MODULE 4

UNIT 1 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

In Unit 1 we discussed the basic concepts related to ordinary differential equations. Further in the introduction to Block 2, we have mentioned that the governing differential equations in many physical or biological problems are not necessarily of first order. Besides the differential equations arrived at, in discussing the above said models may be linear or non-linear. Even among linear differential equations, the coefficients of the differentials may be constants or a function of an independent variable. In this unit we classify the general linear differential equations into two broad categories:

i) homogeneous and non-homogeneous
ii) equations with constant coefficients and variable coefficients.

For a general linear differential equation with variable coefficients, we shall state the conditions under which a unique solution can be found. Further, we shall learn methods of finding the complete solutions of homogeneous differential equations with constant coefficients.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify linear differential equations with constant as well as with variable coefficients

- identify homogeneous and non-homogeneous linear differential equations
- describe the conditions under which a unique solution of a linear differential equation exists
- write the complete primitive of a given differential equation when its various independent integrals are known
- classify solutions of non-homogeneous equations into complementary function and particular integral
- obtain a solution for a homogeneous linear differential equation with constant coefficients.

3.0 MAIN CONTENT

3.1 General Equation

We begin our discussion by considering the most general linear differential equation which is of the form

$$a_0 (x) \frac{d^n y}{dx^n} + a_1 (x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots.+ a_{n-1} (x) \frac{dy}{dx} + a_n (x) y = b (x) \qquad \ldots(1)$$

For, $a_0(x) = 0$, the differential equation is of nth order. The coefficients $a_0(x)$, $a_1(x)$, …., $a_n(x)$ are functions of independent variable x. Eqn. (1) is called **general linear differential equation of nth order with variable coefficients.**

In case coefficients $a_0(x)$, $a_1(x)$,….$a_n(x)$ are all constants and do not depend on x, then Eqn. (1) will be termed as **general linear differential equation of nth order with constant coefficients.** For example, equation $\frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} + y = x^2$ is a third order linear differential equation with constant coefficients.

Further, the right hand side of Eqn. (1), i.e., b(x) may assume one of the following forms:

i) b(x) = 0
ii) b(x) = constant
iii) b9x) a function of x.

when b(x) = 0, Eqn. (1) is classified as the **general homogeneous linear differential equation.** This is also known as the **reduced equation** of Eqn. (1). For example, equation

$$\frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$$

is a third order linear differential equation. But if b(x) in Eqn. (1) is a constant or a function of x, then Eqn. (1) is called **general non-homogeneous linear differential equation.**

Equation $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + 3y = x^2 + 1$ is a linear non-homogeneous equation of 4^{th} order with constant coefficients; where equation $x^3 \frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy = 2$ is a second order non-homogeneous linear differential equation with variable coefficient.

Now suppose that we are required to find the solution of Eqn. (1) on some interval 1 which also satisfy, at some point $x_0 = \in 1$ the conditions,

$$y(x_0) = y_0, y'(x_0) =, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \qquad \ldots(2)$$

Note: Depending on the context, i could represent [a, b] [0, ∞], [-∞,∞[ ]and so on.

Where y_0, y_0', …., $y_0^{(n-1)}$ are arbitrary constants, then Eqns. (1) and (2) together constitute an **initial-value problem (IVP).** The values $y(x_0) = y_0$ $y'(x_0) = y_0'$,….., $y_0^{(n-1)}$ $(x_0) = y_0^{(n-1)}$ are called **initial conditions.**

In the case of a linear second order equation, we can interpret geometrically a solution to the initial value problem

$$a_{n-2}(x) \frac{d^2y}{dx^2} + a_{n-1}(x) \frac{dy}{dx} + a_n(x)\, y = b(x)$$

$y(x_0) = y_0$, $y'(_0) = y_0'$
as a function defined on I whose graph passes through (x_0, y_0) such that the slope of the curve at the point is the number y_0'.

You may **note** here that an equation of the form (1) may not always have a solution. Moreover, even if its solution exists it may not be unique.

Let us now study the conditions under which the solution of Eqn. (1), if it exists shall be unique.

3.1.1 Conditions for the Existence of a unique solution

We may write the general non-homogeneous linear differential Eqn. (1) in the form

$$L(y) = [a_0(x) D^n + a_1(x) D^{n-1} + \ldots + a_{n-1}(x) D + a_n(x)]\, y = b(x) \qquad \ldots(3)$$

Where $D = \frac{dy}{dx}$, $D^2 = \frac{d^2}{dx^2}$, …, $D^n = \frac{d^n}{dx^n}$.

The expression in the parentheses in Eqn. (3) is termed as a **symbolic polynomial** or **operator polynomial** or simply a **differential operator.**

Thus we have herein introduced linear differential operator L of order n given by the expression

$$L = a_0(x)\, D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)\, D + a_n(x) \qquad \ldots(4)$$

In unit 8, we shall learn, in more details, about the differential operators and their properties.

We now choose an interval I = $[\alpha, \beta]$ for α, β real and assume that the coefficients $a_0(x)$, $a_1(x)$, …., $a_n(x)$ and the function are continuous one-valued functions of x throughout the interval and that $a_0(x)$ does not vanish at any point of the interval.

We know that the complete solution of Eqn. (3) shall involve arbitrary constants whose number is equal to the order of the highest derivative involved in it, i.e., n in this case. In order to obtain a unique solution of Eqn. (3), it is necessary to specify n initial conditions in terms of constant values of

$$y,\ \frac{dy}{dx},\ \ldots.,\ \frac{d^{n-1}y}{dx^{n-1}}$$

at any point x_0 of the interval $[\alpha, \beta]$.

We now state a theorem which gives the conditions whose fulfillment guarantee the existence and uniqueness of the solution of Eqn. (3).

Theorem 1: If the functions $a_0(x)$, $a_1(x)$, …., $a_n(x)$ and b(x) are continuous function of x in the interval $[\alpha, \beta]$ and $a_0(x)$ does not vanish at any point of that interval, then the initial Eqn. (3) admits of a unique solution of the form y = f(x), which together with its first (n-1_ derivatives, is continuous in $[\alpha, \beta]$ and satisfies the following initial conditions:

$$y(x_0) = y_0 \quad \left(\frac{dy}{dx}\right)_{x=x_0} = y_0',\ldots.,\ \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_{x=x_0} = y_0^{(n-1)},$$

where x_0 is a point of the interval $[\alpha, \beta]$.

We shall not be proving this theorem here as it is beyond the scope of the present course. However, if the functions $a_0(x)$, $a_1(x)$,…., $a_n(x)$ are constants, we shall give the solution of the corresponding equation in Sec. 5.4 when b(x) = 0 and in units 6, 7 and 8 when $b(x) \neq 0$.

We now illustrate this theorem with the help of a few examples.

Example 1: Show that $y = 3e^{2x} + e^{-2x} – 3x$ is a unique solution of the initial value problem

$y'' – 4y = 12x$
$y(0) = 4\ h'(0) = 1.$

Solution: We have $y = 3e^{2x} + e^{-2x} – 3x$, therefore,

$y' = 6e^{2x} – 2e^{-2x}\ 3$ and $y'' = 12e^{2x} + 4e^{-2x}$

Now $y'' = 4y = 12e^{2x} + 4e^{-2x} – 4(3e^{2x} + e^{-2x} – 3x)$
$= 12e^{2x} + 4e^{-2x} – 12e^{2x} – 4e^{-2x} + 12x$
$= 12x$

Also, $y(0) = 3e^{2.0} + e^{-2.0} – 3.0 = 4$
$Y'(0) = 6e^{2.0} – 2e^{-2.0} – 3 = 1.$

Thus, $y = 3e^{2x} + e^{-2x} – 3x$ is a solution of the given initial value problem. Moreover, the given differential equation is linear and the coefficients as well as $b(x) = 12x$ are continuous on any interval containing $x = 0$. We conclude from Theorem 1 that the given function is the unique solution of the given initial vlue problem.

Remember that both the requirements in Theorem 1, that is $a_i(x)$, $i = 0, 1, …., n$ be continuous and $a_0(x) \neq 0$ for every x in some interval say I are important. Specifically, if $a_0(x) = 0$ for some x in the interval then the solution of a liner initial value problem may not be unique or may not even exist.

We now illustrate this through an example.

Example 2: Obtain the value of c for which the function

$$y = cx^2 + x + 3$$

is a unique solution of the initial value problem

$x^2y'' – 2xy' + 2y = 6,$
$y(0) = 3,\ y'(0) = 1$

On the interval $[-\infty, \infty]$.

Solution: Since $y' = 2cx + 1$ and $y'' = 2c$, it follows that

$X^2y'' – 2xy' + 2y = x^2(2c) – 2x(2cx + 1) + 2(cx^2 + x + 3)$
$= 2cx^2 – 4cx^2 – 2x + 2cx^2 + 2x + 6$
$= 6$

Also, $y(0) = c.(0)^2 + 0 + 3 = 3$
and $y'(0) = 2c.0 + 1 = 1$

Thus, $y = cx^2 + x + 3$ is a solution of the given problem for all values of c in the given interval. The problem does not have a unique solution. In this case although the given equation is liner and its coefficients and b(x) = 6 are continuous everywhere but the coefficient of y" i.e., x^2 is zero at x = 0.

You may now try the following exercise.

You might be familiar with the liner dependence and independence of a set of functions on an interval. Before we study some elementary properties of the solution of linear differential equations, etc us recall these two concepts which are basic to the study of linear differential equations.

3.1.2 Linear Dependence and Independence

We begin with the following two definitions.

Definition: A set of function $y_1(x)$, $y_2(x)$,…., $y_n(x)$ is said to be **linearly independent** on an interval I if there exist constants. c_1, c_2, …., $c_n(x)$ not all zero, such that

$c_1y_1(x) + c_2y_2(x) +….+ c_ny_n(x) = 0$

For every x in the interval.

Definition: A set of functions y1(x), $y_2(x)$,…., $y_n(x)$ is said to be **linear independent** on an interval I, if it is not linearly dependent on the interval.

In other words, a set of functions is linearly independent on an interval if the only constant for which

$$c_1y_1(x) + c_2y_2(x) + ….. + c_ny_n(x) = 0,$$

For every x in the interval, are $c_1 = c_2 = …. = c_n = 0$.

It is easy to understand these definitions in the case of two functions $y_1(x)$ and $y_2(x)$. If the functions are linearly dependent on an interval, then there exists constants c_1 and c_2, both are not zero, such that for every x in the interval

$$c_1y_1(x) + c_2y_2(x) = 0$$

Since $c_1 \neq 0$, it follows that

$$y_1(x) = -\frac{c_2}{c_1}y_1(x),$$

That is, **if two functions are linearly dependent, then one is a constant multiple of the other.** Conversely, if $y_1(x) = c_2y_2(x)$ for some constant c_2, then

$$(-1)\,y_1(x) + c_2y_2(x) = 0$$

for every x on some interval. Hence the functions are linearly dependent, at least one of the constants (namely, $c_1 = -1$) is not zero. We thus conclude that **two functions**

are linearly independent when neither is a constant multiple of the other on an interval.

Functions, $y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$ are linearly dependent on the interval $[-\infty, \infty]$ since $c_1 \sin 2x + c_2 \sin x \cos x = 0$ is satisfied for every real x with

$C_1 = \frac{1}{2}$ and $c_2 = -1$.

In the consideration of linear dependence or linear independence, the interval on which the functions are defined is important. We now illustrate it through an example.

Example3: Show that the function $y_1(x) = x$ and $y_2(x) = |x|$ are

i) linearly independent on the interval $[-\infty, \infty]$.

ii) linearly dependent on the interval $[0, \infty]$.

Solution:

(i) it is clear that in the interval $[-\infty, \infty]$ neither function is a constant multiple of the other (see Fig. 1)

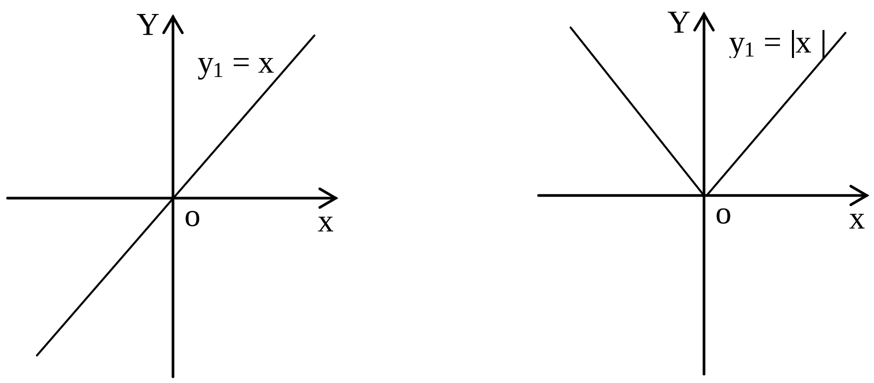

Fig. 1

Thus in order to have $c_1y_1(x) + c_2y_2(x) = 0$ for every real x, we must have $c_1 = 0$ and $c_2 = 0$.

(ii) For $y_1(x) = x$ and $y_2(x) = |x|$ in the interval $[0, \infty]$

$c_1x + c_2 |x| = c_1x + c_2x = 0$

is satisfied for any non zero choice of c_1 and c_2 for which $c_1 = - c_2$.

Thus $y_1(x)$ ands $y_2(x)$ are linearly dependent on the interval $[0, \infty]$.

You may try the following exercises:

The procedure given for examining the linear dependence or independence of a set of functions appears to be quite involved. We, therefore, outline below sufficient condition of examining the linear independence of a set of n functions.

Suppose that $y_1(x)$, $y_2(x)$,…., $y_n(x)$ are n functions on an interval I possessing derivative upto (n – I)th order. If the determinant.

$$W(y_1(x), y_2(x), \ldots, y_n(x)) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ . & . & \cdots & . \\ . & . & \cdots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Is not zero for at least one point in the interval I, then the functions $y_1(x)$, …., $y_n(x)$ are linearly independent on the interval.

This provides a **sufficient condition** for the linear independence of n functions on an interval. The determinant $W(y_1(x), y_2(x),\ldots, y_n(x))$ is called the **Wronskian** of the functions. It is named after a Polish mathematician Josef Maria Hosene Wornski (1778 – 1853).

