MODULE 2

UNIT 1 LINEAR DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

In unit 2, we have discussed methods of solving some first order first degree differential equations, namely,

i) differential equations which could be integrated directly i.e., separable and exact differential equations,

ii) equations which could be reduced to these forms when direct integration is not possible. These includes homogeneous equations, equations reducible to homogeneous form and equations that become exact when multiplied by an I.F.

in this unit, we focus our attention on another very important type of first order first degree differential equations known as **linear equations**. These equations are important because of their wide range of applications, for example, the physical situations we gave in Sec. 1.5 of unit 1 are all governed by linear differential equations. In this unit, we shall solve some of these physical problems.

The problem of integrating a linear differential equation was reduced to quadrature by Leibniz in 1692. In December, 1695, James Bernoulli proposed a solution of a non-linear differential equation of the first order, now known as Bernoulli's equation.

In 1696, Leibniz pointed out Bernoulli's equation may be reduced to a linear differential equation by changing the dependent variable. We shall discuss this equation in the later part of this unit along with some other equations, which may not be of first order or first degree but which can be reduced to linear to linear differential equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify a linear differential equation
- distinguish between homogeneous and non-homogeneous linear differential equation
- obtain the general solution of a linear differential equation
- obtain the particular integral of a linear equation by the methods of undetermined coefficients and variation of parameters
- use general properties of the solutions of homogeneous linear equations for finding their solutions
- obtain the solution of Bernoulli's equation
- obtain solution to linear equations modeled for certain physical situations.

3.0 MAIN CONTENT

3.1 Classification of First Order Differential Equations

We begin by giving some definitions in this section. You may recall that in Unit 1 we defined the general form of first order differential equation to be

$$g\left(x, y\frac{dy}{dx}\right) = 0$$

and if the equation is of first degree, then it can be expressed as

$$\frac{dy}{dx} = f(x, y)$$

In the above equation if the function f(x, y) be such that it contains dependent variable y in the first degree only, then it is called a linear differential equation. Formally, we have the following definition.

Definition: We say that a differential equation is linear if the dependent variable and all its derivatives appear only in the first degree and also there is no term involving the product of the derivatives or any derivative and the dependent variable.

For example, equation $\frac{dy}{dx} + \frac{2y}{x} = x^3$ and $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x \sin x$ are linear differential equations. However $\frac{dy}{dx} + x^2 = 10$ is not linear equation of the presence of the term y $\frac{dy}{dx}$.

The general form of the linear differential equation of the first order is

$$a(x) \frac{dy}{dx} = b(x)y + c(x) \qquad \ldots(1)$$

where a(x), b(x) and c(x) are continuous real valued functions in some interval $I \subseteq R$.
If c(x) is identically zero, then Eqn. (1) reduces to

$$a(x) \frac{dy}{dx} = b(x)y \qquad \ldots(2)$$

Eqn. (2) is called **a linear homogeneous differential equation.**
When c(x) is zero, Eqn. (1) is called **non-homogeneous** (or **inhomogeneous**) linear differential equation

Note: You may note that the word homogeneous as it is used here has a very different meaning from that used in Sec. 2.3, unit 2.

Any differential equation of order one which is not of type (1) or (2) is called a **non-linear differential equation.**

On dividing Eqn. (1) by a(x) for x s.t $a(x) \neq 0$, it can be put in the more useful form

$$\frac{dy}{dx} + P(x)\, y = Q(x), \qquad \ldots(3)$$

where P a. I Q are functions of x alone or are constants. Consider, for instance, the equation $\frac{dy}{dx} = y$

It is a linear homogeneous equation. Here a(x) = 1 and b(x) = 1. Similarly,
$\frac{dy}{dx} = 0$, $\frac{dy}{dx} = e^x y$ are also linear homogeneous equation of order one with

$a(x) = 1$, $b(x) = e^x$ and $c(x) = x$.

Next consider the differential equation $\frac{dy}{dx} = |y|$.

You know that $|y| = y$ for ≥ 0 and $|y| = -y$ for $y < 0$. Hence, in order to solve this equation, we will have to square it and the resulting equation is neither of type (1) nor of (2). It is a case of non-linear equation. Similarly, $\left|\frac{dy}{dx}\right| = y$ is a non-linear equation because of the term $\left|\frac{dy}{dx}\right|$. Again $\frac{dy}{dx}$ = cos y is a non-linear equation (as cos y can be expressed as an infinite series in powers of y).

You may now try this exercise.

You will realize the need for classification of linear differential equation into homogeneous and non-homogeneous equations when we discuss some properties involving the solution of linear homogeneous differential equations. But first let us talk about the general solution of linear non-homogeneous equation of type (1) or (3).

3.2 General Solution of Linear Non-Homogeneous Equation

Consider Eqn. (3) , viz.,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In the discussion that follows, we assume that Eqn. (3) has a solution. You can see that in general, Eqn (3) is not exact. But we will show that we can always find an integrating factor $\mu(x)$, which makes this equation exact – a useful property of linear equations.

Let us suppose that Eqn. (3) is written in the differential form

$$dy + [P(x)y - Q(x)]\, dx = 0 \qquad \ldots(4)$$

Suppose that $\mu(x)$ is an I.F. of Eqn. (4). Then

$$\mu(x)\, dy + \mu(x)\, [P(x)y - Q(x)]\, dx = 0 \qquad \ldots(5)$$

is an exact differential equation. By Theorem 1 of Unit 2, we know that Eqn. (5) will be an exact differential if

$$\left|\frac{\partial}{\partial x}(\mu(x) = \frac{\partial}{\partial y}(\mu(x)[P(x)y - Q(x)]\}\right. \qquad \ldots(6)$$

or $\frac{d\mu}{dx} = \mu P(x)$

This is a separable equation from which we can determine $\mu(x)$. we have

$$\frac{d\mu}{\mu} = P(x)dx$$

or In $|\mu| = \int P(x)\, dx$ $\ldots(7)$

so that $\mu(x) = e^{\int P(x)dx}$ is an integrating factor for Eqn. (4).

Note that we need not use a constant of integration in relation (7) since Eqn. (5) is unaffected by a constant multiple. Also, you may note that Eqn. (4) is still an exact differential equation even when Q(x) = 0. In fact Q(x) plays no part in determining

$\mu(x)$ since we see from (6), that $\frac{\partial}{\partial y}\ \mu(x)\,Q(x) = 0$. Thus both

$e^{\int P(x)dx}dy + e^{\int P(x)dx}\,[P(x)y - Q(x)]\,dx$ and

$e^{\int P(x)dx}dy + e^{\int P(x)dx}\,P(x)y\,dx$

are exact differentials.

We, now, write Eqn. (3) in the form

$$e^{\int Pdx}\left(\frac{dy}{dx} + y\right) = Q\,e^{\int Pdx}$$

This can also be written as

$$\frac{d}{dx}\left(y\,e^{\int Pdx}\right) = Q\,e^{\int Pdx}$$

Integrating the above equation, we get

$y\ e^{\int Pdx} \int Q\ e^{\int Pdx}\ dx + \alpha$, where α is a constant of integration

or $y = e^{\int Pdx} \int Q e^{\int Pdx}\ dx + \alpha\, e^{\int Pdx}$ …(8)

For initial value problem, the constant α in Eqn. (8) can be determined by using initial conditions. Relation (8) gives the general solution of Eqn. (3) and can be used as a formula for obtaining the solution of equation of the form (3). As a matter of advice we may put it that one need not try to learn the formula (8) and apply it mechanically for solving linear equations. Instead, one should use the procedure by which (8) is derived: **multiply by** $e^{\int Pdx}$ **and integrate.**

In case of linear homogeneous equation, the general solution can be obtained by putting Q = 0 in Eqn. (8) as

$$y = \alpha\, e^{-\int Pdx}$$

Note that the first term on the right hand side of Eqn. (8) is due to non-homogeneous term Q of Eqn. (3). It is termed as the **particular integral** of the linear non-homogeneous differential equation, that is, particular integral of Eqn. (3) is

$e^{-\int Pdx} \int Q e^{\int Pdx} dx$.

The particular integral does not contain any arbitrary constant.

The solution of linear non-homogeneous equation and its corresponding linear homogeneous equation are nicely interrelated. We give the first result, in this direction, in the form of the following theorem:

Theorem 1: In $I \subseteq R$, if y_1 be a solution of linear non-homogeneous differential Eq. (3), that is,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and if z be a solution of corresponding linear homogeneous differential equation

$$\frac{dy}{dx} + +(x)y = 0, \qquad \ldots(9)$$

then the function $y = y_1 + z$ is a solution of Eqn. (3) on I.

$$\therefore \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dz}{dx} \qquad ..(10)$$

Since y_1 $P(x)y_1 = Q(x)$

$$\frac{dy_1}{dx} + P(x)y_1 = Q(x) \qquad \ldots(11)$$

Also since z is a solution of (9), therefore

$$\frac{dz}{dx} + P(x)z = 0 \qquad \ldots(12)$$

On combining Eqns. (10) – (12), we get

$$\frac{dy}{dx} = [Q(x) - P(x)y_1] + [-P(x)z]$$
$$= Q(x) - P(x)\,[y_1 + z]$$
$$= Q(x) - yP(x) \qquad \text{as } (y_1 + z = y),$$

i.e., $\frac{dy}{dx} + P(x)y = Q(x)$.

Hence $y = y_1 + z$ is a solution of Eqn. (3) and this completes the proof of the theorem.

From this theorem, it should be clear that any solution of Eqn. (3) must contain solution of Eqn. (9) (corresponding linear homogeneous equation).

In case, the function Q(x) on the right-hand side of Eqn. (3) is a linear combination of functions, then we can make use of the following theorem:

Theorem 2: Let y_i be a particular solution of

$$\frac{dy}{dx} + P(x)y = Q_i(x),$$

where $Q_i(x)$ are continuous functions defined on an interval I for I = 1, 2,…., n. then the function $y_p = y_1 + y_2 + \ldots + y_n$, defined on I, is a particular solution of

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $Q(x) = Q_1(x) + Q_2(x) + \ldots + Q_n(x),\ \forall x \in I$.

The proof of this theorem is simple and is left as an exercise for you.

We now take up some examples and illustrate the method of finding the solution of linear non-homogeneous differential equations.

Example 1: Solve $x\frac{dy}{dx} + y = x^3$

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{1}{x} y = x^2 \qquad \ldots(13)$$

it is a linear equation. Comparing it with Eqn (3), we have

$P = \frac{1}{x}$. So I.F. $= e^{\int P dx} = e^{\int (1/x) dx} = e^{Inx} = x$

Multiplying Eqn. (13) by x, we get

$$x\frac{dy}{dx} + y = x^3$$

i.e., $\frac{d}{dx}(xy) = x^3$, which is exact.

Integrating, we get

$$xy = \frac{x^4}{4} + c,$$

c being a constant, as the required solution.

Example 2: Solve $x\frac{dy}{dx} - ay = x + 1$

Solution: Clearly the given equation is linear and can be written in the form

$$\frac{dy}{dx} - \frac{a}{x}y = \frac{x+1}{x}$$

$\therefore$ I.F. = $e^{\int(-a/x)dx} = e^{-a\ln x} = e^{\ln x^{-a}} = \frac{1}{x^a}$

Multiplying the given equation by $\frac{1}{x^a}$, we get

$$\frac{1}{x^a}\frac{dy}{dx} - \frac{a}{x^{a+1}}y = \frac{x+1}{x^{a+1}},$$

i.e., $\frac{d}{dx}\left(\frac{y}{x^a}\right) = \frac{x+1}{x^{a+1}}$

Integrating the above equation w.r.t.x, we get

$\frac{y}{x^a} = \int \frac{x+1}{x^{a+1}}dx + c$ (c is constant)

$$= \frac{x^{-a+1}}{-a+1} + \frac{x^{-a}}{-a} + c.$$

Thus $y = \frac{x}{1-a} - \frac{1}{a} + cx^a$ is the required solution.

Let us look at another example in which the role of x and y has been interchanged.

