MODULE 3

UNIT 1 FAMILIES OF CURVES ORTHOGONAL AND OBLIQUE TRAJECTORIES; APPLICATION TO MECHANICS AND ELECTRICITY

CONTENTS

1.0 INTRODUCTION

In this section we shall learn how to use differential equations for finding curve that intersect given curves at right angles, a task that arises rather often in applications.

If for each fixed real value of c the equation

(1) $F(x, y, c) = 0$

represents a curve in the xy-plane and if for variable c it represents infinitely many curves, then the totality of these curves is called a **one-parameter family of curves,** and c is called the parameter of the family.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- to find family of curves
- to be able to solve differential equation of family of curves
- One should be able to apply orthogonal trajectories to
  (i) Electrical field. (2) Mechanical field. (3) Temperature
- also to be able to find approximate solutions to directions fields iteration.

Example 1: Families of curves
The equation

(2) $F(x, y, c) = x + y + c = 0$
represents a family of parallel straight lines; each line corresponds to precisely one value of the parameter c. The equation

(3) $F(x, y, c) = x^2 + y^2 – x^2 = 0$

Represents a family of concentric circles of radius c with center at the origin.

The general solution of a first-order differential equation involves a parameter c and thus represents a family of curves. This yields a possibility for representing many one-parameter families of curves by first-order differential equations. The practical use of such representations will become obvious from our further considerations.

Example 2: Differential equations of families of curves

By differentiating (2) we see that

$$y' + 1 = 0$$

is the differential equation of that family of straight lines. Similarly, the differential equation of the family (3) is obtained by differentiation, $2x + 2yy' = 0$, that is,

$$y' = x/y.$$

if the equation obtained by differentiating (1) still contains the parameter c, then we have to eliminate c from this equation by using (1). Let us illustrate this by a simple example.

Example 3: eliminate of the parameter of a family

The differential equation of the family of parabolas

(4) $y = cx^2$
is obtained by differentiating (4),

(5) $y' = 2cx,$
and by eliminating x from (5). From (4) we have $c = y/x^2$, and by substituting this into (5) we find the desired result

(6) $y' = 2y/x.$
note that we may also proceed as follows. We solve (4) for c, finding $c = y/x^2$, and differentiate

3.0 MAIN CONTENT

3.1 Orthogonal Trajectories

In many engineering and other applications, a family of curves is given, and it is required to find another family whose curves intersect each of the given curves at right angles.[14] The curves of the two families are said to be mutually orthogonal, they form an orthogonal net, and the curves of the family to be obtained are called the **orthogonal trajectories** of the given curves (and conversely); cf. fig. 1.

Let us mention some familiar examples. The meridians on the earth's surface are the orthogonal trajectories of the parallels. On a map the curves of steepest descent are the orthogonal trajectories of the contour lines. In electrostatics the equipotential lines and the lines of electric force are orthogonal trajectories of each other. An illustrative example is shown in Fig. 2. We shall see later that orthogonal trajectories are important in various fields of physics, for example, in hydrodynamics and heat conduction.

Given a family of curves $F(x, y, c) = 0$ that can be represented by a differential equation

(7) $y' = f(x, y)$

we may find the corresponding orthogonal trajectories as follows. From (7) we see that a curve of the given family that passes through a point (x_0, y_0) has the slop $f(x_0, y_0)$ at this point. The slop of the orthogonal trajectory through (x_0, y_0) at this point should be the negative reciprocal of (x_0, y_0), that is, $-1/f(x_0, y_0)$, because this is the condition for the tangents of the two curves at $x_0, y_0)$ to be perpendicular. Consequently, the differential equation of the orthogonal trajectories is

(8) $$y' = -\frac{1}{f(x, y)}$$

and the trajectories are obtained by solving this new differential equation.