The functions $y_1(x) = \sin^2 x$ and $y_2(x) = 1\text{-}\cos 2x$, for instance are linearly dependent on $[\infty, \infty]$ because

$$\begin{vmatrix} \text{sun}^2 x & 1-\cos 2x \\ 2\sin x \ \text{cox} & 2\sin 2x \end{vmatrix} = 2\sin^2 x \sin 2x - 2\sin x \cos x + 2\sin x \cos x\ 2x$$

$$= \sin 2x\,[2\sin^2 x - 1 + \cos 2x]$$
$$= \sin 2x\,[2\sin^2 x - 1 + \cos^2 x - \sin^2 x]$$
$$= \sin 2x\,[\sin^2 x + \cos^2 x - 1]$$
$$= 0$$

in example 3 we saw that $y_1(x)= x$ and $y_2(x) = |x|$ are linearly independent on $[-\infty,\infty]$. However, we cannot compute the Wronskian as y_2 is not differentiable at x = 0.

Remember that in the above condition the non-vanishing of the Wronskian at a point in the interval provides only a sufficient condition. In other words, if $W(y_1 y_2,..,y_n) =0$ for x in an interval, it does necessarily mean that the functions are linearly dependent on the interval. We leave it for you to verify it yourself.

With above background in mind we are now set to study the elementary properties of the solutions of linear differential equations.

3.2 Elementary Properties of the Solutions

The general homogeneous linear differential equation corresponding to Eqn. (3) is

$$L(y) = [a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)]\, y = 0$$

i.e., $L(y) = \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ y = 0$ …(5)

We can clearly think of the form of the solutions of linear differential equations by making use of the following elementary theorems:

Theorem 2: If $y = y_1$ is a solution of Eqn. (5) on an interval 1, then $y = cy_1$ is also its solution on I, where c is any arbitrary constant.

Proof: We know that

$D^r (cy^1) = cD^r y^1$

Also, $L(cy^1) = \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ (cy_1)$

$= c \sum_{s=0}^{n} a_s(x)\ D^{n-s}\ y_1$

$= c\ L(y_1).$

$= 0\ (\therefore L(y_1) = 0)$

Thus, if $y = y_1$ is a solution of Eqn (5), so $y = cy_1$. for instance, the function $y = x^2$ is a solution of the homogeneous linear equation.

$X^2y'' – 3xy' + 4y = 0$ on $]0, \infty[$.

Hence $y = cx^2$ is also solutions. For various of c, we see that $y = 3x^2$, $y = ex^2$, $y = 0$ … are all solutions of the equation on the given interval.

Have you notice that **a homogeneous linear differential equation always possesses the trivial solution y = 0?** If not, you can check it now.

Now let us look at another property of the solutions of linear differential equations.

Theorem 3: If $y = y_1, y_2, …, y_m$ are m solutions of homogeneous linear differential Eqn. (5) on an interval 1, then $y = c_1y_1 + c_2y_2 + …. + c_my_m$ is also a solution of Eqn. (5) on 1, where $c_1, …., c_m$ are arbitrary constants.

Proof: If y_1 (i = I, …, m) are solutions of Eqn. (5) then

$L(y_1) = 0$ (for i = 1, 2, …., m) …(6)

We know that

$D^r [c_1y_1 + c_2y_2 + …+ c_my_m]$

$= D^r (c_1y_1) + D^r (c_2y_2) + …+ D^r (c_my_m)$

$= c_1 D^r (y_1) + c_2 D^r (y_2) + …+ c_m D^r (y_m)$

now, $L(c_1y_1 + c_2y_2 + …+ c_my_m)$

$= \sum_{r=0}^{n} a_r(x)\ D^{n-r} (c_1y_1 + c_2y_2 + … + c_my_m)$

$= c_1 \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ y_1 + c_2 \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ y_2 + … + c_m \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ y_m$

$= c_1 L(y_1) + c_2 L(y_2) + \ldots + c_m L(y_m)$ (using Eqn. (5))
$= c_1.0 + c_2.0 + \ldots + c_m.0$ (using Eqn. (6))
$= 0$

Hence if $y_1, y_2 \ldots. y_m$ are solutions of Eqn. (5), then $y = c_1y_1 + c_2y_2 + \ldots + c_my_m$ is also a solution of Eqn. (5).
Theorem 3 is known as the **superposition principle.**

Let us now consider an example.

Example 4: Show that if $y_1 = x^2$ and $y_2 = x^2$ Inx are both solutions of the equation $x^3y'' = 2xy' + 4y = 0$ on the interval $]0, \infty[$. Then $c_1x^2 + c_2x^2$ Inx is also a solution of the equation on the given interval.

Solution: We have $y = c_1x^2 + c_2x^2$ Inx
Now $y' = 2c_1x + 2c_2x \text{ Inx} + c_2x$
$y'' = 2c_1 + 2c_2x \text{ Inx} + 3c_2$

$$y'' = \frac{2c_2}{x}$$

therefore, $x^3y'' - 2xy' + 4y$

$$= x^3 \left(\frac{2c_2}{x}\right) - 2x\,(2c_1x + 2c_2x \text{ Inx} + c_2x) + 4c_1x^2 + 4c_2x^2 \text{ Inx}$$
$$= 2c_2x^2 - 4c_1x^2 - 4c_2x^2 \text{ Inx} - 2c_2x^2 + 4c_1x^2 + 4c_2x^2 \text{ Inx}$$
$$= 0$$

Thus, $y = c_1x^2 + c_2x^2$ Inx is also a solution of the equation on the interval.

Theorem 2 and 3 represent properties that non-linear differential equations, in general, do not possess. This will become more clear to you after you have done the following exercises.
Let us now consider the following definition which involves a linear combination of solutions.

Definition: Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of homogeneous linear differential Eqn. (5) of degree n on an interval I. Then

$$y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x),$$

where c_1, i = 1, 2, …, n are arbitrary constants is defined to be the **general solution** or the **complete primitive** of Eqn. (5) on I.

The above definition automatically generates our interest in knowing when n solutions, $y_1, y_2, \ldots, y_n$ of the **homogeneous** differential Eqn. (5) are linearly independent. Surprisingly, the nonvanishing of the Wronskian of a set of n such solutions on an interval I is both **necessary and sufficient** for linear independence.
That is,

If $y_1, y_2, \ldots, y_n$) be n solutions of homogeneous linear nth order differential EQn. (5) on an interval I, then the set of solutions is linearly independent on I if and only if

$W(y_1, y_2, \ldots, y_n) \neq 0$

For every x in the interval. Such a set $y_1, \ldots., y_n$ of n linearly independent solutions of Eqn. (5) on I is said to be a **fundamental set of solutions** on the interval.

For instance, the second order equation y" – 9y = 0 possesses two solutions

$y_1 = e^{3x}$ and $y_2 = 3^{-3x}$

Since $W(3^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$

For every value of x, y_1 and y_2 form a fundamental set of solution on $]-\infty, \infty[$. The general solution of the differential equation on the interval is

$y = c_1 e^{3x} + c^2 e^{-3x}$

so far we have discussed the properties pertaining to the solution of homogeneous linear equations. We now turn our attention to the **non-homogeneous linear equation.** To this effect, we consider a theorem due to D' Alembert (1762 – 1765) which defines the general solution of a non-homogeneous linear equation.

Theorem 4: If $y = Y_0(x)$ is any solution of the non-homogeneous linear differential Eqn. (3) on an interval I and if y = Y(x) is te complete primitive of the corresponding homogeneous linear differential Eqn. (5) on the interval, then

$Y_0(x) + Y(x)$

Is the general solution of Eqn. (3) on the given interval.

Proof: Since $Y_0(x)$ is a solution of Eqn. (3),

$\therefore L[Y_0(x)] = b(x)$ ...(7)

also, Y(x) is the complete primitive of Eqn. (5),

$\therefore L[Y_0(x)] = 0$ …(8)

further, in Theorem 2 and 3 above, we have seen that the operator Dr and linear differential operator L are distributive. Thus, using relations (7) and (8), we get

$$\begin{aligned} L(y) &= L[Y_0(x) + Y(x)], \\ &= L[Y_0(x)] + L[Y(x)] \\ &= B(x) + 0 \end{aligned}$$

Thus, $y = Y_0(x) + Y(x)$ is a solution of Eqn. (3).

Since $y(x) = Y_0(x) + Y(x)$ involves n arbitrary constants (due to presence of n arbitrary constants in Y(x)), it is, therefore, the general solution of Eqn. (3).

If Y(x) is chosen as to satisfy the condition (2) and if $Y_0(x)$, for some point x_0 of the interval I, is such that

$$Y_0(x_0) = 0 = \left(\frac{dY_0}{dx}\right)_{x=x_0} = \left(\frac{d^2Y_0}{dx^2}\right)_{x=x_0} = \ldots = \left(\frac{d^{=-1}Y_0}{dx^{n-1}}\right)_{x=x_0},$$

Which is possible provided that b(x) is not identically zero, then the solution

$$y = Y_0(x) + Y(x) \qquad \ldots(9)$$

also satisfies the conditions.

$$y(x_0) = y_0, \left(\frac{dy}{dx}\right)_{x=x_0} = y_0', \ldots, \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_{x=x_0} = y_0^{n-1}$$

we usually refer the solution of Eqn. (5) in the form (9) as the **general solution** of the non-homogeneous linear differential Eqn. (3) and it consist of two parts:

i) The complete primitive of Eqn. (5) (the corresponding homogeneous part of Eqn. (3)) in the form

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \ldots. + c_ny_n(x).$$

Which contains n arbitrary constants. The solution y = Y(x) of Eqn. (5) is known as **complementary function** of Eqn (3). We denote the complementary function
Y)x) by $y_c(x)$.

ii) Any solution $y = Y_0(x)$ of Eqn. (3), (which cannot be obtained by assigning any particular value to the arbitrary constants in $y_c(x)$ is known as **particular integral** of Eqn. (3). We denote $Y_0(x)$ by $y_p(x)$.

Thus, we may write Eqn. (9) in the form $y(x) = y_p(x) + y_c(x)$.

You may then ask the natural question – how to find the solution y(x) of Eqn. (3)?

In the next section we give you the methods of finding the complementary function $y_c(x)$ of the given linear equation with constant coefficients. Since the complementary function refer to the solution of the homogeneous equation corresponding o the given equation, we consider the general nth order homogeneous linear differential equation with constant coefficients.

You may recall that in Sec. 5.2, we had mentioned that if the coefficients of y and its derivatives in Eqn (1) are constants and $a_0 \neq 0$, then Eqn (1) is termed as linear differential equation of nth order with constant coefficients. Further, we had mentioned that if the right hand side of Eqn. (1) is zero, then it will be classified as homogeneous linear differential equation for this reason

Eqn. (1) i.e. the function b(x) is also called non-homogeneous term of Eqn. (1). Thus, the general nth order homogeneous linear differential equation with constant coefficients may be expressed as

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots. + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \ldots(10)$$

where the coefficients $a_1, a_2, \ldots., a_n$ are constants.

we would like to mention here that in writing Eqn. (10) we have taken the coefficient of $\frac{d^n y}{dx^n}$ as unity. Even if it is not so, dividing throughout by the coefficient of $\frac{d^n y}{dx^n}$ (which is also assumed to be constant), the equation can be reduced to the form (10).

Let us now discuss the methods of solving EQn. (10).

3.3 Method of Solving Homogeneous Equation with Constant Coefficients

The method of solving Eqn. (10) was given in the year 1739 by Leonhard Euler (1707 – 1783) who was born in Basel, Switzerland and was one of the most distinguished mathematicians of the eighteenth century.

The method is as follows:

Assume that $y = e^{mx}$ is a solution of Eqn. (10). On replacing y and its derivatives upto order n by e^{mx} and $m^n e^{mx}$ in Eqn. (10), we get

$$(m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n) e^{mx} = 0 \qquad \ldots(11)$$

since $e^{mx} \neq 0$ for real values of x, Eqn. (11) is satisfied if

$$m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0 \qquad \ldots(12)$$

Eqn. (12) is called an **auxiliary equation or characteristic equation** corresponding to differential Eqn (10).

You might have observed that an auxiliary equation of a homogeneous or non-homogeneous linear differential operator on replacing D by some finite constant m and equating it to zero.

You may wonder why we assumed the solution of Eqn. (10) in the exponential form. This is because we know that the linear first order equation $\frac{dy}{dx} + ay = 0$.

Where a is a constant, has the exponential solution $y = c_1 e^{-ax}$ on $[-\infty, \infty]$. Therefore, it is natural to determine whether exponential solution exist on $[-\infty, \infty]$. For higher order equations of the form (10).

In the discussion to follow, you will be surprised to see that all solutions of Eqn. (10) are exponential functions or constructed out of exponential functions.

Let us now consider the following examples:

Example 5: Write auxiliary equation corresponding to the differential equation
$(D^6 + 12D^4 + 48^2 + 64)\ y = 0$
Solution: Replacing D by m in the linear differential operator of the given equation, the auxiliary equation becomes
$m^6 + 12m^4 + 48m^2 + 64 = 0$

Example 6: Write the characteristic equation corresponding to the differential equation
$(D^2 + 2aD + b^2)\ y = c \sin wx.$

Solution: On replacing D by m in the homogeneous part of the given equation and equating it to zero, we arrive at the following characteristic equation
$m^1 + 2am + b^2 = 0$

remember that while writing the auxiliary equation for non-homogeneous differential equation, he non-homogeneous part is neglected.

Auxiliary Eqn. (12) is a polynomial in m of degree n and, it can have at the most n roots.

Let $m_1, m_2, \ldots, m_n$ be the n roots. Then the following three possibilities arises;

I) Roots of auxiliary equation may be **all real** and distinct,
II) Roots of auxiliary equation may be **all real,** but **some** of the roots may be **repeated.**
III) Auxiliary equation may have **complex roots.**

We now proceed to find the solution of Eqn. (10) for these three cases one by one.

Case 1: Auxiliary equation has real and distinct roots:
Let the roots $m_1, m_2, \ldots, m_n$ of auxiliary Eqn. (12) be real and distinct.

Now suppose $m = m_1$. Since m_1 is a root of auxiliary Eqn. 912), clearly e^{m_1x} is an integral of Eqn. (10) and satisfies it on the interval $[-\infty, \infty]$.

Similarly, for $m = m_2$, e^{m_2x} is a solution of Eqn. (10) and e^{m_1x} and e^{m_2x} are also linearly independent on the interval since

$$W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$$

$$= (m_2 - m_1)\ e^{(m_1+m_2)x} \neq 0 \text{ for } m_1 \neq m_2.$$

Now, the n roots of Eqn (12), namely $m_1, m_2, \ldots, m_n$ are real and distinct solutions, e^{m_1x}, e^{m_2x}, …, e^{m_nx} are all distinct and linearly independent solutions of Eqn. (10).

Sin Eqn. (10) is of nth ordr and we have n distinct and linearly independent solutions, therefore, we can express the complete solution of Eqn (10) as

$$y = c_1 e^{m_1x} + c_2 e^{m_2x} + \ldots + c_n e^{m_nx}, \quad \ldots(13)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
We now illustrate this case with the help of a few examples.