Example 3: Solve $y \ln y \frac{dx}{dy} + x - \ln y = 0$.

Solution: This equation is of first degree in x and $\frac{dx}{dy}$. Hence it is a linear equation with y as independent variable and x as dependent variable.

Then given equation can be written as

$$\frac{dx}{dy} + \frac{x}{y\ln y} = \frac{1}{y}, \qquad \ldots(14)$$

$\therefore$ I.F. $= e^{\int 1/(y \ln y)\, dy}$

$= e^{\ln(\ln y)}$

$= \ln y$

Multiplying Eqn. (14) by Iny, we get

$$\ln y \frac{dx}{dy} + \frac{1}{y}x = \frac{1}{y}\ln y,$$

i.e., $\frac{d}{dy}(x \ln y) = \frac{1}{y}\ln y.$

Integrating the above equation w.r.t.y, we get

$x\ln y = \int \frac{1}{y}\ln y\, dy + c,$

$= \frac{(\text{In}y)^2}{2} + c$, c is a constant,

or 2x Iny = $(\text{In}y)^2 + c_1$, is the required solution where $c_1 = 2c$.

let us consider another example.

Example 4: solve the equation ydx + (3x – xy + 2) dy = 0

Solution: Since the product y dy occurs here, the equation is not linear in dependent variable y. it is, however, linear if we treat variable y as independent variable and x as dependent variable. Therefore, we arrange the terms as

y dx + (3 – y) x dy = - 2 dy,

and write it in the standard form

$$\frac{dx}{dy} + \left(\frac{3}{y} - 1\right) x = -\frac{2}{y}, \text{ for } y \neq 0 \qquad \ldots(15)$$

Now, $\int \left(\frac{3}{y} - 1\right) dy = 3 \text{ In}|y|$,

So that an integrating factor for Eqn. (15) is

$e^{(3\text{In}|y|-y)} = e^{-y} e^{3\text{In}|y|}$

$= e^{\text{In}|y|^3} e^{-y}$

$= |y|^3 e^{-y}$

If follows that for y > 0, y^3e^{-y} is an integrating factor and for y > 0, $-y^3 e^{-y}$ serves as an integrating factor for the given equation. In either case, we are led to the exact equation

$y^3 e^{-y} dx + y^2 (3 – y e^{-y} x dy = - 2y^2e^{-y} dy$,

i.e., $d(xy^3y^{-y}) = - 2y^2e^{-y} dy$.

Integrating the above equation w.r.t.y, we get

$xy^3e^{-y} = - 2\int y^2 e^{-y} dy$

$= 2y^2 e^{-y} + 4y e^{-y} + c$ (Integrating by parts)

Thus, we can express the required solution as

$xy^3 = 2y^3 + 4y + 4 + ce^y$, where c is an arbitrary constant.

You may try the following exercises.

We have seen that general solution of a linear non-homogeneous differential Eqn. (3) is given by Eqn. (8), which involves integrals. We remark that an equation y' = f(x, y) is said to be solve when its solution is reduced to the expression of the form $\int h(x)\, dx$ or $\int ø(y)\, dy$ for some h(x) and ø(y) even if it is impossible to evaluate these integrals in terms of known functions. Further, the reduction of the solutions from one form to a simpler form may require as much labour as the solving of the equations. In solution (8) of Eqn. (3), $e^{-\int P dx} \int Q(x) e^{\int P dx}\, dx$ is the particular integral of Eqn. (3) and the evaluation of this integral will depend on the form of Q(x). This evaluation may sometimes turn out to be a tedious task. But, there are other methods by which particular integral in some cases can be obtained without carrying rigorous integration. We shall briefly discuss these methods now. As these methods are more helpful for higher order differential equations, we shall discuss them in greater detail in next course book.

3.2.1 Method of Undetermined Coefficients

This method is applicable when in Eqn. (3), i.e.,

$$\frac{dy}{dx} + P(x)y = Q(x),$$

P(x) is a constant and Q(x) is any of the following forms:

i) an exponential
ii) A polynomial in x
i) of the form cos βx or sin βx
ii) a linear combination of i), ii) and iii) above.

The general procedure is to assume the particular solution with arbitrary or unknown constants and then determine the constants.

We know that on differentiating functions such as $e^{\alpha x}$ (α constant), x^r (r > 0 is an integer), sin βx or cos βx (β constant), we again obtain an exponential, a polynomial or a function which is a linear combination of sine or cosine function. Hence if the non-homogeneous term Q(x) in Eqn. (3) is in any of the forms (i) – (iv), above, then we can choose the particular integral accordingly to be a suitable combination of the terms n(i) – (iv).

We now take up different cases according to the forms of Q(x).

Case 1: $Q(x) = k\,e^{mx}$, k and m are real constants, that is, Q(x) is an exponential function. In this case, we prove the result in the form of the following theorem.

Theorem 3: If a, k and m are real constants, then a particular solution of

$$\frac{dy}{dx} + ay = k\,e^{mx}$$

is given by

$$y_p(x) = \begin{cases} \dfrac{k}{(a+m)} e^{mx} \text{ if } m \neq -a \\ kxe^{mx} \text{ if } m = -a \end{cases}$$

Proof: in this case, since Q(x) is an exponential function, we assume $y_p(x) = re^{mx}$ to be a particular solution of Eqn. (16), where r is some constant to be determined. Now $y_p(x)$ must satisfy Eqn. (16).

Thus, we get

Rm e^{mx} + ar e^{mx} = ke^{mx}

Or r = $\frac{1}{a+m}$ if m $\neq$ - a.

Therefore, $y_p(x) = \frac{1}{a+m} e^{mx}$ if m $\neq$ - a.

In case m + a = 0, i.e., m = - a, then you may verify that $y_p(x)$ = kx e^{mx} satisfies Eqn. (16). The reasoning for this sort solution will be given when we discuss this method in detail in Block 2. However, we illustrate this case by the following example.

Example 5: Solve y' – y = $2e^x$

Solution: On comparing the given equation with Eqn. (16), we find that a = - 1, k = 2, and m = 1

Also, m + a = -1 + 1 = 0 $\Rightarrow$ m = -a.

$\therefore$ By Theorem 3, a particular integral is $2xe^x$.

Further, I.F. = $e^{-\int Pdx} = e^{-\int dx} = e^{=x}$.($\because$ P = –1)

Therefore, required solution, following relation (8), is

y = P.I + c e^x,

i.e., y = $2xe^x$ + c $e^{x.}$

You may now try this exercise.

Case II: Q(x) = $\sum_{i=0}^{n} a_i x^i$

That is, Q(x) is a polynomial of degree n. In this Eqn. (3) reduces to

$$\frac{dy}{dx} + ay = \sum_{i=0}^{n} a_i x^i \qquad \ldots(17)$$

If a = 0 in Eqn. (17), then particular solution is

$Y_p(x) = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}$, which follows by direct integration.

If in Eqn. (17), a $\neq$ 0, then we assume

$y_p(x) = \sum_{i=0}^{n} P_i x^i$ (Q(x) being a polynomial in this case),

And determine real numbers P_0, P_1, …., P_n so that particular solution $y_p(x)$ satisfies Eqn. (17).

Substituting this value of $y_p(x)$ in Eqn. (17) (with y replaced by $y_p(x)$), we have

$$\sum_{i=1}^{n} iP_i x^{i-1} + \sum_{i=0}^{n} aP_i x^i = \sum_{i=0}^{n} a_i x^i \ (a \neq 0) \qquad \ldots(18)$$

Equating the coefficients of like power of x on both sides of Eqn. (18), we get

$$\left.\begin{array}{l}\text{Coeff. Of } x^i: (I+1)P_{i+1} + aP_i = a_i \text{ for } I = 0, 1, 2, \ldots, (n-1) \\ \text{Coeff. Of } x^n: \mathbf{aP_n = a_a}\end{array}\right\} \quad \ldots(19)$$

Since Q(x) is a polynomial of degree n, thus $a_n \neq 0$ and we can solve EQn. (19) for P_0 $P_1, \ldots. P_n$. from Eqn. (19), we get

$P_n = a_n/a$

$P_{n-1} = \left(a_{n-1} - \frac{na_n}{a}\right)\frac{1}{a}$,

$P_{n-2} = a_{n-2} - \frac{n-1}{a}\left(a_{n-1} - \frac{n}{a}a_n\right)\frac{1}{a}$, and so on.

We illustrate this method with the help of following example.

Example 6: Find the particular solution of $\frac{dy}{dx} + 2y = 2x^2 + 3$.

Solution: We note that, in this case, Q(x) is a polynomial of degree 2. Assume a particular solution of the form

$$y_p(x) = \sum_{i=0}^{n} P_i x^i = P_0 + P_1x + P_1x^{2.}$$

Substitution of $y_P(x)$ in the given equation yields

$$(P_1 + 2P_2x) + 2(P_0 + P_1x + P_2x^2) = 2x^2 + 3. \quad \ldots(20)$$

Equating the coefficients of like powers of x on both side of Eqn. (20), we get

Coeff. of x^2: $2P_2 = 2$ or $P_2 = 1$.

Coeff. of x^2: $2P_2 = 2$ or $P_2 = 1$.

Coeff. of x^0: $P_1 + 2P_0 = 3$ or $P_0 = 2$.

Hence, required particular solution is

$y_P(x) = x^2 - x + 2$.

And now an exercise for you.

Case III: Q(x) = sin βx or cos βx or a βx + b cos βx

Where β, a and b are real constants.

In all these cases, we assume a particular solution of the form c sin βx + d cos βx.

On substituting this solution in the given equation and equating the coefficients of sin βx and cos βx on both sides, we determine the constants c and d.

Let us illustrate this case by an example.

Example 7: Find the particular integral of

$$\frac{dy}{dx} + y = \cos 3x$$

solution: Here Q(x) = cos 3x.

Hence, any particular solution of the given differential equation must be a combination of sin 3x and cos 3x. let the particular solution be
$y_P(x) = c \cos 3x + d \sin 3x$
On substituting this value of $y_P(x)$ in the given equation, we get

$$(-3c \sin 3x + 3x \cos 3x) + (c \cos 3x + d \sin 3x) = \cos 3x \qquad \ldots(21)$$

comparing the coefficients of cos 3x and sin 3x on both sides of Eqn. (21), we get

$c + 3d = 1$ and $d - 3c = 0$
or $c = \frac{1}{10}$ and $d = \frac{3}{10}$

Hence, the particular solution is
$$y_p(x) = \frac{1}{10}(3 \sin 3x + \cos 3x)$$
we now take up an example which is a combination of all the three cases discussed above.

Example 8: Compute the general solution of
$$\frac{dy}{dx} + y = e^x + x + \sin x \qquad \ldots(22)$$

Solution: Here $Q(x) = Q_1(x) + Q_2(x) + Q_3(x)$,
With $Q_1(x) = e^x$, $Q_2(x) = x$ and $Q_3(x) = \sin x$.
You may recall Theorem 2; if y_1, y_2 and y_3 are particular solutions of
$$\frac{dy}{dx} + y = e^x \qquad \ldots(23)$$
$$\frac{dy}{dx} + y = x \qquad \ldots(24)$$
and
$$\frac{dy}{dx} + y = \sin x \qquad \ldots(25)$$
respectively, then $y_p = y_1 + y_2 + y_3$ is a particular solution of the given equation.
Consider Eqn. (23). Let the particular solution be
$Y_1 = re^x$.