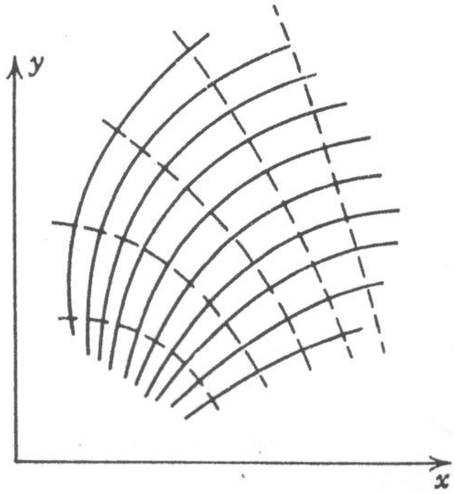

Fig. 1: Curves and their orthogonal trajectories

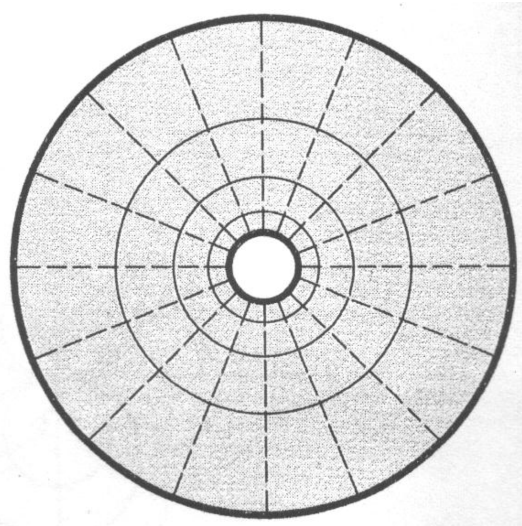

Fig. 2: Equipotential lines and lines of electric force (dashed) between two concentric cylinders

[14]Remember that the **angle of intersection** of two curves is defined to be the angle between the tangents of the curves at the point of intersection.

Example 4: Orthogonal trajectories
Find the orthogonal trajectories of the parabolas in Example 3.

Solution: From (6) we see that the differential equation (8) of the orthogonal trajectories is

$$y' = -\frac{1}{2y/x} = -\frac{x}{2y}$$

By separating variables and integrating we find that the orthogonal tracjectories are the ellipses

$$\frac{x^2}{2} + y^2 = e^* \qquad \ldots\text{(fig. 3)}$$

Example 5: Orthogonal trajectories
Find the orthogonal trajectories of the circles

9) $x^2 + 2(y - c)y' = 0.$

Solution: we first determine the differential equation of the given family, by differentiating (9) with respect to x we obtain

10) $2x + 2(y - c)y' = 0.$

We must eliminate c. solving (9) fir x, we have

11) $c = \frac{x^2 + y^2}{2y}$

By inserting this into (10) and simplifying we get

$$x + \frac{y^2 - x^2}{2y} y' = 0 \text{ or } y' = \frac{2xy}{x^2 - y^2}.$$

From this and (8) we see that the differential equation of the orthogonal trajectories is

$$y' = -\frac{x^2 - y^2}{2xy} \text{ or } 2xyy' - y^2 + x^2 = 0.$$

The orthogonal trajectories obtained by solving this equation (cf. example 1 in sec. 1.4) are the circles (fig. 4)

$$(x - c)^2 + y^2 = c$$

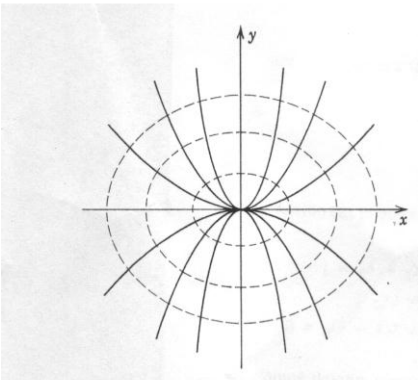

Fig. 3: Parabolas and their orthogonal Example 4

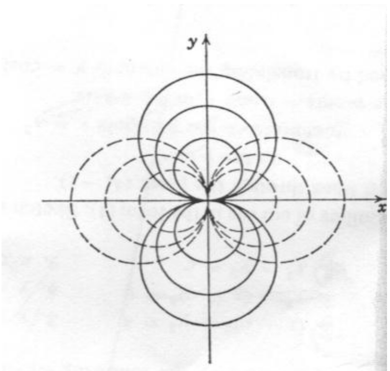

Fig. 4: Circle and their trajectories in orthogonal trajectories (dashed) in Example 5