Example 7: Solve $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

Solution: The given equation can be written as
$(2D^2 + 5D - 12)\, y = 0$
the auxiliary equation is
$2m^2 + 5m - 12 = 0$
$\Rightarrow (2m - 3)(m + 4) = 0$
$\Rightarrow m = 3/2, -4$

here the roots are real and distinct.

Hence complete solution of the given differential equation is $y = c_1e^{(3/2)x} + c_2e^{-4x}$, where c_1 and c_2 are arbitrary constants.

Let us look at another example.

Example 8: If $\frac{d^2y}{dx^2} - a^2y = 0$, show that y = Acosh ax + Bsinh ax is the complete solution.

Solution: The auxiliary equation corresponding to the given differential equation is
$m^2 - a^2 = 0$
$\Rightarrow (m - a)(m + a) = 0$
$\Rightarrow m = a, - a.$

roots being real and distinct, the general solution of the given equation is
$y = c_1e^{ax} + c_2e^{-ax}$

From the definition of hyperbolic functions, we know that

$$\text{Cosh } ax = \frac{1}{2}(e^{ax} + e^{-ax}) \quad \ldots(14)$$

$$\text{and sinh } ax = \frac{1}{2}(e^{ax} - e^{-ax}) \quad \ldots(15)$$

adding relations (14) and (15), we get
$e^{ax} = \cosh ax + \sinh ax$
Substrating relation (15) from (14), we get
$e^{-ax} = \text{coshax} - \text{sinhax}$

the general solution of given differential equation can thus be written as
y = c_1 (cosh ax + sinh ax) c_2 (cosh ax – sinh ax)
$\Rightarrow$ y = (c_1 + c_2) cosh ax +(c_1 – c_2) sinh ax
$\Rightarrow$ A cosh ax + B sinh ax,
where A = c_1 + c_2 and B = c_1 – c_2 are two arbitrary constants.

We now consider an initial value problem.

Example 9: Solve the equation

$$\frac{d^2x}{dx^2} - 4x = 0$$

with the conditions that when t = 0, x = 0 and $\frac{dx}{dt}$ = 3.

Solution: The auxiliary equation corresponding to the given equation is
$m^2 – 4 = 0$
$\Rightarrow$ (m – 2) (m + 2) = 0
$\Rightarrow$ m = 2, -2

hence the general solution of the differential equation is
x = c_1e^{-2t}.
we now apply the given conditions at t = 0.we have

$$\frac{dx}{dt} = 2c_1e^{-2t}$$

Condition that x = 0 when t = 0 gives
0 = c_1 + c_2,
and the condition that $\frac{dx}{dt}$ = 3 when t = 0 gives
3 = $2c_1 – 2c_2$

From the two equations for c_1 and c_2, we conclude tht
$C_1 = \frac{3}{4}$ and $C_2 = -\frac{3}{4}$. Therefore,

$$X = \frac{3}{4}(e^{2t} – e^{-2t})$$

Which can also be put in the form
x = sinh 2t.
Now you may try the following exercises.

We now take up the case when the roots of auxiliary equation are all real but some of them are repeated.

Case II: Auxiliary Equation has real and repeat roots:

Let two roots of auxiliary Eqn. (12) be equal, say $m_1 = m_2$. then solution (13) of Eqn. (10) becomes
$y = (c_1 + c_2)\ e^{m_1 x} + c_3\ 3^{m_3 x} + \ldots + c_n\ e^{m_n x}$

Since $(c_1 + c_2)$ can be replaced by a single constant, this solution will have (n -1) arbitrary constants.

We know that the general or complete solution of an nth order linear differential equation must contain n arbitary constants; hence the above solution having (n -1) arbitrary constants is not the general solution.

To obtain general solution in this case let us reqrite Eqn. (10) in the form

$L_1 (y) = (D^n + a_1 D^{n-1} + \cdot\cdot\ a_{n-1} D + a_n y = 0$ …(16)

Where $D = \frac{d}{dx}$ and $D^n = \frac{d^n}{dx^n}$ and L_1 is a linear differential operator.

If $m_1, m_2, \ldots, m_n$ are the roots of auxiliary equation corresponding to, Eqn. (16), then Eqn.(16) can be written as

$D - m_1) (D - m_2) \ldots (D - m_n) y = 0$ (17)

It is clear that when all the n roots $m_1, m_2, \ldots, m_n$ are real and distinct the complete solution of Eqns. (16) or (17) is constituted by the solutions of the n equations.

$(D - m_1) y = 0, (D - m_2) y = 0 \ldots. (D - m_n) y = 0$
in case the two roots are equal say $m_1 = m_2$, then Eqn. (17) takes the form
$(D =- m_1)^2 (D - m_3) \ldots (D - m_n) y = 0$
and then solutions corresponding to two equatl roots are the solutions of
$(D - m_1)^2 y = 0$
$\Rightarrow (D - m_1) [(D - m_1) y] = 0$ …(18)
Let $(D - m_1) y = v$ …(19)
Then Eqn. (18) reduces to
$(D - m_1) V = 0$

$$\Rightarrow \frac{dV}{dx} - m_1 V = 0$$

it is a linear differential equation of the first order and its solution (ref. Sec. 3.3 of unit 3) is
$V = c_1 e^{m_1 x}$
With this value of V, Eqn. (19) becomes
$(D - m_1) y = c_1 e^{m_1 x}$
which is again a linear differential equation of the first order and is solution is
$y = e^{m_1 x} (c + c_1 x)$,
c_1, c_2 being constants.

Similarly, the solution of Eqn. (17) corresponding to three equal roots say $m_1 = m_2 = m_3$, are the solutions of
$(D – m_1)^3y = 0$
$\Rightarrow$ $(D – m_1) [(D = m_1)^2y] = 0$

te $(D – m_1)^2$ y = z in the above equation. Solving the equation for z and putting the value of z obtained in the above equation, we have
$(D – m_1)^2y = c_1 e^{m_1x}$

Substituting gain $(D – m_1)$ y = t and proceeding as before, we get
$(D – m_1) y = e^{m_1x} (c_2 + c_1x)$
The solution of above linear differential equation of first order is

$$y = e^{m_1x} \left(\frac{c_1}{2}x^2 + c_2x + c_3\right)$$

thus, it is clear that if a root m_1 of Eqn. (16) is repeated r times, then solution corresponding to this root will be of the form

$y = e^{m_1x} (A_1 + A_2x + A_3x^2 + \ldots + A_rx^{r-1})$
and the general solution of Eqn. (16) will then be
$y = e^{m_1x} (A_1 + A_2x + A_3x^2 + \ldots + A_rx^{r-1}) + A_{r+1} e^{m_{r+1}x} + \ldots + A_n e^{m_nx}$ …(20)
We now illustrate the above discussion with the help of a few examples.

Example 10: Solve $\frac{d^4y}{dx^4}$ - $\frac{d^3y}{dx^3}$ - 9$\frac{d^2y}{dx^2}$ - 11 $\frac{dy}{dx}$ - 4 y = 0

Solution: The given differential equation can be written as

$(D^4 – m^3 – 9D^2 – 11D – 4) y = 0$
Auxiliary equation of the given equation is
$M^4 – m^3 – 9m^2 – 11m – 4 = 0$
$\Rightarrow$ $(m + 1)^3 (m – 4) = 0$
$\Rightarrow$ m = - 1, - 1, - 1, 4

here the root – 1 is repeatred three times and root 4 is distinct. Hence, using Eqn. (20), the general solution of the given differential equation is
$y = (A + Bx + Cx^2) e^{-x} + De^{4x}$,
where A, B, C and D are arbitrary constants.

Let us consider another example.

Example 11: Find the complete solution of
$(D^4 – 8D^2 + 16)\, y = 0$

Solution: In this case the auxiliary equation is
$m^4 – 8m^2 + 16 = 0$
$\Rightarrow (m^2 – 4)^2 = 0$
$\Rightarrow\ (m- 2)^2 (m + 2)^2 = 0$
$\Rightarrow m = 2,2, - 2, - 2$

here 2 and -2 are both repeated. Therefore, the method of repeated real roots will be separately applied to each repeated root. Hence the complete solution of the given differential equation is

$y = (A + Bx)\, e^{2x} + (C + Dx)\, e^{-2x}$.

and now some exercises for you.

Now we shall discuss the case when the auxiliary equation may have complex roots.

Case III: Auxiliary Equation has complex roots:

If the roots of auxiliary Eqn. (12) are not all real, then some or, may be, all the roots are complex. We know from the theory of equations that if all the coefficients of a polynomial equation are real, then its complex roots occur in conjugate pairs. In Eqn. (12), all the coefficients are assumed to be real constants and hence complex roots, if any, must occur in conjugate pairs.

Let one such pair of complex roots of Eqn 912) be $m_1 = \alpha - i\beta$, where α and β are real and $i^2 = - 1$. Formally, there is no difference between this case and case I, and hence the corresponding terms of solution are

$$y = c_1\, e^{(\alpha+i\beta)x} + c_2\, e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}\, [c_1\, e^{i\beta x} + c_2 e^{-i\beta x}] \qquad \ldots(21)$$

however, in practice we would prefer to work with real functions instead pf complex exponentials. To achieve this, we make use of the **Euler's formula**, namely,
$e^{i\theta} = \cos\theta i \sin\theta$ and $e^{-i\theta} = \cos\theta - i \sin\theta$,

where θ is any real number. Using these results, the expression (21), which is the part pf the solution corresponding to complex roots, becomes

$$e^{\alpha x}\, [c_1 (\cos\beta x + i \sin\beta x) + c_2 (\cos\beta x – i \sin\beta x)]$$
$$= e^{\alpha x}\, [(c_1 + c_2) \cos\beta x + (c_1 – c_2)\, i \sin\beta x]$$

Since $c_1 + c_2$ are arbitrary constants, we may write

$A = c_1 + c2$ and $B = i(c_1 – c_2)$,

So that A and B are again arbitrary constants, though not real. Expression (21) now takes the form

$$e^{\alpha x}\ [A\cos\beta x + B\sin\beta x] \qquad \ldots(22)$$

Further, if the complex root is repeated, then the complex conjugate root will also be repeated and the corresponding terms in the solution can be written, using the form (20), as

$e^{x(\alpha+i\beta)}\ (c_1 + c_2x) + e^{x(\alpha-i\beta)}\ (c_3 + c_4x)$

Proceeding as above and writing

$A = c_1 + c_3$ $B = i(c_1 - c_3)$ $C = c_2 + c_4$ $D = i(c_2 - c_4)$,

The above expression can be written as

$$e^{\alpha x}\ [(A + Cx)\cos\beta x + (B + Dx)\sin\beta x] \qquad \ldots(23)$$

in the case of multiple repetition of complex roots, the results are obtained analogous to those in the case of multiple repetition of real roots.

We now illustrate this case of complex roots with the help of a dew examples.

Example 12: For the differential equation

$\frac{d^4y}{dx^4} - m^4y = 0$, show that its solution can be expressed in the form

$y = c_1 \cos mx + c_2\sin mx + c_3 \cosh mx + c_4 \sinh mx$.

Solution: The given differential equation can be expressed as

$(D^4 - m^4)\ y = 0$

In this case since m is used as a constant in the given differential equation, we can replace D by some other letter, λ say.

So, the auxiliary equation is

$(\lambda^4 - m^4) = 0$

$\Rightarrow (\lambda^2 - m^2)(\lambda^2 + m^2) = 0$

$\Rightarrow \lambda = m, - m, \pm im$

Now the solution corresponding to roots + m and – m can be obtained as we have done in Example 8 and write it as

$C_3 \cosh mx + c_4 \sinh mx$

Solution corresponding to imaginary roots + im and –im will be

$Ae^{imx + Be}$-imx

Which can be written as

$A(\cos mx + i \sin mx + B(\cos mx - i \sin mx)$

$= (A + B) \cos mx + i (A - B) \sin mx$

$= c_1 \cos mx + c_2 \sin mx$

where $c_1 = (A + B)$ and $c_2 = i(A – B)$ are constants.
Hence the general solution of the given differential equation is

$Y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$

Let us look at another example.

Example 13: Solve $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

Solution: in this case is
$M^4 – 4m^3 + 8m^2 – 8m + 4 = 0$
$\Rightarrow (m^2 – 2m + 2)^2 = 0$
$\Rightarrow [m – (1 + i)]^2 [m – (1 – 9)]^2 = 0$
$\Rightarrow m = 1 + i, 1 + i, 1 – i.$

Roots are complex and repeated in this case.
Hence the general solution can be written as
$Y= (c_1 + xc_2) e^{(1+i)x} + (c_3 + xc_4) e^{(1-i)x}$
$= e^x [(c_1 + xc_2) e^{ix} + (c_3 + xc_4) e^{-ix}]$
$= e^x [(c_1 + xc_2) (\cos x + i \sin x) + (c_3 + xc_4) (\cos x – i\ \text{cin}x)]$
$= e^x \{[(c_1 + xc_2) + x(c_2 + c_4)] \cos x + i [(c_1 – c_3) + x(c_2 – c_4)] \sin x\}$
$= e^x [(A + Bx) \cos x + (C + Dx) \sin x]$
where $A = (c_1 + c_3)$, $B = (c_2 + c_4)$, $C = i (c_1 – c_3)$ and $D = i)c_2 – c_4)$ are all constants.
You may now try the following exercise.

4.0 CONCLUSION

We now end this unit by given a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have covered the following:

1) The general linear differential equation with dependent variable y and independent variable x is termed as an equation.

   a) with variable coefficients if the coefficients of y and its derivatives are functions of x.
   b) with constant coefficients if the coefficients of y and its derivatives are all constants.
   c) homogeneous if the terms other than those of y and its derivatives are absent.
   d) non-homogeneous if the terms other than those of y and derivatives are present and are constants or functions of independent variable x.

2) A solution of general linear differential equation exists and is unique if conditions of Theorem 1 are satisfied.

3) A set of functions $y_1(x)$, $y_2(x)$, …, $y_n(x)$ defined on an interval I is linearly dependent if for constants c_1, c_2, .., c_n not all zero, we have for every x in I,
$c_1y_1(x) + c_2y_2(x) + \dots, + c_ny_n(x) = 0$.

4) A set of functions $y_1(x)$, $y_2(x)$,…, $y_n(x)$ on I is linearly independent on I if it is not linearly dependent on I.

5) If y = y_1 is a solution of homogeneous linear differential equation on I, so is y = c_1 on I, where c is arbitrary constant.

6) If y = y_1, y_2, …, y_m are solutions of linear homogeneous differential equation on I, so is $y = c_1y_1 + c_2y_2 + \dots + c_my_m$ on I, where c_1, c_2, …, c_m are arbitrary constants.