Substituting this in Eqn. (23), we get
$$re^x + re^x = e^x \Rightarrow r = \frac{1}{2}$$

$\therefore y_1 = \frac{1}{2} e^x$ …(26)

For Eqn. (24) we assume the particular solution as
$y_2 = a_1x + a_0$.
Substituting this in Eqn. (24), we get
$a_1 + a_1x + a_0 = x$ …(27)

comparing coefficients of like powers of x on both side of Eqn. (27), we get
$$\left.\begin{array}{l} a_0 + a_1 = 0 \\ a_1 = 1, \end{array}\right\} \Rightarrow a_0 = -1, a_1 = 1$$
Hence $y_2 = x-1$ …(28)

In the case of Eqn (25), assume particular solutions as
$y_3 = c \sin x + d \cos x$.
Substituting this in Eqn. (25), we get
$c \cos x – d \sin x + c \sin x + d \cos x = \sin x$ …(29)

On equating the coefficients of sin x and cos c on both sides of Eqn. (29), we get
$c – d = 1$
$c + d = 0 \quad \Rightarrow c = \frac{1}{2}$ and $d = - \frac{1}{2}$
$\therefore y^3 = \frac{1}{2} (\sin x – \cos x)$ …(30)

Hence, particular solution of Eqn. (22) can be obtained from Eqn. (26), (28) and (30) as
$$y_p(x) = y_1 + y_2 + y_3 = \frac{1}{2} e^x + x - 1 + \frac{1}{2} (\sin x – \cos x)$$

The solution of homogeneous pat of Eqn. (22), i.e.
$$\frac{dy}{dx} + y = 0$$
is given by
$$\frac{1}{y} \frac{dy}{dx} + 1 = 0$$
Integrating the above equation, we get
In $y + x =$ In α, for some constant α,
i.e., $\frac{y}{\alpha} = e^{-x}$
or $y = \alpha e^{-x}$

Hence complete solution of Eqn. (22) is given by
$$y = \alpha e^{-x} + \frac{1}{2} e^x + x-1 + \frac{1}{2} (\sin x – \cos x)$$

How about trying an exercise now?

We thus studied the method of undetermined coefficients for finding the particular integral of the non-homogeneous linear differential EQn. (3). We saw that this method would be applicable only for a certain class of differential equations – those for which P(x) is a constant and Q(x) assumes either of the forms $e^{\alpha x}$, x^r, $\sin \beta x$ or $\cos \beta x$, or their combinations. We shall, now, study a method that carries no such restrictions.

3.2.2 Method of Variation of Parameters

Consider the non-homogeneous linear Eqn. (3), namely,

$$\frac{dy}{dx} + P(x)y(x) = Q(x).$$

the homogeneous equation corresponding to the above linear equation is

$$\frac{dy}{dx} + P(x)\, y(x) = 0.$$

Further we know , from Eqn. (8), that the solution $y_h(x)$ of the homogeneous linear equation is given by

$$y_h(x) = \alpha\ e^{-\int P dx}, \qquad ..(31)$$

where α is a constant.

In tis method we assume that α, in Eqn. (31), is not a constant but a function of x. that is, we vary α with x and assume that the resulting function

$$y(x) = \alpha(x)\ e^{-\int P dx} \qquad \ldots(32)$$

is a solution of Eqn. (3). That is, we try to determine $\alpha(x)$ such that y given by Eqn. (32) solves Eqn. (3). In other words, we determine a necessary condition on $\alpha(x)$ so that y defined by relation (32), is a solution of Eqn. (3).

On combining Eqns. (3) and (32), we get

$$\frac{d}{dx}[\alpha(x)\ e^{-\int P(x)dx}] + P(x)\,[\alpha(x)\ e^{-\int P(x)dx}] = Q(x).$$

i.e., $\alpha(x) = Q(x)\ e^{-\int P dx}] + \alpha'(x)\ e^{-\int P(x)dx} + P(x)\ \alpha(x)\ e^{-\int P(x)dx} = Q(x)$,

i.e., $\alpha'(x) = Q(x)\ e^{\int P(x)dx}$

Integrating w.r.t.x, we get

$$\alpha(x) = \beta + \int Q(x)\ e^{\int P(x)dx} dx \qquad \ldots(33)$$

where β is a constant of integration.

Substituting the value of $\alpha(x)$ from eQn. (33) in relation (32), the solution to Eqn. (3) can be expressed as

$$y(x) = \beta\ e^{-\int P(x)dx} + e^{-\int P(x)dx} \int Q(x)\ e^{-\int P(x)dx} dx$$

You may note here that the solution obtained above is same as the one given by Eqn. (8) that has been obtained directly. Further, the method of variation of parameter neither simplifies any integration/solution nor provides any other form of the solution for first order first degree differential equation. It only provides an alternative approached to arrive at the general solution in this case. However, as we shall see later in next course book, this method turns out to be quite powerful in discussing equations of higher order.

Using the method of undetermined coefficients/variation of parameters or otherwise you may now try this exercise.

Now let us discuss some properties of linear homogeneous differential equations, which give us some insight into qualitative theorem rather than quantitative solutions.

3.3 Properties of the Solution of Linear Homogeneous Differential Equation

In this section we shall discuss certain properties enjoyed by linear homogeneous differential equation. We start with a very important property called **superposition principle.**

Theorem 4: (Superposition Principle)
If y_1 and y_2 are any two solution of the linear homogeneous Eqn. (9), i.e.,

$$\frac{dy}{dx} + P(x)\, y(x) = 0.$$

Then y1 + y2 and cy_1 are also solutions of Eqn. (9), where c is a constant.

Proof: Since y_1 and y_2 are both solutions of Eqn. (9), therefore

$$\frac{dy_1}{dx} + P(x)y_1 = 0 \quad \text{and} \qquad \ldots(34)$$

$$\frac{dy_2}{dx} + P(x)y_2 = 0, \qquad \ldots(35)$$

Let $h(x) = y_1 + y_2$

$$\therefore \frac{dh}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx},$$

$= - P(x)y_1 – P(x)y_2$, (using Eqn. (34) and (35))

$= - P(x)\,(y_1 + y_2)$,

$= - P(x)\, h(x)$,

i.e., $\frac{dh}{dx} + P(x)\, h(x) = 0$, which shows that $h(x) = y_1 + y_2$ is indeed a solution of Eqn. (9).

Next, multiplying Eqn. (34) by c (a constant), we get

$$c\frac{dy_1}{dx} + cP(x)y_1 = 0,$$

i.e. $\frac{d}{dx}(cy_1) + P(x).(cy_1) = 0$,

which shows that (cy_1) is also a solution of Eqn. (9).

In many branches of sciences, $y_1 + y_2$ is called superposition of y_1 and y_2 and hence the name superposition principle for Theorem 4.

The conclusions of Theorem 4 can be reframed as – the set of real (or complex) solution of Eqn. (9) forms a real (or complec) vector space (ref. Block 1, MTE-02, a course on linear algebra).

Do you think Theorem 4, holds for non-homogeneous linear equation? Consider the non-homogeneous equation $y' = 2x$.

The functions $(1 + x^2)$ and $(2 + x^2)$, $x \in R$ are two solution of

$$y' = 2x \qquad \ldots(36)$$

their sum $(2+x^2) + (2+x^2) = 3 + 2x^2$, $x \in R$ does not satisfy Eqn. (36) since $\frac{d}{dx}(3 + 2x^2) = 4x \neq 2x, \ \forall x \in R$.

Thus, Theorem 4 need not be true for a non-homogeneous linear equation does it work for a non-linear equation? Let us look at

$$y' = - y^2 \qquad \ldots(37)$$

which has solutions $y_1(x) = \frac{1}{(1+x)}$ and $y_2 = \frac{1}{1+2x}$ on an interval

$I = [0, \infty]$. This is true because

$$y_1'(x) = - \frac{1}{(1+x)^2} = - y_2^1,$$

and

$$y_2'(x) = - \frac{2.2}{(1+2x)^2} = - y_2^2,$$

Let $y = y_1 + y_2$. Here y is well defined on I. Also, by simple computation, we have

$$y'(x) = - \frac{1}{(1+x)^2} - \frac{2.2}{(1+2x)^2} = - \frac{(8x^2+12x+5)}{(1+x)^2(1+2x)^2} \qquad \ldots(38)$$

on the other hand,

$$- y^2(x) = - \frac{(16x^2+24x+9)}{(1+x)^2(1+2x)^2} \qquad \ldots(39)$$

from relations (38) and (39), it is clear that $y = y_1 + y_2$ is not a solution of (37).

In the next exercise we ask you to show an example of a non-linear equation whose solution is y_1 but cy_1 is not a solution, i.e., the later part of Theorem 4 need not be true for a non-linear equation.

Mostly we study the real solutions of Eqn. (1). You may recall that the functions a and b in Eqn. (1) (defined on I) are assume to be valued. The reason for restricting the study to real solutions will be clear from the following theorem.

Theorem 5: If y = p + iq is a complex valued function defined on I, which satisfies Eqn. (2), that is, $a(x)\frac{dy}{dx} = b(x)y(x)$, then the real part p of y and the imaginary part q of y are also solutions of Eqn. (2) on I.
(recall here that a and b are real valued continuous functions)

proof: By definition y = p+iq and so y' = p' + iq'. Since y satisfies Eqn. (2), we have
$a(x)\{p'(x) + iq'(x)\} = b(x)\{p(x) + iq(x)\}$ …(40)

since a and b are real valued, on equating the real and imaginary parts in Eqn. (4), we get
$a(x)p(x) = b(x)p(x)$,
and
$a(x)q'(x) = b(x)q(x)$,
which shows that p and q are solution of Eqn. (2) on I.

Theorem 5 is also true for higher order linear homogeneous equations which will be discussed in our later blocks and the proof is virtually on the same lines. But the theorem may fail if we replace Eqn. (2) by an arbitrary non-linear equation or a linear non-homogeneous equation. For instance, consider the first order non-linear equation
$yy' = -2x^3$ …(41)

the function $y(x) = ix^2$, $x \in R$ is a complex valued solution of Eqn. (41), since
$y'(x) = 2ix$
and $y(x)y'(x) = (ix)(ix^2 = -2x^3$.

The real par p of y is the zero function. i.e., p(x) = 0. But p is not a solution of Eqn. (41) (since $2x^3 \neq 0$ for all $x \in R$).

The following exercise shows that Theorem 5 may fail in the case of non-homogeneous linear equations.

We shall now be giving another interesting result concerning linear homogeneous equation $a(x)\frac{dy}{dx} = b(x)y(x)$, which can also be written as
$y' = g(x)y$, …(42)

where $g(x) = \frac{b(x)}{a(x)}$, is a real valued continuous function defined on I. Result which we are going to state is a consequence of the uniqueness of solutions of initial value problem for linear equations.

Theorem 6: Let y be a solution of the Eqn. (42) on the interval such that $y(x_1) = 0$ for some x_1 in I. Then y = 0 on I.

Proof: Consider the initial value problem
$y' = g(x)\, y,$
$y(x_1) = 0$

By hypothesis, y is a solution of Eqn. (42). But the function z, defined by z(x) = 0 for all $x \in I$), also satisfies Eqn. (42) (because z'(x) = 0, g(x)z(x) = 0 and $z(x_1) = 0$). By the uniqueness theorem for the initial value problem for linear equation (refer Theorem 1, Unit 1), we have z = y or in other words, y(x) = 0 for $x \in I$. This completes the proof. Just as we have seen in the case of Theorem 4 and 5, Theorem 6 may not be true for non-linear or linear non-homogeneous equations. Consider, for instance, the following non-linear differential equation.

$y' = 2\sqrt{y},\ x \in [0, \infty]$ …(43)

Let c>0. we define the function y on [0, ∞] by

$$y(x) = \begin{cases} 0 \text{ if } 0 \le x < c \\ 2(x-c)^2 \text{ if } c \le x < \infty \end{cases}$$

from the definition of y, we have

$$y'(x) = \begin{cases} 0 \text{ if } 0 \le x < c \\ 2(x-c) = 2\sqrt{y(x)}, \text{if } c \le| x < \infty \end{cases}$$

(Note that y is differentiable at x = c and, in fact, its right as well as left derivative is zero at x = c).

$$y'(x) = \begin{cases} 0 = 2\sqrt{y(x)}, 0 \le| x < c \\ 2(x-c) = 2\sqrt{\ }\ y(x) \text{if } c \le| x < \infty \end{cases}$$

which shows that y satisfies Eqn (43) for all x > 0. We notice, here, that y vanishes on the interval [0, c] and yet y is a non zero function on [0, ∞]; which shows that the conclusions. For example, y(x) = cos x + sin, $x \in R$ is a solution of Theorem 6 may not be true for non-linear equations.

similarly, we can show that Theorem 6 is not valid for linear non-homogeneous equation. for example, y(x) = cos x + sin x, $x \in R$ is a solution of

$y' = y - 2\sin x,\ x \in R$. …(44)

But, y(x) is zero at many points (like x = $-\frac{\pi}{4}$, $-\frac{\pi}{4} + 2\pi$, …)

and assume both negative and positive values. Yet y is a non-linear function which solves Eqn. (44).