In the next section we motivate and discuss two methods of obtaining approximate solutions without actually solving a given differential equation. The first method, called the method of **direction fields,** can relatively easily produce a general picture of the solutions (with limited accuracy) and is of great practical interest. The second method, **Picard's iteration,** is more theoretical; its practical value is limited, since it involved integrations.

SELF ASSESSMENT EXERCISES

i. $4y - x + c = 0$
ii. $(x - c)^2 + y^2 = 4$
iii. $(x - c)^2 + y^2 = c^2/2$
iv. $x^2 - y^2 = c$

Represent the following families of curve in the form (1), sketch some of the curves.

v. All nonvertical straight lines through the point (4, – 1).
vi. The catenaries obtained by translating the catenary $y = \cosh x$ in the direction of the straight line $y = - x$.

Using differential equations, find the orthogonal trajectories of the following curves. Graph some of the curves and the trajectories.

1) $y = 2x + c$
2) $y = cx^3$

3.2 applications of Orthogonal Trajectories

3) **(Electric field)** If an electrical current is flowing in a wire along the z-exist, the resulting equipotential lines in the xy-plane are concentric circles about the origin, and the electric lines of force are the orthogonal trajectories of these circles. Find the differential equation of these trajectories and solve it.

4) **(Electric field)** Experiments show that the electric lines of force of two opposite charges of the same strength at (– 1, 0) and (1, 0) are the circles through (- 1, 0) and (1, 0). Show that these circles can be represented by the equation $x^2 + (y - c)^2 = 1 + c^2$. Show that the equipotential lines (orthogonal trajectories) are the circles $(x + c^*)^2 + y^2 = c^{*2} - 1$, which are dashed in Fig. 5 on the next page.

Other forms of the differential equations Isogonal trajectories

5) Show that (8) may be written in the following form and use this result for determining the orthogonal trajectories of the curves $y = \sqrt{x + c}$.

$$\frac{dx}{dy} = - f(x, y).$$

6) Show that the orthogonal trajectories of a given family $g(x, y) = c$ can be obtained from the differential equation

$$\frac{dy}{dx} = \frac{\partial g/\partial y}{\partial g/\partial x}$$

7) **Isogonal trajectories** of a given family of curves are curves that intersect the given curves at a constant angle θ. Show that at each point the slops m_1 and m_2 of the tangents to the corresponding curves satisfy the relation

$$\frac{m_2 - m_1}{1 + m_1 m_2} = \tan\theta = \text{const.}$$

Using this formula, find curves which intersect the circles $x^2 + y^2 = c$ at an angle of 45^o.

8) Using the Cauchy – Riemann equations (Prob. 43), find the orthogonal trajectories of $e^x \cos y = c$.

3.3 Approximate Solutions: Direction Fields, Iteration

In applications, it will often be impossible or not feasible or not necessary to solve differential equation exactly. Indeed, there are various differential equations, even of the first order, for which one cannot obtain formulas for solutions. [15] There are other differential equations for which such formulas can be derived, but they are so complicated that they are practically useless. Finally, since a differential equation is a model of a physical or other system, and in modeling we disregard factors of minor influence in order to keep the model simple, the differential equation will describe the given situation only approximately, and an approximate solution will often be practically as informative as an exact solution.

Approximate solutions of differential equations can be obtained by **numerical methods.** These are discussed in sec 20.1 and 20.2. at present we shall consider the method of direction field, which is a geometric procedure, and then the so-called Picard iteration, which gives formulas for approximate solutions.

