{\int f(x)\,dx}$ **is an I.F. of the equation ady + bdx = 0**.

Proof: Consider the equation $e^{\int f(x)\,dx}$ (ady + bdx) = 0 …(48)

Let c = b $e^{\int f(x)\,dx}$ and d = a $e^{\int f(x)\,dx}$
Then Eqn. (48) reduces to c dx + d dy = 0
Now, $\frac{\partial c}{\partial y}=\frac{\partial b}{\partial y}\; e^{\int f(x)\,dx}$
and $\frac{\partial d}{\partial x}=\frac{\partial a}{\partial x}\; e^{\int f(x)\,dx} + a\; e^{\int f(x)\,dx}\; f(x)$

$$= e^{\int f(x)\,dx}\left[\frac{\partial a}{\partial x}+a\;f(x)\right]$$

$$= e^{\int f(x)\,dx}\left[\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right]\left[\text{because}\,\frac{1}{a}\left(\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right)=f(x)\right]$$

$$= \frac{\partial b}{\partial y}\; e^{\int f(x)\,dx}$$

$$= \frac{\partial c}{\partial y}$$

therefore, the equation c dx + d dy = 0 is exact.

Hence $e^{\int f(x)\,dx}$ is an I.F. of the equation ady + bdx = 0.

We illustrate this rule with the help of the following example.

Example 17: Solve $(x^2 + y^2)\,dx - 2xydy = 0$.

Solution: Here a = 2xy, $b = x^2 + y^2$

$\therefore \frac{\partial b}{\partial y}$ = 2y and $\frac{\partial a}{\partial x}$ = - 2y. thus, $\frac{\partial b}{\partial y} \neq \frac{\partial a}{\partial x}$.

Here $\frac{1}{a}\left(\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right) = \frac{1}{-2xy}$ (2y + 2y) = - $\frac{2}{x}$, which is a function of x alone.

$$\therefore \text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

Multiplying the given equation by $\frac{1}{x^2}$, we get

$\frac{1}{x^2}(x^2 + y^2)$ dx - $\frac{2y}{x}$ dy = 0,

i.e., dx $\frac{y^2}{x^2}$ dx - $\frac{2y}{x}$ dy = 0,

i.e., dx + d$\left(-\frac{y^2}{x}\right)$ = 0 …(49)

integrating Eqn. (49), the required solution is obtained as

x - $\frac{y^2}{x}$ = c (a constant)

You may now try this exercise.

Rule V: When $\frac{1}{b}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right)$ is a function of y alone, say f(y), then $e^{\int f(x)\, dy}$ is an I.F. of the differential equation
a dy + b dx = 0.

The proof of this rule is similar to the proof of Rule IV above and we leave this as an exercise for you (see E 13).

We however illustrate the use of Rule V with the help of following example.

Example 18: Solve $(3x^2y^4 + 2xy)$ dx + $(2x^3y^3 - x^2)$ dy = 0.

Solution: Here a = $2x^3y^3 - x^2$ and b = $3x^2y^4 + 2xy$

$\therefore \frac{\partial b}{\partial y} = 12x^2y^3 + 2x$ and $\frac{\partial a}{\partial x} = 6x^2y^3 - 2x$

Here $\frac{1}{b}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right) = \frac{1}{3x^2y^4 + 2xy}(12x^2y^3 + 2x - 6x^2y^3 + 2x)$

$= \frac{2(3x^2y^3 + 2x}{y(3x^2y^3 + 2x} = \frac{2}{y}$, which is a function of y alone.

Hence, I.F. = $e^{\int -\frac{2}{y} dy|}$ $|e^{-2Iny} = y^{-2} = \frac{1}{y^2}$.

Multiplying the given equation by the I.F. = y^{-2}, and on rearranging th terms, we get

$(3x^2y^2\, dx + 2x^3y dy) + \left(\frac{2x}{y}dx - \frac{x^2}{y^2}dy\right) = 0$

i.e., $d(x^3y^2) + d\left(\frac{x^2}{y}\right) = 0$

Integrating the above equation, we get

$x^3y^2 + \frac{x^2}{y} = c$, where c is a constant of integration.

i.e., $x^3y^3 + x^2 = cy$, which is the required solution.

And now an exercise for you.

Rule VI: If the differential equation is of the form
$x^\alpha y^\beta$ (mydx + nxdy) = 0, where α, β, m and n are certain constants, then $x^{km-1\alpha} y^{kn-1-\beta}$ is an integrating factor, where k can assume any value.

Proof: Multiplying the given equation by I.F., we get

$X^{km-1} y^{kn-1}$ (mydx + nxdy) = 0,
or km $x^{km-1} y^{kn}$ dx + kn $x^{km}y^{kn-1}$ dy = 0
or $(x^{km-1}y^{kn}) = 0$, which is an exact differential.