Did you notice that in Theorem 6 we did not take a general linear homogeneous equation? we only considered linear homogeneous initial value problem. Why?

Well, consider the linear homogeneous equation

sin x y'(x) = cos x y(x) …(45)

The function y(x) = sin x, $x \in R$ is a non-zer solution of Eqn. (45), which vanishes at many points of R (like x = 0, $\pm\pi$, $\pm 2\pi$,…..) and also changes sign.

You may now try this exercise.

In Theorem 4 to 6, we have given some properties of linear homogeneous equations and corresponding initial value problems. But, none of the results stated asserts the existence of solutions of linear equations or corresponding initial value problem. Such results are called qualitative properties of solution of linear equations and their corresponding initial value problems.

Sometimes equations which are not linear can be reduced to the linear form by suitable transformations of the variables.

In the next section we shall take up such equations.

3.4 Equations Reducible to Linear Equations

Let us consider an equation of the form

$$f'(y)\,\frac{dy}{dx} + P(x), f(y) = Q(x) \qquad \dots(46)$$

where f'(y) is the differential coefficient of f(y).

An interesting feature of Eqn. (46) is that it is a non-linear differential equation of the first order that can be reduced to the linear form by putting v = f(y). With this substitution Eqn. (46) reduced to

$$\frac{dy}{dx} + P(x)v = Q(x), \left(\because \frac{dy}{dx} = f'(y)\frac{dy}{dx}\right),$$

which is a linear equation with v as dependent variable and x as independent variable.

A very important and famous equation of this form, about which we have already mentioned in Sec 3.2, is known as **Bernoulli's Equation,** named after James Bernoulli, who studied it in 1695 for finding its solution. The equation is or the form.

$$\frac{dy}{dx} + Py = Qy'', \qquad \dots(47)$$

where P an Q are functions of x above and n is neither zero nor one. Dividing Eqn (47) by y^n, we get

$$y^{-n}\,\frac{dy}{dx} + P\,y^{1-n} = Q \qquad \dots(48)$$

in the year 1696, Leibniz pointed out that Eqn. (48) can be reduced to a linear equation by takeny^{1-n} as the new dependent variable.

On putting $v = y^{1-n}$ in Eqn. (48), it reduces to

$$\frac{1}{1-n}\frac{dy}{dx} + P\,v = 0. \qquad \ldots(49)$$

which is a linear differential in v and x. Eqn. (49) can now be solve by the known methods.

Note that when n= 0 Eqn. (47)is a linear non-homogeneous equation and when n = 1, Eqn. (47) is a linear homogeneous equation. we now illustrate this method with the help of a few examples.

Example 9: Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$ …(50)

Solution: Dividing Eqn. (50) by sec y, we get

$$\cos y\,\frac{dy}{dx} = \frac{\sin y}{1+x} = (1+x)e^x \qquad \ldots(51)$$

If we put sin y = f(y), then f'(y) = cos y and hence Eqn. (51) is of the form

$$f'(y)\frac{dy}{dx} - \frac{1}{1+x}f(y) = (1+x)e^x \qquad \ldots(52)$$

which is of the type (46). To reduce it to linear from, we put

v = f(y) = sin y

Then Eqn (52) reduces to

$$\frac{dv}{dx} - \frac{1}{1+x}v = (1+x)e^x$$

it is a linear equation with I.F. = $e^{-\int\frac{1}{1+x}dx} = e^{-\text{In}(1+x)} = \frac{1}{1+x}$

Multiplying the above equation by I.F., we get

$$\frac{d}{dx}\left(v.\frac{1}{1+x}\right) = \frac{1}{1+x}(1+x)\,e^x.$$

Integrating w.r.t.x, we have

$v\,\frac{1}{1+x} = e^x + c$, c being a constant

i.e., $v = (1+x)e^x + c(1+x)$.

Substituting sin y for v, the required solution of the given Eqn. (50) is $\sin y = (1+x)e^x + c(1+x)$

Let us look at another example in which n is neither 0 nor 1.

Example 10: Solve $y(axy + e^x)\,dx - e^x\,dy = 0$

Solution: The given equation can be rearranged as

$$e^x \frac{dy}{dx} = e^x y + axy^{2,}$$

i.e., $$\frac{dy}{dx} - y = axy^{-x}y^2 \qquad \ldots(53)$$

it is a Bernoulli's equation with n = 2.

To solve it, let $y^{1-2} = v$, i.e., $v = \frac{1}{y}$.

$$\therefore \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

Consequently, Eqn. (53) reduces to

$$-\frac{dv}{dx} - v = axe^{-x} \qquad \ldots(54)$$

it is a linear equation with I.F. = $e^{\int 1dx} = e^x$

Multiplying both sides of Eqn. (54) by I.F., we get

$$\frac{d}{dx}(ve^x) = -ax$$

Integrating w.r.t.x, we get

$$V e^x = -\int ax\,dx + c.$$

$$= -\frac{ax^2}{2} + c$$

Replacing v by $\left(\frac{1}{y}\right)$, the required solution can be expressed as

$$E^x = y\left(c - \frac{ax^2}{2}\right),$$

Remark: There are many second or higher order linear equations which can be solved easily by reducing them to linear first order equations by making some transformation of the variables. We shall take up such equations later in next course book when we discuss second order equations.

You may now try the following exercises.

You may recall that in Unit 1, we discussed some physical situations expressed in terms of differential equations. In the following section we have attempted to solve some of them.

Applications of Linear Differential Equations

Let us consider the problem discussed in Unit 1 one by one.

I. Population model

You may recall that while studying the equation for population problem we had arrived at the initial value problem. (ref. Eqns. (32) and (33) of unit 1)

$$\left.\begin{aligned} \frac{d}{dt} N(t) &= k\,N(t) \\ N(t_0) &= N_0 \end{aligned}\right\} \qquad \ldots(55)$$

Since k is a constant in Eqn. (55), the first of the equations in (55) is a linear differential equation of order one. From Sec. 3.3 we know that its solution is

$$N(t) = N(t_0) \exp(k(t-t_0)) \qquad \ldots(56)$$

In Eqn. (56), we normally assume that $N(t_0)$ is specified. If k is known then we can find the solution using relation (56). In reality, it is too hard to measure k (which gives the rate of growth). In a particular case, we can actually find the exact value of k if we know the value of N at t_1 ($t_1 \neq t_0$). The details are shown in the following example.

Example 11: Assuming that the rate of growth of a species is proportional to the amount N(t) present at time t, find the value of N(t) given that N(0) = 100 and after one unit of time, the size of the specie has grown to 200.

Solution: In this case $t_0 = 0$, N(0) = 100. The solution of the problem is given by
$N(t) = 100 \exp(kt),\ t \geq 0$

We determine k from the additional condition N(1) = 200 (N(1) = size of population at time t = 1).
Thus $200 = 100 \exp(k) \Rightarrow k = \text{In}2$

Hence the solution is

$N(t) = 100 \exp(t\,\text{In}2) = 100 \exp(\text{In}2^t)$
Or $N(t) = (100)\,2^t$

In this problem the constant k has been determined from the given date.

In the following exercise we ask you to solve a similar problem.

Let us now discuss the problem of decay of radioactive material.

II. Radioactive Decay

In unit 1, (Ref. Eqn. (35)), we have seen that equation which governs the radioactive decay of a given radioactive material is

$$y'(t) = k\,y(t) \qquad \ldots(57)$$

Note: Half-life is the time needed for the material to reduce itself to half of its original mass.

Where y(t) is the mass of the radioactive material at time t and k < 0 is a real constant. Eqn. (57) can be used to find the half-life of the radioactive material.

In the following example we consider this problem in detail.

Example 12: A radioactive substance with a mass of 50 gms. Was found to have a mass of 40 gms. After 30 years. Find its half-life.

Solution: The mass y(t) of the material satisfies

$$\left.\begin{array}{l} \frac{d}{dt}y(t) = k\ y(t) \\ y(0) = 50\ \text{gms}., \\ y(30) = 40\ \text{gms}. \end{array}\right] \qquad \text{...(58)}$$

The solution of the first two equations in Eqns. (58) can be expressed as
y(t) = 50 exp (kt),

Using the third equation in Eqn. (58), we can write.
y(30) = 40 = 50 exp (30k),
or exp (30k) = 4/5,
i.e., $k = \frac{1}{30}\ \text{In}\left(\frac{4}{5}\right)$

thus, the mass y(t) satisfies

$$y(t) = 50 \exp\left(\frac{t}{30}\text{In}\frac{4}{5}\right) \qquad \text{…(59)}$$

Let t_1 be its half-life, i.e., after time t_1 the mass reduces to $\frac{50}{2}$ = 25 gms.

Then $y(t_1) = 25$ …(60)

We are required to find t_1. using condition (60), Eqn. (59) reduces to

$$25 = 50 \exp\left(\frac{t_1}{30}\text{In}\,4/5\right)$$

or t_1 In (4/5) = 30 In (1/2)
i.e. t_1 = 30 (In (1/2))/In (4/5) …(61)

so after t_1 years (t_1 defined by Eqn. (61)), the mass of the material will be 25 gms.

Let us now deal with the temperature variations of a hot object.

III. Newton's Law of Cooling

The temperature of a hot a hot body kept in a surrounding of constant temperature T_0 has been discussed in Unit 1 and the governing equation of the temperature T of the body is

$$T'(t) = k(T(t) - T_0) \qquad \ldots(62)$$

(Ref. Eqn. (34) of Unit 1)

we illustrate this by the following example.

Example 13: A rod of temperature 100^oC is kept in a surrounding of temperature 20^oC. If the temperature of the rod was found to be 80^oC after 10 minutes, find the temperature T(t) of the rod.

Solution: We are required to solve

$$\frac{d}{dt} T(t) = k(T(t) - 20) \qquad \ldots(63)$$

Let us put y(t) = T(t) – 20. Then y'(t) = T'(t) and Eqn. (63) reduces to

$$\frac{d}{dt} y(t) = k\, y(t) \qquad ..(64)$$

Eqn. (63) is not a linear homogeneous equation whereas Eqn. (64) is which explains the reason for introducing y) Along with Eqn. (64), we have

$$\left.\begin{array}{ll} \text{a)} & y(0) = T(0) - 20 = 100 - 20 = 80^oC, \\ \text{b)} & y(10) = T(10) - 20 = 80 - 20 = 60^oC \end{array}\right] \qquad \ldots(65)$$

the solution of Eqn. (64), with the condition 65(a), is

$y(t) = 80 \exp (kt)$

with this value of y and condition (65b), we have

$y(10) = 60 = 80 \exp (k. 10)$

or, $k = \frac{1}{10}$ In (6/8) = $\frac{1}{10}$ In (3/4)

Hence the value of y is determined by

$$y(t) = 80 \exp \left(\frac{t}{10} \text{In}(.75)\right),$$

and the temperature T is given by

$$T(t) = 80 \exp \left(\frac{t}{10} \text{In}(.75)\right) + 20$$

And now an exercise for you.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have covered the following points:

1) The general form of the linear equation of the first order is $\frac{dy}{dx} + P(x)y = Q(x)$, (see Eqn. (3))
Where P(x) and Q(x) are continuous real-valued functions on some interval $I \subseteq R$.