3.3.1 Method of Direction Fields

In this method we get a rough picture of all solutions of a given differential equation

1) $$y' = f(x, y)$$

without actually solving the equation. The idea is quite natural and simple, as follows.

We assume that the function f is defined in some region of the xy-plane, so that at each point in that region it has one (and only one) value. The [15]Reference [All] in Appendix 1 includes more than 1500 important differential equations and their solutions, arranged in systematic order and accompanied by numbers references to original literature.

Solutions of (1) can be plotted as curves in the xy-plane. We do not know the solutions, but we see from (1) that a solution passing through a point (x_0, y_0) must have the slope $f(x_0, y_0)$ at this point. This suggests the following method.

Ist Step (Isoclines). We graph some of the curves in the xy-plane along which f(x, y) is constant. These curves

$$f(x, y) = k = \text{const}$$

are called curves of constant slope or **isoclines.** Here the value of k differs from isoclines to isocline. So these are not yet the solution curves of (1), but just auxiliary curves.

2nd Step (Direction field). Along the isocline f(x, y) = k we draw a number of parallel short line segments **(lineal elements)** with slope k, which is the slop of solution curves of (1) at any point of that isocline. This we do for all isoclines which we graphed before. In this way we obtain a field of lineal elements, called the **direction field** of (1).

3rd Step (Approximate solution curves). With the help of the lineal elements we can now easily graph approximation curves to the (unknown) solution curves of the given equation (1) and thus obtain a qualitatively correct picture of these solution curves.

It suffices to illustrate the method by a simple equation that can be solved exactly, so that we get a feeling for the accuracy of the method.

Example 1: Isoclines, direction field
Graph the direction field of the first-order differential equation

2) $y' = xy$

and an approximation to the solution curve through the point (1, 2). Compare with the exact solution.

Solution: The isoclines are the equilateral hyperbolas $xy = k$ together with the coordinate axes. We graph some of them. Then we draw lineal elements by sliding a triangle along a fixed ruler. The result is shown in Fig. 6, which also shows an approximation to the solution curve passing through the point (1, 2).

By separating variables, $y = ce^{x^2/2}$. The initial condition is $y(1) = 2$. Hence $2 = ce^{1/2}$, and the exact solution is

$$y = 2e^{(x^2-1)/2}.$$

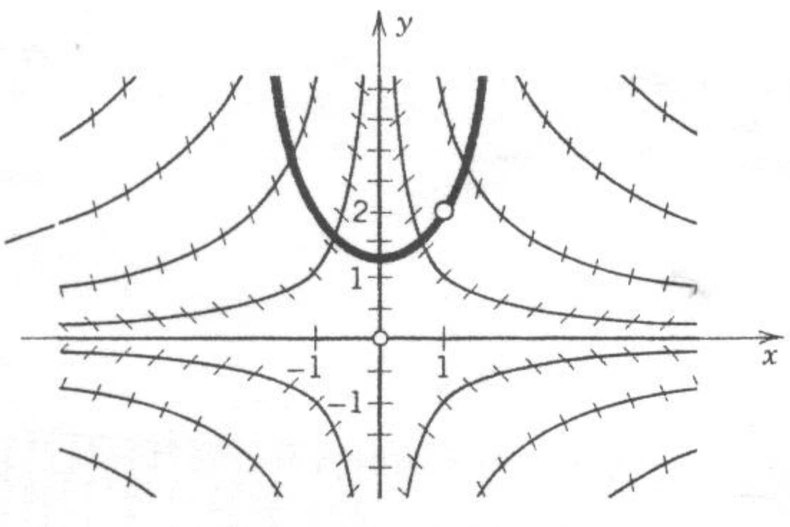

Fig. 7: Direction field of the differential equation (2)

3.3.2 Picard's Iteration method[16]

This method gives approximate solutions of an initial value problem

3) $$y' = f(x, y), \quad y(x_0) = y_0$$

Which is assume to have a unique solution in some interval on the x-axist containing x_0. Picard's existence and uniqueness theorem, which we shall discuss in the next section. Its practical value is limited because it involves integrations that may be complicated.