It may be noted that if the given differential equation is of the form
$x^\alpha y^\beta$ (mydx + nxdy) + $x^{\alpha 1}y^{\beta 1}(m_1ydx + n_1xdy) = 0$
then also I.F. can be determined.

By Rule VI, $x^{km-1-\alpha}y^{kn-1-\beta}$ will make the first term exact, while $x^{k_1m_1-1-\alpha_1} y^{k_1n_1-1-\beta}$ will make the second term exact, where k and k_1 can have any value.

Now these two factors will be identical if

$kn-1-\alpha = k_1m_1 - 1-\alpha_1$
and $kn -1-\beta = k_1n_1-1-\beta_1$.

Values of k and k_1 can be found tp satisfy these two algebraic equations. Then either factor is an integrating factor of the above equation.

We now consider an example to illustrate this rule.

Example 19: Solve $(y^3 - 2yx^2)$ dx + $(2xy^2 - x^3)$ dy = 0.

Solution: On rearranging the term of the given equation, we can write
$y^2(ydx + 2xdy) - x^2(2ydx + xdy) = 0$ …(50)

For the first term, $\alpha = 0, \beta = 2$, m = 1 and n = 2 and hence its I.F.is $x^{k-1}y^{2k-1-2}$
For the second term $\alpha_1 = 2$, $\beta_1 = 0$, $m_1 = 2$, $n_1 = 1$ and hence for the second term I.F. is $x^{2k_1-1-2}y^{k_1-1}$
these two integrating factors will be identical if

$$\left.\begin{array}{l} k-1 = 2k_1 -1-2 \\ \text{and } 2k-1-2 = k_1 - 1 \end{array}\right\} \qquad \ldots(51)$$

solving the system fo Eqn. (51) for k and k_1, we get k = 2 and k_1 = 2 and, therefore, integrating factor for Eqn. (50) is
$x^{2-1}\, y^{4-1-2}$, i.e., xy.

Multiplying Eqn. (50) by xy, we get

$$xy^3 (ydx + 2xdy) – x^3y (2ydx + xdy) = 0$$

$$\Rightarrow xy^4dx + x^2y^3dy – (2x^3y^2dx + x^4ydy = 0$$

$$\Rightarrow \frac{1}{2} (2xy^4dx + 4x^2y^3dx) – \frac{1}{2}(4x^3y^2dx + 2x^4y^4dy) = 0$$

$$\Rightarrow \frac{1}{2} d[2x^2y^4] - \frac{1}{2} d[x^4y^2] = 0 \qquad \ldots(52)$$

Integrating Eqn. (52), we get the required solution as

$$\frac{x^2y^4 – x^4y^2}{2} = c_1 \text{ (a constant)}$$

or $x^2y^2(y^2 – x^2) = 2c_1 = c$ (a constant).

You may now apply your knowledge about these rules and try to solve the following exercises.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have covered the following:

1) An equation $\frac{dy}{dx}$ = f(x, y) is called a **separable equation** or an **equation with separable variables** if f(x, y) = X(x) Y(y). to solve a separable equation, we can write it as

$$a(y) \frac{dy}{dx} + b(x) = 0$$

For some a(y) and b(x). Integrating w.r. to x and equating it to a constant, we get its solution.

2) a) A real-valued function h(x, y) of two variables x and y is called a **homogeneous function** of degree n, if $h(\lambda x, \lambda y) = \lambda^n h(x, y)$, where n is a real number and λ is any constant.

b) A differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a **homogeneous differential equation** of first order when f is a homogeneous function of degree zero.

c) A homogeneous differential equation reduces to separable equation by the substitution y = vx, where v is some function of x.

3) Equations of the form

$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ where a, b, c, a', b', c' are constants and $\frac{a}{a'} \neq \frac{b}{b'}$ can be **reduced to homogeneous equations** by the substitution x' = x + h, y' = y + k, where h and k are such that ah + bk + c = 0 and a'h + b'k + c' = 0.

In case $\frac{a}{a'} \neq \frac{b}{b'} = \frac{1}{m}$, say, then substitution ax + by = v reduces this type of equations to separable equations.

4) An exact differential equation s formed by equating an exact differential to zero.