7) If y_1, y_2,…, y_n are linearly independent solutions of an nth order homogeneous linear differential equation on an interval I,
then
$y = c_1y_1 + c_2y_2 + \dots + c_ny_n$
(where c_1,c_2,…, c_n being arbitrary constants)
is defined as the complete primitive of the given equation on I.

8) For a non-homogeneous equation
   a) the complete primitive of the corresponding homogeneous part is called it complementary function.
   b) particular solution of the non-homogeneous part involving no arbitrary constant is called its particular integral.
   c) Complementary function and particular integral constitute its general solution.

9) Solution y, of an nth order linear differential equation

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

with constant coefficients a_1, …., a_{n-1}, a_n having n roots m_1, m_2, …, m_n, when
   a) roots are real and distinct, is
      $y = c_1 e^{m_1x} + c_2 e^{m_2x} + \dots + c_n e^{m_nx}$
   b) roots are real and repeated, say $m_1 = m_2 = \dots, = m_r$, is
      $y = (c_1 + c_2x + \dots + c_rx^{r-1}) e^{m_1r} + c_{r+1} e^{m_{1+1}x} + \dots, + c_n e^{m_nx}$.
   c) roots are complex and one such pair is $\alpha \pm i\beta$, is
      $y = e^{\alpha x} (c_1 \cos\beta x + c_2 \sin\beta x)$
      corresponding to that pair of roots.

6.0 TUTOR MARKED ASSIGNMENT

1. Verify if the function $y = \frac{1}{4} \sin 4x$ is a unique solution of the initial value problem

   $y' + 16y = 0$
   $y(0) = 0, y'(0) = 1.$

2. In the following problems verify that the given function y_1 and y_2 are the solutions of the corresponding equations. Decide whether the set $\{y_1, y_2\}$ of solutions is linearly dependent or independent.

   a) $y'' - y = 0$, $y_1 = e^x$ and $y_2 = e^{-x}$ over $-\infty < x < \infty$

   b) $y'' + 9y = 0$, $y_1 = \cos 3x$ and $y_2 = \cos\left(3x + \frac{\pi}{2}\right)$ over $-\infty < x < \infty$.

   c) $y'' - 2y' + y = 0$ $y_1 = e^x$ and $y_2 = xe^x$ over $-\infty < x < \infty$.

3. Construct an example to show that a set of functions could be linearly independent on some interval and yet have a vanishing Wronskian.

4. Verify that $y = 1/x$ is a solution of the non-linear differential equation $y'' = 2y^3$ on the interval $]0, \infty[$, but the constant multiple $y = c/x$ is not a solution of the equation when $c \neq 0$, and $c \neq \pm 1$.

5. Functions $y_1 = 1$ and $y_2 = \text{In}x$ are solutions of the non-linear differential equation $y'' + (y')^2 = 0$ on the interval]0, [. Then

   a) is $y_1 + y_2$ a solution of the equation?
   b) is $c_1y_1 + c_2y_2$, a solution of the equation, where c_1 and c_2 are arbitrary constants?

6. Solve the following equations:

   a) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

   b) $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$

   c) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

7. In the following equations find the solution y for x = 1:

   a) $(D^2 - 2D - 3)\,y = 0$; when $x = 0$, $y = 4$ and $y' = 0$
   b) $(D^3 - 4D)\,y = 0$, when $x = 0$, $y = 0$ $y' = 0$ and $y'' = 2$.

8. Find the complete primitive of the following equations:

   a) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$

   b) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4Y = 0$

c) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

d) $\frac{d^3y}{dx^3} -3 \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

9. Find the general solution of the following equations subject to the conditions mentioned alongside:

a) $(D^2 + 4D + 4)y = 0$; when x = 0, y = 1 and y' = - 1
b) $(D^3 – 3D – 2) y = 0$; when x = 0, y' = 9 and y'' = 0
c) $(D^4 + 3D^3 + 2D^2) y = 0$; when x = 0, y = 0, y' = 4, y'' = - 6, y'' = 14.

10. Find the general solution of the following Equations:

a) $\frac{d^2y}{dx^2} - 2\alpha \frac{dy}{dx} + (\alpha^2 + \beta^2) y = 0$

b) $\frac{d^4y}{dx^4} + a^4y^{= 0}$

c) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 METHOD OF UNDETERMINED COEFFICIENTS

CONTENTS

1.0 INTRODUCTION

In Unit 5, we learnt that in order to find the complete integral of a general non-homogeneous linear differential equation, namely

$$L(y) = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = b(x) \qquad \ldots(1)$$

where ao, a,, ... , a, are constants, it is necessary to find a general solution of the corresponding homogeneous equation that is, the complementary function and then add to it any particular solution of Eqn. (1). In Sec. S.3 we discussed the methods of determining complementary function of linear differential equations with constant coefficients having auxiliary equations with different types of roots. But how do we find a particular solution of these equations? We shall now be considering this problem in this unit.

Variety of methods exist for finding particular integral of a non-homogeneous linear differential equations. The simplest of these methods is the method of undetermined coefficients. Basically, this method consists in making a guess as to the form of trial solution and then determine the coefficients involved in the trial solution so that it actually satisfies the given equation. You may recall that we had touched upon this method in Sec. 3.3 of Unit 3 for finding the particular integral of non-homogeneous linear differential equations of the first order having constant coefficients. In this unit we shall be discussing this method in general for finding the particular integral of second and higher order linear differential equations with constant coefficients.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify the types of non-homogeneous terms for which method of undetermined coefficients can be successfully applied.
- write the form of trial solutions when non-homogeneous terms are polynomials, exponential functions or their combinations.
- describe the constraints of this method.

3.0 MAIN CONTENT

3.1 Types of Non-Homogeneous Terms for Which the Method is Applicable

The method of undetermined coefficients, as we have already mentioned in Sec. 6.1, is a procedure for finding particular integral y^ in a general solution y(x) = yc(x) °f equations of the form (1). The success of this method is based on our ability to guess the probable form of particular solutions.

We know that the result of differentiating functions such as x^r (r > 0, an integer) an exponential function $e^{\alpha x}$ (α constant) or sinmx or cosmx (m constant) is again a polynomial, an exponential or a linear combination of sine or cosine functions respectively. Hence, if the non-homogeneous term b(x) in Eqn. (1) is a polynomial an exponential function, or a sine or cosine function then we can choose the particular integral to be a suitable combination of polynomial, an exponential. I sinusoidal function with a number of undetermined constants. These constant! I then be determined so that the trial solution satisfies the given equation.

Note: A function which is a combination of a sine function (or cosine function) with an exponential function and/or a polynomial is a sinusoidal function.

Thus the types of non-homogeneous term for which the method of undetermined coefficients is successfully applicable are

i) polynomials
ii) exponential functions
iii) sine or cosine functions
iv) a combination of the terms of types (i), (ii) and (iii) above.

We shall now discuss the method of undetermined coefficients to find the particular integral for these various types of non-homogeneous terms one by one.

3.1.2 Non-homogeneous term is an Exponential Function:

Let us suppose that the non-homogeneous term b(x) in Eqn. (1) is an exponential function of the form $e^{\alpha x}$ (α a constant).

In other words, suppose we have to solve an equation.

$$L(y) = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = e^{\alpha x} \qquad \ldots(6)$$

The appropriate form of the trial solution can be taken as

$$y_p(x) = A e^{\alpha x} \qquad \ldots(7)$$

provided $e^{\alpha x}$ is not a solution of the homogeneous differential equation corresponding to Eqn. (1) (i.e., α is not a root of the auxiliary equation).

If α is a root of Eqn. (6), then the choice (7) would not give us any information for determining the value of A. in that case, we can take $y_p(x) = Axe^{\alpha x}$ as the trial solution. If α is r-times repeated root of the auxiliary equation, then the suitable form of the trial solution for determining particular integral will be

$$y_p(x) = Ax^r e^{\alpha x} \qquad \ldots(8)$$

substituting this value of y_p in Eqn. (6) and equating coefficients of $e^{\alpha x}$ on both sides, we can find the value of undermined coefficient A and thus find the particular integral (8).

For a better understanding of whatever we have discussed above let us take up a few examples.

Example 3: Find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^x.$$

Solution: Auxiliary equation is

$(m + 1)(m + 2) = 0$

$\Rightarrow m = -1, -2,$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

Since e^x is not a part of the complementary function, hence trial solution for finding a particular integral can be taken as

$y_p(x) = Ae^x$

Substituting this value of y_p in the given differential equation,
we get
$2Ae^x + 3Ae^x + Ae^x = 3e^x$
$\Rightarrow 6Ae^x = 3e^x$

Equating coefficient of e^x on both sides, we get
$6A = 3, \Rightarrow A = \frac{1}{2}$
hence
$P.\ I = \frac{1}{2}\ e^x$
$\therefore$ The general solution fo the given differential equation is
$y = c_1e^{-x} + c_2e^{-2x} + ---- e^x$
let us consider another example which illustrate the case of repeated roots of an auxiliary equation.

Example 4: solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 12e^x$

Solution: Auxiliary equation is
$(m - 1)^3 = 0$
$\Rightarrow m = 1, 1, 1.$
$\therefore C.F. = (c_1 + c_2x + c_3x^2)\ e^x$

Since non-homogeneous term of the given differential equation is e^x which is present in the complementary function and moreover I is 3-times repeated root of the auxiliary equation, we take the form of trial solution to be
$y_p(x) = Ax^3e^x.$

Note that in the selection of the trial solution $y_p(x)$ no smaller power of x will give us the particular integral. Moreover, it is not similar to any term of complementary function of the given equation.

On substituting this value pf y_p in the given differential equation, we get
$- Ax^33^x + 3A\ [x^3e^x + 3x^2e^x] - 3A\ [x^3e^x + 6x^2e^x\ 6xe^x]$
$+ A\ [x^3e^x + 9x^2e^x + 18xe^x + 6e^x] = 12e^x$

Equating coefficients of e^x on both sides, we get
$6A = 12, \Rightarrow a = 2.$
Thus, $P.I. = 2x^3e^x$
$\therefore$ The general solution of the given differential equation is
$y = (c_1 + c_2x + c_3x^2)\ e^x + 2x^3e^x$

And now an exercise for you.

You may also come across the situation when b(x) in Eqn. (1) is a sum of two or more functions. Suppose $b(x) = b_1(x) + b_2(x)$; then from the superposition principle we have the P.I. $y_p(x)$ of L(y) = b(x) to the equal to $y_p = y_{p1} + y_{p2}$, where y_{p1} is a P.I. of $L(y) = b_1(x)$ and y_{p2} is a P.I. of $L(y) = b_2(x)$. This enables us to decompose the problem of solving linear equation L(y) = b(x) into simpler problem an example.

Example 5: find a general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - + y = e^x + 4.$$

Solution: Auxiliary equation is

$M^2 – 2m + 1 = 0$

$\Rightarrow (m – 1)^2 = 0$

$\Rightarrow m = 1, 1.$

$\therefore$ C.F. = $(c_1 + xc_2)\, e^x$

To find the particular solution we first consider equation

$+ y = e^x$ …(9)

1 is a repeated root of the auxiliary equation, we consider the trial solution

$y_{p1} = Ax^2e^x$

on substituting y_{p1} in Eqn. (9), we find that

$(2Ae^x + 4xAe^x + x^2Ae^x) – 2(2xAe^x + x^2Ae^x) + Ax^2e^x = e^x$

comparing the coefficient of e^x on both sides, we have

$2Ae^x = e^x$

$\Rightarrow A = \frac{1}{2}$

$\therefore y_{p1} \frac{x^2}{2} e^x$

Now consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} \quad y = 4 \qquad ...(10)$$

Since no-homogeneous term is a constant, we try y_{p2} = A and find that A = 4 satisfies (10). Hence a particular solution of the given equation is

$$y_p = y_{p1} + y_{p2} = \frac{x^2}{2}e^x = 4$$

A general solution will them be

$y = c_1e^x + c_2xe^x + 4 + \frac{x^2}{2}e^x$

The term b(x) can be a combination of many more terms like this. We may have b(x) = x + e^x, b(x) = x + x^3, b(x) = 3 + x^2 etc. In these cases, we can obtain particular integral using I and II discussed above and by finding y_{p1} and y_{p2} as we have done in Example 5.

We shall give you the general method of finding P.I. when we discuss cases IV and V.

You may try these exercises.

We can now take up the case when b(x) in Eqn. (1) is either a sine or a cosine function.

3.1.3 Non-homogeneous Term is a Sine or a Cosine Function

After going through I and II above and attempting the exercises given so far, you know how to handle b(x) when it is polynomial, an exponential function or a combination of both. Now can you say how this case is handled when b(x) is a sine or a cosine function?

We know that the linear differential operator when applied to $\sin\beta x$ or $\cos\beta x$ will yield a linear combination of $\sin\beta x$ and $\cos\beta x$. Therefore, if the non-homogeneous term b(x) of differential Eqn. (1) is of the form

B(x) = $\alpha_1 \sin\beta x$ or $\alpha_2 \cos\beta x$ or $\alpha_1 \sin\beta x + \alpha_2 \cos\beta x$

We can take the trial solution in the form

$$y_p(x) = A\cos\beta x + B\sin\beta x \quad \ldots(11)$$

provide $\pm i\beta$ are not roots of the auxiliary equation corresponding to the given differential equation.

If $\pm i\beta$ are r-times repeated roots of the auxiliary equation, then we can take the form of trial solution to be

$$y_p(x) = x^r(A\cos\beta x + B\sin\beta x) \quad \ldots(12)$$

We then substitute the value of $y_p(x)$ in the form (11) or (12), whichever is applicable in Eqn. (1) and equate the coefficients of $\sin\beta x$ and $\cos\beta x$ on both sides of the resulting equation. This gives us equations for obtaining the values of A and B in terms of known quantities. Knowing the values of A and B, particular integral of Eqn. (1) is obtained from relations (11) or (12).

We now illustrate this theory with the help of a few examples.

Example 6: Find the general solution of

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = \sin x$$

Solution: Auxiliary Equation is
$(m^4 – 2m^2 + 1) = 0$
$\Rightarrow (m^2 – 1)^2 = 0$
$\Rightarrow m = 1, 1, - 1, -1$
$\therefore$ C.F. = $(c_1 + c_2x) e^x + (c_3 + c_4x) e^{-x}$

Since ------ i is not a root of the auxiliary equation, that is, term sinx does not appear in the complementary function, we can take the trial solution in the form
$y_p(x) = A\sin x + B\cos x$.