When Q(x) = 0 it is called **homogeneous linear differential equation** of order one.
When $Q(x) \neq 0$, it is called **non-homogeneous** (or **inhomogeneous**) **linear differential equation** of order one.
I.F. for this equation of $e^{\int P(x)dx}$ and the **general solution** is given by
$y = e^{-\int P(x)dx} . \int Q(x)\ e^{\int P(x)dx}$
Here, $e^{-\int P(x)dx} \int Q(x)\ e^{\int P(x)dx}$ is the particular solution of the equation.

2) The sum of the solution of linear non-homogeneous differential equation of the form (3) and the solution of its corresponding homogeneous equation is again a solution of the equation.

3) If in the differential equation
$\frac{dy}{dx} + P(x)y = Q(x)$,

P(x) is a constant and Q(x) is any of the forms $e^{\alpha x}$ (α constant), x^r(r > 0, an integer), sin βx or cos βx (β constant) or a linear combination of such functions, then method of undetermined coefficients can be applied to find the particular solution of the equation and the particular integrals for different Q(x) are given by the following table

P(x)	Q(x)	Particular Integral
a(constant	e^{mx} (m constant)	$\begin{cases} \frac{e^{mx}}{m+a} \text{ if } m \neq -a \\ xe^{mx} \text{ if } m = -a \end{cases}$
a	$\sum_{i=0}^{n} a_i x^i$ (i > 0 an integer)	$\begin{cases} \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1} \text{ if } a = 0 \\ \sum_{i=0}^{n} P_i x^i \text{if } a \neq 0 \end{cases}$

with $P_n = \frac{a_n}{a}$, $P_{n-1} = \frac{1}{a}\left(a_{n-1} - \frac{na_n}{a}\right)$,

$P_{n-2} = \frac{1}{a}\left[a_{n-2} - \frac{n-1}{a}(a_{n-1} - \frac{n}{a}a_a)\right]$ and so on

a sin βx or cos βx A linear combination of sin βx and cos βx or A sin βx + B cos βx (β, A, B constants)

---

4) Method of variation of parameters for finding the solution of non-homogeneous linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

5) Some properties of the solution of linear homogeneous differential equation

$$\frac{dy}{dx} + P(x)y = 0 \text{ are}$$

a) (Superposition Principle): If y_1 and y_2 are any two solutions of the equation, then $y_1 + y_2$ and cy_1 are also solution of the equation, where c is a real constant.

b) If a complex valued function y = p+iq, defined on 1, is a solution of the equation, then real part of p of y and imaginary part q of y are also solutions of the equation on 1.

c) If y be a solution of the equation on I such that $y(x_1) = 0$ for some x_1 in I, then y = 0 on I.

6) a) Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)\, y^n,$$

where P and Q are functions of x alone and n is neither zer nor one, reduces to a linear equation by the substitution $v = y^{1-n}$.

b) Equations of the type

$$f'(y)\frac{dy}{dx} + P(x)\, f(y) = Q(x)$$

reduce to linear equations by the substitution f(y) = v.

7) The differential equations governing physical problems such as population model, radioactive decay and Newton's law of cooling have been solved.

6.0 TUTOR MARKED ASSIGNMENT

1. From the following equations, classify which are linear and which are non-linear.

Also state the dependent variable in each case.

a) $\frac{dy}{dx} - y = xy^2$

b) $rdy - 2ydx = (x - 2)\, e^x\, dx$.

c) $\frac{di}{dt} - 6i = 10 \sin 2t$

d) $\frac{dy}{dx} + y = y^2 e^x$

e) $ydx + (xy + x - 3y)\, dy = 0$

f) $(2s - e^{2t})\, ds = 2(se^{2t} - \cos 2t)\, dt$

2. Prove Theorem 2.

3. Solve the following equations:

c) $(x^2 + 1) \frac{dy}{dx} + 2xy = 4x^2$

d) $\frac{dy}{dx} + \frac{2}{x} y = \sin x$

e) $\sec x \frac{dy}{dx} + y = \sin x$

f) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

g) $(2x - 10y^3) \frac{dy}{dx} + y = 0$

4. Solve the following equations.

a) $y' = y + \frac{e^x}{x}$, $e \in [1, \infty\ ]$

b) $y' = y + x + x^3 + x^5$

c) $y' = y + x \sin x\, e^x + x^5$

d) $y' + 3y = |x|$, $y(0) = 1$.

5. Solve $\frac{dy}{dx} + y = 2ae^x$

6. Solve $\frac{dy}{dx} = y + x^2$.

7. Solve the following differential equations:

a) $\frac{dy}{dx} - y = 6 \cos 2x$

b) $\frac{dy}{dx} + 3y = x^2 + 3e^{2x} + 4 \sin x$

8. Solve the following equations:

a) $y' - 2y = \sin \pi x + \cos \pi x$, $y(1) = 1$

b) $y' - y = \cos 2x + e^x + e^{2x} + x$

c) $y' - 3y = x^2 - \cos 3x + 2$ (Hint: Treat 2 as $2e^{0x}$)

d) $y' + y = -x - x^2$, $y(0) = 0$.

e) $Y' - y = e^x$, $y(0) = -3$.

9. Show that $y_1(x) = -\frac{1}{(1+x)}$ is not a solution of Eqn. (37)

10. Show that the solution $y(x) = e^x + ie^x - (1+x)$ of equation $y' = y + x$, for $x \in I = R$ does not satisfy the hypothesis of Theorem 5.

11. Show that $y(x) = \sin 2x - \cos 2x$, $x \in R$ is a solution of $y' = 2y + 4\cos 2x$. why does Theorem 6 fail in this case?

12. Solve the following equations:

    a) $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

    b) $\frac{dy}{dx} + xy = x^3y^3$

    c) $3e^x \tan y + (1-e^x) \sec^2 y \frac{dy}{dx} = 0$

13. Find the solutions of

    a) $\frac{dy}{dx} + x \sin 2y = x^3\cos^2 y$

    b) $\frac{dy}{dx} + y = e^x y^3$

    c) $2x\frac{dy}{dx} + y(6y^2 - x - 1) = 0$

14. A culture initially has N_0 number of bacteria. At t = 1 hour, the number of bacteria is measured to be $\left(\frac{3}{2}\right)$ N_0. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

15. Suppose that a thermometer having a reading of 70^oF inside a house is placed outside where the air temperature is 10^oF. Three minutes later it is found that the thermometer reading is 25^oF. Find the temperature reading T(t) of the thermometer.

7.0 REFERENCE/FURTHER READING

Theoretical Mechanics by Murray R. Spiegel

Advanced Engineering Mathematics by Kreyzig

Vector Analysis and Mathematical Methods by S.O. Ajibola

Engineering Mathematics by PDS Verma

Advanced Calculus Schaum's outline Series by Mc Graw-Hill

Indira Gandhi National Open University School of Sciences Mth - 07

UNIT 2 DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

CONTENTS

1.0 INTRODUCTION

In unit 1, we discussed the nature of differential equations and various types of solutions of differential equations. In Unit 2 and 3, we have given you the methods of solving different types of differential equations of first order and first degree. In this unit we shall consider those differential equations which are of first order but not of first degree.

If we denote $\frac{dy}{dx}$ by P, then the most general form of a differential equation of the first order and nth degree can be expressed in the form

$$P^n + P_1p^{n-1} + P_2p^{n-2} + \ldots..+ P_{n-1}p + P_n = 0, \qquad \ldots(1)$$

Where $P_1, P_2, \ldots.., P_n$ are functions of x and y.

It is difficult to solve Eqn. (1) in its most general form. In this unit we shall consider only those forms of Eqn. (1) which can be easily solved and discuss the methods of solving such equations.

It was Isaac Newton (1642 – 1727), the English mathematician and scientist, who classified differential equations of the first order (then known as fluxional equations) in "Methodus Fluxionum et serierum infinitarum", written around 1671 and published in1736. Count Jacopo Riccati (1676-1754), an Italian mathematician, was mainly responsible for introducing the ideal of Newton to Italy. Riccati was destined to play an important part in further advancing the theory of differential equations. In 1712, he reduced an equation of the second order in y to an equation of first order in p. In 1723, he exhibited that under some restricted hypotheses, the particular equation to which the name of Riccati is attached, can be solved.

Later the French mathematician Alexis Claude Clairaut (1713-1765) introduced the idea of differentiating the given differential equations in order to solve them. He applied it to the equations that now bear his name and published the method in 1734. We shall also be discussing the equations introduced by Riccati and Clairaut in this unit.

Clairaut (1713-1765)

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the solution of the differential equations which can be resolved into rational linear factors of the first degree;
- obtain the solution of equations solvable for y, x or p;
- obtain the solution of the differential equations in which x or y is absent;
- solve the equations which may be homogeneous in x and y;
- identify and obtain the solution of Clairaut's equation;
- identify and obtain the solution of Riccati's equation.

3.0 MAIN CONTENT

3.1 Equations which can be factorized

Let us consider the general form of differential equation of the first order and nth degree given by Eqn. (1) namely,

$P^n + P_1p^{n-1} + P_2p^{n-2} + \ldots..+ P_{n-1}p + P_n = 0,$
Where $P_1, P_2,\ldots.., P_n$ are functions of x and y.

For this equation, we shall consider two possibilities:

a) When the left-hand side of Eqn. (1) can be resolved into rational factors of the first degree.

b) When the left-hand side of Eqn. (1) cannot be factorized.
In this section we shall take up the first possibility.

When Eqn. 91) can be factorized into rational factors of the degree, then it can take the form

$(p – R_1) (p – R_2)\ldots\ldots(p – R_n) = 0$ …(2)
for some $R_1, R_2, \ldots., R_n$, which are functions of x and y.

Eqn. 91) will be satisfied by a value of y that will make any of the factors in Eqn (2) equal to zero. Hence, to obtain the solution of Eqn. (1), we equate each of the factors in Eqn. (2) equal to zero. Thus, we get
$P – R_1 = 0, p – R_2 = 0, \ldots, p – R_n = 0$ …(3)

There are n equations of first degree. Using the methods given in Unit 2 and 3 we can now obtain the solution of the above n equations of first order and first degree.

Let us suppose that the solutions desired for Eqn. (3) are

$$\left.\begin{array}{c} f_1(x, y, c_1) = 0 \\ f_2(x, y, c_2 = 0 \\ ;; \\ f_n(x, y, c_n) = 0 \end{array}\right\} \quad \ldots(4)$$

where $c_1, c_2,\ldots., c_n$ are the arbitrary constants of integration.

Since each of the constants $c_1, c_2,\ldots.., c_n$ can take any one of an infinite number of values, thus these solutions remain general even if

$C_1 = c_2 = \ldots.= c_n = c$, say.
In that case, the n solutions will be

$f_1(x, y, c) = 0$
$f_2(x, y, c) = 0$
$f_3(x, y, c) = 0$
….
….
$F_n(x, y, c) = 0$

These n solutions can be left distinct or we can combine them into one equation, namely,
$F_1(x, y, c). f_2(x, y, c) \dots f_n(x, y, c) =$

The reason of taking all $c_1, c_2, \dots, c_n$ equal in Eqn. (4) is the fact that Eqn. (2) being of first order, its general solution can contain only one arbitrary constant.

We illustrate this method by the following examples:

Example 1: Solve $p^2 + px + py + xy = 0$

Solution: The given equation is equivalent to
$(p+x)(p+y) = 0$
that is, either
$p+x = 0$ or, $p+y = 0$
In other words,
$\frac{dy}{dx} + x = 0$, or $\frac{dy}{dx} + y = 0$

the solutions of the factors are
$2y = -x^2 + c$
and
$x = -\ln |y| + c$, for c being an arbitrary constant.

Therefore, the general solution of the given equation is
$(2y + x^2 - c).(x + \ln |y| - c) = 0.$

Let us look at another example.