5) The differential equation
A(x, y)dy + b(x, y)dx = 0

6.0 TUTOR MARKED ASSIGNMENT

1. Solve the following equations.

a) (1 – x)dy – (1 + y) dx = 0

b) $y - x\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0$

c) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

d) $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$

c) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

2. Solve the following equations satisfying initial condition indicted alongside.

a) $2xy\frac{dy}{dx} = 1 + y^2$, y(2) = 3 ∀ x, y > =

b) $\frac{dy}{dx}$ = - 4xy, y(0) = $y_0 \forall y > 0$

c) $\frac{dy}{dx}$ xe^{y-x^2}, y(0) = 0

d) y $\frac{dy}{dx}$ = g, y $(x_0) = y_0$

3. Solve the following equations.

a) $\frac{dy}{dx} = \frac{y}{x}$ for x $\in [0, \infty]$and for x $\in [-\infty, 0]$

b) $\frac{dy}{dx} = \frac{2x + y}{3x + 2y}$

c) (x sin $\frac{y}{x}$) dy – (y $\sin^{-1} \frac{y}{x}$ - x) dx = 0

d) x $\frac{dy}{dx}$ = y (Iny – Inx + 1)

e) x dy – y dx = $\sqrt{x^2 - y^2}$dx.

4. Solve the following equations subject to the indicated initial conditions.

a) $2x^2 \frac{dy}{dx} = 3xy + y^2$, y(1) = -2

b) $(x + ye^{y/x})$ dx – $xe^{y/x}$ dy = 0, y(1) = 0

c) $(y^2 + 3xy)$ dx = $(4x^2 + xy)$ dy, y(1) = 1.

d) $y^2dx + (x^2 + xy + y^2)$ dy = 0, y (0) = 1.

5. Solve the following equations.

a) $\frac{dy}{dx} = \frac{2y - x - 4}{y - 3x + 3}$

b) (7y – 3x + 3) $\frac{dy}{dx}$ + (3y – 7x + 7) = 0

c) $(2x + y + 1)\, dx + (4x + 2y – 1)\, dy = 0$

d) $(x + y)\, dx + (3x + 3y – 4)\, dy = 0$

6. Prove that the following equations are exact and solve them.

a) $(y\cos(x) + 2x\, e^y) + (\sin(x) + x^2 e^y + 2)y' = 0$
b) $y' = -\frac{ax+by}{bx+cy}$ (a,b,c,d are given real constants).
c) $96x + y/x) + (Inx + y)y' = 0,\ x \geq 1.$

7. Determine the values of k for which the equations given below are exact and find the solution for these values of k.

a) $x + kyy' = 0\ (k \neq 0)$
b) $y + kxy' = 0\ (k \neq 0)$
c) $(2y\, e^{2xy} + 2x) + k\, x\, e^{2xy} y' = 0$

8. In each of the following equations verify that the function F(x, y), indicated alongside is an I.F. of the equation:

i. $6xy\, dx + (4y + 9x^2)\, dy = 0;\ f(x, y) = y^2$
ii. $–y^2\, dx + (x^2 + xy)\, dy = 0;\ f(x, y) = \frac{1}{x^2 y}$
iii. $(- xy \sin x + 2y \cos x)\, dx + 2x \cos x\, dy = 0;\ F(x, y) = xy.$

9. Solve the following equations.

a) $y(2yx + e^x)\, dx – e^x\, dy = 0$
b) $ydx – xdy + Inx\, dx = 0\ \forall x, y > 0.$
c) $(xy – 2y^2)\, dx – (x^2 – 3xy)dy = 0$

10. Solve $(x^4 + y^4)\, dx – xy^3 dy = 0.$

11. Solve $y(x^2y^2 + 2)\, dx + x(2 – 2x^2y^2)\, dy = 0.$

12. Solve $(x^2 + y^2 + x)\, dx + xy\, dy = 0.$

13. Prove Rule V above.

14. Solve $(2xy^4e^y + 2xy^3 + y)\, dx + (x^2y^4e^y – x^2y^2 – 3x)\, dy = 0$

15 Solve the following equations.
a) $(x^2 + y^2 + 2x)\, dx + 2y = 0$
b) $x^2y\, dx – (x^3 + y^3)\, dy = 0$
c) $(y^4 + 2y)dx + (xy^3 + 2y^4 – 4x)\, dy = 0\ \forall x, y > 0.$

d) $(y^2 + 2x^2y)dy + (2x^3 - xy)dy = 0$
e) $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0$

16. Solve the following equations.

a) $(x + y)^2 \dfrac{dy}{dx} = a^2$
b) $ydx + dy = 0$
c) $1 + \left(\dfrac{x}{y} - \sin y\right)\dfrac{dy}{dx} = 0$
d) $(3y^2 + 2xy) = (2xy + x^2)\dfrac{dy}{dx} = 0 \;\forall\, x > 0,\, y > 0.$
e) $y + y_2 + \left(2xy + \dfrac{y}{1+y}\right)\dfrac{dy}{dx} = 0$
f) $2x^2y^2 + 3x(1 + y^2)\dfrac{dy}{dx} = 0$
g) $\dfrac{dy}{dx} + \dfrac{2y}{x} = 0,\; y \geq 0$

7.0 REFERENCES/FURTHER READINGS

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Indira Gandhi National Open University School of Sciences Mth-07