Substituting this value of y_p in the given differential equation, we get
$(A\sin x + B\cos x) – 2(-A\sin x – B\cos x) + (A\sin x + B\cos x) = \sin x$
$\Rightarrow 4A\sin x + 4B\cos x = \sin x$

Equating coefficients of sinx and cosx on both sides, we get
$4A = 1 \Rightarrow A = \frac{1}{4}$
and $4B = 0 \Rightarrow B = 0$
Thus, $y_p(x) = \frac{1}{4}\sin x$
and the complete solution of the differential equation is

$$y =(c_1 + c_2x) e^x + (c_3 + c_4x) e^{-x} + \frac{1}{4}\sin x.$$

Let us look at another example.

Example 7: Solve the initial value problem

$$\frac{d^2y}{dx^2} + y = 2\cos x,\ y(0) = 1,\ y'(0) = 0$$

Solution: The auxiliary equation is
$m^2 + 1 = 0$
$\Rightarrow m = \pm i$
$\Rightarrow$ C.F. = $c_1\cos x + c_2\sin x$

Now since $\pm i$ is a root of the auxiliary equation i.e., cosx itself appears in the complementary function, we take the form of the trial solution as
$y_p(x) = x (A\sin x + B\cos x)$

Substituting the value of $y_p(x)$ in the equation, we get
2(Acosx – Bsinx) + x(- Asinx – Bcosx) + x(Asinx + Bcosx) = 2cosx
$\Rightarrow$ 2Acosx – 2Bsinx = 2cosx

Comparing the coefficients of sinx and cosx on both sides, we get
2A = 2 $\Rightarrow$ A = 1 and B = 0.
Therefore,
$y_p(x)$ = xsinx
and the general solution is
y(x) = $c_1$cosx + $c_2$sinx + xsinx
we now use initial conditions to determine c_1 and c_2
Now y(0) = 1 gives c_1 = 1
And y'(0) = 0 gives c_2 = 0
Thus, y(0) = cosx + xsinx

You may now try the following exercises.

In the example considered so far, did you notice that the function b(x) itself suggested the form of the particular solution $y_p(x)$? in fact, we can expand the list of functions b(x) for which the method of undetermined coefficients can be applied to include products of these functions as well. We now discuss such cases.

3.1.4 Non-homogeneous Term is a Product of an Exponential and a Polynomial

Let us suppose that b(x) is of the form
$b(x) = e^{\alpha x}[b_0x^k + b_1x^{k-1} + \ldots. + b_{k-1}x + b_k] = e^{\alpha x}P_k(x)$
with this form of b(x), Eqn. (1) reduces to

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots. + a_{n-1}\frac{dy}{dx} + a_n = e^{\alpha x}\,[b_0x^k + b_1x^{k-1} + \ldots + b_{k-1}x + b_k] \quad \ldots(13)$$

We now take the trial solution in the form

$$y_p(x) = e^{\alpha x}\,[A_0x^k + A_1x^{k-1} + \ldots. + A_{k-1}x + A_k] \quad \ldots(14)$$

provide α is not a root of the auxiliary equation corresponding to Eqn. (13). If α is a root of the auxiliary equation, say, it is r-times repeated root of the auxiliary equation then we modify the trial solution as

$$y_p(x) = x^r e^{\alpha x}\,[A_0x^k + A_1x^{k-1} + \ldots + A_{k-1}x + A_k] \quad \ldots(15)$$

Remember that in Eqn. (15) no smaller power of x will yield a particular integral. Here r is the smallest positive integer for which every term in the trial solution (15) will differ from every term occurring in the complementary function corresponding to Equ (13).

In order to determine the constants $A_0, A_1, \ldots, A_k$ we substitute $y_p(x)$ in the form (14) or (15) as the case may be in eqn. (13) and then compare the coefficients of $e^{\alpha x}$ on both sides. For a better understanding of whatever we have discussed above, let us consider an example.

Example 8: Solve $\frac{d^3y}{dx^3} - \frac{dy}{dx} = xe^{-x}$

Solution: Auxiliary equation is

$m^3 - m = 0$

$\Rightarrow m(m^2 - 1) = 0$

$\Rightarrow m = 0, -1, 1$

$\therefore$ C.F. $= c_1 + c_2e^{-x} + c_3e^{x}$

Here the non-homogeneous term is xe^{-x} appears in the complementary function. Further, (-1) is a non-repeated root of the auxiliary equation. Thus, we take the form of trial solution as

$y_p(x) = x[B + Ax]e^{-x} = Ax^2e^{-x} = Ax^2e^{-x} + Bxe^{-x}$

substituting this value of y_p in the given differential equation, we get

$$- A[-x^2e^{x} + 2xe^{-x}] + A[-x^2e^{-x} + 6xe^{-x} - 6e^{-x}] - B(-xe^{-x} + e^{-x}) + B(-xe^{-x} + 3e^{-x}) = xe^{-x}$$

Comparing the coefficients of xe^{-x} and e^{-x} on both sides, we get

$4A = 1 \Rightarrow A = \frac{1}{4}$

and $-6A + 2B = 0 \Rightarrow B = \frac{3}{4}$

Hence $y_p(x) = \frac{1}{4}x^2e^{-x} + \frac{3}{4}xe^{-x} = \frac{e^{-x}}{4}(x^2 + ex)$

And the general solution is

$$y = c_1 + c_2e^{-x} + \frac{e^{-x}}{4}(x^2 + 3x).$$

You may now try the following exercise.

Lastly, we take up the case when b(x) is a product of a polynomial, an exponential function and a sinusoidal function.

3.1.5 Non-homogeneous Term is a Product of a Polynomial, an Exponential and a Sinusoidal function

Let us suppose that the non-homogeneous term b9x) in Eqn. (1) has one of the following two forms:

$$b(x) = e^{\alpha x}P_k(x)\sin\beta x \text{ or } b(x)\ e^{\alpha x}\ P_k(x)\cos\beta x, \quad \ldots(16)$$

where $P_k(x)$, as given by Eqn. (2), is apolynomial of degree k or less and α and β are any real numbers. You may recall Euler's formula and write

$e^{(\alpha+i\beta)x} = e^{\alpha x} \cos\beta x + i\, e^{\alpha x} \sin\beta x$

or, equivalently, we have

$$e^{\alpha x} \cos\beta x = \text{Real}\,(e^{(\alpha+i\beta)x}) = \frac{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}}{2}$$

and $$e^{\alpha x} \sin\beta x = \text{Imaginary}\,(e^{(\alpha+i\beta)x}) = \frac{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}}{2i}$$

Hence b(x) in Eqn. (16) reduces to

$$b(x) = (b_0x^k + b_1x^{k-i} + \ldots + b_k)\left\{\frac{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}}{2i}\right\}$$

or $$b(x) = (b_0x^k + b_1x^{k-1} + \ldots + b_k)\left\{\frac{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}}{2}\right\}$$

In either of the above two cases, we take the trial solution in the form

$$y_p(x) = (A_0x^k + A_1x^{k-1} + \ldots + A_k)\,(e^{(\alpha+i\beta)x})$$
$$(B_0x^k + B_1x^{k-1} + \ldots + B_k)\,(e^{(\alpha-i\beta)x})$$

or, equivalently,

$$y_p(x) = (A_0x^k + A_1x^{k-1} + \ldots + A_k)\, e^{\alpha x} \cos\beta x +$$
$$(B_0x^k + B_1x^{k-1} + \ldots + B_k)\, e^{\alpha x} \sin\beta x,$$

provided $\alpha \pm i\beta$ is not a root of the auxiliary equation.

If $(\alpha \pm i\beta)$ is r-times repeated root of the auxiliary equation, we can then modify the trial solution by multiplying it by x^r. We then substitute the trial solution in Eqn. (1) and equate the coefficients of like terms on both sides to determine $A_0, A_1, \ldots, A_k$ and $B_0, B_1, \ldots, B_k$. Substituting these values of undetermined coefficients in the trial solution, we get the particular integral.

Let us now illustrate the above case with the help of a few examples.

Example 9: find the appropriate form of trial solution for the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 3e^x + 2xe^{-x} + e^{-x}\sin x$$

Solution: Auxiliary equation is

$m^4 + 2m^3 + 2m^2 = 0$

$\Rightarrow m^2(m^2 + 2m + 2) = 0$

$\Rightarrow m = 0, 0, -1 \pm i$

$\therefore$ C.F. $= c_1 + c_2x + e^{-x}(c_3\sin x + c_4\cos x)$

Here the non-homogeneous term is $3e^{x} + 2xe^{-x} + e^{-x}\sin x$

Since the term $e^{-x}\sin x$ also appear in C.F., the appropriate form of the trial solution is

$y_p = Ae^{x} + (Bx + C)e^{-x} + xe^{-x}(D\cos x + E\sin x)$.

We now take up an example in which b(x) is a product of a polynomial an exponential and a sinusoidal function.

Example 10: Write down the form of the trial solution for the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\ 5y = x^2e^{-x}\sin x$$

Solution: the auxiliary equation is
$m^2 + 2m + 5 = 0$
$\Rightarrow m = -1 \pm 2i$
The roots are not equal to $-1 \pm i$. Hence the for of trial solution is
$y_p = (A_0x^2 + A_1x + A_2)e^{-x}\cos x + (B_0x^2 + B_1x + B_2)e^{-x}\sin x$

Note that the form of trial solution taken in Case v above is the most general form. This is because the trial solutions taken in Cases I –IV are particular forms of Case V.

And now some exercise for you.

After going through the Cases I – V above and attempting the exercises given, you must have understood the method of undetermined coefficient quite well. Did you make certain observations about the method? Let us now summarize the observations and constraints of this method.

3.2 Observations and Constraints of the Method

1) Method is straight forward in application.
2) It can be used by any learner who is not familiar with more elegant techniques of finding the solutions f the differential equations such as inverse operators and variation of parameters, which involve integrations and which we shall be discussing in the subsequent units.
3) Success of this method depends to a certain extent on the ability to guess an appropriate form of the trial solution.
4) If the non-homogeneous term is complicated and the trial solution involves a large number of terms, then determination of coefficients in the trial solution becomes laborious.
5) This method is not a general method of finding the particular solution of differential equations. It is applicable to linear non-homogeneous equations with **constant coefficients and with restricted** forms of the no-homogeneous terms.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit, we have covered the following:

1) Method of undetermined coefficients is applicable if
   a) The equation is a linear equation with constant coefficients.
   b) The non-homogeneous term is either a polynomial, an exponential function, a sinusoidal function or a product of these functions.

2) The results giving trial solutions corresponding to different non-homogeneous terms in the equation L(y) = b(x), where the equation L(y) = 0 has r-times repeated roots are summarized in the following table.

Non-homogeneous term b(x)	Trial solution, $y_p(x)$
$P_p(x) = b_0x^k + b_1x^{k-1} + \dots + b_{k-1}{}^{x+b}{}_k$	$x^r(A_0x^{k-1} + \dots + A_k)$
$e^{\alpha x}$	$x^r(A\,e^{\alpha x})$
$\begin{cases} \sin\beta x \\ \cos\beta x \end{cases}$	$x^r(A\sin\beta x + B\cos\beta x)$
$e^{\alpha x}\ P_k(x)$	$x^r e^{\alpha x}(A_0x^k + \dots + A_k)$
$e^{\alpha x} P_k(x) \begin{cases} \sin\beta x \\ \cos\beta x \end{cases}$	$x^r[(A_0x^k + \dots + A_k)e^{\alpha x}\sin\beta x$ $+(B_0x^k + \dots) + B_x)e^{\alpha x}\cos\beta x]$

3) Observations and constrains of the method.

6.0 TUTOR MARKED ASSIGNMENT

1. Find a form of particular integral of the following equations

   a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 + 1$

   b) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$

2. Determine the general solution of the following equations.

   a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$

b) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = x$

3. Find a particular integral of the following differential equations.

a) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = e^{-2x}$

b) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = e^{-x}$

4. Find a general solution of the following differential equations:

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 (e^x + e^{-x})$

b) $2\frac{d^2y}{dx^2} + 8y = x^3 + e^{2x}$

5. Solve the following initial value problems:

a) $\frac{d^2y}{dx^2} - y = e^{2x}$. y(0) = -1, y'(0) = 1.

b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y + e^{2x} = 0$, y(0) = y'(0) = 0.

6. Solve the following equations:

a) $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = \sin 2x$

b) $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 2\cos x$

7. Solve the following initial value problems:

a) $\frac{d^2y}{dx^2} + 4y = \sin x$, y(0) = 2, y'(0) = -1

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \cos x - \sin 2x$, $y(0) = \frac{-7}{20}$, $y'(0) = \frac{1}{5}$

8. Solve the following equations:

a) $\frac{d^2y}{dx^2} + 9y = x^2 e^{3x}$

b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4x\, e^{2x}$

9. Write the form of the trial solution for each of the following:

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = x\cos 3x - \sin 3x$

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = xe^{-x}\cos 2x$

c) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^{x}\cos 2x$

d) $\frac{d^2y}{dx^2} + y = x^2 \sin x$

e) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = xe^{2x}\sin x$

10. Find the general solution of the following equations.

a) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x + 3\cos x + e^{-2x}$

b) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = x^2 + 4 + x\sin x$

c) $\frac{d^3y}{dx^3} + \frac{dy}{dx} = x^3 + \cos x$

7.0 REFERENCES/FURTHER READINGS

Theoretical Mechanics by Murray, R. Spiegel

Advanced Engineering Mathematics by Kreyzig

Vector Analysis and Mathematical Methods by S.O. Ajibola

Engineering Mathematics by PDS Verma

Advanced Calculus Schaum's outline Series by Mc Graw-Hill

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UNIT 3 METHOD OF VARIATION OF PARAMETERS

CONTENTS

1.0 INTRODUCTION

In unit 6, we discussed the method of undetermined coefficients for determining particular solution of the differential equation with constant coefficients when its non-homogeneous term is of a particular form (viz, a polynomial, an exponential, a sinusoidal function etc).

In this unit we familiarize you with an alternative approach fir determining a particular solution that can be applied even when the coefficients of the differential Equation are functions of the independent variable and the non-homogeneous term may not be of a particular form. Such an approach is due to Joseph Louis Lagrange (1736 – 1813) and is termed as variation of parameters. Even though the approach is quite general bit is limited in its scope in the sense that it can be utilized in situations where the fundamental solution set for the reduced equation is known. Also, it can be used for first and higher order equations alike though its appreciation can be well understood for the later set of equations. The method requires for its applicability the complete knowledge of fundamental solution set of the reduced equation and for equations with variable coefficients the determination of this set may be extremely difficult. In the case of linear differential equations with variable coefficients, at times, it may not be possible to find all linearly independent solutions of the reduced but at least one or more may be obtainable. For such situations Jean le Rond d'Alembert (1717 – 1783), a French mathematician and a physicist, developed a method that is often called the method of **reduction of order.** When one or more solutions of reduced equation are known that D'Alembert's method can be used to derive an equation of order lower than that of a given equation and obtain the rest of the solutions of a reduced equation as well as the particular integral of the non-homogeneous term. We shall be discussing the method of reduction of order in Sec. 7.3 of the unit. For some particular forms of the second order linear differential equations with variable coefficients, we have also listed some rule by which one integral of the homogeneous equation an be guessed.