Example 2: Solve $p^3 (x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$

Solution: The given equation is equivalent to
$p[p^2(x + 2y) + 3p(x + y) + (y + 2x)] = 0$
$\Rightarrow p[p^2(x + 2y) + p\{(y + 2x) + (x + 2y)\} + (y + 2x)] = 0$
$\Rightarrow p(p + 1) [(x + 2y)p + (y + 2y)] = 0$

its component equations are
$p = 0, p + 1 = 0, (x + 2y)p + (y + 2x) = 0$
Now $p = 0 \Rightarrow \frac{dy}{dx} = 0$, which has the solution
$y = c$ …(5)

Now $p+1 = 0 \Rightarrow \frac{dy}{dx} + 1 = 0$
i.e., $dy + dx = 0$
which has the solution
$y+x = c$ …(6)
Further, $(x + 2y)p + (y + 2x) = 0$
$\Rightarrow (x + 2y)dy + (y + 2x)dx = 0$
$\Rightarrow d(xy + x^2 + y^2) = 0.$

Which has the solution
$xy + {}^2 + y^2 = c$ …(7)

Therefore, the general solution of the given equation, from Eqns. (5), (6) and (7), is
$(y - c). (y+x - c). (xy + x^2 + y^2 - c) = 0.$

You may now try the following exercise.

As you know from algebra, every equation over Q need not have all its roots in Q, i.e., it need not be factorizable in Q.

We now take up those equations of form (1) which cannot be factorized into rational factors of the first degree.

3.2 Equations which cannot be Factorized

in this case, let the form of Eqn. (1) be
$f(x, y, p) = 0$ ….(8)

eqn. (8) is not solvable in its most general form

we shall discuss only those equations of type (8) which possess one or more of the following properties.

i) It may be solvable for y.
ii) It may be solvable for x.
iii) It may solvable for p.
iv) Either it may not contain y or it may not contain x, that is, either x or y is absent from the differential equation.
v) It may be homogeneous in x and y.
vi) It may be of first degree in x and y.
vii) It may be Riccati's equation.

We now discuss these cases one by one.

3.2.1 Equation Solvable for y

Consider an equation
$xp^2 - yp - y = 0$ …(9)

we can write Eqn. (9) in the form
$y(p+1) = xp^2$,
$\Rightarrow y = \dfrac{xp^2}{p+1}$

That is, Eqn. (9) can be solvable for y in terms of x and p.

Similarly when Eqn. (8), i.e., f(x, y, p) = 0 is solvable for y, then it can be pt in the form

$y = F(x, p)$ …(10)

Differentiating Eqn. (10) w.r. to x, we get an equation of the form

$P = ø\left(x, p, \dfrac{dp}{dx}\right)$ …(11)

Eqn. (11) is in two variables x and p; and we may possibly solve and get a relation of the type
$\psi(x, p, c) = 0$ …(12)
for some constant c.

It we now eliminate p between Eqns. (8) and (12), we get a relation involving x, y and c, which is the required solution. In the cases when the elimination of p between Eqns. (8) and (12) is not possible, we then obtain the values of x and y in terms of p as a parameter and these together give us the required solution.

We now illustrate this method with the help of a few examples.
Example 3: The given equation is solvable for y.
Solving it for y, we get

$y = p + \dfrac{x}{p}$ …(13)

Differentiating Eqn. (13) w.r. to x, we get

$p = \dfrac{dp}{dx} + \dfrac{1}{p} + x\left(-\dfrac{1}{p^2}\right)\dfrac{dp}{dx}$

i.e., $\left(p - \dfrac{1}{p}\right)\dfrac{dx}{dp} + \dfrac{1}{p^2}\ x = 1$ …(14)

this is linear equation of the first order if we consider p as independent variable and x as dependent variable.

We can write Eqn (14) as,

$$\frac{dx}{dp} + \frac{1}{p(p-1)(p+1)} x = \frac{p}{p^2-1} \qquad \ldots(15)$$

For Eqn. (15) $e^{\int \frac{1}{p(p^2-1)} dp}$ is an integrating factor.

Now, $e^{\int \frac{1}{p(p^2-1)} dp} = e^{\int \left[\frac{1}{2(p-1)} + \frac{1}{2(p+1)} - \frac{1}{p}\right] dp}$

$$= e^{\ln \frac{(p^2-1)1/2}{p}} = \frac{(p^2-1)^{1/2}}{p}$$

the, the solution of Eqn. (15) is obtained as

$$x \frac{(p^2-1)^{1/2}}{p} = \int \frac{p}{p^2-1} \frac{(p^2-1)^{1/2}}{p} dp = \int \frac{1}{\sqrt{p^2-1}} dp = c + \cos h^{-1}p,$$

or $x = p(c + \cos h^{-1}p)\,(p^2 -1)^{-1/2}$ …(16)

you may notice that elimination of p between Eqns. (13) and (16) is not easy. However, by substituting for x from Eqn. (16) in Eqn. (13),
we get

$y = p+(c + \cosh^{-1}p)\,(p^2 -1)^{-1/2}$ …(17)

Eqns. (16) and (17) are two equations for x and y in terms of p. These are the parametric equations of the solution of the given differential equat.

Let us look at another example.

Example 4: Solve $y = 2px + p^4x^2$, $x > 0$.

Solution: The given equation

$y = 2px + p^4x^2$ …(18)

is in itself solvable for y.

differentiating it w.r. to x, we get

$$p = 2p + 2x \frac{dp}{dx} + 2xp^4 + 4x^2p^3 \frac{dp}{dx}$$

$$\Rightarrow p(1 + 2xp^3) + 2x\frac{dp}{dx}(1+2xp^3) = 0,$$

$$\Rightarrow (1+2xp^3) + (p+2x \frac{dp}{dx}) = 0 \qquad ...(19)$$

Eqn. (19) holds when either of the factors $(1+2xp^3)$ or $(p + 2x \frac{dp}{dx})$ is zero.

First consider the factor

$$P+ 2x \frac{dp}{dx} = 0$$

$$\Rightarrow \frac{2\,dp}{p\,dx} + \frac{1}{x} = 0$$

Integrating the above equation w.r.t.x, we get
2 In |p| + In |x| = constant.
$\Rightarrow p^2x = c$, (c an arbitrary constant)
or p = $\sqrt{\frac{c}{x}}$.

Substituting this value of p in the given Eqn. (18), we get

$Y = 2\ \sqrt{cx + c^2}$
Which is the required solution.

If we consider the factor $1 + 2xp^3 = 0$ in Eqn. (19), then by eliminating p between this factor and given Eqn. (18), we get another solution. This solution will not contain any arbitrary constant and is the singular solution of the given equation.

How about trying an exercise now?

We next consider the case when Eqn. (8) is solvable for x.

3.2.2 Equations Solvable for x

Consider an equation of the form
$P^3 – 4xyp + 8y^2 = 0$ ...(20)
It is difficult to solve Eqn. (20) for y whereas it is easy to solve it for x as a function of y and p and write

$$x = \frac{p^3 + 8y^2}{4yp}.$$

In such cases when equation of the form (8) is solvable for x, and can be put in the form
x = g(y,p) …(21)
then to solve it, we differentiate Eqn. (21) w. r. to y, and get an equation of the form

$$\frac{1}{p} = ø\left(y,p,\frac{dp}{dx}\right)$$

on solving this equation we obtain a relation between p and y in the form
f(y,p,c) = 0, …(22)
where c is an arbitrary constant.

Now, we may eliminate p between Eqn (21) and (22) to obtain the solution or, x and y may be expressed in terms of p as we have done in Sec. 4.3.1.

Remark: Note that when Eqn. (8) is solvable for y, we differentiate it w.r. to x, whereas, when it is solvable for x, we differentiate it w. r. to y.

We illustrate this method by the following examples.

Example 5: Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Solution: The given equation can be written as

$$X = \tan^{-1}p + \frac{p}{1+p^2} \qquad ...(23)$$

Differentiating Eqn. (23) w. r. y, we get

$$\frac{1}{p} = \frac{p}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)-p(2p)}{(1+p^2)^2}\frac{dp}{dy}$$

$$= \frac{1+p^2+1+p^2-2p^2}{(1+p^2)^2}$$

$$= \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow dy = \frac{2p}{(1+p^2)^2}\,dp \qquad \ldots(24)$$

Note that Eqn. (24) is in variable separable form.
Integrating Eqn. (24), we get

$$y = c - \frac{1}{1+p^2}, \qquad \ldots(25)$$

c being an arbitrary constant.

It is not possible to eliminate p between Eqns. (23) and (25). Thus, Eqns. (23) and (25) together constitute the solution of the given equation in terms of parameter p.

Let us look at another example.

Example 6: Solve $p^2y + 2px = \ \forall x$, y and $p > 0$

Solution: We can write the given equation in the form

$$X = \frac{y}{2p} - \frac{py}{2} \qquad ...(26)$$

Differentiating Eqn. (26) w.r. to y, we get

$$\frac{1}{p} = \frac{1}{2p} + \frac{y}{2}\left(-\frac{1}{p^2}\right)\frac{dp}{dy} - \frac{p}{2} + \frac{y}{2}\frac{dp}{dy},$$

$$\Rightarrow \frac{1}{2p} + \frac{p}{2} + \frac{y}{2}\ \frac{dp}{dy}\ \left(1 + \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \frac{1+p^2}{2p} + \frac{y}{2}\left(\frac{1+p^2}{p^2}\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{1+p^2}{2p}\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0 \qquad \ldots(27)$$

In Eqn. (27), we may have

$$\frac{1+p^2}{2p} = 0$$

or $\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0.$

If we have first factor equals zero, then $p^2 = -1$.

Thus real solution of the given problem is obtained when

$$\frac{1}{y} + \frac{1}{p}\frac{dp}{dy} = 0$$

Here variables are separable. Integrating, we get

Iny + Inp = Inc

$\Rightarrow$ py = c

or $p = \frac{c}{y}$ …(28)

Eliminating p between Eqn.s (26) and (28), we get

$$X = \frac{y^2}{2c} - \frac{c}{y}\ \frac{y}{2}$$

or $x = \frac{y^2}{2c} - \frac{c}{2}$,

which is the required solution.

Note that you could also have solved Example 6 by taking $y = \frac{2p}{1-p^2}$ and then proceeding as in sec. 4.3.1.

You may now try the following exercise.

We now consider Eqn. (8) with the property that Eqn. (8) may be solvable for p. in that case Eqn. (8) which is of nt degree in p, in general, is reduced to n equations of the first degree and this case has been considered in Section 4.2.

We next take up the case when Eqn. (8) may not contain either independent variable x, or, dependent variable y explicitly.

3.2.3 Equations in which Independent Variable or Dependent Variable is Absent

We shall consider the two cases separately.

Case 1: Equations not containing the independent variable:

When Eqn. (8) does not contain independent variable explicitly then the equation has the form

$$f(y,p) = 0 \qquad \ldots(29)$$

For instance, consider the equation

$$y - \frac{1}{\sqrt{1+p^2}} = 0.$$

This equation does not contain x explicitly. Also, it is readily solvable for y, since it can be written in the form

$$y = \frac{1}{\sqrt{1+p^2}} \qquad \ldots(30)$$

Eqn. (30) can, now, be solved by the method discussed in Sec. 4.3.1. In case Eqn. (29) is solvable for p, then we can write it in the form

$$p = \frac{dy}{dx} = ø(y) \qquad \ldots(31)$$

The integral of Eqn. (31) will, then, give us the solution of Eqn. (29).
To be more clear, let us consider the following example.

Example7: Solve $y = 2p + 3p^2$

Solution: We have

$y = 2p + 3p^2$
which is already in the form y = F(p). Following the method discussed in sec. 4.3.1, we differentiate it w. r. t. x, so that

$$p = 2\frac{dp}{dx} + 6p\frac{dp}{dx}$$

$$\text{or } \frac{p}{2+6p} = \frac{dp}{dx}$$

Here variable are separable and we have

$$dx = \left(\frac{2}{p} + 6\right) dp$$

Integrating, we get

$x = 6p + 2In |p| + c.$ … (33)

c being an arbitrary constant.

Since it is not possible to eliminate p from Eqns. (32) and (33), these equations together yield the required solution in terms of the parameter p.