The basic idea of Picard's method is very simple. By integration we see that (3) may be written in the form

4) $$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt$$

Where t denotes the variable of integration. In fact, when $x = x_0$ the integral is zero and $y = y_0$, so that (4) satisfies the initial condition in (3); furthermore, by differentiating (4) we obtain the differential equation in (3).

To find approximations to the solution y(x) of (4) we proceed as follows. We substitute the crude approximation $y = y_0$ = const on the right; this yields the presumably better approximation

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt\,.$$

In the next step we substitute the function $y_1(x)$ in the same way to get

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\, dt$$

etc. The nth step of this iteration gives an approximating function

5) $$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt\,.$$

In this way we obtain a sequence of approximations.

$$y_1(x), \qquad y_2(x), \ldots\ldots\ldots.. \qquad y_n(x), \ldots\ldots\ldots,$$

and we shall see in the next section that the conditions under which this sequence converges to the solution y(x) of (3) are relatively general.

[16]EMILE PICARD (1856 – 1941), French mathematician, professor in Paris since 1881, also known for his important contributions to complex analysis (see Sec 14.10 for his famous theorem).

An **iteration method** is a method that yields a sequence of approximations to an (unknown) function, say, y_1, y_2 …., where the nth approximation, y_n, is obtained in the nth step by using one (or several) of the previous approximations, and the operation performed in each step is the same. This is a practical advantage, for instance in programming for numerical work.

In the simplest case, y_n is obtained from y_{n-1}; denoting the operation by T, we may write

$$y_n = T(y_{n-1}).$$

Picard's method is of this type, because (5) may be written

$$y_n(x) = T(y_{n-1}(x)) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\ dt\ .$$

To illustrate the method, let us apply it to an equation we can readily solve exactly, so that we may compare the approximations with the exact solution. The example to be discussed will also illustrate that the question of the convergence of the method is of practical interest.

Example 2: Picard iteration
Find approximate solution to the initial value problem

$$y' = 1 + y^2, \quad y(0) = 0$$

Solution: In this case, $x_0 = 0$, $f(x, y) = 1 + y^2$, and (5) becomes

$$y_n(x) = \int_0^x [1 + y_{n-1}^2(t)] = x + \int_0^x y_{n-1}^2(t)\ dt.$$

Starting from $y_0 = 0$, we thus obtain (cf. Fig. 8)

$$y_1(x) = x + \int_0^x 0\ dt = x$$

$$y_2(x) = x + \int_0^x t^2 dt = x + \frac{1}{3}x^3$$

$$y_3(x) = x + \int_0^x \left(t + \frac{t^3}{3}\right)^2 dt = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

etc. of course, we can obtain the exact solution of our present problem by separating variables (see Example 2 in Sec 1.2), finding

6) $$y(x) = \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right).$$

The first three terms of $y_3(x)$ and the series in (6) are the same. The series in (6) converges for $|x| < \pi/2$. This illustrates that the study of convergence is of practical importance.

The next section, the last of Chap. 1, concerns the problems of **existence** and **uniqueness** of solutions of first-order differential equations. These problems are of greater relevance to engineering applications than one would at first be inclined to believe. This is so because modeling involves the discarding of minor factors, and in more complicated situations it is often difficult to see whether some physical factor will have a minor or major effect, so that one may not be sure whether a model is faithful and does have a solution, or a unique solution, even though the physical system can be expected to behave reasonably. The matter becomes even more crycial in connection with numerical methods: make sure that the solution exists before you try to compute it.