However, there exist linear differential equation with variable coefficients if second and higher order for which we may not be able to guess any integral of its complementary function. But, among such equations is a class of equations known as Euler's equation or homogeneous linear differential equations, where, by certain substitution, it is possible to find all the integrals of its complementary function. In Sec. 7.4, we shall be discussing the method of solving Euler's equations and those equations which are reducible to Euler's form.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Use the method of variation of parameters to find particular integral of non-homogeneous linear differential equations with constant or variable coefficients.
- Use the method of reduction of order to find the complete integral of the linear non-homogeneous equation of second order when one integral of the corresponding homogeneous equation is known.
- Write down one integral for second order linear homogeneous differential equation with variable coefficients in certain cases merely through inspection.
- Solve Euler's equations.

3.0 MAIN CONTENT

3.1 Variation of Parameters

Let us not discuss the details of the method by considering the non-homogeneous second order linear equation.

$L[y] = y'' + a_1(x)\, y' + a_2(x)y = b(x),$ …(1)

Where we have taken the coefficients of y'' to be 1 and $a_1(x)$, $a_2(x)$, and b(x) are defined and continuous on some interval J. Let $[y_1(x),\ y_2(x)]$ be a fundamental solution set for the corresponding homogeneous equation

$L[y] = 0$ …(2)

Then we know that the general solution of (2) is given by
$Y_c(x) = c_1y_1(x) + c_2y_2(x),$ …(3)

Where c_1 and c_2 are constants. To find a particular solution of the non-homogeneous equation, the idea associated with the method of variation of parameters is to replace the constants in Eqn. (3) by function of x. That is, we seek a solution of Eqn. (1) of the form
$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$ …(4)

where $u_1(x)$ and $u_2(x)$ are unknown functions to be determined. Since we have introduced two unknowns, we need two equations involving these functions for their determination.

In other words, we impose two conditions which the functions u_1 and u_2 must satisy in order that relation (4) is a solution of Eqn. (1). We call these conditions the **auxiliary conditions.** These conditions are imposed in such a way that the calculations are simplified. Let us see how this is done.

Now if relation (4) is a solution of Eqn. (1), then it must satisfy it. Thus, first we compute $y'_p(x)$ and $y''_p(x)$ from Eqn. (4).

$$y'_p = (u'_1 y_1 + u'_2 y_2) + (u_1 y'_1 + u_2 y'_2) \qquad \ldots(5)$$

To simplified the computation and to avoid second order derivatives for the unknown u_1, u_2 in the expression for y''_p, let us choose the first **auxiliary condition** as

$$u'_1 y_1 + u'_2 y_2 = 0 \qquad \ldots(6)$$

Thus relation (5) becomes

$$y'_p = u_1 y'_1 + u_2 y'_2 \qquad \ldots(7)$$

and

$$y'_p = u'_1 y'_1 + u'_1 y''_1 + u'_2 y'_2 + u_2 y''_2 \qquad \ldots(8)$$

Substituting in Eqn (1), the expressions for y_p, y'_p and y''_p as given b Eqn. (4), (7) and (8), respectively, we get

$b(x) = L[y_p]$

$= (u'_1 y'_1 + u_1 y''_1 + u'_2 y'_2 + u_2 y''_2) + a_1(u_1 y'_1 + u_2 y'_2) + a_2(u_1 y_1 + u_2 y_2)$

$= (u'_1 y'_1 + u'_2 y'_2) + u_1(y''_1 + a_1 y'_1 + a_2 y_1) + u_2(y''_2 + a_1 y'_2 + a_2 y_2)$

$$= (u'_1 y'_1 + u'_2 y'_2) + u_1 L[y_1] + u_2 L[y_2] \qquad \ldots(9)$$

since y_1 and y_2 are the solution of the homogeneous equation, we have

$L[y_1] = L[y_2] = 0$

Thus Eqn. (9) becomes

$$u'_1 y'_1 + u'_2 y'_2 = b(x) \qquad \ldots(10)$$

which; is the **second auxiliary condition.**

Now if we can find u_1 and y_2 satisfying Eqns. (16) and (10), viz.,

$$\left.\begin{aligned} y_1 u'_1 + y_2 u'_2 &= 0 \\ y'_1 u'_1 + y'_2 u'_2 &= b(x) \end{aligned}\right] \qquad \ldots(11)$$

then y_p given by Eqn. (4) will be a particular solution of Eqn. (1). In order to determined u_1, u_2 we first solve the linear system of Eqns (11) for u_1' and u_2'.
Algebraic manipulations yield

$$u_1'(x) = \frac{-b(x)y_2(x)}{W(y_1, y_2)},\ u_2'(x) = \frac{b(x)\ y_1(x)}{W(y_1, y_2)}, \qquad \ldots(12)$$

where

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\ y_2' - y_2\ y_1'$$

is the **Wronskian** of $y_1(x)$ and y_2 (x).

Note that this Wronskian is never zero on J, because $\{y_1, y_2\}$ is a fundamental solution set.
On integrating $u_1'(x)$ and u_2' given by Eqn. (12), we obtain

$$y_p(x) = \int \frac{-b(x)\ y_2(x)}{W(y_1, y_2)}\, dx,\ u_2(x) = \int \frac{b(x)\ y_1(x)}{W(y_1, y_2)}\, dx \qquad \ldots(13)$$

Hence

$$y_p(x) = y_1(x) \int \frac{-b(x)\ y_2(x)}{W(y_1, y_2)}\, dx + y_2(x) \int \frac{b(x)\ y_1(x)}{W(y_1, y_2)}\, dx \qquad \ldots(14)$$

is a particular integral of Eqn. (1).

We now sum up the various steps involved in determining a particular solution of Eqn. (1).

Step 1: Find a fundamental solution set $\{y_1(x),\ y_2(x)\}$ for the corresponding homogeneous equation.

Step II: Assume the particular integral of Eqn (1) in the form
$y_p(x) = u_1(x)\ y_1(x) + u_2(x)\ y_2(x)$
and determine $u_1(x)$ and $u_2(x)$ by using the formula (13) directly or by first solving the system of Eqns. (11) for ------- and then integrating.

Setp III: Substitute $u_1(x)$ and $u_2(x)$ into the expression for $y_p(x)$ to obtain a particular solution.

We now illustrate these steps with the help of the following examples.

Example 1: Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sec x,\ 0 < x < \frac{\pi}{2}$$

Solution: Step I: The auxiliary equation corresponding to the given equation is
$m^2 + 1 = 0$
$\Rightarrow m = \pm i$
and the two solutions of the reduced equation are
$y_1(x) = \cos x$
and
$y_2(x) = \sin x$.
Hence the complementary function is given by
$Y_c(x) = c_1\cos x + c_2\sin x$.

Step II: To find particular integral, we write

$$y_p(x) = u_1(x) \cos x + u_2(x)\sin x \qquad \ldots(15)$$

$$\therefore \frac{dy_p}{dx} = [-u_1(x)\sin x + u_2(x)\cos x] + \frac{du_1}{dx} \cos x + \frac{du_2}{dx} \sin x$$

Let us take the first auxiliary condition as

$$\frac{du_1}{dx} \cos x + \frac{du_2}{dx} \sin x = 0 \qquad \ldots(16)$$

So that

$$\frac{dy_p}{dx} = - u_1(x)\sin x + u_2(x)\cos x$$

Differentiating the above equation once again, we get

$$\frac{dy_p}{dx} = - u_1(x)\cos x = u_2(x)\sin x - \sin \frac{du_1}{dx} + \cos x \frac{du_2}{dx} \qquad \ldots(17)$$

Since $y_p(x)$ must satisfy the given equation, we substitute in the given equatin the expression for y_p and ----- fom Eqns. (15) and (17), respectively, and obtain

$$- \sin x \frac{du_1}{dx} + \text{xosx} \frac{du_2}{dx} = \sec x \qquad \ldots(18)$$

On solving Eqns. (16) and (18) for $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$, we get

$$\frac{du_1}{dx} = - \tan x,\ \frac{du_2}{dx} = 1,$$

which on integration yields
$u_1(x) = \text{In}(\cos x)$ and $u_2(x) = x$

Step III: Substituting the values of $u_1(x)$ and $u_2(x)$ in Eqn. (15) we obtain a particular solution of the given equation in the form
$y_p(x) = \cos x \, In(\cos x) + x\sin x$
and the general solution is
$y = c_1\cos x + c_2\sin x + x\sin x + \cos x In(\cos x)$

Note that in Eqn. (1) we have taken the coefficients of y" to be 1. if the given equation is of the form $a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)$, then before applying the method it must be put in the form $y'' + p(x)y' + q(x)y = g(x)y = g(x)$ as we have done in the following example.

Example 2: Find the general solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = f(x) \qquad \ldots(19)$$

Solution: Step 1: We first rewrite the given equation in the form

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\frac{dy}{dx} = \frac{f(x)}{(1-x^2)}$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\frac{dy}{dx} = 0 \qquad \ldots(20)$$

This is a first order equation in $\frac{dy}{dx}$. To solve this we put $\frac{dy}{dx}$ = p. Then Eqn. (20) reduces to

$$\frac{dp}{dx} - \frac{1}{x(1-x^2)}p = 0$$

or $$\frac{1}{p}dp = \frac{dx}{x(1-x^2)} \qquad \ldots(21)$$

Now Eq. (21) is in variable separable form and can be expressed as

$$\frac{dp}{p} = \left[\frac{1}{x} + \frac{x}{1-x^2}\right]dx$$

Integrating we get

$$P = \frac{c_1x}{\sqrt{1-x^2}}$$

or $$\frac{dy}{dx} = \frac{c_1x}{\sqrt{1-x^2}} \qquad \ldots(22)$$

Integrating Eqn. (22), once again, we get the solution of Eqn. (20) in the form

$$Y_c(x) = -\sqrt{1-x^2} + c_2 \qquad \ldots(23)$$

where c_1 and c_2 are arbitrary constants.

Step II: For the given differential equation, assume a particular solution in the form

$y_p(x) = u_1(x)\sqrt{1-x^2} + u_2(x)$

$$\therefore \frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}} u_1 + \left[\sqrt{1-x^2}\frac{du_1}{dx} + \frac{du_2}{dx}\right]$$

We choose the first auxiliary condition as

$$\sqrt{1-x^2}\frac{du_1}{dx} + \frac{du_2}{dx} = 0 \qquad \ldots(24)$$

Then

$$\frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}} u_1$$

and $\dfrac{d^2y_p}{dx^2} = -\dfrac{1}{(1-x^2)^{3/2}} u_1 - \dfrac{1}{\sqrt{1-x^2}}\dfrac{du_1}{dx}$

Substituting, from above, the expressions for y_p' and y_p'' in Eqn. (19), we get

$$-x\sqrt{1-x^2}\frac{du_1}{dx} = f(x) \qquad \ldots(25)$$

as our second auxiliary condition.

Solving Eqns. 924) and (25) for u_1' and u_2' and integrating, we get

$U_1(x) = -\int \dfrac{f(x)}{x\sqrt{1-x^2}} dx$ and $u_2(x) = \int \dfrac{f(x)}{x} dx$

Step III: The expressions for $u_1(x)$ and $u_2(x)$ when substituted in $y_p(x)$ gives a particular integral in the form

$$Y_p(x) = -\sqrt{1-x^2}\int \frac{f(x)}{x\sqrt{1-x^2}} dx + \int \frac{f(x)}{x} dx$$

Hence a general integral o fthe given differential equation is

$$Y = -c_1\sqrt{1-x^2} + c_2 - \sqrt{1-x^2}\int \frac{f(x)}{x\sqrt{1-x^2}} dx + \int \frac{f(x)}{x} dx$$

You may now try the following exercises.

If you have carefully gone through Example 1 and 2 above, and also attempted 1. And 2., you will find that the results of second order non-homogeneous linear differential equations can be put in the form of the following theorem.

Theorem 1: If the functions $a_0(x)$, $a_1(x)$, $a_2(x)$ and b(x) are continuous on some interval J and if y_1 and y_2 are the linearly independent solutions of the homogeneous equations associated with the differential equation

$$A_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = b(x), \qquad \ldots(26)$$

then a particular solution of Eqn. (26) is given by

$y_p(x) = -y_1(x) \int \frac{y_2(x)\ b(x)}{a_0(x)\ W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x)\ b(x)}{a_0(x)\ W(y_1, y_2)} dx$ ...(27)

where $w(y_1, y_2)$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

Remark: In using the method of variation of parameters for finding a particular integral of a given equation, it is advisable to choose a particular integral $y_p(x) = u_1(x)\ y_1(x) + u_2(x)\ y_2(x)$, and then proceed to find $u_1(x)$ and $u_2(x)$ as we have done in Examples 1 and 2 above. It is usually avoided to memorise formulas given by Eqns. (13) or (27). But since the procedure involved is somewhat long and complicated and moreover, it may not always be easy or even possible to evaluate the integrals involved, these formulas turn out to be useful. In such cases, the formulas for $y_p(x)$ provide a starting point for the numerical evaluation of $y_p(x)$.

The method of variation of parameters which we have discussed for non-homogeneous second order equations can be easily generalized to nth order equations of the form

$$A_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n(x)y = b(x)$$

where $a_0(x)$, $a_1(x)$,……, $a_n(x)$, $b(x)$ are continuous in some interval J. The learner interested into the details of the method for a higher order equation may refer to the Appendix at the end of the Unit. We shall not be giving the details at this stage but, however, illustrate it through an example.