Let us look at another example

Example 8: Solve $y^2 = a^2 (1 + p^2)$ … (34)

Solution: The given equation is an equation in y and p only. It can be written as

$$P^2 = \frac{y^2}{a^2} - 1$$

Solving for p, we get

$$p = \pm\sqrt{\frac{y^2}{a^2 - 1}}$$

$\therefore$ Either $p = \sqrt{\frac{y^2}{a^2-1}}$ or $p = -\sqrt{\frac{y^2}{a^2-1}}$,

Now $p\ \sqrt{-1+\frac{y^2}{a^2}}$ gives

$$\frac{a}{\sqrt{y^2 - a^2}}\, dy = dx.$$

Integrating the above equation, we get

$A\ In\ |y + \sqrt{y^2 - a^2}\ | = x + c,$

c being an arbitrary constant.

Similarly, $p = -\sqrt{-1+\frac{y^2}{a^2}}$, on integration, yields

$a\ In\ |y + \sqrt{y^2 - a^2}\ | = -x + c$ (c being a constant).

Hence, the general solution of the given equation is

$$[a\ In \left|y + \sqrt{y^2 - a^2}\right| - x - c]\ [a\ In \left|y + \sqrt{y^2 - a^2}\right| + x - c] = 0$$

Note that we solve Eqn. (34) for p. You could also have integrated it by solving it for y.

We next consider the equations in which the dependent variable is absent.

Case II: Equations not containing the dependent variable:
In this case Eqn. (8) has the form

$$g(x, p) = 0 \text{ or } x = F(p) \qquad \ldots(35)$$

As in case 1, Eqn. (35) is either solvable for p or solvable for x. if it is solvable for p, then it can be written as

$$p = \psi(x)$$

which, on integration, gives the solution of Eqn. (35)

If Eqn. (35) is solvable for x, then it corresponds to the case discussed in Section 4.3.2.

We give below examples to illustrate the theory.

Example 9: Solve $x(1+p^2) = 1$

Solution: The given equation can be written as

$$x = \frac{1}{1+p^2} \qquad \ldots(36)$$

Differentiating Eqn. (36) w.r. to y, we get

$$\frac{1}{p} = \frac{-2p}{(1+p^2)^2}\frac{dp}{dy},$$

i.e., $dy \; \frac{-2p^2}{(1+p^2)^2} dp$

i.e., $dy = 2\left[\frac{-1}{1+p^2} + \frac{1}{(1+p^2)^2}\right] dp$

Here variables are separable. Integrating, we get

$$y = -2\tan^{-1}p + 1\int \frac{dp}{(1+p^2)^2} + c \qquad \ldots(37)$$

c is a constant.
Eqns. (36) and (37) together yield the required solution with p as parameter.

Note that problem in example 9 could have also been done by solving it for p. we illustrate this method in the next example.

Example 10: Solve $p^2 - 2xp + 1 = 0$

Solution: The given equation is
$P^2 - 2xp + 1 = 0$
Solving for p, we get

$$p = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$\therefore$ Either $p = x + \sqrt{x^2-1}$ or $p = x - \sqrt{x^2-1}$
Now $p = x + \sqrt{x^2-1}$, on integration yields

$$y = \frac{x^2}{2} + \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2} \text{In} \left| x + \sqrt{x^2-1} \right| + c,$$

c being an arbitrary constant.

Similarly, $p = x - \sqrt{x^2-1}$ yields

$$y = \frac{x^2}{2} - \frac{1}{2} x\sqrt{x^2-1} + \frac{1}{2} \text{In} \left| x + \sqrt{x^2-1} \right| + c,$$

Hence, the general solution of the given equation is
$[x^2 + x\sqrt{x^2-1} - \text{In} \left| x + \sqrt{x^2-1} \right| - 2y + c_1] [x^2 - x\sqrt{x^2-1} + \text{In} \left| x + \sqrt{x^2-1} \right| -2y + c_1] = 0.$
Where $c_1 = 2c$ is an arbitrary constant.

And now some exercise for you.

We next discuss the case when Eqn. (8) may be homogeneous in x and y

3.2.4 Equations Homogeneous in x and y

In this case, Eqn. (8) can be expressed in the form

$$\varnothing \left(p, \frac{y}{x} \right) = 0 \qquad \ldots(38)$$

For solving Eqn. (38), we can proceed in two ways. In case Eqn. (38) is solvable for p then it can be expressed as

$$P = \frac{dy}{dx} = f\left(\frac{y}{x} \right) \qquad \ldots(39)$$

We already know from our knowledge of unit 2 that equations of the type (39) can be solved by using the substitution y = vx.

The second possibility, that is, when Eqn. (38) is solvable for y/x, then it can be put in the form

$$\frac{y}{x} = \psi(p) \text{ or } y = x\psi(p).$$

In this case we can proceed as in Sec. 4.3.1. Differentiating the above equation w.r. to x, we get

$$p = \psi(p) + x\psi'(p)\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{x} = \frac{\psi'(p)dp}{p - \psi(p)} \qquad \ldots(40)$$

Eqn. (40 is in variable separable form. On integrating, it yields

$$\text{In}|x| = c + \int \frac{\psi'(p)}{p - \psi(p)} dp$$

$$= c + ø(p), \text{ say.}$$

The elimination of p between this equation and y = xψ(p) will give us the required solution. But it is not always easy to eliminate p, so it may be retained as the parameter.

To understand the theory, we take an example.

Example 11: Solve $y^2 + xyp - x^2p^2 = 0 \ \forall x, y, p > 0$.

Solution: The given equation is homogeneous in y and x and it may be written as

$$p^2 - \left(\frac{y}{x}\right) p - \left(\frac{y}{x}\right)^2 = 0 \qquad \ldots(41)$$

Solving Eqn. (41) for p, we get

$$p = \frac{(y/x) \pm \sqrt{(y/x)^2 + 4(y/x)^2}}{2} = (y/x)\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

$$\text{Thus } \frac{dy}{dx} = \frac{y}{x}\left(\frac{1 \pm \sqrt{5}}{2}\right) \text{or } \frac{dy}{dx} = \left(\frac{1 \pm \sqrt{5}}{2}\right)\frac{y}{x}$$

$$\text{Let } y = vx, \text{ then } \frac{dy}{dx} = v + x\frac{dy}{dx}$$

$\therefore v + x\,\frac{dv}{dx} = v\left(\frac{1+\sqrt{5}}{2}\right)$ and $v + x\,\frac{dv}{dx} = \left(\frac{1-\sqrt{5}}{2}\right)v$

$\Rightarrow x\,\frac{dv}{dx} = \left(\frac{\sqrt{5}-1}{2}\right)v$ and $x\,\frac{dv}{dx} = \left(\frac{-1-\sqrt{5}}{2}\right)v$

Integrating, we get

In xc = In $v^{2/(\sqrt{5}-1)}$ and In xc = In $v^{-2/(\sqrt{5}+1)}$
Or xc = $(y/x)^{2/(\sqrt{5}-1)}$ and xc = $(y/x)^{-2/(\sqrt{5}+1)}$
or y = x $(xc)^{(\sqrt{5}-1)/2}$ and y = $x(xc)^{-(\sqrt{5}+1)/2}$

Hence the general solution is

$[y-x(xc)^{(\sqrt{5}-1)/2}].\,[y-x(xc)^{-(\sqrt{5}+1)/2}] = 0$
i.e., $y^2 - xy\,[(xc)^{(\sqrt{5}-1)/2} + (xc)^{-(\sqrt{5}+1)/2}] + x^2\,(xc)^{-1} = 0$

Now you may try the following exercise.

We next discuss the case when Eqn. (8) may be of first degree in x and y.

3.2.5 Equations of the First Degree in x and y – Clairaut's equation

When Eqn. (8) is of first degree in x and y, it is solvable for x and y both and hence can be put in either of the following forms.

$y = xf_1(p) + f_2(p)$ …(42)
or $x = yg_1(p) + g_2(p)$ …(43)

Hence, we can use the methods discussed in Sec. 4.3.1 and 4.3.2 to solve equations of the for (42) and (43), respectively.

However, if in Eqn. (42), $f_1(p) = p$, then we get one particular form of this equation known as **Clairaut's Equation** and about which we have already mentioned in sec. 4.1.

Thus the Clairaut's equation is of the form
$y = px + f(p)$ …(44)

In Eqn. (44), f(p) is a known function which contains neither x nor y explicitly. also, note, that Eqn. (44) can be non-linear. For instance, $y = px + p^2$ and $y = x + e^p$ are examples of Caliraut equation. But equations $y = xy + p^2$ or $y = xp + yp^2$ are not of the Clairaut's form.

On differentiating Eqn. (44) w.r. to x, we have
$p = p + p'x + f'(p)p'$ …(45)

$\Rightarrow$ p' [x + f'(p)] = 0

Then either $p' = \frac{dp}{dx} = 0$ …(46)

Or, x + f'(p) = 0 …(47)

The solution of Eqn. (46) is p = c, where c is an arbitrary constant. Thus, we can write the general solution of Eqn. (44) as

y = cx + f(c) …(48)

Note that Eqn. (48) is an equation of a family of straight lines.

Now consider Eqn. (47). Since f(p) and f'(p) are known functions of p, Eqns. (47) and (44) together constitute a set of parametric equations giving x and y in terms of the parameter p.

If we can eliminate p from Eqn. (44) and (47) and if the resulting equation satisfies Eqn. (44), we get another solution of Eqn. (44) (could be an implicit solution). This solution does not contain an arbitrary constant and is a singular solution of Eqn. (44).

We give you some examples to help you understand this method.

Example 12: Solve $(y')^2 + 4xy' – 4y = 0$ …(49)

Solution: With p = y', Eqn. (49) can be written as

$y = px + \frac{1}{4} p^2$, …(50)

which is in the Clairaut's form. Differentiating Eqn. (50) w.r. to x, we get

$p = p + p'x + \frac{p}{2}p'$

$\Rightarrow p'(x + \frac{p}{2}) = 0$

then either p = 0 which gives p = c (a constant) …(51)

or $x + \frac{p}{2} = 0$ …(52)

From Eqns. (50) and (51), we obtain

$$y = cx + \frac{c^2}{4}$$

as the solution of Eqn. (50). Eliminating p from Eqns. (50) and (52), we get

$y = x(-2x) + \frac{1}{4}(-2x)^2$,

i.e., $y(x) = - x^2$,

which contains no arbitrary constant. Since this value of y satisfies Eqn. (50), it is the singular solution of Eqn. (50).

Let us look at another example.

Example 13: Solve $y = xp + \frac{1}{p}$

Solution: If we compare the given equation with Eqn. (44) we notice that in the case $f(p) = \frac{1}{p}$ and $f'(p)\ \frac{1}{p^2}$. From Eqn. (48) then the solution is given by

$y = ax + \frac{1}{a}$

where a ($\neq 0$) is an arbitrary constant.
Also in this case, equation corresponding to Eqn. (47) is

$0 = x\ \frac{1}{p^2}$

The elimination of p between the above equation and the given equation yields
$y^2 = 4x$,
which is a singular solution of the given equation.

You may, now, try the following exercises.

Finally, we take up in the next solution, another non-linear equation known as Riccati's equation, which we mentioned in Sec. 4.1

3.2.6 Riccati's Equation

Originally, this name was given to the first order differential equation

$$\frac{dy}{dx} + by^2 = cm^m, \qquad \ldots(53)$$

where b, c and m are constants. This is known as the **special Riccati equation.** Eqn. (53) is solvable in finite terms only if the exponent m is – 2 or, of the form $\frac{-4k}{(2k+1)}$ for some integer k. Riccati merely discussed special cases of this equation without offering any solutions. Now a days Riccati's equation is usually understood by an equation of the form.

$$y' = a(x) + b(x)\, y + c(x)y^2 \qquad \ldots(54)$$

where a, b and c are given functions of x on an interval 1 (of R). Equations $y' = 1 + xy + e^x y^2$ and $y' = x + x^2 y + \sin(x) y^2$ are example of Riccati's equations whereas, equations $y' = 1 + y + y^3$, and $y' = 1 + y + 2y^2$ are not of Riccati's type.