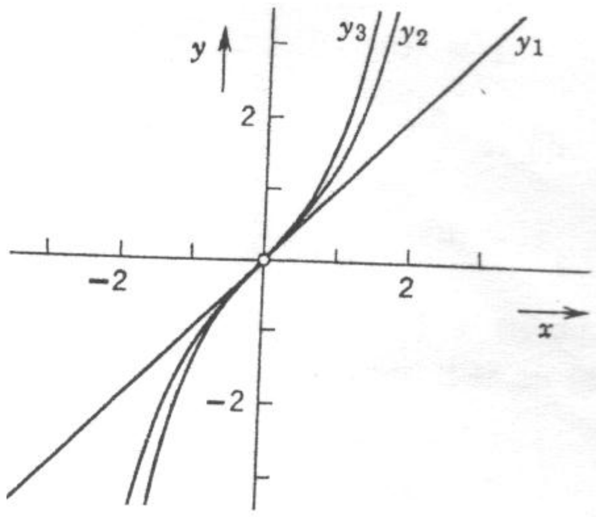

Fig 8: Approximate solutions in Example 2

SELF ASSESSMENT EXERCISES

Direction fields

In each case draw a good direction field. Plot several approximate solution curves. Then solve the equation analytically and compare, to get a feeling for the accuracy of the present method.

i. $y' = - y/x$
ii. $y' = - x/y$
iii. $y' = x + y$
iv. $4yy' + x = 0$
v. **(Verhulst population model)** Draw the direction field of the differential equation in Prob. 54 of Sec. 1.7, with a = 0.03 and b = 1.6. 10^{-4} and use it to discuss the general behavior of solutions corresponding to initial greater and smaller than 187.5
vi. Apply Picard's method to $y' = y$, $y(0) = 1$, and show that the successive approximations tend to $y = e^x$, the exact solution.
vii. In Prob. 12, compute the values y_1, $y_2(1)$, $y_3(1)$ and compare them with the exact value $y(1) = e = 2.718\cdots$.

Apply Picard's method to the following initial value problems. Determine also the exact solution. Compare.

viii. $y' = xy$, $y(0) = 1$
ix. $y' = 2y$, $y(0) = 1$
x. $y' – xy = 1$, $y(0) = 1$.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

Applications are included at various places. The unit part entirely devoted to applications are 3.3.1 and 3.3.2. on separable and linear equation respectively. And are applied to electrical circults and on orthogonal trajectories, that is curves that intersect given curves at right angles.

Direction field (3.3.1) help in sketching families of solutions curves, for instance, in order to gain an impression of their general behaviour.

Picard's iteration method gives approximate solutions of initial value problems by iteration.

6.0 TUTOR MARKED ASSIGNMENT

In each case draw a good direction field. Plot several approximate solution curves then solve the equation analytically and compare, to get a feeling for the accuracy of the present method;

1) $y' = -\dfrac{y}{x}$

2) $y' = -\frac{x}{y}$

3) $y' - x + y$

4) $4yy' + x = 0$

5) draw the direction field of the differential equation $\frac{dy}{dt} = ay - by^2$ a>0, b> 0. with a = 0.03 and b= 1.6 x 10^{-1} and use it to discuss the general behaviour of solutions corresponding to initial conditions greater and smaller than 187.5.

6) apply Picard's method to $\frac{dy}{dx} = y$, y(0) = 1 and show that the successive approximations tends to $y = e^x$, the exact solutions.

7) In the i.e. $\frac{dy}{dx} + 3y = e^{2x} + 6$, compute the values $y_1(1)$, $y_2(1)$, $y_3(1)$ and compute them with the exact value y(1) = e = 2.718.

Apply Picard's method to the following initial value problems. Determine also the exact solution.
Compare

8) a) $\frac{dy}{dx} = y$, y(0) = 1

9) b) $\frac{dy}{dx} = 2y$, y(0) = 1.

10) c) $\frac{dy}{dx} - xy = 1$, y(0) = 1.

7.0 REFERENCES/FURTHER READINGS

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Advanced Calculus Schaum's outline Series by Mc Graw-Hill

Indira Gandhi National Open University School of Sciences Mth-07