Example 3: Find the general solution of

$$\frac{d^3y}{dx^3} - 6 \frac{d^2y}{dx^2} + 11 \frac{dy}{dx} - 6y = e^{2x}$$

Solution: Step I: The auxiliary equation corresponding to the given equation is
$m^3 – 6m^2 + 11m – 6 = 0$
$\Rightarrow (m – 1) (m^2 – 5m + 6) = 0$
$\Rightarrow (m – 1) (m – 2) (m – 1) = 0$
thus the linearly independent solutions are
$y_1(x) = e^x$, $y_2(x) = e^{2x}$, $y_3(x) = e^{3x}$,
and the complementary function is given by
$y_c(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}$ …(28)

Step II: To find particular integral, we write
$y_p(x) = u_1(x)e^x + u_2(x)e^{2x} + u_3(x)e^{3x}$ …(29)

$\therefore \frac{dy_p}{dx} + (u_1' e^x + u_2' e^{2x} + u_3' e^{3x}) + (u_1 e^x + 2u_2e^{2x} + 3u_3e^{3x})$

Let the **first auxiliary condition** be

$u_1' ex + u_2' e^{2x} + u_3' e^{3x}) = 0$ …(30)

Thus

$y_p' = u_1e^x + 2u_2e^{2x} + 3u_3e^{3x}$

and

$y_p'' = u_1' e^x + 2u_2' e^{2x} + 3u_3' e^{3x}) + (u_1e^x + 4u_2e^{2x} + 9u_3e^{3x})$

Let us choose the **second condition** as

$u_1' e^x + 2u_2' e^{2x} + 3u_3' e^{3x} = 0$ …(31)

Then

$y_p'' = u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}$

$\therefore y_p'' = (u u_1' e^x + 4u_2' e^{2x} + 9u_3' e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$

Substituting the values of y_p, y_p', y_p'' and y_p''' in the given equation, we get

$(u_1' e^x + 4u_2' e^{2x} + 9u_3' e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$
$- 6(u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}) + 11(u_1e^x + 2u_2e^x + 2u_2e^{2x} + 3u_3e^{3x})$
$- 6(u_1e^x + u_2e^{2x} + u_3e^{3x}) = e^{2x}$ …(32)

$\Rightarrow u_1' e^x + 4u_2' e^{2x} + 9u_3' e^{3x} = e^{2x}$,

which is our **third auxiliary condition**

Thus, we get the system of equations

$$\left.\begin{aligned} u_1'e^x + u_2'e^{2x} + u_3'e^{3x} &= 0 \\ u_1'e^x + 2u_2'e^{2x} + 3u_3'e^{3x} &= 0 \\ u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x} &= e^{2x} \end{aligned}\right] \quad \text{…(33)}$$

Solving Eqns. (33) for u_1', u_2' and u_3', we get

$u_1' = \frac{1}{2} e^x$, $u_2' = -1$ and $u_3' = \frac{1}{2} e^{-x}$

Integrating, we get

$u_1 = \frac{1}{2}e^x$, $u_2 = - x$ and $u_3 = \frac{1}{3} e^{-x}$

Step III: We get a particular integral in the form

$y_p(x) = \frac{1}{2} e^{2x} - xe^{2x} - \frac{1}{2} e^{2x} = - xe^{2x}$,

and the general solution is

$y = c_1e^x + c_2e^{2x} + c_3e^{2x} - c_3e^{3x} - xe^{2x,}$

You may now try this exercise.

Clearly the method of variation of parameters has an advantage over the method of undermined coefficients in the sense that it always yields a particular solution y_p provided all the solutions of the corresponding homogeneous equation are known. Moreover, its application is not restricted to particular forms of the non-homogeneous term. In the next section we will discuss a technique which is very similar to the method of variation of parameter.

3.2 Reduction of Order

For a given nth order linear homogeneous differential equation, if one nontrivial solution is known, then the method of reduction of order, as the name suggests, reduces the equation to an (n- 1)th order equation. Thus, if we can find in some way, one or more linearly independent solutions of the reduced equation, we can accordingly reduce the order of the given differential equation. In other words, if p independent solutions of a homogeneous linear corresponding to an nth order equation are known, where $p < n$, then the technique can be used to obtain a linear equation of order (n- p). This fact is particularly interesting when $n = 2$, since the resulting first order equation ca always be solved by the methods we have done in Block 1. That is, if we know one solution of the homogeneous linear differential equation of the second order, we can solve the non-homogeneous equation by the method of reduction of order and obtain both a particular solution and a second linearly independent solution of the homogeneous equation. Let us now see how method works for a second order linear equation.

Consider a second order non-homogeneous equation of the form (1), viz.,

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = b(x),$$

where $a_1(x)$, $a_2(x)$ and $b(x)$ are continuous on some interval J. Suppose that $y = y_1(x)$ is a nontrivial solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \ldots(34)$$

Then $y = cy_1(x)$ is also a solution of Eqn. (34) for some constant c. We now replace the constant c by an unknown function v(x) and take a second trial solution in the form

$y = v(x)y_1(x)$

Now,

$y' = v'y_1 + vy_1'$

$y'' = v''y_1 + 2v'\,y_1' + vy_1''$

substituting from above the expression for y, y' and y'' in the given equation, we get

$(v''y_1 + 2v'y_1' + vy_1'') + a_1(v'y_1 + vy_1') + a_2vy_1 = b(x)$

$$\Rightarrow v''y_1 + v'\,(2y_1' + a_1y_1) + v(y_1'' + a_1y_1' + a_2y_1) = b(x) \qquad \ldots(35)$$

Since y_1 is a solution of Eqn. 934), the last term on the l.h.s. of Eqn. (35) is zero. Therefore Eqn. (35) reduces to

$$v''y_1 + v'(2y_1' + a_1y_1) = b(x) \qquad \ldots(36)$$

Let $\frac{dv}{dx}$ = p(x), so that Eqn. (36) becomes

$$\frac{dp}{dx} + \frac{2y_1' + a_1y_1}{y_1}\,p = \frac{b(x)}{y_1} \qquad \ldots(37)$$

This is a first order linear differential equation with integrating factor

$$\text{I.F.} = \text{EXP}\left[\int \frac{2y_1' + a_1y_1}{y_1}\,dx\right]$$

Now $\int \frac{2y_1' + a_1y_1}{y_1}\,dx = 2\text{In}y_1 + \int a_1(x)\,dx$

$\therefore$ I.F. = $y_1^2\, e^{\int a_1(x)dx} = y_1^2\, h(x)$, where $h(x) = e^{\int a_{--}(x)dx}$

Thus Eqn. (37) reduces to

$$y_1^2\, h(x)p(x) = c_1 + \int b(x)\, y_1h(x)\, dx$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{y_1^2 h(x)}\left[c_1 + \int b(x)\ y_1h(x)\ dx\right]$$

Integrating the above equation once again, we obtain

$$V(x) = c_2 + c_1 \int \frac{1}{y_1^2 h(x)}\,dx + \int \frac{1}{y_1^2 h(x)}\left[\int b(x)\ y_1h(x)\ dx\right]dx$$

Thus the general solution of the given equation can be expressed as

$$y = v(x)y_1(x) = c_2y_1(x) + c_1y_1(x)\int \frac{1}{y_1^2 h(x)}$$

$$+\ y_1(x)\int \frac{1}{y_1^2 h(x)}\left[\int b(x)\ y_1h(x)\ dx\right] dx \qquad \ldots(38)$$

Note that the function $y_1(x) \int \frac{1}{y_1^2 h(x)}$, in the second term on the r.h.s. of Eqn. (38), is the 2nd linearly independent solution of Eqn. (34) and the last term on the r.h.s. is a particular integral of the given non-homogeneous equation.

We now take up an example to illustrate the theory.

Example 4: Find the general solution of

$x^2y'' - xy' + y = x^{1/2}$, $0 < x < \infty$,

Given that $y_1 = x$ is a solution of the corresponding homogeneous equation.

Solution: The given equation is

$x^2y'' – xy' + y = x^{1/2}$ …(39)

Let us take y = xv(x) as a trial solution for Eqn. (39). So that

$y' = v + xv'$

$v'' = 2v' + xv''$

Substituting for y, y' and y'' from above in Eqn. (39), we obtain

$x^2(2v' + xv'') – x(v + xv') + xv = x^{1/2}$

$\Rightarrow x^3v'' + x^2v' = x^{1/2}$

$\Rightarrow v'' + \text{-----}\ v' = x^{-5/2}$ …(40)

Eqn. (40) is a linear differential equation in v'. ts integrating factor is

$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\text{In}x} = x$

Therefore, Eqn. (40) yields

$V'\ x = \int x.x^{-5/2}\ dx + c_1$

$\Rightarrow v' = c_1\ x^{-1} – 2x^{-3/2}$

Integrating once again, we have

$v = c_1\ \text{In}x + 4x^{-1/2} + c_2$

Thus,

$y = xv = c_1x\ \text{In}x + c_2x + 4x^{1/2}$

is the general solution of Eqn. 939).

And now some exercise for you.

From that above it is seen that if one solution of the second order linear homogeneous Eqn. (34) is known, then the second linearly independent solution and a particular integral of the associated non-homogeneous equation can be determined.

We now give some rules, which will help you to find one integral included in the complementary function merely by inspection.

For a homogeneous equation of the form (34) if

Rule I: $1 + a_1(x) + a_2(x) = 0$, then $y = e^x$ is an integral of the Eqn. (34).

For instance, consider an equation

$xy'' – y' + (1 – x)\ y = x^2e^{-x}$ …(41)

To bring it to the form (34), we write it a

$y'' - \frac{1}{x}y' + \left(\frac{1-x}{x}\right) y = xe^{-x}$

Thus, $a_1(x) = - \frac{1}{x}$ and $a_2(x) = \frac{1}{x} - 1$

Now $1 + a_1(x) + a_2(x) = 1 - \frac{1}{x} + \frac{1}{x} - 1 = 0$

Thus, according to Rule 1, $y = e^x$ is an integral of the equation. You can verify your result by substituting $y = e^x$ in the given equation and check if it satisfies the given equation.

Rule II: $a_1(x) + xa_2(x) = 0$ then y = x is an integral of the Eqn. (34)
Consider the equation,
$91 - x^2) y'' + xy' - y = x((1 - x^2)^{3/2}$
This equation can be written as

$$y'' + \frac{x}{1-x^2} y' - \frac{1}{1-x^2} y = x\sqrt{1-x^2}$$

Comparing the above equation with Eqn. (34), we have

$$a_1(x) = \frac{x}{1-x^2} \text{ and } a_2(x) = -\frac{1}{1-x^2}.$$

Here $a_1(x) + xa_2(x) = 0$, hence by the above rule y = x is an integral of the homogeneous equation corresponding to the equation.

Rule III: $1 - a_1(x) + a_2(x) = 0$, then $y = e^{-x}$ is an integral of the Eqn. (34)

Rule IV: $2 + 2xa_1(x) + x^2 a_2(x) = 0$, then $y = x^2$ is an integral of the Eqn. (34).

Rule V: $1 + \frac{a_1(x)}{\alpha} + \frac{a_2(x)}{\alpha^2} = 0$, $\alpha > 0$, then $y = e^{\alpha x}$ is an integral of the Eqn. (34).

Note that in applying Rules I – V the given equation should be first put in the form of Eqn. (34).

You may now try the following exercise.

So far you have seen that the method of variation of parameters can be used only for those differential equation for which we know all the linearly independent solutions of the corresponding homogeneous equation. Method of reduction of order is helpful for finding complete solution of the second order non-homogeneous linear equations even if **one** solution of the corresponding homogeneous equation is known. There exists certain rules which, at once, give **one** solution merely through an inspection, included in the complementary function of the second order linear equations with constant coefficients. But, no rules exist which may help to guess **one or more** integrals included in the complementary function when the equation is or order higher than two and is having variable coefficients. However, there exists a class of linear differential equations with variable coefficients known as Euler's **equations** for which it is possible to find all the linearly independent integrals of the complementary function. In the next section we take up the method of solving Euler's Equations.

3.3 Euler's Equations

Consider the following differential equations

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + 4x + 2y = e^x \qquad \ldots(42)$$

$$x \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2y = e^x \qquad \ldots(43)$$

$$(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1) \frac{dy}{dx} - 2y = \sin x \qquad \ldots(44)$$

All the three equations given above are linear as the dependent variable y and its derivative appear in their first degree and moreover there is no term involving the product of the two. Out of the three equations, only Eqn. (42) is such that the **powers of x in the coefficients are equal to the orders of the derivatives associated with them.** This type of equation known as **homogeneous linear differential equation or Euler's Equation.** Eqn (43) is linear but not homogeneous. Eqn. (44) is not of Euler's from form but can be reduced to Euler's form by the substitution $X = 2x - 1$. Here we shall consider only equations of the form (42) and (44).

The general form of Euler's equation of nth order is

$$x^n \frac{d^ny}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = f(x), \qquad \ldots(45)$$

where $P_1, P_2, \ldots, P_n$ are constants and right hand side is a constant or a function of x alone.

Eqn. (45) can be transformed to an equation with constant coefficients by changing the independent variable through the transformation

$z = \text{In}x$ or $x = e^z$

with this substitution, we have

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = D_1 y, \text{ where } D_1 = \frac{d}{dz}$$

$$\text{Also } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dx^2}\frac{dz}{dx}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (D_1^2 - D_1)\, y = D_1 (D_1 - 1)\, y$$

Proceeding as above, we shall, in general, get

$$x^n \frac{d^n y}{dx^n} = D_1 (D_1 - 1)(D_1 - 2) \ldots (D_1 - \overline{n-1})\ y$$

Thus, Eqn (45) is transformed to the equation

$$[D_1 (D_1 - 1) - (D_1 - \overline{n-1}) + P_1 D_1(D_1 - 1) \ldots (D_1\ \overline{n-2}) + \ldots$$
$$\ldots + P_{n-2} D_1 (D_1 - 1) + P_{n-1} D_1 + P_n]\ y = f(e^z) \qquad \ldots(46)$$

Eqn. (46) is an equation with constant coefficients and its complementary function can be determined by the methods given in Unit 5. For obtaining its particular integral either the method of undetermined coefficients (as given in Unit 6 subject to the form of $f(e^z)$), or the method of variation of parameters can be utilized if the solution of Eqn. (46) is

$y = g(z)$,

then the solution of Eqn. (45) will be

$y = g(\text{In}x)$

we illustrate this method by the following examples

Example 5: Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \text{In}x$

Solution: It is Euler's equation of order 2. To solve it, let

$x = e^z$ or $z = \text{In}x$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = D_1 y, \text{ where } D_1 = \frac{d}{dz}$$

$$\therefore \frac{d^2y}{dx^2}\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = D_1 (D_1 - 1)\ y$$

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$[D_1 (D_1 - 1) - D_1 + 1]\ y = z$

$\Rightarrow [\text{---} - 2 D_1 + 1]\ y = z \qquad \ldots(47)$

A.E. is

$m^2 - 2m + 1\ \ 0$

$\Rightarrow m = 1, 1$

$\therefore$ C.F. $= (c_1 + cz)\ e^z$

To find P.I. of Eqn. (47), let us assume that

$y_p = u_1(z)\ e^z + u_2(z)\ ze^z \qquad \ldots(48)$

$$\therefore \frac{dy_p}{dz} = u_1' e^z + u_2' ze^z + u_1 e^z + u_2 (ze^z + e^z)$$