It is difficult to obtain a solution of Riccati's Eqn. (54) containing an arbitrary constant. But, the general solution of Eqn. (53) can be obtained if we have the knowledge of a particular solution of Eqn. (53). This can be done as follows:

Let y_1 be a solution of Eqn (53) Then we determine a function v so that y defined bu the relation

$$y = y_1 + \frac{1}{v} \qquad \ldots(55)$$

is a solution of Eqn. (54).
Differentiating Eqn. (55) w.r. to x, we get

$$y' = y_1 + v\left(-\frac{1}{v^2}\right)$$

Since y and y_1 satisfy eqn. (54), we have
$y_1' = a(x) + b(x)y_1 + c(x)y_1^2$
and $y_1' - \frac{v'}{v^2} = a(x) + b(x)y + c(x)\, y^2$

subtracting the second equation from the first, we have

$$v'\left(\frac{1}{v^2}\right) = b(x)\,(y_1 - y) + c(x)\,(y_1^2 - y^2)$$

or $v' = b(x)\, v^2(y_1 - y) + c(x)\,(y_1 - y)\,(y_1 + y)v^2$ …(56)
From Eqn. (55), we have
$(y - y_1)\, v = 1$ or $(y_1 - y)\, v = -1$ …(57)
Also, $(y_1 + y)v = (2y_1 + y - y_1)v = 2y_1v + 1\,(y - y_1)v = 2y_1v + 1$ (using Eqn (57))
Now $(y_1^2 - y^2)v^2 = (y_1 - y)v.(y_1 + y)v$
$= (-1)\,(2y_1v + 1) = -1 - 2y_1v$ …(58)

Substituting from Eqn (58) in Eqn (55), we get
$v' = -(b(x) + 2c(x)y_1)v - c(x),$ …(59)
which is a linear (non-homogeneous) equation for determining a function v.

the general solution of Eqn. (59) contains an arbitrary constant and the substitution of this general solution in Eqn. (54 gives us the solution of Eqn. (53) containing an arbitrary constant.

Let us now go through some examples to understand the above theory.

Example 14: Solve $y' = -y + x^2y^2$

Solution: On comparing this equation with Eqn (53) we find that in this case a = 0, b = -1 and $c = x^2$ (1 = R). The given equation is a Riccati's Equation which has a (particular) solution $y_1 = 0$. by using the substitution

$y = y_1 + \frac{1}{v} = 0 + \frac{1}{v} = \frac{1}{v}$

in Eqn (58), we have $v' = v - x^2$ …(60)

Eqn. (60) is a linear first order equation with

I.F. = $e^{\int -1 dx} = e^{-x}$

Hence, the general solution of Eqn. (60) is

$v = - e^x \int e^{-x} x^2 + dx + Ae^x$

$= (x^2 + 2x + 2) + Ae^x$

and the solution of the given equation is

$y = \frac{1}{Ae^x + x^2 + 2x + 2}$,

which contains an arbitrary constant.

Let us look at another example.

Example 15: Solve $y' = - 1 - x^2 + y^2$.

Solution: By inspection, we see that $y_1(x) = - x$ is a solution of the given equation. Comparing the given equation with Eqn. (53), we get $a = -1- x^2$, $b = 0$ and $c = = 1$.

We look for a function v, so that $y = y_1 + \frac{1}{v} = - x + \frac{1}{v}$.

In this case Eqn. (58) reduces to

$v' = 2xv - 1$

$\Rightarrow \frac{dv}{dx} - 2xv = -1$

The integrating factor for this equation is e^{-2x}, and so it can be written as

$\frac{d}{dx}[e^{-2x}v] = - e^{-2x}$

Thus, on integration, we write

$e^{-2x} v = -\int e^{-2x} dx + c$

or, $v = e^{2x} [-\int e^{-2x} dx + c]$

where c is an arbitrary constant.

So, the required solution is

$y(x) = - x + \frac{e^{-2x}}{-\int e^{-2x} + c}$

Now the integral $\int e^{-2x}\,dx$ cannot be evaluated in terms of elementary functions. When an initial condition is specified, then integral of the form $\int_{x_0}^{x} e^{-t^2}\,dt$ can be used.

You may now try the following exercises:

3.3 Bernoulli Equation

Reduction of non-linear equation to linear form.

Here, we shall illustrate that certain non-linear first order differential equations may be reduced to linear form by a suitable change of the dependent variable.

The differential equation
$y' + p(x)y = g(x)y^a$
where 'a' is a real number is call the "Bernoulli Equation".

For a = 0 and a= 1 the equation is linear, and otherwise it is non-linear.

Set $\{y(x)\}^{1-a} = u(x)$ and show that the equation assume the linear form.

$U' + (1 - a)\,p(x)\,u = (1-a)\,g(x)$.

Solve the following Berboulli equation

1) $y' + y = \frac{x}{y}$
2) $y' + xy = xy^{-1}$
3) $3y' + y = *1 - 2x(y^4$
4) (A population model, the logistic law). Matheus's law states that the time rate of change of a population y(t) is proportional to x(t). This holds for many populations as they are not too large. A more refined model is the logistic law given by

   $\frac{dy}{dt} = ay - by^2$……… (a > 0, b> 0)
   where the "breaking term" = by^2 has the effect that the population cannot grow indefinitely.

   Solve this Bernoulli equation. What is the limit of y(t) as t as ∞? For the united state, vertalst predicted in 1845 the values a = 0.03 and b = 1.6 x 10^{-4} where t is measured in years and y(t) in millions find the particular solution satisfying y(0) = 5.3 (corresponding to the year 1800) and compare the values of this solution with some actual values.

1800	1830	1860	1890	1920	1950	1980
5 – 3	13	31	63	105	150	230

5) Apply the suitable substitutions, reduce to linear form and solve:

   a) $y' \cos y + x \sin y = 2x$
   b) $e^x y^1 + 2e^x y + x = 0$

4.0 CONCLUSION

We end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

in this unit we have covered the following:

1) the general differential equation of first order and nth degree is given by Eqn (1), namely
   $p^n + p_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0$
   where $P_1, P_2 \ldots\ldots, P_n$ are functions of x and $p = \frac{dy}{dx}$.

2) If Eqn. (1) can be resolved into rational linear factors of the first order, then Eqn. (1) takes the form
   $(p-R_1 (p- R_2) \ldots\ldots (p - R_n) = 0$
   for some $R_1, R_2, \ldots\ldots R_n$ which are functions of x and y, and
   if $f_1(x, y, c) = 0$, $f_2(x, y, c) = 0 \ldots.., f_n(x, y, c) = 0$ are the solutions of $p - R_1 = 0$, $p - R_2 = 0 \ldots..$, $p - R_n = 0$ respectively, then
   $f_1(x, y, c) . f_2(x, y, c) \ldots\ldots f_n(x, y, c) = 0$
   is the general solution of Eqn. (1).

3) If Eqn. (1) cannot be factorized into rational linear factors, the
   a) it is said to be solvable for y if we can express it as
      $y = F(x,p)$ (see Eqn. (10)).
      To solve Eqn. (10), differentiate it with respect to x, and it may be possible to solve resulting differential equation in x and p. Elimination of p between the solution of resulting differential equation and EQn. (10) gives the solution of Eqn. (10).

   b) it is said to be solvable for x if we can express it as
      $x = g(x, p)$ (see Eqn. (21)).
      To solve Eqn. (21), differentiate it w.r.t.y and it may be possible to solve the resulting differential equation in y and p. Elimination of p between the solution of the resulting equation and Eqn. (21) gives the solution of Eqn. (21).

4) If Eqn. (1) does not contain independent variable or dependent variable explicitly and can be put in the form

$f(y, p) = 0$ (See Eqn. (29))

or $g(x, p) = 0$ (see Eqn. (35))

then it may either be possible to factorize Eqn. (29) into linear factors or it may be solvable for y.

Similarly, eqn. (35) can either be factorized or it may be solvable for x.

5) If Eqn. (1) is homogeneous in x and y then either substitution $y = vx$ may reduce it to separable equation or it may be put as $y = x\psi(p)$, which is solvable for y or x.

6) Clairaut's equation is an equation of first order and of any degree if it can be put in the form
$y = xp + f(p)$ (see Eqn. (44))
This equation is solvable for y and its solution is
$y = cx + f(c)$

7) Riccati's equation is an equation of the form
$\frac{dy}{dx} = a(x) + b(x)y + c(x(y^2$ (see Eqn. (53))
where a(x), b(x) and c(x) are given functions on an interval ! of R.

The general solution of Eqn. (53) can be obtained if we know a particular solution y_1 of Eqn. (53(and then we determine a function v defined by relation

$y = y_1 + \frac{1}{v}$, (see Eqn 54). …(see Eqn 54)
so that y is solution of Eqn. (54).

6.0 TUTOR MARKED ASSIGNMENT

1. Solve the following equations:

a) $p^2y + p(x - y) - x = 0$
b) $p^2 - 5y + 6 = 0$
c) $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$
d) $\left(\frac{dy}{dx}\right)^3 = ax^4$
e) $x + yp^2 = p(1 + xy)$

2. Solve the following equations:

   a) $y = x + a \tan^{-1}p$
   b) $x = y + \text{In}p$
   c) $p^3 + p = e^y$
   d) $y = p \tan p + \text{In} \cos p$

3. Solve the following equations.

   a) $p^2 – py + x = 0$
   b) $x = y + a\ \text{In}p$
   c) $x = y + p^2$
   d) $y^2 \text{In}y = xyp + p^2$

4. Solve the following equations:

   a) $(y')^2 – 4 = 0$
   b) $\sin(y') = 0$
   c) $(y')^2 + 4y' – x^2 = 0$

5. Obtain the solution of the following equations:

   a) $\exp (y' + (1 + x^2)) = 1$
   b) $(y')^2 + 2 (x + y)y' + 4xy = 0$
   c) $p^2 – (3x +2y)p + 6xy = 0$

6. Solve the following equations.

   a) $y = yp^2 + 2px$
   b) $x^2p^2 + 4xyp – 8y^2 = 0$

7. Solve the following equations:

   a) $y = xp + \frac{a}{p}$ $(a \neq 0$, is a constant)
   b) $y = xp + p^2$
   c) $y = xp + p – p^2$

8. Solve $e^{4x} (p – 1) + e^{2y} p^2 = 0$

9. Solve $y = x^4p^2 – px$

10. Solve $xy (y – px) = x + py$

11. Which of the following are Riccati’s equation, Clairaut’s equation or neither.

a) $y = 2xp + y^2p^3$
b) $y' = e^x + e^y + y^2$
c) $y' = (1 + \sin 2x) + \frac{2}{1+x^2} y + e^x y^2$
d) $y = 3px + 6y^2p^2$
e) $y' = \sin x + \sin y$

12. Find a solution, containing an arbitrary constant (given a particular solution), of the following Riccati's equations:

a) $y' = 1 - xt + y^2$ $(y_1(x) = x)$
b) $y' = 2 + 2x + x^2 - y^2$ $(y_1(x) = 1 + x)$
c) $y' = 2x - x^2 - x^2 - x^4 + y + y^2$ $(y_1(x) = x^2)$

13. By eliminating arbitrary constant c from the equation

$$y = \frac{cg(x) + G(x)}{cf(x) + F(x)}$$

obtain the Riccati's equation:

$(gF - Gf)y' = (gG' - g'D) + (Gf' - gF' + g'F)y + (fF' - fF)y^2$.

14. Show that, when m = 0, Riccati's equation

$$\frac{dy}{dx} + by^2 = cx^m$$

7.0 REFERENCES/FURTHER READINGS

Theoretical Mechanics by Murray, R. Spiegel

Advanced Engineering Mathematics by Kreyzig

Vector Analysis and Mathematical Methods by S.O. Ajibola

Engineering Mathematics by PDS Verma

Advanced Calculus Schaum's outline Series by Mc Graw-Hill

Indira Gandhi National Open University School of Sciences Mth-07