As first auxiliary condition, assume that

$u_1' e^z + u_2' ze^z = 0, \qquad \ldots(49)$

so that

$$\frac{dy_p}{dz} = u_1e^z + u_2(z + 1)\, e^z \qquad \ldots(50)$$

Differentiating once again, we have

$$\frac{dy_p}{dz} = u_1^{'} e^z + u_2^{'} (z + 1)\, e^z + u_1e^z + u_2e^z\, (z + 1) + u_2e^z \qquad \ldots(51)$$

If $y_p(z)$ is a solution of Eqn. (47), it must satisfy it. Hence substituting the expression for y_p, and --- and --- from Eqn. (48), (50) and (51), respectively, in Eqn. (47), we obtain the second auxiliary condition as

$$u_1^{'} e^z + (z + 1)\, e^z = z \qquad \ldots(52)$$

Solving Eqn. (49) and (52) for-------, we get

$u_2^{'} e^z = z$ and $e^z\ u_1^{'} = - z^2$

$\Rightarrow\ u_1^{'} = - z^2\, e^{-z}$ and --- $= ze^{-z}$

Integrating the above equation, we get

$u_1 = - \int z^2\, e^{-z}\, dz$

$= - \left[z^2 \frac{e^{-z}}{-1} + 2\int z\ e^{-z}dz \right]$

$= + z^2\, e^{-z} - 2 \left[z^2 \frac{e^{-z}}{-1} + \int e^{-z}dz \right]$

$= z^2\, e^{-2} + 2z\, e^{-z} + 2e^{-z}$

and $u_2 = \int z\, e^{-z}\, dz = - ze^{-z} + \int e^{-z}\, dz = - z\, e^{-z}\ \ e^{-z}$

Substituting the values of $u_1(z)$ and $u_2(z)$ in Eqn. (48), a particular integral of Eqn. (47) can be expressed in the form

$y_p(z) = (z^2 + 2z + 2)\, e^{-z}.\, e^z + (- z – 1)\, e^{-z}\, ze^z$

$= (z^2 + 2z + 2) – z(z + 1)$

$= z^2 + 2z + 2 – z^2 – z$

$= z + 2$

and the general solution of Eqn. (47) is

$y = (c_1 + c_2z)\, e^z + z\ z + 2$

Replacing z by In x, the general solution f the given equation is

$y = (c_1 + c_2\text{Inx}).\ x + \text{Inx} + 2$

The complementary function of Euler's Eqn. (45) can also be found by assuming $y = x^m$ in the homogeneous part of the equation and then finding the values of m. we illustrate it through the following example.

Example 6: Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} - 6x \frac{dy}{dx} + 18y = 0$

Solution: Let $y = x^m$

$\therefore \frac{dy}{dx} = m\, x^{m-1}$

$\frac{d^2y}{dx^2} = m\,(m-1)\,(m-2)\, x^{m-3}$

Substituting the above values in the given equation, we get

$[m\,(m-1)\,(m-2) \ldots m(m-1) - 6m + 18]\, x^m = 0$

$\Rightarrow (m^3 - 4m^2 - 3m + 18)\, x^m = 0$

thus, if

$m^3 - 4m^2 = 3m + 18 = 0,$ …(53)

then $y = x^m$ satisfies the given equation.

Equ. (53) is an algebraic equation of 3rd degree in m and its root are

M = - 2, 3, 3.

Thus, $y = x^{-2}$, $y = x^3$ and $y = x^3$ are the solutions of the given equation. hence the general solution of the given equation is

$y = c_1x^{-2} + x^3(c_2 + c_3\,(Inx))$

Note: Had all the roots of Eqn. (53) been real and different, the solutions corresponding to these roots would have been independent solutions and the general solution would have been pf the form

$y = c_1\, x^{m_1} + c_2\, x^{m_2} + c_3\, x^{m_3}$

In the case of repeadted real roots of Eqn. (53), if a root m_1 is repeated r times, the integral corresponding to root m_1 is

$[c_1 + c2Inx + c3\,(Inx)2 + \ldots + cr\,(Inx)^{r-}1]\ x^{m_1}$

Further, if Eqn. (53) had a pair of complex roots, say $\alpha \pm i\beta$, then the corresponding part of the complementary function would have been

$x^{\alpha}\ [c_1 \cos(\beta\, Inx) + c_2 \sin(\beta\, Inx)]$

we illustrate the case of complex roots by the following example.

Example 7: Solve $x^2y'' + xy' + 4y = 0$

Solution: Substituting $y = x^m$ in the given equation, we get

$[m(m-1) + m + 4]\, x^m = 0$

Thus $y = x^m$ satisfies the given equation if

$m(m-1) + m + 4 = 0$

$\Rightarrow m^2 + 4 = 0$

$\Rightarrow m = \pm\, 2i$

Hence, the general solution of the given equation is

$y = c_1 \cos(2\text{In}x) + c_2 \sin(2\text{In}x)$

You may now try the following exercises.

Earlier we mentioned that Eqn. (44) is not Euler's equation, but can be reduced to Euler's form by the substitution X = 2x – 1. We now consider such equations which are reducible to Euler's form.

Equations Reducible to Euler's form

Consider the general nth order equation

$$(ax + b)^n \frac{d^n y}{dx^n} + (ax + b)^{n-1} P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + (ax + b) P_{n-1} \frac{dy}{dx} + P_n y = f(x), \qquad \ldots(54)$$

where a, b, $P_1, \ldots, P_n$ are all constants.

Equations of the form (54) can be reduced to Euler's equations by substituting X = ax + b.

With this substitution

$$\frac{dy}{dx} = a\frac{dy}{dX}, \frac{d^2y}{dx^2} = a^2 \frac{d^2y}{dx^2}, \ldots, \frac{d^n y}{dx^n} = a^n \frac{d^n y}{dX^n}$$

and Eqn. (54) reduces to the equation,

$$a^n X^n \frac{d^n y}{dx^n} + a^{n-1} X^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + aX P_{n-1} \frac{dy}{dx} + P_n y = g(X), \qquad \ldots(55)$$

where g is transformed form of the function f.

Eqn. (55) is now in Euler's form and can solved by the methods given earlier.

However, Eqn. (54) can be directly reduced to an equation with constant coefficients by substituting $ax + b = e^z$, instead of first substituting ax + b = X and then $X = e^z$.

We illustrate the above theory with the help of following example.

Example 8: Solve $(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solution: The given equation is an equation reducible to Euler's equation. We can, however, reduce it to an equation with constant coefficients by a single substitution.

$3x + 2 = e^z$ or $z = \text{In}(3x + 2)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{3x+2} . 3\frac{dy}{dz} \Rightarrow (3x + 2)\frac{dy}{dx} = 3\frac{dy}{dz}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3}{3x+2}\frac{dy}{dz}\right] = \frac{-3^2}{(3x+2)^2}\ \frac{dy}{dx} + \frac{3}{3x+2}\ \frac{d^2y}{dz^2}\ \frac{dz}{dx}$

$$= \frac{3^2}{(3x+2)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

Substituting y' and y'' from above in the given equation, we get

$$9\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right] + 3.3\ \frac{dy}{dz} - 36y = \frac{1}{3}\ [e^{2z} - 1]$$

$$\Rightarrow \frac{d^2y}{dz^2} - 4y = \frac{1}{27}\ (e^{2z} - 1) \qquad \ldots(56)$$

A.E is

$m^2 - 4 = 0 \Rightarrow m = \pm\ 2$

Hence C.F. = $y_c = c_1e^{2z} + c_2e^{-2z}$

To find a particular integral, we write

$$y_p(z) = u_1(z)\ e^{2z} + u_2(z)\ e^{-2z} \qquad \ldots(57)$$

$$\therefore \frac{dy_p}{dz} = u_1' e^{2z} + u_2'\ e^{-2z} + 2\ (u_1e^{2z} - u_2e^{-2z})$$

As the first auxiliary condition, let

$$u_1'\ e^{2z} + u_2'\ e^{-2z} = 0 \qquad \ldots(58)$$

so that

$$\frac{dy_p}{dz} = 2\ (u_1e^{2z} - u_2e^{-2z}) \qquad \ldots(59)$$

Differentiating Eqn. (59) once again, we get

$$\frac{dy_p}{dz} = 2(u_1' e^{2z} - u_2' e^{-2z}) + 4u_1e^{-2z} + 4u_2e^{-2z} \qquad \ldots(60)$$

Since $y_p(z)$ must satisfy Eqn. (56), hence on combining Eqn. (57), (59) and (60) we get the second auxiliary condition as

$$2(u_1'\ e^{2z} - u_2'\ e^{-2z}) = \frac{1}{27}\ (e^{2z} - 1)$$

$$\Rightarrow u_1'\ e^{2z} - u_2'\ e^{-2z} = \frac{1}{54}\ (e^{2z} - 1) \qquad \ldots(61)$$

Solving Eqns. (58) and (61) for u_1' and u_2', we get

$$u_1' = \frac{1}{108}\ (1 - e^{-2z}) \text{ and } u_2' = -\frac{1}{108}\ (1 - e^{2z})$$

Integrating u_1' and u_2', we get

$u_1 = \frac{1}{108}\left(z + \frac{e^{-2z}}{2}\right) e^{2z}$ - $\frac{1}{108}\left(z - \frac{e^{-2z}}{2}\right)$

on substituting the values of $u_1(z)$ and $u_2(z)$ in relation (57), a particular solution of Eqn. (56) is obtained in the form.

$y_p = \frac{1}{108}\left(z + \frac{e^{-2z}}{2}\right) e^{2z}$ - $\frac{1}{108}\left(z - \frac{e^{-2z}}{2}\right) e^{-2z}$

$= \frac{1}{108}\ z\ (e^{2z} - e^{-2z}) + \frac{1}{108}$

$\therefore$ The general solution of Eqn. (56) is

$y = c_1e^{2z} + c_2e^{-2z} + \frac{1}{108}\ z\ [e^{2z} - e^{-2z}] + \frac{1}{108}$

and the required solution of the given equation is

$y = c_{1\ (3x+2)}{}^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108}\ \text{In}(3x+2)\left[(3x+2)^2 - \frac{1}{(3x+2)^2}\right] + \frac{1}{108}$

You may not try the following exercises.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have studied the details concerning the following results:

1) Let y_1 and y_2 be the linearly independent solutions of the reduced equation of a non-homogeneous second order linear differential equation with constant or variable coefficients. Then on substituting $y = y_1u_1(x) + y_2u_2(x)$ and imposing the conditions (11), the particular integral of the given equation can be found.

2) If $y = y_1(x)$ is one solution of the reduced equation, then on substituting
$y = y_1(x)\ v(x)$
the second solution of the reduced equation and a particular integral of the corresponding non-homogeneous equation can be determined.

3) Rules for finding one integral included ion the complementary function of equations of the form (34) by mere inspection are given by the following table:

Condition satisfied	One integral
$1 + a_1(x) + a_2(x) = 0$	$y = e^x$
$1 - a_1(x) + a_2(x) = 0$	$y = e^{-x}$
$a_1(x) + xa_2(x) = 0$	$y = x$
$2 + 2x\ a_1(x) + x^2\ a_2(x) = 0$	$y = x^2$
$1 + \frac{a_1(x)}{\alpha} + \frac{a_2(x)}{\alpha^2} = 0,\ \alpha > 0$	$y = e^{\alpha x}$

4) Differential equation with variable coefficient of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1} x \frac{dy}{dx} + P_n y = f(x),$$

where P_1, P_2,…, P_n are constants and in which the powers of x in the coefficients are equal to the orders of the derivatives associated with them, is known as Euler's equation. This equation can be reduced to an equation with constant coefficients by using the substitution $x = e^z$.

6.0 TUTOR MARKED ASSIGNMENT

1. Determine a particular integral, using the method of variation of parameter for the following differential equations:

   a) $y'' + y = \text{cosec } x, 0 < x < \frac{\pi}{2}$
   b) $y'' - 2y' + y \; xe^x \text{ Inx}, x > 0$
   c) $y'' + y = \tan x, 0 < x < \frac{\pi}{x}$

2. Find a general solution of the following differential equations, given that the functions $y_1(x)$ and $y_2(x)$ for $x > 0$ are linearly independent solutions of the corresponding homogeneous equations.

   a) $x^2y'' - 2xy' + 2y = x + 1; y_1(x) = x, y_2(x) = x^2$
   b) $x^2y'' + xy' - y = x^2e^x; y_1(x) = x, y_2(x) = \frac{1}{x}$
   c) $xy'' - (x + 1)y' + y = x^2; y_1(x) = e^x, y_2(x) = x + 1$

3. Using the method of variation of parameters, find the general solution of the following equations:

   a) $y'' - y' = x^2$
   b) $y'' - 2y'' - y' + 2y = e^{3x}$

4. Solve the following differential equations:

   a) $x^2y'' - 2xy' + 2y = 4x^2, x > 0; y_1(x) = x$
   b) $x^2y'' + 5xy' - 5y = x^{-1/2}, x > 0; y_1(x) = x$

5. A solution of the differential equati0n

$$X^2 (1 - x^2) \frac{d^2y}{dx^2} - x^3 \frac{dy}{dx} - 2y = 0$$

Is $y_1 = \frac{\sqrt{1-x^2}}{x}$. Use the method of reduction of order to find a general solution.

6. Solve equation

$$X(x\cos x - 2\sin x)\frac{d^2y}{dx^2} + (x^2 + 2)\sin x\frac{dy}{dx} - 2(x\sin x + \cos x)y = 0$$

Given that $y = x^2$ is a solution.

7. Verify that $y_1(x) = e^x$ is a solution of the homogeneous equation corresponding to Eqn. (41).

8. Find an integral included in the complementary function of the following equations, merely by inspection:

a) $y'' - \cot x\, y' - (1 - \cot x)\, y = e^x \sin x$
b) $(x \sin x + \cos x)\, y'' + x(\cos x)\, y' - y\cos x = x$
c) $(3 - x)\, y'' - (9 - 4x)\, y' + (6 - 3x)\, y = 0$

9. Solve the following equations:

a) $(x^2D^2 + 3xD)\, y = \frac{1}{x}$
b) $(x^2D^2 + xD - 1)\, y = x^m$
c) $\left(D^3 - \frac{4}{x}D^2 + \frac{5}{x^2}D - \frac{2}{x^3}\right) y = 1$

10. Solve the following equations.

a) $[(x + a)^2 D^2 - 4(x + a)D + 6]\, y = x$
b) $[(1 + x)^2 D^2 + (1 + x)D + 1]\, y = 4\cos[\ln(x + 1)]$.

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